Preface

The study of the motions of our oceans and atmosphere has a long and honourable history, driven by our need to understand and predict the properties of these most elemental components of our planet. Indeed, much that we experience, on a daily basis, is dominated by the weather, which is a fundamental characteristic of the atmosphere and which, in turn, provides a significant control over the oceans. The earliest studies (of both the ocean and the atmosphere) can be traced back at least to Aristotle (384–322 BC) and Strabo (63 BC–23 AD), although in-depth attempts to collect and use information about the oceans (and prevailing winds) were probably not undertaken until the 15th century by Portuguese sailors. The first truly world oceanographic expedition was by the Danish ship *Grønland* (to Arabia in 1761–1767), although this was little concerned with currents, waves or the generation of storms. On the other hand, Benjamin Franklin (1706–1790) measured water temperatures on his trips across the Atlantic, and James Cook (1728–1779) collected information about the currents in the Pacific. Significant explorations continued into the 19th century (Ross, FitzRoy and many others), but it was from the end of the 19th century/beginning of the 20th that competent theoretical treatments started to appear – and this is our interest here.

The birth of modern oceanography, it can be argued, was the Challenger Expedition of 1872–1876, funded by the British Government following pressure from the Royal Society. A vast amount of data was collected from around the world and, on the back of this, one of the prime movers (Sir John Murray, 1841–1914) founded the Edinburgh Marine Laboratory in 1883 – he even coined the term 'oceanography'. The observations noted on this expedition, and on many others by different nations over the next 20 or 30 years, led to a blossoming of theoretical ideas. Pervading most of the work was the development of the correct representation of rotating systems (due to Gaspard-Gustave Coriolis, 1792–1848) applied to the description of a fluid (although George Hadley, 1685–1768, predated this work by recognising the connection between the Earth's rotation and atmospheric circulation). Of the many researchers who have contributed to theories of oceanic movements, the most familiar are probably: V. W. Ekman (1874–1954), C.-G. A. Rossby (1898–1957), H. M. Stommel (1920–1992) and H. Sverdrup (1888–1957).

When we turn to the atmospheric sciences, following the period of classical antiquity, the first attempts to understand (in the modern scientific sense) large-scale weather systems were made by Edmond Halley (1656–1742), of comet fame, and George Hadley. These two scientists introduced us, for example, to the rôles of solar heating and Coriolis effects. These ideas were extended by William Ferrel (1817–1891) and, later, by Vilhelm Bjerknes (1862–1951) and his Norwegian group, and also by Gilbert Walker (1968–1958) who developed a model for air flow in the tropics – the Walker cell. Further, the work of Ekman and Rossby naturally carry over to the atmosphere (indeed, Rossby was arguably the first to use the ideas of classical fluid dynamics to explain large-scale atmospheric flows); contributions were also made by D. Brunt (1886–1965) and V. Väisälä (1889–1969), and many others including Lord Kelvin, William Thomson (1824–1907), remembered for numerous contributions to the dynamics of fluids.

In this brief overview of the historical background (which might be of interest to non-specialists in the field), we have emphasised the dynamical aspects of these systems, but there is much more. Studies in oceanography naturally include: marine life, chemical composition of the seas (e.g. salinity), properties (and mapping) of the ocean depths, and so on. Correspondingly, atmospheric sciences include: the composition of our atmosphere, the generation of clouds and of precipitation, the mechanics of internal heat transfer (latent heat, etc.), the generation and effects of electrical storms, to mention just a few. However, most of the above is predicated on an understanding of the underlying dynamics of the ocean and of the atmosphere. If we can provide a firm foundation for this then, we argue, we may proceed with confidence to include other physical attributes. With this general background in mind, let us turn to a rationale and plan for this text.

Standard theories for the motions of the ocean and the atmosphere are based, almost exclusively, on the direct application of physical principles, often tailored-made for the phenomenon under discussion: the modelling process. This approach has produced many important – indeed, often ground-breaking – descriptions of specific flow patterns and flow structures. However, these methods tend to oversimplify the problem by invoking unnecessary assumptions – an obvious example being the use of f- or β-plane geometry, which ignores important geometrical properties of the flow. Furthermore, modelling tends to hide the assumptions needed, and is also very vague about the errors involved. At the most basic level, the dynamical problems associated with the ocean and the atmosphere are examples of flows governed by the general equations of fluid dynamics (which have been known for almost 200 years). However, obvious though it may seem, very little has been done that follows the classical route taken by applied mathematicians with a background in fluid mechanics; this is the approach that we advocate for these problems. This has, we believe, two very significant advantages: (i) it lays a solid foundation for theories and associated predictions (which, in turn, provides the groundwork for numerical solutions), which should be of interest to those already working in these fields; (ii) it shows mathematicians, not conversant with these studies, what is involved – and thereby, perhaps, excites their interest and encourages them to delve more deeply.

This text aims to present a careful mathematical treatment (in the applied sense) of the dynamical aspects of motion in the ocean and in the atmosphere (but we will not pursue the problem of the interaction between them, and so we do not discuss surface waves, important though they are). Because of the vastly greater number of different types of flow in the atmosphere, and the additional complication of perturbing a background state, the emphasis here is on the atmosphere rather than the ocean. Furthermore, this is intended to be a mathematical introduction to this material: many elements that are important in a comprehensive discussion are set aside because it is essential that we first have a clear understanding of the dynamics before moving on. Our view is that, once we have a robust and detailed dynamical theory, other ingredients (chemical composition, cloud formation, topography and so on) can be added. Also, because we want to provide a new perspective for the experts, and an accessible basis for the interested mathematician, we limit our discussion to the careful presentation of the underlying formulation, linked to some important applications. It must be emphasised, therefore, that we are not providing an introduction to oceanography or to the atmospheric sciences – there are many very good texts that describe the relevant background and associated physics. At its most fundamental level, we will show how the general equations of fluid mechanics (e.g. the Navier–Stokes equation) can be manipulated, invoking minimal assumptions, to produce a form relevant for a description of the ocean and of the atmosphere. With all this in mind, the plan for the text is as follows.

After a brief introduction (Chapter 1) to the classical equations of fluid dynamics – detailed derivations are not appropriate for a text such as this – the rotating-frame version is developed, together with a suitable correction to accommodate a description of the Earth's geoid. The next stage (Chapter 2) involves the introduction of a suitable non-dimensionalisation (which can vary somewhat, depending on the nature of the fluid and the class of problems being considered). This leads directly to the identification of various parameters and, most significantly for us, the thin-shell parameter that is relevant to both the oceans and the atmosphere on our planet. We will show that, assuming the ocean and the atmosphere constitute thin shells (as they do) on a (nearly) spherical Earth, no other assumptions are required in order to proceed. This leads to manageable versions of the governing equations, retaining all other physical attributes, and with well-defined error terms. In Chapter 3 we describe a number of oceanic flows (most of which are steady), based on our generic thin-shell equations, some of which recover classical results, but now based on a systematic approach; these results also include a few important generalisations, not accessible by the familiar route. Then we turn to our main interest here, mainly because of its complexity and the many and varied applications: the atmosphere. We start with the steady atmosphere which is discussed in Chapter 4, which includes a detailed mathematical description of the background state of the atmosphere (within our fluid model), and many examples of the moving atmosphere, in the absence of waves. This discussion is predominantly about the troposphere, but the higher regions – stratosphere and mesosphere – are mentioned, showing how our approach can also apply here. We conclude with the unsteady version of our thin-shell equations (Chapter 5), on suitable time scales, which is used to present an extensive selection of wave-like phenomena that occur in our atmosphere. We aim, throughout the presentation of all these various applications, to emphasise the underlying mathematical structure, showing how analyses of this type lay the foundations for more detailed studies.

The majority of the new ideas developed here are the culmination of a sustained period of investigation undertaken by the author and Prof. Adrian Constantin. Some of the material describing oceanic flows formed part of a course given by the author at ESI, October–December 2016. This course also included an introduction to the ideas of mathematical fluid dynamics, the theory of water waves and the techniques that underpin the use of asymptotic expansions.

The various calculations, and the production of graphical solutions, were performed using Maple.