Chapter 1

Introduction

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This is a detailed overview of the content of all the chapters of this book. At the same time, it is a survey of the mathematical world of Norbert A'Campo.

The present volume consists of a collection of essays dedicated to Norbert A'Campo on the occasion of his 80th birthday. The topics discussed include hyperbolic and super hyperbolic geometry, 3-manifolds, metric geometry, mapping class groups, linear groups, Riemann surfaces, Teichmüller spaces, high-dimensional complex geometry, differential topology, symplectic geometry, singularity theory, number theory, algebraic geometry, dynamics, mathematical physics and philosophy of mathematics. These topics are very diverse, but they are all part of Norbert's interests, and the whole set is a sign of the broadness of his mind.

I often say, with Norbert, that as mathematicians, we have the chance of choosing the topics on which we work, and, perhaps more importantly, the people with whom we work. Most of the chapters that constitute this book are written by friends of Norbert or friends of mine, and several among them are common friends. Independently of this fact, I am pleased that this collective volume turns out to be a glimpse into a good number of interesting geometrical topics, old and new.

I will now give a rather detailed overview of these chapters. I have tried to organise them in sections, but this was not easy to realize. There is however a certain logic in the order I chose.

Chapter 2, which immediately follows this introduction, is a *Vita* of Norbert, in the form of recollections of facts I learned on him, during conversations we had, spread over a long period of friendship. About this friendship, only one thing I want to say here: it was constantly at the same level, there were never ups and downs.

The next three chapters contain personal recollections and thoughts on mathematics, by three mathematicians.

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Chapter 3, by Dennis Sullivan, is titled *Learning about dynamics, Kleinian groups, quasiconformality, and period doubling universality: Orsay Chapter*. In this chapter, Sullivan recounts a series of episodes that were important in his mathematical life and that took place in Orsay or Bures-sur-Yvette in the 1970s, which is the epoch where Norbert was present there. I would like to emphasize, without further comment, the word "learning" in the title. The topics that Dennis touches upon include foliations, dynamics, monodromy of isolated singularities, quasiconformality, circle diffeomorphisms, period doubling, universality, ergodicity, KAM theory, Thurston's theory of surface diffeomorphisms and the Ahlfors–Bers theory of quasiconformal deformations of Kleinian groups. Sullivan also mentions animal vision and works by physicists. This overview, showing only one part of Sullivan's broad interests, is intended to give a taste of the questions that were discussed in Orsay and Bures among him, Norbert and others.

Valentin Poénaru was another colleague of Norbert at Orsay. In Chapter 4, titled *My mathematical world*, he gives a personal account, written in an informal philosophical tone, of his approach to the two problems that are haunting him since his youth: the 3-dimensional Poincaré conjecture and the 4-dimensional Schoenflies problem. At the same time, he reviews some important notions he encountered in his mathematical life and he mentions a few conjectures (which are still open) that he proposed while he was working on the two major problems we mentioned. These notions include simple connectivity at infinity, the QSF (quasi-simply filtered) and GSC (geometric simply connected) properties for groups. About the latter, he writes: "I feel that the ubiquity of GSC gives a certain sense of unity to that mathematical world of mine." Poénaru, throughout this short article, mentions several mathematicians with whom he interacted and who played a significant role in his life.

Chapter 5, by Alexey Sossinsky, is written in French, and is titled *Le Diable, le Bon Dieu et la Sphère de dimension n* (The Devil, the Good Lord and the n-dimensional Sphere). It consists of variations on the theme: *Why is it so hard to see any logic or any beauty in the sequences of integers associated with constructions involving n-dimensional spheres?* Indeed, one may wonder why, for instance, the sphere packing problem is solved in only five cases: dimensions 1, 2, 3, 8, and 24. We recall that this problem is part of Hilbert's Problem VIII (which contains 3 distinct questions); in fact, Hilbert asks for the densest packing of solids of an unspecified shape, but it is usually assumed that it concerns packings of spheres. The problem goes back at least to Kepler. In fact, Sossinsky discusses not exactly this problem, but a related one, viz. the problem of calculating the maximum number of spheres of the same diameter that can be arranged around an n-dimensional sphere (again of the same diameter) in the Euclidean $(n + 1)$-dimensional space. This is the famous "kissing number" in which, as Sossinsky recalls, Isaac Newton and David Gregory were interested towards the end of the 17th century. This sequence of numbers, the kissing numbers in dimension n, is known for its very first terms: n =1, 2, 3, 4, and then only for $n = 8$ and 24. Isn't that awkward? Among the other sequences of integers that arise in geometry with no apparent reason, Sossinsky mentions the set of integers n such that the n-dimensional sphere admits a unique triangulation up to PL-equivalence. Here, the set of integers satisfying the given property includes $n = 1, 2$ and 3, it does not include any $n \geq 5$, and the problem is still open in the case $n = 4$, a case which turns out to be much more difficult than all the others. Any human being with some sense of order would have asked: Why these numbers? Where is the logic behind this? There is also the sequence of integers n such that the n-sphere admits an exotic differentiable structure: It is known that the 7-dimensional sphere admits 28 different differentiable structures, while the 8-dimensional sphere admits only two and the 9-dimensional sphere eight different such structures; we also know that the 12-dimensional sphere admits a unique differentiable structure, that the sphere of dimension 15 has more than 16 000 such structures, and there are a few more other known cases, where some numbers occur for no apparent reason. There is no clear logic in that anarchy. Another problem that puzzles the author is the apparent lack of order in the sequence of homotopy groups of n-spheres. In view of this persistent disorder, Sossinsky addresses the philosophical question of the existence of order and beauty in Creation. This is the main problem addressed in Chapter 5.

When Sossinsky sent me a first version of his paper, in French, I asked him whether he wanted it to be published in French. He said yes, because of the title – reference to a play by Sartre. This play, titled *Le Diable et le Bon Dieu*, is a drama in which the French philosopher puts into action the dilemma between good and evil, which, in fact, he considers to be a false dilemma, because of man's inability to do either one or the other. The main character of the play, in his attempts to do either good or evil, ends up, in both cases, destroying human lives.

I know another reason for which Sossinsky prefers French; this is because this article is also a literary composition and his French is superb, more suitable than his English (which by the way is excellent) for such an exercise.

The mention of n-dimensional spheres also reminds us of Norbert's first important result, namely, his construction of a foliation on the 5-sphere (*Un feuilletage de* S^5, 1971).

Chapter 6, by Bob Penner, titled *Super Hyperbolic Law of Cosines: same formula with different content*, is an immersion into the world of supergeometry, a modern generalization of differential and algebraic geometry which is also the geometry of the Standard Model of high energy physics. The word supersymmetry, which appeared in physic in the 1970s, is only one predecessor of several other words with the prefix "super" (super-gauge transformation, superalgebra, supermanifold, super Minkowski space, super Teichmüller space, super Lie group, etc.). I know that Bob is unhappy with the attribute *super* for such a geometry. He tells me (half seriously and half humorously) that the reason is the perhaps pejorative connotation it gives to the "ordinary" non-super case! For him, supergeometry is just geometry. But the word became standard.

Working in the super Minkowski model, Penner answers a purely geometrical question: "What are the Laws of Cosines and Sines for triangles in the hyperbolic superplane?" It is good to remember here that the trigonometric formulae, in any geometry, constitute the heart of that geometry. This was particularly stressed by Nikolay Lobachevsky who, in his *Pangeometry* and in other writings, deduced all the basic theorems of his new geometry from the trigonometric formulae he first established. This makes me say that the result of this chapter is a fundamental step toward understanding super Minkowski geometry.

The next six chapters are concerned with various aspects of surfaces, their mapping class groups, their complex moduli and their Teichmüller spaces.

Chapter 7, by Hugo Parlier, is titled *The topological types of length bounded multicurves*. In this chapter, the author presents inequalities involving lengths of closed geodesics or systems of disjoint closed geodesics on hyperbolic surfaces. There are two classical such inequalities, namely, an upper bound on systoles (the lengths of a shortest curve on the surface), and the so-called Bers inequality, which is an existence theorem for "short pants decompositions". More precisely, the Bers inequality says that an arbitrary closed hyperbolic surface carries a pair of pants decomposition by closed geodesics whose lengths are bounded above by a certain constant (the "Bers constant") which depends only on the topological type of the surface. The results discussed in Chapter 7 follow this tradition. They include a characterization of the topological types of closed curves and systems of closed curves that are homotopic to a closed geodesics or systems of closed geodesics satisfying certain given length inequalities.

Chapter 8, by Öykü Yurttaş, titled *A recipe for the dilatation of families of pseudo-Anosov braids*, is a survey of the author's work on the computation of the dilatation and the characterization of the invariant measured foliations of each member of a certain family of pseudo-Anosov braids. The methods use what she calls the Dynnikov coordinates. The author notes that the family of braids considered is of interest in the study of braids with low dilatations. Results on the dilatations of this family were previously obtained by E. Hironaka and E. Kin, using the more familiar train track techniques, in their paper *A family of pseudo-Anosov braids with small dilatation* (2006). The author, in Chapter 8, gives an alternative way of computing these dilatations.

Chapter 9, by Marc Burger and Alessandra Iozzi, is titled ℓ^2*-stability and homomorphisms into the mapping class group*. In this chapter the authors formulate a new cohomological vanishing condition, in the bounded cohomology of a group, which they call ℓ^2-*stability* and which implies a superrigidity type result for homomorphisms from that group into a mapping class group. We recall that the term "superrigidity" designates the fact that a certain homomorphism (e.g., a linear representation of a discrete group in an algebraic group) can, under certain circumstances, be enhanced to (for instance) a representation of the algebraic group itself. The first typical situation where the term "superrigidity" was used is the case where the famous Margulis theorem holds. This is in the context of homomorphisms from irreducible lattices in real semisimple Lie groups of rank ≥ 2 into simple Lie groups. The term was extended later to the setting of surface mapping class groups, when a fruitful analogy between algebraic properties of these groups and results on lattices in higher rank groups was exploited. In Chapter 9, Burger and Iozzi give new examples of ℓ^2-stable groups. At the same time they provide a unifying setting for some existing superrigidity results for mapping class groups. Talking about superrigidity, I take this opportunity to mention the paper by Marc Burger and Norbert A'Campo, *Réseaux arithmétiques et commensurateur d'après G. A. Margulis* (1994).

Chapter 10, by Christian Blanchet, is titled *Heisenberg homology of surface configurations via ribbon graphs*. In this chapter, the author reviews the Heisenberg homology of the configuration space of unordered points on an oriented surface with boundary. This is a homology with local coefficients that arises from a representation of the Heisenberg group. In the case of a surface with one boundary component, the topic was introduced and studied in a previous joint paper by Blanchet, Martin Palmer and Awais Shaukat (*Heisenberg homology on surface configurations*, 2021). One interesting feature of this homology is that it carries a twisted action of the mapping class group of the surface.

After a review of the theory of Heisenberg homology for a surface with one boundary component, Blanchet extends it to oriented compact surfaces with an arbitrary positive number of boundary components. He then considers the surface associated with a ribbon graph and shows that its Heisenberg homology can be extracted from this graph, and more generally from what he calls a relative ribbon graph. *Tête-à-tête* twists appear in this theory. These are mapping class group elements that generalize Dehn twists; they are associated with graphs instead of simple closed curves. These objects were introduced by A'Campo as a combinatorial tool for describing mapping classes of surfaces with boundary arising as monodromies of curve singularities.

Chapter 11, by Hiroshige Shiga, is titled *Quasicircles and Dirichlet finite harmonic functions on Riemann surfaces*. Here, a *quasidisc* is the image of the unit disc in the complex plane by a quasiconformal self-mapping of this plane. A quasicircle is the image of the unit circle by such a quasiconformal self-mapping. Quasicircles and quasidiscs admit several characterizations. In particular, there is a classical characterization of quasidiscs involving the existence of a double inequality on Dirichlet finite harmonic functions on a Jordan domain. The author in Chapter 11 is interested in the extendability of such a characterization to quasicircles on general Riemann surfaces, of finite or infinite type. In this setting, a simple closed curve on a Riemann surface is called a quasicircle if one can find an annular neighborhood of this curve and a conformal mapping from this annulus to the complex plane such that the image of the curve by the mapping is a quasicircle in the complex plane. The author notes that his work is motivated by a paper by Schippers and Staubach, *Transmission of harmonic functions through quasicircles on compact Riemann surfaces* (2020). The theory developed in Chapter 11 works for general open Riemann surfaces including surfaces of infinite type.

Chapter 12 by Tadashi Ashikaga and Yukio Matsumoto, titled *Universal degeneration of Riemann surfaces and fibered complex surfaces*, is concerned with the complex structure of the Teichmüller space of a surface. The authors start by recalling a canonical construction by Kodaira of elliptic surfaces with given monodromies and J-invariants (K. Kodaira, *On compact analytic surfaces II*, 1963). Their aim is to generalize this construction to the case of fibered complex surfaces of genus ≥ 2. In doing so, they introduce a new orbifold fiber space, obtained by patching Kuranishi families of stable curves, which has the property that any fibered complex surface can be pulled back from this fibering by a certain orbifold moduli map which they construct. Because of this universal property, the authors call the orbifold fibration they obtain the *universal degenerating family of Riemann surfaces*. Their construction is inspired by a description given by Arbarello and Cornalba in their book *Geometry of Algebraic Curves* (2011) of a bordification of Teichmüller space using real blow-ups and methods of the so-called log geometry developed by Kato–Nakayama and Usui.

The work in Chapter 12 is based on previous work by Matsumoto in which he constructed a new orbifold structure over the Deligne–Mumford compactification of the moduli space of Riemann surfaces, using a certain bordification of Teichmüller space (*The Deligne–Mumford compactification and crystallographic groups*, 2020).

The next two chapters are concerned with 3-manifolds.

Chapter 13, by Ken'ichi Ohshika, is titled *Surface bundles in 3-dimensional topology*. Surface bundles over the circle play a very important role in 3-dimensional geometry and topology. As Ohshika recalls, such manifolds already appear in the work of Poincaré who, in his 1895 *Analysis situs*, studied torus bundles over the circle. By the middle of the 1970s, Robert Riley and Troels Jørgensen gave examples (considered as the first examples) of hyperbolic surface bundles over the circle. Soon later, Thurston showed that in a precise sense most of surface bundles over the circle admit such structures; for instance, mapping tori of pseudo-Anosov homeomorphisms are all hyperbolic. By the end of the 1970s, Thurston made the important conjecture that any closed hyperbolic 3-manifold has a finite-sheeted covering which is fibered over the circle. This conjecture, which became known as the "virtually fibered conjecture", was proved in several steps over a period of a few decades. The final step, which settled the case of all closed hyperbolic 3-manifolds, was achieved by Ian Agol in 2012. An excellent survey of Thurston's work on hyperbolic manifolds fibered over the circle, until 1980, is Sullivan's Bourbaki seminar, *Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur* S^1 (1980).

In Chapter 13, Ohshika gives an overview, with sketches of proofs, of the development of the theory of 3-dimensional surface bundles, starting from Poincaré's work, passing through Stalling's theorem on fibrations over the circle, Thurston's classification of surface mapping classes and their action on Teichmüller space, his double limit theorem, and the existence of hyperbolic structures on mapping tori with pseudo-Anosov monodromy. The chapter ends with an exposition of recent results on volumes of hyperbolic surface bundles. A substantial part of this survey is dedicated to the important work of Thurston.

Chapter 14, by Charalampos Charitos, is titled *The complex of incompressible surfaces of a handlebody*. In this chapter, the author associates with a 3-dimensional handlebody of genus ≥ 2 a simplicial complex called its complex of incompressible surfaces. As the name suggests, the vertices of this complex are the isotopy classes of incompressible surfaces in the handlebody, and for every $k \geq 0$, $k+1$ vertices form a k-simplex if they can be represented by a collection of disjoint incompressible surfaces. Charitos proves that for $g \geq 3$, any automorphism of this complex is induced by a homeomorphism of the handlebody. This rigidity result follows several rigidity results of the same kind for surfaces and their mapping class groups, obtained starting in the 1980s. Typically, in the surface case, one studies simplicial complexes whose k-faces are homotopy classes of $k+1$ homotopically non-trivial and pairwise non-homotopic disjoint simple closed curves on which some property may be imposed (to separate the surface into two connected components, etc.). After several works on surfaces, with simplicial complexes built using such collections of simple closed curves, several complexes appeared in the realm of 3-manifolds, including the complex of essential discs (which, in the case where the 3-manifold is a handlebody, coincides with the complex of meridians), introduced by McCullough (*Virtually geometrically finite mapping class groups of 3-manifolds*, 1991). A rigidity result for this complex was obtained by Korkmaz and Schleimer (*Automorphisms of the disk complex*, 2009). Other rigidity results for complexes associated with surfaces in 3-manifolds were obtained by Charitos–Papadoperakis–Tsapogas (*A complex of incompressible surfaces for handlebodies and the mapping class group*, 2012 and *On the complex of separating meridians in handlebodies*, 2022). The work in Chapter 14 is a sequel to these works.

Chapter 15, by Krishnendu Gongopadhyay, Tejbir Lohan and Chandan Maity, is titled *Reversibility and real adjoint orbits of linear maps*. The authors extend classical results on the classification of the elements of the general linear groups over $\mathbb{R}$ and $\mathbb{C}$ that are reversible (that is, conjugate to their inverse) to the case of the group $\mathrm{GL}(n, \mathbb{H})$. They also provide a new proof of the known classification results for the groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. At the same time, they give a classification of the real adjoint orbits in the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{C})$.

The next four chapters are on knot theory, and the first two of them are concerned with Seifert surfaces, that is, surfaces embedded in the sphere whose boundary is the given knot (or link).

Chapter 16, by Mikami Hirasawa, Ryota Hiura and Makoto Sakuma, is titled *Invariant Seifert surfaces for strongly invertible knots*. The setting is that of strongly invertible knots, that is, smooth knots in the 3-sphere for which there exists a smooth involution of this sphere which preserves the knot and fixes a simple loop intersecting it in two points. A related and older notion is that of (cyclically) periodic knot with period n, that is, a knot in the 3-sphere for which there exists a periodic diffeomorphism of period n of the ambient sphere which leaves the knot invariant and fixes a simple loop in its complement. The importance of periodic knots was realized in the 1960s–1980s, in works of Trotter and Murasugi, then Edmonds and Livingston. Edmonds and Livingston proved that every periodic knot admits an invariant incompressible Seifert surface. In the present chapter, Hirasawa, Hiura and Sakuma study the general question of existence of invariant Seifert surfaces for strongly invertible knots. They prove that for such a knot, the gap between the equivariant genus, that is, the minimum of the genera of invariant Seifert surfaces, and the (usual) genus may be arbitrarily large. This result is in sharp contrast with a result of Edmonds, obtained in 1984, stating that every periodic knot admits an invariant incompressible minimal genus Seifert surface. Edmonds used this result in his proof of Fox's conjecture stating that any nontrivial knot has only finitely many periods. Hirasawa, Hiura and Sakuma, in Chapter 16, obtain variants of Edmonds' theorem which are useful in the study of invariant Seifert surfaces for strongly invertible knots. On the same occasion, they report on the relations between their work and a construction of fibered links in the 3-sphere using divides, that is, immersions of 1-manifolds in the disc, discovered by A'Campo in his study of isolated singularities of complex hypersurfaces and which has been a source of inspiration for their work (cf. A'Campo's *Generic immersions of curves, knots, monodromy and Gordian number*, 1998).

Chapter 17, by Sebastian Baader, Pierre Dehornoy and Livio Liechti, titled *Minor theory for quasipositive surfaces*, is concerned again with Seifert surfaces in link complements. Given a pair of Seifert surfaces Σ_1 and Σ_2 for a knot in $\mathbb{R}^3$, Σ_1 is said to be a minor of Σ_2 if Σ_1 is isotopic to an incompressible subsurface Σ_1' of Σ_2, that is, Σ_1' is contained in Σ_2 such that the complement $\Sigma_2 \setminus \Sigma_1'$ has no disc component. The authors in Chapter 17 note that the word "minor" originates in graph theory, where a minor refers to a graph obtained from a finite graph by a finite number of operations of vertex and edge deletions and edge contractions. Baader, Dehornoy and Liechti, in Chapter 17, study the order relation on Seifert surfaces induced by the property of incompressible inclusion. They consider quasipositive surfaces, that is, Seifert surfaces associated with quasipositive links. Such surfaces originate in the study of complex plane curves, and they are also used in contact geometry. A quasipositive link is the braid closure of a quasipositive braid, that is, a product of conjugates of the standard generators of the Artin braid group. Quasipositivity is related to the fact that the braid closure admits a canonical Seifert surface of minimal genus. The authors show that the set of quasipositive surfaces is closed under the relation of incompressible inclusion. They also prove that the order induced by incompressible inclusion on fiber surfaces of positive braid links containing a fixed root of a full twist is a well-quasi-order, that is, it has the property that every infinite family contains two comparable elements.

Chapter 18 by Rinat Kashaev, titled *The Alexander polynomial as a universal invariant*, is concerned with universal quantum knot invariants. These constitute an algebraic tool that is used for encoding in a representation-independent way the multitude of quantum invariants associated with a given Hopf algebra. Kashaev addresses the question of identifying the universal invariant of long knots in one of the simplest cases of non-trivial Hopf algebras, namely, the case of the commutative complex algebra $\mathsf{B}_1 = \mathbb{C}[a^{\pm 1}, b]$ with its structure of complex Hopf algebra induced from its interpretation as the algebra of regular functions on the affine linear algebraic group of complex invertible upper triangular 2x2 matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$. He proves that the universal invariant of a long knot K associated with B_1 is the reciprocal of the canonically normalized Alexander polynomial $\Delta_K(a)$. The main result of this chapter is then a proof of a conjecture which Kashaev proposed in 2019, which gives a new point of view on the Melvin–Morton–Rozansky conjecture saying that the Alexander–Conway polynomial of a knot can be retrieved from the coefficients of the Jones polynomials of its cables. This conjecture was settled by Bar-Nathan and Garoufalidis in 1996, and later, in an analytic form, by Garoufalidis and Lê, in 2011. It has applications to the so-called Generalized volume conjecture, an important conjecture formulated by Kashaev in 1997 and by Hitoshi and Jun Murakami in 2001, connecting two different approaches to knot theory, namely Topological Quantum Field Theory and hyperbolic geometry.

Chapter 19, by Eva Bayer, is concerned with high-dimensional knots. It is titled *Alexander polynomials and signatures of some high-dimensional knots*. The general question addressed in this chapter is to find the possibilities for an integer to be the signature of a knot with a given Alexander polynomial. This question was already answered by the same author for classical knots, that is, 1-dimensional knots in the 3-sphere. In this chapter, she studies the same question for high-dimensional knots, and more especially, for m-dimensional knots K^m in the sphere S^{m+2} with

$m \equiv -1 \pmod 4$. The case $m = 3$ (that is, 3-dimensional knots in the 5-sphere) requires a special discussion.

Now we pass to dimension four.

Chapter 20, by Anna Beliakova and Marco de Renzi, is titled *Refined Bobtcheva–Messia invariants of 4-dimensional 2-handlebodies*. The authors develop, in the setting of 4-manifolds, an analogue of the theory of quantum invariants of 3-manifolds. More precisely, they deal with 4-dimensional smooth 2-handlebodies, that is, smooth 4-manifolds with boundary obtained from the 4-ball by attaching finitely many 1-handles and 2-handles. They consider these 4-manifolds up to a certain equivalence relation called 2-deformation, or 2-equivalence. They define invariants of pairs (W, ω) where W is such a 4-manifold and ω a relative cohomology class in $H^2(W, \partial W; G)$ and where G is some abelian group. The algebraic input required for this construction is a unimodular ribbon Hopf G-coalgebra. The authors discuss in detail these invariants for the restricted quantum group $U = U_q\mathfrak{sl}_2$ at a root of unity q of even order $2p$, which is a unimodular Hopf $\mathbb{Z}/2\mathbb{Z}$-coalgebra which contains the small quantum group $\bar{U}$ as its degree zero part, and for its braided ribbon extension $\tilde{U} = \tilde{U}_q\mathfrak{sl}_2$, which fits in this setting where $G = \mathbb{Z}/2\mathbb{Z}$. They deduce formulae that generalize a well-known decomposition of the Witten–Reshetikhin–Turaev invariants in terms of spin structures and cohomology classes. The term "Bobtcheva–Messia invariants" in the title of Chapter 20 refers to an invariant of 4-thickenings of 2-dimensional CW Complexes that was introduced by I. Bobtcheva and M. Messia in their article *HKR-type invariants of 4-thickenings of 2-dimensional CW complexes* (2003). The authors work with an extended version of this invariant.

The next two chapters constitute an excursion in the world of differential topology.

Chapter 21, by François Laudenbach, is titled *Conic singularities and immediate transversality*. The author starts by recalling the notion of submanifold with C^1 conic singularities. Such an object appears in the closure of invariant submanifolds of Morse gradients (that is, gradients of Morse functions) under some assumptions on the simplicity of this gradient. He proves a result concerning *immediate transversality by flow*, a notion which has other potential applications and which he introduced in a recent work on A_∞-structures on Morse complexes (H. Abbaspour and F. Laudenbach, *Morse complexes and multiplicative structures*, to appear). Here, given a smooth manifold M with two smooth submanifolds S and Σ, an ambient isotopy ϕ^t on M is said to be of immediate transversality of S with respect to Σ if $\phi^t(S)$ is transverse to Σ for every small enough positive real t. The important case studied in the paper quoted is when $S = \Sigma$. In Chapter 21, this theory is extended, and the author shows that the notion of immediate transversality is useful in the setting of arbitrary compact submanifolds with C^1 conic singularities and not only in that of Morse theory. These details provide, for Laudenbach, an opportunity for reviewing several elements of singularity theory. References to the work of Thom arise naturally in this discussion.

Chapter 22, by Yakov Eliashberg and Dichant Pancholi, titled *Honda–Huang work on contact convexity revisited*, is based on some recent important work by Ko Honda and Yang Huang on contact convexity in high dimensions. In this setting, a hypersurface in a contact manifold is said to be convex if it admits a transverse contact vector field. In their article titled *Convex hypersurfaces in contact topology* (2019), Honda and Huang, generalizing work of Emmanuel Giroux in dimension three, proved that in any manifold with a co-oriented contact structure, an arbitrary co-oriented hypersurface can be C^0-approximated by an isotopic convex surface. In Chapter 22, Eliashberg and Pancholi provide a shorter and more accessible proof of this result, clarifying the original proof at some delicate points.

The terms Milnor fiber, Milnor fibration, etc. refer to a device, introduced by Milnor in the 1960s, in the study of the germ of a complex analytic function f at a critical point x whose image (the critical value) is assumed to be 0, by examining nearby fibers, after normalizing the map f by dividing it by $|f|$. The geometry of such a nearby fiber turns out to be a valuable information on the singularity at x. The Milnor fibration of a singularity of a complex polynomial, and the study of the monodromy of this fibration, have been probably the most useful tools for studying the structure of such a singularity. This construction has several variants which depend on the setting, and it led to important developments in the study of isolated hypersurface singularities and more generally in differential topology and (complex) algebraic geometry. In the next seven chapters, the notion of Milnor fiber and some other related notions play a key role. A'Campo's contribution is visible in this setting.

The first one in this series of chapters, Chapter 23, is written by François Loeser and it is titled *Rambling around the Milnor fiber*. The author's goal in this chapter, as he puts it himself, is "to present the pervading influence of Norbert's works on monodromy and the Milnor fiber in current research, and their interplay with other topics like non-archimedean geometry, finite fields or symplectic geometry." The developments accounted for include the computation of the Lefschetz numbers of monodromies, based on two early works by A'Campo titled *Le nombre de Lefschetz d'une monodromie* (1973) and *La fonction zêta d'une monodromie* (1975). Loeser then discusses versions of the Milnor fiber in the setting of non-Archimedean geometry, and the computation of the monodromy zeta functions for discriminants of finite Coxeter groups. The connection with symplectic geometry stems from the fact that a Milnor fiber is viewed as a symplectic manifold, with its boundary endowed with a contact structure.

In Chapter 24, titled *Singular fibrations over surfaces*, Louis Funar studies smooth maps from compact connected oriented 4-manifold onto compact oriented surfaces with finitely many critical points. Such a map is said to be a *singular fibration* if all its critical points are regular and its singularities cone-like. Funar presents several constructions of singular fibrations, including ones with a unique singularity, and the so-called achiral singular fibrations of the 4-sphere over the 2-sphere, originating in work of Yukio Matsumoto. He establishes classification results for singular fibrations which are similar to those known for achiral Lefschetz fibrations. He shows that relatively minimal singular fibrations are determined by their monodromies. The work presented in this chapter owes a lot to A'Campo's study of isolated singularities of planar curves and his construction of fibered links from divides. At the same time, Funar outlines works of Hirzebruch and Hopf on 2-plane fields with finitely many singularities, making connections between these works and those of Neumann and Rudolph on the Hopf invariant. He uses these results to prove that a closed orientable 4-manifold with large first Betti number and vanishing second Betti number does not admit any singular fibration. He discusses several open problems, and in particular the question of whether any smooth closed simply connected oriented 4-manifold is the total space of a singular fibration over some surface.

Chapter 25 by Masahaku Ishikawa, Yuya Koda and Hironobu Naoe, titled *Presentation of the fundamental groups of complements of shadows*, is a continuation of work started by A'Campo on the relation between divides, the links they generate, and the associated shadowed polyhedra. Here, a shadowed polyhedron is a polyhedron with some extra structure encoded by half integers assigned to some regions called gleams. The polyhedron is embedded in a compact, oriented, smooth 4-manifold as a spine of that manifold. A shadowed polyhedron represents its ambient manifold in some precise sense. The notion of shadow was introduced by Turaev, and has played an essential role in 3- and 4-dimensional topology, and it is also intimately related to the theory of singularities of maps from 3- and 4-dimensional manifolds to surfaces developed by A'Campo. The relation between A'Campo's divides and Turaev's shadows was already highlighted in a 2020 paper by Ishikawa and Naoe, titled *A'Campo's divide and Turaev's shadow*.

In Chapter 25, Ishikawa, Koda and Naoe consider more particularly contractible shadows obtained from the unit disk by attaching annuli along some closed curves generically immersed in this disk. In this context, the underlying 4-manifold is the 4-ball. Milnor fibers of plane curve singularities can be represented in this way. In fact, the union of a Milnor fiber of a plane curve singularity and the disks bounded by its vanishing cycles is a polyhedron embedded in the so-called Milnor ball, a small 4-ball in $\mathbb{C}^2$ centered at the singular point. In this case, the polyhedron becomes the union of the unit disk with a finite number of annuli attached to it along some curves immersed in that disc. The resulting polyhedron is simple and contractible. The main result in Chapter 25 is a presentation of the fundamental group of the complement of a sub-polyhedron of a shadowed polyhedron in its ambient 4-manifold, in the case where the shadow consists of the unit disk and of annuli attached to it along immersed curves, so that the polyhedron is simple and contractible. The authors apply this theory to polyhedra of fibrations of divides, and in particular to polyhedra of Milnor fibrations and to complexified real line arrangements.

Chapter 26, by Sabir Gusein-Zade, is titled *A'Campo type equations and integrals with respect to the Euler characteristic*. In 1975, A'Campo established equations for the Euler characteristic of the Milnor fiber of a germ of a holomorphic function and for its monodromy zeta function at a singular point, in terms of a resolution. (This is A'Campo's paper *La fonction zêta d'une monodromie*.) Equations of this kind are called A'Campo type equations, and they constitute a predecessor of Viro's notion of integral with respect to Euler characteristic, which Viro introduced in 1988. In Chapter 26, Gusein-Zade shows that A'Campo type equations arise from some integrals with respect to the Euler characteristic over infinite-dimensional spaces such as projectivizations of spaces of function germs and spaces of divisors on a singularity. He also shows that in some cases the values of these integrals coincide with the zeta functions of certain monodromy operators. The results obtained in this chapter are instances of situations where analytic invariants (the integrals with respect to the Euler characteristics) coincide with topological ones (zeta functions of monodromies or Alexander polynomials). This chapter is also the occasion of presenting some beautiful mathematics discovered by Norbert.

Chapter 27, by Mutsuo Oka, is titled *Almost non-degenerate functions and a Zariski pair of links*. In this chapter, the author gives a generalization of a formula due to Varchenko for the zeta function of the Milnor fibration of a Newton non-degenerate function. This generalization concerns germs of analytic functions that have some Newton degenerate faces. This work uses in an essential way A'Campo's 1975 paper, *La fonction zêta d'une monodromie*, in which the latter gave a formula for the zeta function of the Milnor monodromy of the germ of an analytic function of n complex variables at a singular point, given a local resolution of the singularity. This paper is probably the most quoted work by A'Campo. As an application, Oka obtains an example of a pair of hypersurfaces with the same Newton boundary and the same zeta function but with different tangent cones.

Walter Neumann and Nathalie Wahl, in a paper published in 2002 and titled *Universal abelian covers of surface singularities*, introduced the class of *splice type surface singularities*, a class which contains all known examples of integral homology spheres that appear as links of isolated complete intersections of dimension two. Such singularities are determined, up to equisingularity, by decorated trees called *splice diagrams*. The *Milnor fiber conjecture*, formulated by the same authors in 2005, says that any choice of an internal edge of a splice diagram determines a special kind of decomposition into pieces of the Milnor fibers of the associated singularities.

In Chapter 28, titled *The Milnor fiber conjecture of Neumann and Wahl and an overview of its proof*, Maria Angelica Cueto, Patrick Popescu-Pampu and Dmitry

Stepanov provide an overview of the conjecture mentioned in the title together with a detailed outline of a proof they obtained of it. The proof uses techniques from toric, tropical and log geometry in the sense of Fontaine and Illusie. The latter geometry is reviewed in detail in this chapter. A central ingredient is the operation of rounding a complex logarithmic space, introduced in 1999 by Kato and Nakayama. This is a functorial generalization of an operation introduced by A'Campo in 1975, in his study of Milnor fibrations, called *real oriented blowup*. In the same chapter, Cueto, Popescu-Pampu and Stepanov show that A'Campo's operation gives canonical representatives of the Milnor fibration over the circle of a smoothing, provided an embedded resolution of this smoothing is given. The outline of the proof of the Milnor fiber conjecture is presented in 8 steps in the introduction to Chapter 28 and in 28 steps in the last section of the same chapter. The detailed proof is announced to appear in one or several papers.

Chapter 29, by Vladimir Fock, is titled *Singularities and clusters*. A correspondence between singularities and cluster varieties was observed recently by Sergey Fomin, Pavlo Pilyavsky, Dylan Thurston, and Eugenii Shustin in their paper titled *Morsifications and mutations* (2022). This correspondence is based on certain real forms of deformations of singularities introduced by A'Campo in his paper *Le groupe de monodromie du déploiement des singularités isolées de courbes planes* (1975) and by Gusein-Zade in his paper *Dynkin diagrams of the singularities of functions of two variables* (1974). Fomin, Pilyavsky, Thurston, and Shustin showed that different resolutions of the same singularity give the same cluster variety. In Chapter 29, Fock describes a geometric relation between simple plane curve singularities, classified by simply laced Cartan matrices, and cluster varieties of finite type, classified by the same matrices. He constructs certain varieties of configurations of flags from Dynkin diagrams and from singularities, and he shows that they coincide if the Dynkin diagram corresponds to the singularity. In particular, the author describes a map from the base of a versal deformation of a singularity to the corresponding cluster variety. The result of this chapter makes Fomin, Pilyavsky, Thurston, and Shustin's correspondence more geometrical and less mysterious.

Chapter 30, by İsmail Özkaraca and Muhammed Uludağ, is concerned with dynamics and measure theory. It is titled *Deformations of Lebesgue's measure on the boundary of the Farey tree*. Based on joint work between the second author and Hakan Ayral, the authors study deformations of the Lebesgue measure on the interval $(0, 1)$. The latter is seen as a measure on the boundary of the Farey tree realized in the usual way in the hyperbolic plane: the vertices of this tree are arranged using an operation on the rationals in $(0, 1)$ where the tree-structure is a result of using an "addition" which assigns to two irreducible rational fractions a fraction whose numerator is the sum of the numerators and whose denominator is the sum of the denominators. The measures obtained on the boundary of the Farey tree appear then as deformations of the Lebesgue measure using two involutions which the authors call K and $\mathfrak{z}$. The authors prove that these new measures are singular with respect to Lebesgue's measure and they compute special values of their cumulative distribution functions. It turns out that these measures possess a subtle symmetry involving an outer automorphism of the group $\mathrm{PGL}(2, \mathbb{Z})$, which was introduced by Joan Dyer in the late 1970s and which induces an involution of the real line that preserves the set of quadratic irrationals, permuting them in a non-trivial way and commuting with the Galois action on them. Dyer's outer automorphism conjugates the Gauss continued fraction map to the so-called Fibonacci map and it has other interesting features. The properties that Özkaraca and Uludağ obtain, together with experimental data they provide, show that these measures constructed on the boundary of the Farey tree have an arithmetic significance.

The next three chapters are on algebraic geometry, with relations with mathematical physics.

Chapter 31, by Noémie Combe, Yuri Manin and Matilde Marcolli, is titled *Birational maps and Nori motives*. The theory of Nori motives, introduced by Modhav Nori, is an approach to the theory of mixed motives. The latter is a conjectural abelian tensor category (whose existence was conjectured by Beilinson) taking values on all varieties, which is related to several conjectures in algebraic geometry. Nori's original writings in this domain consist of a set of unpublished notes of lectures given at Bombay's TIFR and at the University of Chicago. The theory has been later developed from different points of view by several authors including Huber, Ayoub, Kontsevich, Connes, Marcolli, Manin, Levine and others. Nori motives appear in various domains of algebraic geometry, in particular in geometries in characteristic 1, in the theory of persistence formalism, in the study of the absolute Galois group, and in the context of Kontsevich's conjectures on the Grothendieck–Teichmüller group introduced by Drinfeld and Ihara (cf. Kontsevich's *Operads and motives in deformation quantization*, 1999). In the recent monograph *Periods and Nori motives* by A. Huber and S. Müller-Stach (2017), this theory was developed systematically and studied as a universal (co)homology theory of algebraic varieties or schemes in the sense of Grothendieck. In Chapter 31, Combe, Manin and Marcolli present a sketch of an approach to the problems of equivariant birational geometry developed by Kontsevich and Tschinkel, in which the Burnside invariants were introduced, making explicit the role of the Nori constructions in the latter setting.

Chapter 32, by Alexander Varchenko, is titled *Dwork-type congruences and p-adic KZ connection*. The Knizhnik–Zamolodchikov (KZ) equations are differential equations that appear in conformal field theory, representation theory and enumerative geometry. In a previous work (*Arrangements of hyperplanes and Lie algebra homology*, 1991), Varchenko, together with V. Schechtman, showed that the solutions of the KZ equations take the form of multidimensional hypergeometric functions. In this chapter, Varchenko discusses analogues of hypergeometric solutions of these equations in a setting where a p-adic field replaces that of the complex numbers. In doing so, he develops new matrix Dwork-type congruences for Hasse–Witt matrices of KZ equations.

Chapter 33, by Toshitake Kohno, is titled *Temperley–Lieb–Jones category and the space of conformal blocks*. In this chapter, Kohno starts by reviewing the relationship between homological representations of the braid groups, that is, the action of these groups on the homology of abelian coverings of certain configuration spaces, and the monodromy representations of the KZ connection. This leads to a topological method for computing the monodromy of the space of conformal blocks. Using this method, the author shows that there is an isomorphism between the space of conformal blocks and the space of morphisms of the Temperley–Lieb–Jones category which is equivariant under the action of the braid group. As a result, he recovers the unitarity of the braid group action on the space of conformal blocks by means of the positivity of the Markov trace.

Chapter 34, by Sumio Yamada and the present author, is titled *On the timelike Hilbert geometry of a simplicial simplex*. Timelike geometry is a metric geometry developed by Herbert Busemann in his paper *Timelike spaces* (1967). The theory of timelike spaces is a generalization of Riemannian geometry in which the quadratic form defining the metric infinitesimally is not required to be positive definite. Busemann introduced this theory as a metric setting for general relativity. In the axioms of timelike spaces, one starts with a topological space equipped with two basic objects: a distance function, which plays the role of the indefinite metric, and a partial order relation $<$. This order relation corresponds to the causality property of the space-time of relativity theory. (One thinks of the relation $x < y$ as meaning that y is in the future of x.) In a timelike space, the distance between a point and a second one is defined only in the case where the second point is in the future of the first one. Triples of points x, y, z such that $x < y$ and $y < z$ satisfy the reverse triangle inequality (called time inequality): $d(x, z) \geq d(x, y) + d(y, z)$. The motivation comes again from the theory of relativity, where triples of point satisfying the causality relation are subject to the reverse triangle inequality. Busemann's theory of timelike spaces is parallel to the one called "chronogeometry", which was developed at about the same time by A. D. Alexandrov in Russia.

From the purely mathematical viewpoint, it is natural to ask what are the analogues in timelike geometry of the usual notions, properties and results that are known in classical metric geometry. For instance, we know that there are timelike analogues of the classical Funk and Hilbert geometries associated with convex subsets of n-dimensional Euclidean space, of the n-sphere, and of the hyperbolic n-space. As a matter of fact, in the timelike setting, one rather talks about "exterior" Funk and Hilbert geometries, rather than Funk and Hilbert geometries. Yamada and myself have expanded these theories, in the paper *Timelike Hilbert and Funk geometries* (2019). Several natural questions arise in the timelike setting, and among them the existence of analogues of the rigidity results that hold for the classical Funk and Hilbert geometries. For instance, a well-known result (due to Busemann) says that in the classical Hilbert geometry, a convex set equipped with a Hilbert metric is isometric to a finite-dimensional vector space if and only if the underlying convex set is a simplex. In Chapter 34, we study an analogous question in the timelike spherical Hilbert setting, that is, we study the exterior Hilbert geometry of the union of two disjoint antipodal simplices on the sphere. This question of characterizing the exterior Hilbert geometry of the union of two disjoint antipodal spherical simplices arose naturally after we noticed (in the paper mentioned above) that the exterior Hilbert geometry of a union of two disjoint antipodal geometric discs in the sphere is the familiar de Sitter geometry.

The last three chapters of this volume are concerned with philosophy of mathematics. Topics like those that are covered in these three chapters, that constitute the culmination of the book, reflect an important aspect of the discussions I have regularly with Norbert.

Chapter 35 by Victor Pambuccian, is titled *The single intuition of a move of time*. The author discusses Brouwer's claim that mathematics, in its development, needs only the basic intuition of time. In particular, it does not need an intuition of space. More generally, Brouwer stated that "the only a priori element in science is time." This was against Kant's view on the "subjective constitution of our mind", which is based on two forms of intuition, time and space. Pambuccian explains that the arguments that Brouwer presents against the validity of a spatial intuition playing a major role in the foundations of mathematics stems from physics, in particular from the existence of spaces of constant curvature that are different from Euclidean space. At the same time, the author examines the role played by geometry in Brouwer's philosophical work, and in particular in his intuitionistic approach to mathematics.

Chapter 36, by Arkady Plotnitsky, is titled *Continuity and discreteness, between mathematics and physics*. The subject is classical; indeed, the reader might know that the notions of continuity and discreteness were thoroughly discussed by mathematicians, physicists and philosophers in Greek antiquity, that is, long before the rise of modern topology and the definitions that are given to these notions in terms of set theory. The author's treatment of this subject is very fresh, and it sheds a new light on the fundamental problems of the philosophy of science.

Plotnitsky, in this chapter, elaborates on the place of discreteness and continuity in modern mathematics and physics, especially in light of the advent of new theories like quantum physics. This discussion is also the occasion for the author to reflect on the more general question of the relation between mathematics and physics, and in particular on the limitations of the mathematical representation of nature in modern physical theories like quantum electrodynamics, quantum mechanics, relativity and quantum field theories. While he discusses the introduction of probability theory, of infinite-dimensional Hilbert spaces and of non-commutative algebra in quantum mechanics and quantum field theories, he addresses the more general question of the use of mathematical thinking and its limitations in the description of nature, stressing nevertheless the fact that mathematics and physics have always been connected. In doing so, he develops a new point of view on the relationship between reality and representation, both in mathematics and physics. In particular, he introduces two philosophical notions, that of “reality without realism” and that of “ideality without idealism”. The important philosophical questions of “what is reality” and “what is being” are also addressed.

Part of the discussion is based on an analysis and an original interpretation of ideas of prominent mathematicians and physicists who were also philosophers, including Riemann, Poincaré, Einstein, Weyl, Grothendieck, Dirac, Schrödinger and others. At the same time, the author comments on important passages from Riemann’s Habilitation lecture, *On the hypotheses that lie at the foundation of geometry*, a piece of literature which, in Plotnitsky’s words, revolutionised the mathematical foundations of spatiality and geometry. He emphasizes in particular the places where Riemann talks about the reality of space and about the discrete vs. the continuous (and Plotnitsky points out the fact that the idea, emphasized by Grothendieck, among others, that the continuous may serve as an approximation of the discrete, rather than the other way around, originates in a remark by Riemann), actualizing these passages with comments by later authors, including Heisenberg, Grothendieck and others. The notions of space, determinism, causality, and the infinitely small (which the author prefers to call “immeasurably small”) are discussed. Plotnitsky also comments on passages from Einstein’s work *Physics and reality* and on the philosophical debates between Einstein and Bohr concerning the usage of mathematics (algebra, geometry, probability) in the study of nature. Thom’s ideas on science which were against the stream, and in some places revolutionary, are highlighted. Philosophical comments by Grothendieck, Cartier and others involving the roles of the motivic Galois group in renormalization and QFT, of that of the cosmic Galois group, and of the “symmetries with geometric origin” pointed out by Connes and Marcolli, are also included in the debate. An interesting distinction is made between Plato’s philosophy and mathematical Platonism, the latter being, according to the author, a twentieth-century invention. The reader interested in such ideas may also want to read Plotnitsky’s recent book *Reality without realism: Matter, thought and technology in quantum physics* (Springer, 2021).

Chapter 37, by Stelios Negrepontis, is the last chapter of this book. It is titled *Zeno’s arguments and paradoxes are not against motion and multiplicity but for the separation of true Beings from sensibles*. I would like to linger a bit on this chapter.

Let me first recall who Zeno is.

Zeno of Elea (c. 490–430 BC) is a Presocratic Greek philosopher who is known through accounts of Plato, Aristotle and their commentators, mainly Proclus and Simplicius. According to Diogenes Laërtius' *Lives of the eminent Philosophers*, Aristotle considered that Zeno was the inventor of dialectic, the art of asking questions and defending successively opposite theses with the aim of finding the truth. At the heart of his speculations are his theories on the infinite, the unlimited and the infinitely divisible. According to Aristotle (from his *Metaphysics*), Zeno was also interested in the nature of the line, whether it is a collection of points or not – an important subject of discussion at Plato's Academy.

Zeno is famous for his arguments and paradoxes which were reported on mainly by Plato in the *Parmenides* and in other dialogues, by Aristotle in the *Physics*, and later by Simplicius in his *Commentaries on Aristotle's Physics*. One of these paradoxes, known under the name *Achilles and the Tortoise*, involves Achilles, who was known to be a very fast runner, and a tortoise he is chasing. The argument says that it is impossible for Achilles to overtake a turtle. Indeed, while he is running to the point where the tortoise is at a certain moment, the tortoise has continued to move forward, and so on, so that the tortoise will always be ahead of Achilles. Another paradox involves the sound made by falling grains of millet. The problem originates in that, when falling to the ground, the content of a bushel of grain produces noise, while the fall of each individual grain produces no noise. This poses several problems at the same time: Oneness/Multiplicity, Discreteness/Continuity, Motion/Stillness, Theory/-Experience, and there are others. Some paradoxes concern the numbering of infinities, and there are also others. These paradoxes have fascinated scientists and common people for millennia. According to Plato's *Parmenides*, the paradoxes were meant to give support to Zeno's teacher, the great philosopher Parmenides (c. 520–400 BC), defending his basic philosophical thesis saying that everything is a single unified and unchanging whole ("the way of truth"), and that all apparent change and multiplicity is merely an illusion ("the way of opinion"). Parmenides' theories reached us through (a very substantial) fragment of a poem he wrote, whose original title is unknown and which is known by the name *On Nature*.

Modern commentators generally consider that Zeno's arguments and paradoxes are meant to show the impossibility of physical motion and multiplicity, in accordance with Parmenides' theories.

As Plato describes in his dialogue *Parmenides*, Zeno's "Basic Argument" entails the simultaneous presence of a variety of dyads of opposite properties, such as dissimilarity and similarity, infinite and finite, many and one, great and small, motion and rest. A common feature of modern interpretations is that Zeno meant these dyads of opposing properties to be self-contradictory, so that no entity whatsoever can satisfy simultaneously these properties. This is what Negrepontis calls the "standard interpretation of Zeno's Basic Argument", namely, that the compresence of these opposites properties is a formal contradiction, and this contradiction shows the partless Oneness of the true Being.

In Chapter 37, Negrepontis offers a radically new interpretation of Zeno's paradoxes. Based on a new interpretation of ancient sources, including Plato, Aristotle, Eudemus, Proclus, Simplicius and others, he explains that the coexistence of opposite properties is most definitely not excluded, but on the contrary is specifically satisfied by Zeno's true Being. In fact, this coexistence of opposites in the true Being is precisely what makes it different from the sensible entities. This analysis dismantles the standard interpretation of Zeno's basic arguments. It shows that Zeno means to help his teacher Parmenides, not by showing that physical change and multiplicity are impossible, but that the sensible entities (the Parmenides way of opinion) are different and inferior to the true Being (the Parmenides way of truth), because they are characterized by change only without permanence, and by multiplicity only, without Oneness, while a true Being is changeless as it changes, and is One as it is infinitely Many, in fact a self-similar and not a partless One.

Negrepontis' interpretation of Zeno's arguments and paradoxes is based (a) on his prior interpretation of Plato's intelligible Being, as the philosophical analogue of a dyad of lines in periodic anthyphairesis, and (b) on several arguments showing the close connection of Plato's intelligible Being with Zeno's true Being, including the identification of both with the so-called indivisible lines and the rejection of both by Aristotle in his work *Physics*, for employing the (unacceptable to him) actual infinite. It turns out that Zeno's paradoxes were inspired by the fundamental Pythagorean mathematical discovery of incommensurability of the diameter to the side of a square, and its proof, using the method of infinite anthyphairesis, "finitized" by the conservation of application of areas/Gnomons. It also turns out that Plato's intelligible Being was greatly influenced by Zeno's true Being, modified by Theaetetus' further fundamental discoveries on general quadratic incommensurabilities, including his change from the conservation of application of areas to the Logos Criterion.

It may be useful to recall that Zeno, Plato, Aristotle, Eudemus, Proclus, Simplicius and several other philosophers from ancient Greece were fully aware of the mathematics of their epoch, discussing important mathematical problems in a language which is obviously different from today's language, but where the fundamental ideas are the same as ours. The reading and commentary of the texts and fragments of Ancient Greece by someone of Negrepontis' stature, at the same time mathematician and historian of Greek Mathematics – probably the person who has contributed most to our understanding of the mathematics in Plato's writings – is always a refreshing draught of air, both for mathematics and for the philosophy of science.

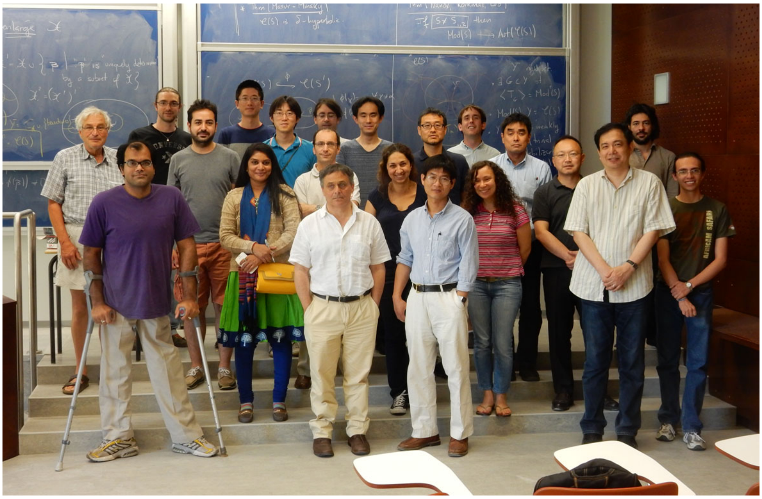
The reader interested in these questions of philosophy of mathematics may also want to read the chapter by Farmaki and Negrepontis, titled "The Paradoxical Nature of Mathematics”, in the volume dedicated to V. Turaev, published in the same series (EMS Publishing house, 2021). The anthyphairetic reasoning used by Plato in his philosophy is also discussed in the chapter “Plato on Geometry and the Geometers” by Negrepontis in the book *Geometry in history* which I co-edited with S. G. Dani (Springer, 2019).

Finishing this Introduction, I feel a bit nostalgic, as at a mathematics conference; it’s like a family gathering that is ending.

Strasbourg, February, 2023

Weixu Su, Lüping Jiang and Norbert A'Campo, Conference on group actions, Sanya, December 17, 2014

April 27, 2015, Norbert's house at Witterswil, with students and friends



French–Japanese workshop on Teichmüller spaces and surface mapping class groups, Strasbourg, June 4, 2015
First row: Krishnendu Gongopadhyay, Athanase Papadopoulos, Lizhen Ji, Ken'ichi Ohshika, N. N.; Second row: Darshana Prajapati, Valentina Disarlo, Elena Frenkel, Sumio Yamada; Third row: Norbert A'Campo; Firat Yasar, Jean-Marc Schlenker, Hideki Miyachi, Nariya Kawazumi; Fourth and fifth row: Vincent Alberge, Hengnan Hu, Yusuke Kuno, Gwénaël Massuyeau, Takuya Sakasai, Javier Aramayona, Andres Sambarino

S. J. Dani, Jyotsna Dani, Noémie Combe, Norbert A'Campo, Toshikazu Sunada, Sumio Yamada. Strasbourg, Conference "Geometry in history," June 11, 2015

From left to right: Alena Zhukova, Brijesh Kumar Tripathi, Ivan Ismestiev, Olga Kharlampovich, Athanase Papadopoulos, Kalmesh Kumar Dubey, Norbert A'Campo. On the Ganga river at Varanasi, Winter school on Finsler geometry and applications, Banaras Hindu University, December 8, 2018

Norbert A'Campo, Athanase Papadopoulos, Sumio Yamada, Oberwolfach (Research in pairs program), October 16, 2021

Bob Penner, Norbert A'Campo, Athanase Papadopoulos, Strasbourg, September 8, 2017

Athanase Papadopoulos, Xenia Semenova, Yuri Manin, Norbert A'Campo, Strasbourg, September 13, 2019

Norbert A’Campo, 2015

Basel, Leonhard Euler’s house, in Riehen (Basel), Krishnendu Gongopadhyay, Norbert A’Campo, Annette A’Campo, Ken’ichi Ohshika, Yoshiko Ohshika, March 2022

Strasbourg, Hôtel de Ville (City Hall) reception celebrating Norbert's 70th birthday. Norbert A'Campo, Annette A'Campo, Athanase Papadopoulos

Norbert and Annette A'Campo, Mittag-Leffler Institut, February 2015

May 22, 2020, The Julier Pass in the Swiss Alps

Norbert at his home in Witterswil, October 4, 2022. The figure is a representation of a so-called "wall crossing" between singularities. This figures, engraved in the marble, illustrates a result observed by Viro on Arnol'd's modulo 8 congruence of the number of ovals. On the left, the picture gives four odd ovals, on the right 4 even ovals. The difference is 8, as it should be. The two level sets are separated by a wall crossing. This singularity is called E_8.