Chapter 1

Introduction

Arithmetic analogues of ordinary differential equations were introduced in [6]. The role of functions of one variable t was played by elements of the completed valuation ring R of the maximal unramified extension of $\mathbb{Q}_p$. The role of the derivation $\frac{d}{dt}$ was played by the *p-derivation* $\delta_p = \delta : R \to R$ defined by

$$\delta a := \frac{\phi(a) - a^p}{p}, \quad a \in R,$$

where $\phi : R \to R$ is the unique Frobenius lift on R. One interpreted δ as an "arithmetic differentiation" with respect to the "arithmetic direction" p. This theory was successfully applied to a series of problems in Diophantine geometry [7–10, 13]. For example a special case of the main results of [13] states that for the modular curve $X := X_1(N)$ over R (with $N \geq 4$ coprime to p) and an elliptic curve E over R if $\Theta\colon X \to E$ is any surjective morphism then the intersection the CL-locus of X with the inverse along Θ of any finite rank subgroup of $E(R)$ must be finite. This is morally done by producing interesting homomorphisms $\psi\colon E(R) \to R$ and showing that $\mathrm{CL} \cap \Theta^{-1}(\mathrm{Ker}(\psi))$ is finite where CL is the CL-locus of canonical lift points (which are the analogues, in the local setting, of CM points). The functions ψ are arithmetic versions of Manin maps ("δ-characters") of order 2 while the theory of "δ-modular forms" provides functions that vanish on CL. The δ-characters, as well as the δ-modular forms, are basic examples of *arithmetic ODEs* of order $\leq r$ where the latter are defined as R-valued functions on sets of R-points of schemes which are locally given, in the Zariski topology, by restricted power series in the *p-derivatives* $\delta^i a_j$ of the affine coordinates a_j of the points with $i \leq r$. In particular, this theory only applies to unramified settings and concerns a single Frobenius lift.

In this memoir, we describe a significant enhancement of the foundational theory of arithmetic differential equations. On the one hand, for any prime element π in a Galois extension of $\mathbb{Q}_p$, we consider the ramified setting $R_\pi := R[\pi]$, where the elements of R_π are viewed as analogues of functions of several variables. Also, we will consider several Frobenius lifts $\phi_{\pi,1}, \dots, \phi_{\pi,n}$ on R_π and their corresponding *π-derivations*, $\delta_{\pi,i} : R_\pi \to R_\pi$,

$$\delta_{\pi,i} a := \frac{\phi_{\pi,i}(a) - a^p}{\pi}, \quad a \in R_\pi, \quad i \in \{1, \dots, n\},$$

leading to an arithmetic PDE theory. We will present a series of applications. As an example for $n = 2$ and E an elliptic curve over R_π, we produce a genuinely new homomorphism $\psi\colon E(R_\pi) \to R_\pi$ which is an order 1 arithmetic PDE analogue of a Manin map, for which we can prove the following finiteness theorem replacing CL with the locus of quasi-canonical lifts considered in [20]. For an open set $X \subset X_1(X)_{R_\pi}$, denote by $\mathrm{QCL}(X(R_\pi))$ the set of *quasi-canonical lift points* in $X(R_\pi)$ i.e., points corresponding to ordinary elliptic curves whose Serre–Tate parameter is a root of unity. We prove the following finiteness result (cf. Corollary 7.69).

Theorem 1.1. *Consider a surjective morphism of R_π-schemes $\Theta : X_1(N) \to E$ and denote by $\Theta_{R_\pi} : X_1(N)(R_\pi) \to E(R_\pi)$ the induced map. There exists an open set $X \subset X_1(N)$ with non-empty reduction mod π such that for all except finitely many cosets C of* $\mathrm{Ker}(\psi)$ *in $E(R_\pi)$ the set* $\mathrm{QCL}(X(R_\pi)) \cap \Theta_{R_\pi}^{-1}(C)$ *is finite.*

As a further application we will show that the map ψ above (and other maps similar to it) satisfy a remarkable "Reciprocity theorem" as follows. Let E_0 be an ordinary elliptic curve over $R_\pi/\pi R_\pi$. For every $\alpha, \beta \in R_\pi$ with absolute value less than $p^{-\frac{1}{p-1}}$ let E_α and E_β be the elliptic curves over R_π with reduction E_0 and with logarithms of the Serre–Tate parameters equal to α and β respectively. Furthermore, let $P_{\alpha,\beta} \in E_\beta(R_\pi)$ and $P_{\beta,\alpha} \in E_\alpha(R_\pi)$ be the points whose elliptic logarithms equal α and β, respectively. Finally, let ψ_α and ψ_β be the corresponding arithmetic PDE Manin maps of order 1 attached to E_α and E_β, respectively. The Reciprocity theorem for our arithmetic Manin maps referred to above is the following statement (cf. Theorem 7.44).

Theorem 1.2. *The following equality holds,*

$$\psi_\beta(P_{\alpha,\beta}) + \psi_\alpha(P_{\beta,\alpha}) = 0.$$

1.1 Background

The present work is essentially self-contained. However, for convenience, we explain its background in what follows.

As already mentioned above, a theory of arithmetic ordinary differential equations (ODEs) was initiated in [6] and had a series of Diophantine applications; cf. [6–8, 13]. In particular, in [6] arithmetic analogues of the classical Manin maps [27] were constructed and in [6, 8] arithmetic analogues of Manin's theorem of the kernel were proved. We recall that, for a function field F, the classical Manin maps are F-valued non-linear differential operators of order 2 defined on the set of F-rational points of an abelian F-variety. Similarly, the arithmetic Manin maps in [6] had order 2 and were defined on the set of points of an abelian variety over a p-adic field. Other basic ODEs were shown to have arithmetic analogues. This is the case for Schwarzian-type ODEs satisfied by classical modular forms, cf. [3, 9] and [10, Chapter 8] where a theory of differential modular forms was developed.

In the framework of [6, 10] the only solutions of arithmetic ODEs that were defined were “unramified solutions” i.e., solutions (with coordinates) in the completion R of the maximum unramified extension of $\mathbb{Z}_p$. Subsequently, the δ-overconvergence machinery in [12, 14] allowed one to define “ramified solutions” to the main arithmetic ODEs of the theory, i.e., solutions in the ring of integers R^{alg} of the algebraic closure K^{alg} of $K := R[1/p]$ and sometimes even in the ring of integers $\mathbb{C}_p^\circ$ of the complex p-adic field $\mathbb{C}_p$.

A theory of arithmetic PDEs with two “directions” one of which was arithmetic and the other geometric was then developed in [17, 18]. This theory combined an arithmetic differentiation δ_p in the “arithmetic direction p” with usual differentiation $\delta_q := \frac{d}{dq}$ with respect to a “geometric direction” defined by a variable q. The two operators δ_p and δ_q were viewed as acting on the power series ring $R[\![q]\!]$ and solutions were well defined (and extensively studied) in this ring. A somewhat surprising outcome of [15] was that, in this arithmetic PDE context, analogues of Manin maps exist that have order 1 (rather than 2) and interesting interactions were found between the order 2 ODE Manin maps (both arithmetic and geometric) and the newly discovered order 1 PDE Manin maps. In some sense the existence of order 1 Manin maps was an effect of the arithmetic direction p and the geometric direction q “conspiring” to create lower order Manin maps. In [18] a theory of differential modular forms in this setting was developed. This version of the theory was an “unramified theory” in the sense that solutions were defined in $R[\![q]\!]$ and did not make sense in $R^{\text{alg}}[\![q]\!]$.

It is reasonable instead to hope for a “purely arithmetic” PDE theory i.e., a PDE theory in which all the directions are “arithmetic.” Along these lines a theory of arithmetic PDEs with $n \geq 2$ arithmetic directions was developed in [4, 16] in which n arithmetic differentiation operators were attached to n distinct prime integers. In this version of the theory, the solutions of arithmetic PDEs were only defined in number fields that were unramified at the primes in question. Arithmetic Manin maps were constructed in this context using a technique introduced in [16] called *analytic continuation between primes*. The order of the arithmetic Manin maps in this setting was $2n \geq 4$; hence, in some sense, the primes involved acted as if they obstructed each other in the process of creating Manin maps.

There is a basic version of the theory that is missing from the above series of approaches, namely a purely arithmetic PDE theory where the *several arithmetic differentiations* are all attached to *one* fixed prime p. For such a theory to be relevant one needs to make sense of “ramified solutions,” i.e. of solutions in R^{alg}. It is the aim of the present work to systematically develop such a theory and provide new applications. We have already made clear in the introduction that tangible Diophantine applications will come out of this enhancement. However, there is even more. Additionally, certain ODE versions of the PDEs appearing in classical Riemannian geometry related to Chern and Levi-Civita connections have been developed (cf. [11]). The fundamental framework we describe here will have ramifications in that theory as well, and will be explored in upcoming work.

1.2 Framework of this memoir

Our starting point is the observation that in the ramified p-adic world one should envision not one but many arithmetic directions reflecting the fact that the absolute Galois group of $\mathbb{Q}_p$ does not have one but several (in fact one can take 4) topological generators (cf. [22]). These topological generators can be chosen to be *Frobenius automorphisms of* $\mathbb{Q}_p^{\text{alg}}$; cf. Definition 2.3. One can then develop the theory starting from an arbitrary finite collection $\phi_1, \dots, \phi_n$ of Frobenius automorphisms of K^{alg}. Remarkably this approach, combined with the δ-overconvergence technique in [12, 14], allows one to define solutions to our equations in R^{alg}. As an application we will again construct arithmetic Manin maps which (as in [17] but unlike in [16]) have order 1; so the various arithmetic differentiation operators at p conspire, again, to create lower order arithmetic Manin maps. On the other hand for $n = 2$ one can introduce a remarkable order 2 arithmetic PDE Manin map that can be viewed as the "Laplacian" of our context. This is very different from the order 4 arithmetic Laplacian in the context of [16]. An arithmetic PDE version of the theory of differential modular forms in [3, 9, 10] will also be developed in this memoir and a series of new "purely PDE" phenomena will be put forward.

We summarize our discussion above in the following table. Here N_{pr} below is the number of primes involved, N_{ari} is the number of arithmetic directions, and N_{geo} is the number of geometric directions.

Reference	N_{pr}	N_{ari}	N_{geo}	Ramified solutions defined
[6]	1	1	0	NO
[12]	1	1	0	YES
[17]	1	1	1	NO
[16]	n	n	0	NO
This work	1	n	0	YES

1.3 Terminology

In this memoir, unless otherwise stated, all rings will be commutative with identity. A morphism of Noetherian rings will be called *smooth* if it is of finite type and is 0-smooth in the sense of [28, page 193]. Throughout this memoir we fix an odd prime $p \in \mathbb{Z}$ and for any ring S and any Noetherian scheme X we denote by $\widehat{S}$ and $\widehat{X}$ the respective p-adic completions. The superscript "alg" will mean algebraic closure.

The superscript "ur" will mean maximum unramified extension. By an *elliptic curve* over a ring we mean an abelian scheme of relative dimension one. There are two contexts in which the word "ordinary" appears in this memoir: one as in "ordinary versus partial differential equation"; and the other as in "ordinary versus supersingular elliptic curve." To avoid confusion we will always say "ODE" instead of "ordinary" in the first situation. Also, we will often use "ODE" and "PDE" as adjectives as in "ODE arithmetic Manin maps," "PDE differential modular forms," etc.

1.4 Main results

In what follows we explain some of our main results including the previously mentioned theorems in more context. For the precise definitions of our concepts we refer to the body of the memoir. For simplicity we assume, for the rest of this introduction, that the number of Frobenius automorphisms is $n = 2$. Some of the results below have variants that will be proved for arbitrary n.

Let Π be the set of all prime elements π in all finite Galois extensions of $\mathbb{Q}_p$. With $R = \widehat{\mathbb{Z}_p^{\text{ur}}}$ and $K = R[1/p]$ and $\pi \in \Pi$ as above let $R_\pi := R[\pi]$ and $K_\pi := K(\pi)$. Recall from [6] that a *π-derivation* on a flat R_π-algebra A is a map $\delta_\pi : A \to A$ such that the map $\phi : A \to A$ defined by $\phi(x) = x^p + \pi\delta_\pi(x)$ is a ring homomorphism which is then referred to as a *π-Frobenius lift*. We fix a pair $\Phi = (\phi_1, \phi_2)$ of Frobenius automorphisms of K^{alg}; the automorphisms ϕ_1, ϕ_2 induce *π-derivations* on R_π. For any smooth scheme X over R_π we will define a sequence of p-adic formal schemes $J^r_{\pi,\Phi}(X)$ called the *partial π-jet spaces* of X. The ring of functions on $J^r_{\pi,\Phi}(X)$ will be referred to as the ring of (purely) *arithmetic PDEs* on X order $\leq r$ (cf. Definition 2.25). We will then define its subring of *totally δ-overconvergent* elements (cf. Definition 2.28). There is a natural action of ϕ_1, ϕ_2 on the colimit as $r \to \infty$ of these rings. Every arithmetic PDE f on X defines a map of sets $f_{R_\pi} : X(R_\pi) \to R_\pi$. If f is totally δ-overconvergent then the map f_{R_π} extends to a map of sets $f^{\text{alg}} := f^{\text{alg}}_{R_\pi} : X(R^{\text{alg}}) \to K^{\text{alg}}$ and the preimage of 0 under this map is the *set of solutions* in R^{alg} of the arithmetic PDE f.

Let E be an elliptic curve over R_π. We define a *partial δ_π-character* of order $\leq r$ on E to be an arithmetic PDE of order $\leq r$ which, viewed as a morphism $J^r_{\pi,\Phi}(E) \to \widehat{\mathbb{G}_a}$, is a group homomorphism; cf. Definition 3.1. Extending terminology from [6] "partial δ_π-characters" is the name for our "arithmetic Manin maps" in our PDE setting. To each E and every basis ω for the 1-forms on E we will attach two families of elements in R_π called (primary, respectively secondary) *arithmetic Kodaira–Spencer classes*; cf. Definitions 5.5 and 5.15. Finally, to each partial δ_π-character of E we will attach a *Picard–Fuchs symbol* which is a formal K_π-linear combination of non-commutative monomials in ϕ_1, ϕ_2; cf. Definition 3.7. The arithmetic Kodaira–Spencer classes appear then as coefficients of the symbols of certain distinguished partial δ_π-characters. Among the primary Kodaira–Spencer classes a special role will be played by elements denoted by $f_1, f_2 \in R_\pi$. One of our main results will be the following (cf. Corollaries 5.14, 3.9 and Proposition 3.13). This can be viewed as a simultaneous generalization of the main results in [6] and [12].

Theorem 1.3. *Let E be an elliptic curve over R_π.*

(1) *If $f_1 \neq 0$ or $f_2 \neq 0$ then the R_π-module of partial δ_π-characters of order $\leq r$ has rank equal to $2^{r+1} - 3$.*
(2) *If $f_1 = f_2 = 0$ then the R_π-module of partial δ_π-characters of order $\leq r$ has rank equal to $2^{r+1} - 2$.*
(3) *Every partial δ_π-character ψ is totally δ-overconvergent and the induced group homomorphism $\psi^{\text{alg}} : E(K^{\text{alg}}) \to K^{\text{alg}}$ can be extended to a continuous homomorphism $\psi^{\mathbb{C}_p} : E(\mathbb{C}_p) \to \mathbb{C}_p$. If ϕ_1, ϕ_2 are monomially independent then ψ is uniquely determined by ψ^{alg}.*

The homomorphisms ψ^{alg} are not given by algebraic (or even by analytic) functions in the coordinates but rather by analytic (in fact rigid analytic) functions in the coordinates and their various "$\delta_{\pi'}$-derivatives" for various π''s dividing π. The recipe for defining these maps involves the notion of total δ-overconvergence which is analogous to the one in [12] and will be explained in the body of the memoir. Note that since E is projective over R_π we have $E(K^{\text{alg}}) = E(R^{\text{alg}})$; however, it is an important feature of the theory that the images of the maps ψ^{alg} are not contained in R^{alg}.

For the case of order ≤ 2 we have more precise results. Consider the subset $\mathbb{M}_2^{2,+} := \{1, 2, 11, 22, 12, 21\}$ of non-empty words of length ≤ 2 in the free monoid with identity generated by the set $\{1, 2\}$. There are 6 primary Kodaira–Spencer classes of order ≤ 2,

$$f_\mu, \quad \mu \in \mathbb{M}_2^{2,+}. \tag{1.1}$$

The classes f_1, f_2, f_{11}, f_{22} come from the ODE theory [10]. On the other hand the classes f_{12}, f_{21} are "genuinely PDE" (not "reducible to ODEs"). The secondary Kodaira–Spencer classes will be denoted by

$$f_{\mu,\nu}, \quad \mu, \nu \in \mathbb{M}_2^{2,+}, \quad \mu \neq \nu. \tag{1.2}$$

They satisfy $f_{\mu,\nu} + f_{\nu,\mu} = 0$. The classes $f_{11,1}, f_{22,2}$ come from the ODE theory while the others classes $f_{\mu,\nu}$ are, again, "genuinely PDE". By the theory in [10] the secondary classes $f_{ii,i}$, $i \in \{1, 2\}$, are known to be expressible in terms of the primary ones f_i as $f_{ii,i} = p\phi_i f_i$; cf. Remark 7.17. Note that if E has ordinary reduction then $f_i = 0$ for some i if and only if $f_\mu = f_{\mu,\nu} = 0$ for all μ and ν, if and only if "the" Serre–Tate parameter of E is a root of unity; cf. Proposition 7.39. Finally, note (cf. Theorem 5.33) that there exist π, ϕ_1, ϕ_2 and a pair (E, ω) over R_π such that E has ordinary reduction and all classes (1.1) and (1.2) attached to (E, ω) are non-zero.

In case $\pi = p$ we can be more specific. Indeed, for all (E, ω) over R we have $f_1 = f_2$, $f_{11} = f_{22}$ and $f_{1,2} = f_{11,22} = 0$. (In [3] and [10] f_i was denoted by f^1 and f_{ii} was denoted by f^2.) In this case we have that $f_i = 0$ if and only if E has ordinary reduction and is a canonical lift of its reduction; cf. Remark 5.9. Also, if E comes from a curve $E_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ and has ordinary reduction but is not a canonical lift then $f_{ii} = a_p f_i \neq 0$ where $a_p \in \mathbb{Z}$ is the trace of Frobenius on the reduction mod p of $E_{\mathbb{Z}_p}$; cf. Remark 5.27.

Going back to the general situation when π is arbitrary we let $N(\pi)$ be the smallest integer $N \in \mathbb{Z}$ such that for all integers $n \geq 1$ we have $\pi^n/n \in p^{-N}\mathbb{Z}_p$; in particular $N(p) = -1$. If $\mu = i \in \{1, 2\}$ we set $\phi_\mu = \phi_i$ while for $\mu = ij$ with $i, j \in \{1, 2\}$ we set $\phi_\mu = \phi_i \phi_j$. We also set $\tilde{f}_\mu = p^{N(\pi)+1} f_\mu$. We will prove (see Corollaries 5.23 and 5.24) the following summary result, which also may be viewed as a generalization of the main results of [6].

Theorem 1.4. *Assume in Theorem 1.3 that $f_1 f_2 \neq 0$. The following hold:*

(1) *For all $\mu, \nu \in \mathbb{M}_2^{2,+}$ there is a unique δ_π-character $\psi_{\mu,\nu}$ with Picard–Fuchs symbol $\tilde{f}_\nu \phi_\mu - \tilde{f}_\mu \phi_\nu + f_{\mu,\nu}$.*

(2) *A basis modulo torsion of the R_π-module of partial δ_π-characters of order ≤ 1 consists of $\psi_{1,2}$.*

(3) *A basis modulo torsion of the R_π-module of partial δ_π-characters of order ≤ 2 consists of the elements $\psi_{1,2}$, $\phi_1\psi_{1,2}$, $\phi_2\psi_{1,2}$, $\psi_{11,1}$, $\psi_{22,2}$.*

Here and in the following by a *basis modulo torsion* of an R_π-module M we mean a family of elements in M inducing a basis of the K_π-linear space $M \otimes_{R_\pi} K_\pi$.

One is tempted to view $\psi_{11,22}$ as the "Laplacian" equation in our context while $\psi_{12,21}$ reflects, in some sense, the non-commutation of ϕ_1 and ϕ_2 and can be viewed as a "Poisson bracket operator." In case $f_1 = f_2 = 0$ a result similar to Theorem 1.4 will be proved; cf. Corollary 5.14.

The main flavor of our results above is "global on E". However, by looking at the completion of E at the origin, one obtains in particular the following integrality statement; cf. Corollary 5.19.

Theorem 1.5. *Let E be an elliptic curve over R_π with logarithm $\sum_{N=1}^{\infty} \frac{b_N}{N} T^N$, $b_N \in R_\pi$. Let $\mu, \nu \in \mathbb{M}_2^{2,+}$ and let $r, s \in \{1, 2\}$ be the lengths of the words μ, ν, respectively. Assume $r \geq s$. Then the following relations hold for all $N \geq 1$:*

$$\tilde{f}_\nu \frac{\phi_\mu(b_N)}{N} - \tilde{f}_\mu \frac{\phi_\nu(b_{p^{r-s}N})}{p^{r-s}N} + f_{\mu,\nu} \frac{b_{p^r N}}{p^r N} \in pR_\pi. \tag{1.3}$$

This integrality statement can be viewed as an analogue (for several "conjugates" of an elliptic curve) of the integrality statement of Atkin and Swinnerton-Dyer for a given elliptic curve [1, 34].

The next step in the theory will be to extend some of the theory of δ-modular forms [10] to the PDE case by defining *partial δ-modular forms* (whose weights are $\mathbb{Z}$-linear combinations of non-commutative monomials in ϕ_1, ϕ_2) and *isogeny covariance* for such forms; cf. Chapter 7. We will also attach *symbols* to isogeny covariant partial δ-modular forms for weights of degree -2; these symbols are, again, K-linear combinations of non-commutative monomials in ϕ_1, ϕ_2.

To state our main result we need to consider the standard modular curve $Y_1(N) = X_1(N) \setminus \{\text{cusps}\}$ over R_π (with $N \geq 4$ coprime to p) and the natural $\mathbb{G}_m$-bundle $B = B_1(N)$ over the $Y_1(N)$; so B classifies pairs consisting of an elliptic curve with $\Gamma_1(N)$-structure and a basis for the 1-forms. Let B_{ord} be the preimage in B of the ordinary locus in $Y_1(N)$. We will show (cf. Theorems 7.11, 7.13, 7.18, 7.19, 7.34, Proposition 7.38 and Corollary 7.30) the following characterization of these forms.

Theorem 1.6. *The following hold:*

(1) *The classes f_μ and $f_{\mu,\nu}$ are induced by isogeny covariant partial δ-modular forms, denoted by f_μ^{jet} and $f_{\mu,\nu}^{\text{jet}}$, of weight $-1-\phi_\mu$ and $-\phi_\mu - \phi_\nu$, respectively.*

(2) *There exists $c \in \mathbb{Z}_p^\times$ such that for every distinct words $\mu, \nu \in \mathbb{M}_2^{2,+}$ of lengths $r, s \in \{1, 2\}$, respectively, the symbols of f_μ^{jet} and $f_{\mu,\nu}^{\text{jet}}$ are equal to $c(\phi_\mu - p^r)$ and $c(p^s\phi_\mu - p^r\phi_\nu)$, respectively.*

(3) *The forms f_μ^{jet} and $f_{\mu,\nu}^{\text{jet}}$ naturally induce totally overconvergent arithmetic* PDE*s on B and the induced maps $B(R^{\text{alg}}) \to K^{\text{alg}}$ restricted to $B_{\text{ord}}(R^{\text{alg}})$ extend to continuous maps $B_{\text{ord}}(\mathbb{C}_p^\circ) \to \mathbb{C}_p$.*

(4) *The form $f_{1,2}^{\text{jet}}$ is a basis modulo torsion of the module of isogeny covariant partial δ-modular forms of order ≤ 1 and weight $-\phi_1 - \phi_2$.*

The forms f_μ^{jet}, $f_{\mu,\nu}^{\text{jet}}$ in the theorem satisfy a series of cubic and quadratic relations (cf. Theorems 7.18 and 7.19). We will use these relations to determine the *Serre–Tate expansions* of the forms involved (cf. Theorem 7.28) which is what, in particular, leads to the determination of the corresponding symbols in part 2 of the theorem above. As a consequence of these computations we will derive some explicit formulae for the values of δ-characters in terms of Serre–Tate parameters. These formulae will exhibit a rather unexpected antisymmetry property that translates into a *Reciprocity theorem* similar to Theorem 1.2 and valid for arbitrary δ-characters $\psi_{\mu,\nu}$ where $\mu, \nu \in \mathbb{M}_2^{2,+}$; cf. Theorem 7.44 for details and a precise statement. The critical map ψ in Theorems 1.1 and 1.2 is the following. For $\pi \in \Pi$ and an elliptic curve E over R_π we recall our δ_π-character $\psi_{1,2}$. The map ψ in Theorems 1.1 and 1.2 is the induced group homomorphism

$$\psi_{R_\pi} := (\psi_{1,2})_{R_\pi} : E(R_\pi) \to R_\pi.$$

We also note that the proof of Theorem 1.1 utilizes a version of a classic theorem of Strassman, see Lemma 7.68. It would be immediate to conclude a uniform version of Theorem 1.1 from a uniform version of Lemma 7.68.

In addition, our considerations above will lead to an explicit description of the kernel of $\psi^{\text{alg}}_{\mu,\nu,\beta}$ in terms of β. This result can be viewed as an arithmetic PDE *Theorem of the kernel* analogue of Manin's theorem of the kernel [27] and extending the arithmetic ODE results in [6, Theorems A and B] and [8, Theorem 1.6]. This utilizes an interesting pairing defined as follows.

For μ, ν of lengths r and s respectively define the $\mathbb{Q}_p$-bilinear map

$$\langle \ , \ \rangle_{\mu,\nu} : K^{\text{alg}} \times K^{\text{alg}} \to K^{\text{alg}}$$

by the formula

$$\langle \alpha, \beta \rangle_{\mu,\nu} = \beta^{\phi_\nu}\alpha^{\phi_\mu} - \beta^{\phi_\mu}\alpha^{\phi_\nu} + p^s(\alpha\beta^{\phi_\mu} - \beta\alpha^{\phi_\mu}) + p^r(\beta\alpha^{\phi_\nu} - \alpha\beta^{\phi_\nu}).$$

The version of the Theorem of the kernel (cf. Theorem 7.42) is as follows.

Theorem 1.7. *We have a natural group isomorphism*

$$\text{Ker}(\psi^{\text{alg}}_{\mu,\nu,\beta}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \{\alpha \in K^{\text{alg}} \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}.$$

Note that for E_β ordinary with β not a root of unity we have that $\text{Ker}(\psi^{\text{alg}}_{\mu,\nu,\beta})$ (which always contains the torsion group of $E_\beta(R^{\text{alg}})$) does not reduce to the torsion group.

Another application of our theory of δ-modular forms is the construction, for every weight w, of a *δ-period map*

$$\mathfrak{p}_w : Y_1(N)(R^{\text{alg}})^{\text{ss}}_w \to \mathbb{P}^{N_w}(R^{\text{alg}})$$

where $Y_1(N)(R^{\text{alg}})^{\text{ss}}_w \subset Y_1(N)(R^{\text{alg}})$ is a natural set of *semistable* points; cf. Definition 7.35. The terminology is motivated by the following analogy with geometric invariant theory. Group actions are replaced, in our setting, with the action of Hecke correspondences and the "components" of our δ-period maps are given by isogeny covariant δ-modular forms which should be viewed as analogues of invariant sections of line bundles in geometric invariant theory. As we shall see the δ-period maps are rather non-trivial already for w of order 2 and degree 4; cf. Example 7.37. On the other hand isogeny covariance will imply the following result (cf. Theorem 7.36).

Theorem 1.8. *The δ-period maps $\mathfrak{p}_w$ are constant on prime to p isogeny classes.*

Next, as in [3, 9, 10], we will construct certain 'crystalline forms' f^{crys}_μ, $f^{\text{crys}}_{\mu,\nu}$ and prove they are proportional to the forms f^{jet}_μ, $f^{\text{jet}}_{\mu,\nu}$; cf. Corollary 7.52. In addition, we will consider *δ-modular forms on the ordinary locus*. (Such forms were called "ordinary" in [10, Chapter 8] but here we will avoid this term so that no confusion arises with its use in the ODE/PDE distinction.) Then using a crystalline construction as in loc.cit. we will completely determine the structure of the spaces of isogeny covariant δ-modular forms on the ordinary locus for the weights of degree 0 and -2; cf. Corollary 7.58.

1.5 Leitfaden

In Chapter 2, we begin by discussing Frobenius lifts and Frobenius automorphisms of K^{alg} after which we introduce partial δ_π-jet spaces which are a PDE analogue of the ODE π-jet spaces in [6]. In Chapter 3, we introduce and study δ_π-characters of group schemes as well as their Picard–Fuchs symbols. Chapter 4, is devoted to analyzing these concepts for the multiplicative group $\mathbb{G}_m$. Chapter 5, does a similar analysis for elliptic curves. Here we introduce and study the arithmetic Kodaira–Spencer classes f_μ, $f_{\mu,\nu}$ and the δ_π-characters $\psi_{\mu,\nu}$. All the above discussion is made in the context of an arbitrary number n of Frobenius automorphisms and an arbitrary order r. We next specialize our discussion of elliptic curves to the case $n = r = 2$, and we derive a series of quadratic and cubic relations satisfied by the arithmetic Kodaira–Spencer classes. Chapter 6, summarily explains how all the above theory can be developed in a "relative setting;" this is necessary for Chapter 7 where we introduce partial δ-modular forms which are a PDE version of the ODE concept introduced in [9]. The relative arithmetic Kodaira–Spencer classes define such forms. We then introduce and compute the Serre–Tate expansions of these forms, we construct our δ-period maps, and we derive the Theorem of the kernel and the Reciprocity theorem for arithmetic Manin maps. We continue by discussing the crystalline side of the story and forms on the ordinary locus, and we present a construction of finite covers defined by δ-modular forms (cf. Theorem 7.64) which is then used to prove our main Diophantine application to modular parameterizations (cf. Corollary 7.69). We end our Chapter 7 by introducing a PDE version of the ODE δ-Serre operators in [3, 10]; these PDE δ-Serre operators lead to genuine (not arithmetic) PDEs satisfied by our arithmetic PDEs and can be viewed as Pfaffian systems of equations on the arithmetic jet spaces. The memoir ends with an Appendix where we briefly discuss a more general theoretical framework in which commutation relations and inversion of Frobenius lifts are "built into" our jet spaces. We will provide there some simple computations illustrating the complexity of this more general framework.