Chapter 2

Purely arithmetic PDEs

2.1 Frobenius automorphisms

We start with the following standard definition.

Definition 2.1. By a *Frobenius lift* for an A-algebra $\varphi : A \to B$ we understand a ring homomorphism $\phi : A \to B$ such that the induced homomorphism $\overline{\phi} : A/pA \to B/pB$ equals the composition of the induced homomorphism $\overline{\varphi} : A/pA \to B/pB$ with the p-power Frobenius on A/pA. If $B = A$ and $\varphi = 1_A$ we say that ϕ is a *Frobenius lift* on A.

For every, not necessarily algebraic, field extension $F \subset L$ we denote by $\mathfrak{G}(L/F)$ the group of all field automorphisms of L that are the identity on F. For every field L we denote by $\mathfrak{G}_L$ the absolute Galois group $\mathfrak{G}(L^{\mathrm{alg}}/L)$, where L^{alg} is an algebraic closure of L.

We recall the main setting in [12]. Consider the field of p-adic numbers with absolute value $|\ |$ normalized by $|p| = p^{-1}$. Let $\mathbb{Q}_p^{\mathrm{alg}}$ be an algebraic closure of $\mathbb{Q}_p$, let $\mathbb{Q}_p^{\mathrm{ur}}$ be the maximum unramified extension of $\mathbb{Q}_p$ inside $\mathbb{Q}_p^{\mathrm{alg}}$, let K be the metric completion of $\mathbb{Q}_p^{\mathrm{ur}}$ and let K^{alg} be the algebraic closure of K in the metric completion $\mathbb{C}_p$ of $\mathbb{Q}_p^{\mathrm{alg}}$. We still denote by $|\ |$ the induced absolute value on all of these fields. We denote by $\mathbb{Z}_p^{\mathrm{ur}}, \mathbb{Z}_p^{\mathrm{alg}}, R, R^{\mathrm{alg}}, \mathbb{C}_p^{\circ}$ the valuation rings of $\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Q}_p^{\mathrm{alg}}, K, K^{\mathrm{alg}}, \mathbb{C}_p$, respectively. In particular, $R := \widehat{\mathbb{Z}_p^{\mathrm{ur}}}$. We set $k := R/pR$; so $k \simeq \mathbb{F}_p^{\mathrm{alg}}$.

Remark 2.2. The natural ring homomorphism

$$\mathbb{Q}_p^{\mathrm{alg}} \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} K \to K^{\mathrm{alg}} \tag{2.1}$$

is an isomorphism. Indeed, this map is surjective because by Krasner's lemma, we have $K^{\mathrm{alg}} := K\mathbb{Q}_p^{\mathrm{alg}}$; cf. [5, Proposition 5, page 149]. To check that the map (2.1) is injective write and $\mathbb{Q}_p^{\mathrm{alg}} = \bigcup F_i$ with $F_i/\mathbb{Q}_p$ finite and let $F_i^0 \subset F_i$ be the maximum unramified extension of $\mathbb{Q}_p$ contained in F_i; so F_i/F_i^0 is totally ramified and $\mathbb{Q}_p^{\mathrm{ur}} = \bigcup F_i^0$. It is enough to check that $F_i \otimes_{F_i^0} K \to K^{\mathrm{alg}}$ is injective for all i. To check this note that F_i/F_i^0 is generated by a root of an Eisenstein polynomial f_i in $F_i^0[x]$; but every such polynomial is an Eisenstein polynomial in $K[x]$ and so $F_i \otimes_{F_i^0} K = K[x]/(f_i)$ is a field, therefore it injects into K^{alg}.

Definition 2.3. Let L be a subfield of $\mathbb{C}_p$ containing $\mathbb{Q}_p$. A *Frobenius automorphism* of L is a continuous automorphism $\phi \in \mathfrak{G}(L/\mathbb{Q}_p)$ such that ϕ induces the p-power Frobenius on the residue field of the valuation ring of L. We denote by $\mathfrak{F}^{(1)}(L/\mathbb{Q}_p)$ the set of Frobenius automorphisms of L.

More generally the theory of the present memoir can be developed based on the set $\mathfrak{F}^{(s)}(L/\mathbb{Q}_p)$ of all continuous automorphisms $\phi \in \mathfrak{G}(L/\mathbb{Q}_p)$ such that ϕ induces the p^s-power Frobenius on the residue field of the valuation ring of L where s is some fixed positive integer; for simplicity we will not consider this more general situation in what follows.

Note that if ϕ is a Frobenius automorphism of K^{alg} then ϕ sends R into R, induces the Frobenius lift on R, and induces an automorphism of R^{alg} (which is however not a Frobenius lift on R^{alg} in the sense of Definition 2.1). Conversely, every automorphism $\phi \in \mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$ extending the Frobenius lift on R is a Frobenius automorphism of K^{alg}. Indeed, for every finite Galois extension L_0 of $\mathbb{Q}_p$, the field $L := L_0 K$ is sent onto itself by ϕ and the absolute values $|\ |$ and $|\phi(\ )|$ on L have the same restriction to K, hence must coincide; cf. [25, page 32]; in particular ϕ is continuous and induces the p-power Frobenius on k.

The set $\mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ is a principal homogeneous space for the absolute Galois group $\mathfrak{G}_K$ under the action given by $(\gamma, \phi) \mapsto \gamma\phi$ for $\phi \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ and $\gamma \in \mathfrak{G}_K$. On the other hand, by the fact that the homomorphism (2.1) is an isomorphism we immediately get that the restriction homomorphism $\mathfrak{G}_K \to \mathfrak{G}_{\mathbb{Q}_p^{\text{ur}}}$ an isomorphism of topological groups and the restriction map $\mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p) \to \mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ is a bijection. Note that $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ has a purely (topological) group characterization as a subset of $\mathfrak{G}_{\mathbb{Q}_p}$; cf. [31, Lemma 12.1.8, page 665]. The elements of $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ are referred to in loc.cit. as *Frobenius lifts* but adopting that terminology here would conflict with our Definition 2.1.

By the way, the absolute Galois group $\mathfrak{G}_{\mathbb{Q}_p}$ is known to have 4 topological generators one of which is in $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$; the relations among these topological generators are also known, cf. [22] or [30, Theorem 7.5.10, page 360]. We say that a subset of a topological group is a set of *topological generators* if the subgroup generated by this set is dense in the group. One can easily see, by the way, that one can find a set of 4 topological generators of $\mathfrak{G}_{\mathbb{Q}_p}$ that is contained in $\mathfrak{F}^{(1)}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$. We will not use this observation in what follows. What we will be interested in is the monoid (rather than the group) generated by our Frobenius automorphisms, as explained in the next subsection.

2.2 Monomial independence

In what follows monoids will not necessarily be commutative. Let $\mathbb{M}_n$ be the free (non-commutative) monoid with identity generated by the set $\{1, \dots, n\}$,

$$\mathbb{M}_n := \{0\} \cup \{i_1 \dots i_s \mid l \in \mathbb{N},\ i_1, \dots, i_s \in \{1, \dots, n\}\};$$

its elements will be referred to as *words*, the *length* $|\mu|$ of a word $\mu := i_1 \dots i_s$ is defined by $|\mu| = s$, 0 is called the *empty word* and its length is defined by $|0| = 0$.

Multiplication is given by concatenation $(\mu, \nu) \mapsto \mu\nu$ and 0 is the identity element. For all $r \in \mathbb{N} \cup \{0\}$ let $\mathbb{M}_n^r$ be the set of all elements in $\mathbb{M}_n$ of length $\leq r$. Set $\mathbb{M}_n^+ := \mathbb{M}_n \setminus \{0\}$ and $\mathbb{M}_n^{r,+} := \mathbb{M}_n^r \setminus \{0\}$.

Definition 2.4. A family of distinct elements $\phi_1, \dots, \phi_n$ in a monoid $\mathfrak{G}$ with identity 1 is called *monomially independent* if the monoid homomorphism

$$\mathbb{M}_n \to \mathfrak{G}, \quad \mu = i_1 \dots i_l \mapsto \phi_\mu := \phi_{i_1} \dots \phi_{i_l}, \quad \text{for } l \in \mathbb{N}, \quad \text{and } 0 \mapsto 1$$

is injective.

Remark 2.5. Note that in our notation above we have the formula $\phi_\mu \phi_\nu = \phi_{\mu\nu}$ for $\mu, \nu \in \mathbb{M}_n$. Note also that if $\mathfrak{G}$ is a group and $\phi_1, \dots, \phi_n \in \mathfrak{G}$ are monomially independent in $\mathfrak{G}$ then the subgroup of $\mathfrak{G}$ generated by $\phi_1, \dots, \phi_n$ is not necessarily freely generated by $\phi_1, \dots, \phi_n$; an example that naturally occurs in our context is given in Remark 2.9.

The following lemma follows trivially from the well known "algebraic independence of field automorphisms" but, for convenience, we provide a proof.

Lemma 2.6. *Let L be a field of characteristic zero and let $\phi_1, \dots, \phi_n$ be monomially independent elements in $\mathfrak{G}(L/\mathbb{Q})$. Let $F = F(\dots, x_\mu, \dots)$ be a polynomial with L-coefficients in the variables x_μ with $\mu \in \mathbb{M}_n$ and consider the function $f : L \to L$ defined by*

$$f(\lambda) = F(\dots, \phi_\mu(\lambda), \dots), \quad \lambda \in L.$$

Let A be a subring of L and assume $f(\lambda) = 0$ for all $\lambda \in A$. Then $F = 0$.

Proof. By Artin's independence of characters, cf. [26, page 283], if $\mathfrak{A}$ is a monoid then every family of distinct monoid homomorphisms $\mathfrak{A} \to L^\times$ is L-linearly independent in the L-linear space of all maps $\mathfrak{A} \to L$. Let $\mathfrak{A} = A \setminus \{0\}$. Then by Artin's independence of characters it is enough to check that for distinct vectors $e := (e_\mu)_{\mu \in \mathbb{M}_n}$ with entries non-negative integers, almost all zero, the maps $f_e : A \to L$ defined by

$$f_e(\lambda) := \prod_\mu (\phi_\mu(\lambda))^{e_\mu}, \quad \lambda \in A$$

are distinct. Assume $f_e = f_{e'}$ and let us show that $e = e'$. For all integers $m \in \mathbb{Z}$ we have

$$\prod_\mu (m + \phi_\mu(\lambda))^{e_\mu} = \prod_\mu (m + \phi_\mu(\lambda))^{e'_\mu}, \quad \lambda \in A.$$

Since L has characteristic zero we have an equality

$$\prod_\mu (t + \phi_\mu(\lambda))^{e_\mu} = \prod_\mu (t + \phi_\mu(\lambda))^{e'_\mu}, \quad \lambda \in A$$

in the ring of polynomials $L[t]$. Looking at degrees in t we get $\sum_\mu e_\mu = \sum_\mu e'_\mu =: d$. Picking out the coefficient of t^{d-1} we get

$$\sum_\mu e_\mu \phi_\mu(\lambda) = \sum_\mu e'_\mu \phi_\mu(\lambda), \quad \lambda \in A.$$

By monomial independence of the ϕ_i's and, again, by Artin's independence of characters, the family $(\phi_\mu)_{\mu \in \mathbb{M}_n}$ is L-linearly independent in the L-linear space of maps $\mathfrak{A} \to L$ so, since L has characteristic zero, we conclude that $e_\mu = e'_\mu$ for all μ. ■

Example 2.7. In what follows we show that the set $\mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ of Frobenius automorphisms of K^{alg} contains large subsets of monomially independent elements that remain monomially independent on "small" (abelian) extensions of K. We recall some standard constructions from Iwasawa theory; cf. [21]. Let $l \neq p$ be a prime. Consider sequences $\pi_m \in K^{\mathrm{alg}}$ and $\zeta_{l^m} \in K$ with $m \geq 0$ such that

$$\pi_0 = p, \ \ \zeta_{l^0} = 1, \ \ \pi_{m+1}^l = \pi_m, \ \ \zeta_{l^{m+1}}^l = \zeta_{l^m}, \ \ m \geq 0.$$

Since the polynomial $x^{l^m} - p$ is Eisenstein over K and π_m is one of its roots we have that the field $K_{\pi_m} := K(\pi_m)$ generated by π_m is isomorphic to $K[x]/(x^{l^m} - p)$ and K_{π_m} is Galois over K with cyclic Galois group of order l^m generated by the automorphism τ_m satisfying $\tau_m \pi_m = \zeta_{l^m} \pi_m$. Define

$$K^{(l)} := \bigcup_{m \geq 0} K_{\pi_m}. \tag{2.2}$$

Clearly the automorphisms τ_m are compatible and yield an automorphism $\tau_{(l)} \in \mathfrak{G}(K^{(l)}/K)$. For all $\gamma \in \mathbb{Z}_l$ one defines $\tau_{(l)}^\gamma \in \mathfrak{G}(K^{(l)}/K)$ as follows: if $\gamma \equiv b_m$ mod l^m with $b_m \in \mathbb{Z}$ then one lets $\tau_{(l)}^\gamma$ to be $\tau_{(l)}^{b_m}$ on $K(\pi_m)$. Then the map $\mathbb{Z}_l \to \mathfrak{G}(K^{(l)}/K)$ given by $\gamma \mapsto \tau_{(l)}^\gamma$ is an isomorphism. On the other hand the fields K_{π_m} possess compatible automorphisms extending the Frobenius lift on R and fixing the π_m's; they induce an automorphism $\phi_{(l)}$ on $K^{(l)}$. One trivially checks that $\phi_{(l)}\tau_{(l)}$ and $\tau_{(l)}^p \phi_{(l)}$ coincide on all roots of unity in K (and hence on K) and also on all π_m's; so $\phi_{(l)}\tau_{(l)} = \tau_{(l)}^p \phi_{(l)}$ in $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$. For each $\gamma \in \mathfrak{B}$ we set $\phi_{(l)}^{(\gamma)} := \tau_{(l)}^\gamma \phi_{(l)} \in \mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$, and we let $\phi^{(\gamma)} \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ be an arbitrary extension of $\phi_{(l)}^{(\gamma)}$.

Proposition 2.8. *The following hold:*

(1) $\phi_{(l)}^{(0)}, \ldots, \phi_{(l)}^{(p-1)}$ *are monomially independent in* $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$. *In particular,* $\phi^{(0)}, \ldots, \phi^{(p-1)}$ *are monomially independent in* $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$.

(2) *Let* $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}_l$ *be* $\mathbb{Z}$*-linearly independent. Then* $\phi_{(l)}^{(\gamma_1)}, \ldots, \phi_{(l)}^{(\gamma_n)}$ *are monomially independent in* $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$*; in particular* $\phi^{(\gamma_1)}, \ldots, \phi^{(\gamma_n)}$ *are monomially independent in* $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$.

Proof. We will prove Part 2. Part 1 is proved similarly. Write $\phi_{i,l} := \phi_{(l)}^{(\gamma_i)}$ for $i \in \{1,\dots,n\}$. Let $\mu = i_1 \dots i_s$ where $i_1,\dots,i_s \in \{1,\dots,n\}$ and similarly $\mu' = i'_1 \dots i'_{s'}$ where $i'_1,\dots,i'_{s'} \in \{1,\dots,n\}$. Assume

$$\phi_{i_1,l} \dots \phi_{i_s,l} = \phi_{i'_1,l} \dots \phi_{i'_{s'},l}$$

and let us prove that $\mu = \mu'$. We first note that for all integers $j \geq 0$ we have $\phi_{(l)}\tau_{(l)}^j = \tau_{(l)}^{pj}\phi_{(l)}$; this follows by induction on j. We conclude that $\phi_{(l)}\tau_{(l)}^{\gamma} = \tau_{(l)}^{p\gamma}\phi_{(l)}$ for all $\gamma \in \mathbb{Z}_l$; this equality holds because it holds on every K_{π_m}. Next note that for all integers $i \geq 1$ and for all $\gamma \in \mathbb{Z}_l$ we have $\phi_{(l)}^i\tau_{(l)}^{\gamma} = \tau_{(l)}^{p^i\gamma}\phi_{(l)}^i$; this follows by induction on i. Using the latter equalities we get

$$\phi_{i_1,l} \dots \phi_{i_s,l} = \tau_{(l)}^{\gamma_{i_1}}\phi_{(l)} \dots \tau_{(l)}^{\gamma_{i_s}}\phi_{(l)} = \tau_{(l)}^{\gamma_{i_1}+p\gamma_{i_2}+\dots+p^{s-1}\gamma_{i_s}}\phi_{(l)}^s$$

and similarly for μ', so we get

$$\tau_{(l)}^{\gamma_{i_1}+p\gamma_{i_2}+\dots+p^{s-1}\gamma_{i_s}}\phi_{(l)}^s = \tau_{(l)}^{\gamma_{i'_1}+p\gamma_{i'_2}+\dots+p^{s'-1}\gamma_{i_{s'}}}\phi_{(l)}^{s'}.$$

Since $\phi_{(l)}$ has infinite order on K we get $s = s'$. Since $\tau_{(l)}$ has infinite order on $K^{(l)}$ we get

$$\gamma_{i_1} + p\gamma_{i_2} + \dots + p^{s-1}\gamma_{i_s} = \gamma_{i'_1} + p\gamma_{i'_2} + \dots + p^{s-1}\gamma_{i'_s} \tag{2.3}$$

in F. We will be done if we prove the following.

Claim. *An equality of the form (2.3) implies that $i_j = i'_j$ for all $j \in \{1,\dots,s\}$.*

The claim can be proved by induction on s. The case $s = 1$ is trivial. The induction step follows if we show that the equality (2.3)implies that $i_1 = i'_1$. Assume $i_1 \neq i'_1$ and seek a contradiction. Recalling that $\gamma_1,\gamma_2,\dots,\gamma_n$ are $\mathbb{Z}$-linearly independent write the left-hand side of (2.3) as a sum $\sum_{i=1}^n c_i\gamma^i$ with $c_i \in \mathbb{Z}$ and write the right-hand side of (2.3) as a sum $\sum_{i=1}^n c'_i\gamma_i$ with $c'_i \in \mathbb{Z}$. So $c_i = c'_i$ for all i. Since $\gamma_{i_1} \neq \gamma_{i'_1}$ we get that $c_{i_1} \equiv 1 \bmod p$ while $c'_{i_1} \equiv 0 \bmod p$, a contradiction. This ends the proof of our claim and hence of our proposition. ∎

Remark 2.9. Note that, in spite of the fact that $s_1 := \phi_{(l)}^{(0)} = \phi_{(l)}$ and $s_2 := \phi_{(l)}^{(1)} = \tau_{(l)}\phi_{(l)}$ are monomially independent in $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$ we have that the subgroup of $\mathfrak{G}(K^{(l)}/\mathbb{Q}_p)$ generated by s_1 and s_2 is not freely generated by s_1 and s_2; indeed we have the following relation:

$$s_1(s_2s_1^{-1}) = (s_2s_1^{-1})^p s_1.$$

2.3 π-Frobenius lifts

Throughout the memoir we denote by Π the set of all elements $\pi \in \mathbb{Q}_p^{\text{alg}}$ such that there exists a finite Galois extension $E/\mathbb{Q}_p$ with the property that π is a prime element in $\mathcal{O}_E$. Note that Π consists exactly of those elements $\pi \in \mathbb{Q}_p^{\text{alg}}$ which are roots of Eisenstein polynomials with coefficients in $\mathbb{Z}_p^{\text{ur}}$ and for which $\mathbb{Q}_p^{\text{ur}}(\pi)/\mathbb{Q}_p$ is Galois. We have $\mathbb{Q}_p^{\text{alg}} = \mathbb{Q}_p^{\text{ur}}(\Pi)$. For any $\pi \in \Pi$ write $K_\pi = K(\pi)$ and let $R_\pi = R[\pi]$ which equals the valuation ring of K_π. We write $\pi'|\pi$ if and only if $K_\pi \subset K_{\pi'}$. Note that $K^{\text{alg}} = K(\Pi)$. Clearly for $\pi \in \Pi$ the field K_π is mapped into itself by every Frobenius automorphism ϕ of K^{alg}. By continuity of ϕ we have an induced automorphism $\phi_\pi : R_\pi \to R_\pi$ (which we sometimes still denote by ϕ) inducing the p-power Frobenius on $R_\pi/\pi R_\pi = k$.

Remark 2.10. We take the opportunity to correct here a typo in [12]: in the definition of Π of Section 2.1 the exponent "ur" in the condition "$\mathbb{Q}_p^{\text{ur}}(\pi)/\mathbb{Q}_p$ is Galois" was inadvertently dropped.

More generally we will need the following.

Definition 2.11. Let A be an R_π-algebra. By a *π-Frobenius lift* for an A-algebra $\varphi : A \to B$ we understand a ring homomorphism $\phi : A \to B$ such that the induced homomorphism $\overline{\phi} : A/\pi A \to B/\pi B$ equals the composition of the induced homomorphism $\overline{\varphi} : A/\pi A \to B/\pi B$ with the p-power Frobenius on $A/\pi A$. If $B = A$ and $\varphi = 1_A$ we say that ϕ is a *π-Frobenius lift* on A.

In particular, for every Frobenius automorphism ϕ of K^{alg} and every $\pi \in \Pi$ the induced automorphism ϕ_π of R_π is a π-Frobenius lift.

2.4 Rings of symbols

Definition 2.12. Consider a family $\Phi := (\phi_1, \ldots, \phi_n)$, $\phi_i \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ of distinct Frobenius automorphisms and let $\pi \in \Pi$. Let $\mathbb{M}_\Phi$ be the free monoid with identity on the set Φ; so we have an isomorphism $\mathbb{M}_n \simeq \mathbb{M}_\Phi$, $i \mapsto \phi_i$. We define the *ring of symbols* $K_{\pi,\Phi}$ to be the free K_π-module with basis $\mathbb{M}_\Phi$ equipped with multiplication defined by

$$\phi_i \cdot \lambda = \phi_i(\lambda) \cdot \phi_i \tag{2.4}$$

for $\lambda \in K_\pi$, $i \in \{1, \ldots, n\}$. If in the above definition we replace K_π we obtain a ring $R_{\pi,\Phi}$.

So every element in $K_{\pi,\Phi}$ (respectively $R_{\pi,\Phi}$) can be uniquely written as

$$\sum_{\mu \in \mathbb{M}_n} \lambda_\mu \phi_\mu$$

with λ_μ in K_π (respectively in R_π). These rings have filtrations "by order" given by the subgroups:

$$K^r_{\pi,\Phi} := \left\{ \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \mid \lambda_\mu \in K_\pi \right\} \subset K_{\pi,\Phi},$$
$$R^r_{\pi,\Phi} := \left\{ \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \mid \lambda_\mu \in R_\pi \right\} \subset R_{\pi,\Phi}.$$

The ring $K_{\pi,\Phi}$ is a K_π-linear space with left multiplication by scalars but, of course, it is not a K_π-algebra. If $\mathrm{End}_{\mathrm{gr}}(K^{\mathrm{alg}})$ denotes the ring of all group endomorphisms of K^{alg} then we have a natural K_π-linear ring homomorphism

$$K_{\pi,\Phi} \to \mathrm{End}_{\mathrm{gr}}(K^{\mathrm{alg}}), \quad \theta \mapsto \theta^{\mathrm{alg}}. \tag{2.5}$$

Remark 2.13. Note that if $\phi_1, \dots, \phi_n \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ are monomially independent in $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ then, by Lemma 2.6 (and in fact directly from Artin's "independence of characters") the natural ring homomorphism (2.5) is injective.

Remark 2.14. One can also consider the free ring $\mathbb{Z}_\Phi$ generated by Φ which we refer to as the ring of *integral symbols*; as an abelian group it is the free abelian group with basis $\mathbb{M}_\Phi$. So every element of this ring can uniquely be written as

$$w = \sum_{\mu \in \mathbb{M}_n} m_\mu \phi_\mu, \quad m_\mu \in \mathbb{Z}.$$

This ring has an order (with non-negative elements defined as those with non-negative coefficients) and has a filtration "by order" given by the subgroups $\mathbb{Z}^r_\Phi$ consisting of $\mathbb{Z}$-linear combinations of elements ϕ_μ with $\mu \in \mathbb{M}^r_n$. Then for all $\lambda \in R^\times_\pi$ and all $w \in \mathbb{Z}_\Phi$ we write

$$\lambda^w = \prod_{\mu \in \mathbb{M}_n} (\phi_\mu(\lambda))^{m_\mu} \in R^\times_\pi.$$

For every $w = \sum m_\mu \phi_\mu \in \mathbb{Z}_\Phi$ we define the *degree* of w to be $\deg(w) = \sum m_\mu$.

2.5 Partial π-jet spaces

For $\pi \in \Pi$ let $C_p(X,Y) \in \mathbb{Z}[X,Y]$ be the polynomial

$$C_p(X,Y) := \frac{X^p + Y^p - (X+Y)^p}{p}.$$

Following [6, 7, 23] a *π-derivation* from an R_π-algebra A into an A-algebra B is a map $\delta_\pi : A \to B$, $x \mapsto \delta_\pi x$, such that $\delta_\pi(1) = 0$ and

$$\delta_\pi(x+y) = \delta_\pi x + \delta_\pi y + \frac{p}{\pi} C_p(x,y),$$

$$\delta_\pi(xy) = x^p \cdot \delta_\pi y + y^p \cdot \delta_\pi x + \pi \cdot \delta_\pi x \cdot \delta_\pi y,$$

for all $x, y \in A$. Given a π-derivation as above and denoting by $\varphi : A \to B$ the structure map of the A-algebra B we always denote by $\phi_\pi : A \to B$ the map $\phi(x) = \varphi(x)^p + \pi\delta_\pi x$; then ϕ_π is a π-Frobenius lift. If π is a non-zero divisor in B then the above formula gives a bijection between the set of π-derivations from A to B and the set of π-Frobenius lifts from A to B.

Definition 2.15. By a *partial δ_π-ring* we understand an R_π-algebra A equipped with an n-tuple $(\delta_{\pi,1}, \dots, \delta_{\pi,n})$ of π-derivations $A \to A$. (We do *not* assume any "commutation relation" between them.)

Assume we are given a family $\Phi := (\phi_1, \dots, \phi_n) \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)^n$ of distinct Frobenius automorphisms of K^{alg}. Note that for every $\pi \in \Pi$ we get an induced tuple $\Phi_\pi = (\phi_{\pi,1}, \dots, \phi_{\pi,n})$ of (not necessarily distinct) π-Frobenius lifts on R_π, called the restriction of Φ to R_π. We therefore get an induced tuple $(\delta_{\pi,1}, \dots, \delta_{\pi,n})$ of π-derivations on R_π and hence a structure of partial δ_π-ring on R_π.

Following the lead of [6] we need to consider the following generalization of the notion of partial δ_π-ring.

Definition 2.16. Define a category $\mathbf{Prol}^*_{\pi,\Phi}$ as follows. An object of this category is a countable family of p-adically complete R_π-algebras $S^* = (S^r)_{r \geq 0}$ equipped with the following data:

(1) R_π-algebra homomorphisms $\varphi \colon S^r \to S^{r+1}$;

(2) π-derivations $\delta_{\pi,j} \colon S^r \to S^{r+1}$ for $1 \leq j \leq n$.

We require that $\delta_{\pi,i}$ be compatible with the π-derivations on R_π and with φ, i.e., $\delta_{\pi,j} \circ \varphi = \varphi \circ \delta_{\pi,j}$. Morphisms are defined in a natural way. We denote by $\phi_{\pi,j} : S^r \to S^{r+1}$ the corresponding π-Frobenius lifts, defined by $\phi_{\pi,j}(x) = \varphi(x)^p + \pi\delta_{\pi,j}x$. Also, for all $\mu := i_1 \dots i_l \in \mathbb{M}_n$ and all $x \in S^r$ we set $\delta_{\pi,\mu}x := (\delta_{\pi,i_1} \circ \dots \circ \delta_{\pi,i_l})(x) \in S^{r+l}$ and $\phi_{\pi,\mu}x := (\phi_{\pi,i_1} \circ \dots \circ \phi_{\pi,i_l})(x) \in S^{r+l}$.

The objects of $\mathbf{Prol}^*_{\pi,\Phi}$ are called *prolongation sequences* (over R_π with respect to Φ or Φ_π). We sometimes identify elements $a \in S^r$ with the elements $\varphi(a) \in S^{r+1}$ if no confusion arises. We sometimes write $S^* = (S^r, \varphi, \delta_{\pi,1}, \dots, \delta_{\pi,n})$. We denote by $\mathbf{Prol}_{\pi,\Phi}$ the full subcategory of $\mathbf{Prol}^*_{\pi,\Phi}$ whose objects are the prolongation sequences (S^r) such that all S^r's are Noetherian and flat over R_π.

Remark 2.17. (1) If S is a p-adically complete partial δ_π-ring whose π-derivations are compatible with those on R_π then the sequence $S^* = (S^r)$ with $S^r = S$ has a natural structure of object of $\mathbf{Prol}^*_{\pi,\Phi}$ with φ the identity and obvious $\delta_{\pi,j}$. If in addition S is Noetherian and flat over R_π then S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$. The initial object in $\mathbf{Prol}^*_{\pi,\Phi}$ (and also of $\mathbf{Prol}_{\pi,\Phi}$) is the sequence $R^*_\pi = (R^r_\pi)$ with $R^r_\pi := R_\pi$.

(2) If $S^* = (S^r, \varphi, \delta_{\pi,1}, \ldots, \delta_{\pi,n})$ is an object of $\mathbf{Prol}^*_{\pi,\Phi}$ then the ring

$$\lim_{\overrightarrow{\varphi}} S^r$$

has a natural structure of partial δ_π-ring.

Remark 2.18. For every $\pi'|\pi$ and every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ the sequence

$$S^* \otimes_{R_\pi} R_{\pi'} := (S^r \otimes_{R_\pi} R_{\pi'})_{r\geq 0}$$

is naturally an object of $\mathbf{Prol}_{\pi',\Phi}$; cf. [12, Section 4.1].

Remark 2.19. For $\mu = i_1 \ldots i_r \in \mathbb{M}_n^r \setminus \mathbb{M}_n^{r-1}$ we define the integral symbol:

$$w(\mu) := 1 + \phi_{i_1} + \phi_{i_1 i_2} + \phi_{i_1 i_2 i_3} + \cdots + \phi_{i_1 i_2 i_3 \ldots i_{r-1}} \in \mathbb{Z}_\Phi.$$

For every object $S^* = (S^r)$ in $\mathbf{Prol}^*_{\pi,\Phi}$, every $r \geq 1$, every $\mu \in \mathbb{M}_n^r \setminus \mathbb{M}_n^{r-1}$ and every $a \in S^0$ there exists $a_\mu \in S^{r-1}$ such that

$$\phi_{\pi,\mu} a = \pi^{w(\mu)} \delta_{\pi,\mu} a + \varphi(a_\mu); \tag{2.6}$$

this is trivially proved by induction on r.

Definition 2.20. Consider two families of distinct Frobenius automorphisms $\Phi' := (\phi'_1, \ldots, \phi'_{n'})$ and $\Phi'' := (\phi''_1, \ldots, \phi''_{n''})$ of K^{alg}. Also let $\pi \in \Pi$. A map of sets

$$\epsilon : \{1, \ldots, n'\} \to \{1, \ldots, n''\} \tag{2.7}$$

is called a *selection map* (with respect to (Φ', Φ'', π)) if for all $j \in \{1, \ldots, n'\}$ we have that $\phi_{\pi,j} = \phi_{\pi,\epsilon(j)}$. Consider next an object of $\mathbf{Prol}^*_{p,\Phi''}$,

$$S^* = (S^r, \varphi, \delta''_{\pi,1}, \ldots, \delta''_{\pi,n}),$$

and let ϵ be a selection map as above. One defines the object S^*_ϵ in $\mathbf{Prol}^*_{p,\Phi'}$ by:

$$S^*_\epsilon := (S^r, \varphi, \delta''_{\pi,\epsilon(1)}, \ldots, \delta''_{\pi,\epsilon(n')}).$$

This construction depends only on the restrictions Φ'_π and Φ''_π of Φ' and Φ'' to K_π.

Motivated by Proposition 2.8, introduce variables denoted by $\delta_{\pi,\mu} y_j$ for $\mu \in \mathbb{M}_n$, $\pi \in \Pi$, $j \in \{1, \ldots, N\}$. Fix an integer N and consider the ring $R_\pi[y_1, \ldots, y_N]$ and the rings

$$J^r_{\pi,\Phi}(R_\pi[y_1, \ldots, y_N]) := R_\pi[\delta_{\pi,\mu} y_j \mid \mu \in \mathbb{M}_n^r, j \in \{1, \ldots, N\}]\widehat{\ }. \tag{2.8}$$

The sequence $J^*_{\pi,\Phi}(R_\pi[y_1, \ldots, y_N]) := (J^r_{\pi,\Phi}(R_\pi[y_1, \ldots, y_N]))$ has a unique structure of object in $\mathbf{Prol}_{\pi,\Phi}$ such that $\delta_{\pi,i} \delta_{\pi,\mu} y := \delta_{\pi,i\mu} y$ for all $i = 1, \ldots, n$. We have an induced evaluation map $F_{R_\pi} : R_\pi^N \to R_\pi$: for $(a_1, \dots, a_N) \in R_\pi^N$ we let $F_{R_\pi}(a_1, \dots, a_N) \in R_\pi$ be obtained from F by replacing the variables $\delta_{\pi,\mu} y_j$ with the elements $\delta_{\pi,\mu} a_j$. Note that the map

$$J^r_{\pi,\Phi}(R_\pi[y_1, \dots, y_N]) \to \mathrm{Fun}(R_\pi^N, R_\pi), \; F \mapsto F_{R_\pi} \tag{2.9}$$

is not injective in general, even if Φ is monomially independent. Here and in the following "Fun" stands for the set of set-theoretic maps. For instance if $\pi = p$ we have $(\delta_{p,i} y - \delta_{p,j} y)_R = 0$. This is in stark contrast with [12]. See, however, Remark 2.34.

Definition 2.21. For every R_π-algebra of finite type $A := R_\pi[y_1, \dots, y_N]/I$, we define

$$J^r_{\pi,\Phi}(A) := J^r_{\pi,\Phi}(R_\pi[y_1, \dots, y_N])/(\delta_{\pi,\mu} I \mid \mu \in \mathbb{M}^r_n).$$

This algebra is called the *partial π-jet algebra* of A of order r.

Note that $J^r_{\pi,\Phi}(A)$ is Noetherian and p-adically complete but generally not flat over R_π, even if $\pi = p$ and A is flat over R_π as one can see by taking $A = R[x]/(x^p)$. It is trivial to see that the sequence $J^*_{\pi,\Phi}(A) := (J^r_{\pi,\Phi}(A))$ has a natural structure of prolongation sequence, i.e., it is an object of $\mathbf{Prol}^*_{\pi,\Phi}$ (but, as just noted, it is not generally an object of $\mathbf{Prol}_{\pi,\Phi}$). Also note that $J^r_{\pi,\Phi}(A)$ depends only on r, π, A and on the restriction Φ_π of Φ to R_π.

Proposition 2.22. *If A is a smooth R_π-algebra, and $u\colon R_\pi[T_1, \dots, T_d] \to A$ is an étale morphism of R_π-algebras, then there is a (unique) isomorphism*

$$A[\delta_{\pi,\mu} T_j \mid \mu \in \mathbb{M}^{r,+}_n, j \in \{1, \dots, d\}]\widehat{\ } \cong J^r_{\pi,\Phi}(A)$$

sending $\delta_{\pi,\mu} T_j$ into $\delta_{\pi,\mu}(u(T_j))$ for all j and μ. In particular, $J^r_{\pi,\Phi}(A)$ is flat over R_π so the sequence $J^*_{\pi,\Phi}(A)$ is an object of $\mathbf{Prol}_{\pi,\Phi}$.

Proof. Similar to [10, Proposition 3.13]. ∎

We have the following universal property.

Proposition 2.23. *Assume A is a finitely generated (respectively smooth) R_π-algebra. For every object T^* of $\mathbf{Prol}^*_{\pi,\Phi}$ (respectively in $\mathbf{Prol}_{\pi,\Phi}$) and every R_π-algebra map $u : A \to T^0$ there is a unique morphism $J^*_{\pi,\Phi}(A) \to T^*$ over S^* in $\mathbf{Prol}^*_{\pi,\Phi}$ (respectively in $\mathbf{Prol}_{\pi,\Phi}$) compatible with u.*

Proof. Similar to [10, Proposition 3.3]. ∎

We next record the existence of "prolongations of derivations." Let S be a ring. Recall that by an *S-derivation* from an S-algebra A to an A-algebra B one understands an S-module endomorphism $A \to B$ satisfying the Leibniz rule.

Proposition 2.24. *Let A be a smooth R_π-algebra equipped with an R_π-derivation $D : A \to A$. Then for every $r \geq 1$ and every $\mu \in \mathbb{M}_n^r$ there exists a unique R_π-derivation $D_\mu : J^r_{\pi,\Phi}(A) \to J^r_{\pi,\Phi}(A)$ satisfying the following properties:*

(1) $D_\mu \phi_\mu a = p^r \cdot \phi_\mu Da$ *for all $a \in A$;*

(2) $D_\mu \phi_\nu a = 0$ *for all $a \in A$ and all $\nu \in \mathbb{M}_n^r \setminus \{\mu\}$.*

Proof. Similar to [10, Proposition 3.43]. We recall the argument. Uniqueness is clear. To prove existence let $u : S := R_\pi[T_1, \dots, T_d] \to A$ be an étale map and let $a_i := DT_i \in A$. Then consider the derivation

$$\frac{p^r}{\pi^{w(\mu)}} \sum_{i=1}^{d} a_i^{\phi_\mu} \frac{\partial}{\partial \delta_{\pi,\mu} T_i} : J^r_{\pi,\Phi}(S) = R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r] \to J^r_{\pi,\Phi}(A).$$

By Proposition 2.22 this derivation extends to a derivation $D_\mu : J^r_{\pi,\Phi}(A) \to J^r_{\pi,\Phi}(A)$. To check properties (1) and (2) it is enough to check them for $a = T_i$ because if (1) and (2) hold for two elements of $J^r_{\pi,\Phi}(S)$ then (1) and (2) hold for their sum and their product. But for $a = T_i$ the equalities (1) and (2) hold in view of formula (2.6). ∎

The jet construction can be globalized as follows.

Definition 2.25. For every smooth scheme X over R_π define the p-adic formal scheme

$$J^r_{\pi,\Phi}(X) = \bigcup \mathrm{Spf}(J^r_{\pi,\Phi}(\mathcal{O}(U_i))),$$

called the *partial π-jet space* of order r of X, where $X = \bigcup U_i$ is (any) affine open cover. The gluing involved in this definition is well defined because the formation of π-jet spaces is compatible with fractions; cf. Proposition 2.22. The elements of the ring $\mathcal{O}(J^r_{\pi,\Phi}(X))$, identified with morphisms of p-adic formal schemes $J^r_{\pi,\Phi}(X) \to \widehat{\mathbb{A}^1}$, are called (purely) *arithmetic PDEs* on X over R_π of order $\leq r$.

For all $\pi'|\pi$ we write $X_{\pi'} := X \otimes_{R_\pi} R_{\pi'}$. Clearly $J^0_{\pi',\Phi}(X_{\pi'}) = \widehat{X_{\pi'}}$. Note also that $J^r_{\pi',\Phi}(X_{\pi'})$ only depends on r, π', X and on the restriction Φ_π of Φ to R_π.

Proposition 2.26. *Assume A is a smooth R_π-algebra. For all $\pi''|\pi'|\pi$ there are natural homomorphisms*

$$\iota_{\pi'',\pi'} : J^r_{\pi'',\Phi}(A) \to J^r_{\pi',\Phi}(A) \otimes_{R_{\pi'}} R_{\pi''} \tag{2.10}$$

such that the homomorphism

$$\iota_{\pi'',\pi} : J^r_{\pi'',\Phi}(A) \to J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi''} \tag{2.11}$$

equals the composition

$$J^r_{\pi'',\Phi}(A) \xrightarrow{\iota_{\pi'',\pi'}} J^r_{\pi',\Phi}(A) \otimes_{R_{\pi'}} R_{\pi''} \xrightarrow{\iota_{\pi',\pi} \otimes 1} (J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi'}) \otimes_{R_{\pi'}} R_{\pi''}, \tag{2.12}$$

where the targets of the maps (2.11) *and* (2.12) *are naturally identified. Moreover, the homomorphisms* (2.10) *are injective.*

Proof. This follows similarly to [12, Proposition 4.1 (1)] and [14, Proposition 2.2]. The map $\iota_{\pi'',\pi'}$ is guaranteed by Proposition 2.23 as $(J^r_{\pi',\Phi}(A) \otimes_{R_{\pi'}} R_{\pi''})$ is naturally an object of $\mathbf{Prol}_{\pi'',\Phi}$; cf. Remark 2.18. The factorization (2.11) arises from naturality of base change. Finally, to address the injectivity of (2.10), pick an étale homomorphism $R_\pi[T_1, \ldots, T_d] \to A$. Both the source and target of (2.10) then embed in the common ring

$$K_{\pi''}[\![\delta_{\pi'',\mu} T_j \mid \mu \in \mathbb{M}^r_n, j = 1, \ldots, d]\!] \cong K_{\pi''}[\![\delta_{\pi',\mu} T_j \mid \mu \in \mathbb{M}^r_n, j = 1, \ldots, d]\!]$$

recovering the natural base change (2.10) from which the injectivity is clear. ∎

Remark 2.27. For every smooth algebra A over R_π and every selection map ϵ with respect to (Φ', Φ'', π) we get (by the universality property of J^r) a natural morphism of prolongation sequences over R_π with respect to Φ', $J^*_{\pi,\Phi'}(A) \to J^*_{\pi,\Phi''}(A)_\epsilon$, cf. Definition 2.20 for the subscript notation. Hence for every smooth scheme X over R_π we get morphisms

$$J^r_{\pi,\Phi''}(X) \to J^r_{\pi,\Phi'}(X). \tag{2.13}$$

We shall be interested later in four special cases of this construction.

(1) Assume $\pi = p$, $\Phi' = \Phi''$, and $\epsilon : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection. Then the above construction defines an action of the symmetric group Σ_n on $J^r_{\pi,\Phi}(X)$.

(2) Assume $n' = s$, $n'' = n$, $\Phi' = (\phi'_1, \ldots, \phi'_s)$, $\Phi'' = \Phi = (\phi_1, \ldots, \phi_n)$,

$$\phi'_1 = \phi_{i_1}, \ldots, \phi'_s = \phi_{i_s}, \ 1 \le i_1 < i_2 < \cdots < i_s \le n, \ \epsilon(1) = i_1, \ldots, \epsilon(s) = i_s.$$

Then we get a natural morphism (referred to as a *face* morphism)

$$J^r_{\pi,\Phi}(X) = J^r_{\pi,\phi_1,\ldots,\phi_n}(X) \to J^r_{\pi,\phi_{i_1},\ldots,\phi_{i_s}}(X).$$

(3) Assume $\pi = p$, $n' = n$, $n'' = 1$, $\Phi' = \Phi = (\phi_1, \ldots, \phi_n)$, $\Phi'' = \{\phi\}$, and hence ϵ is the constant map. Then we get a natural morphism (referred to as the *degeneration* morphism):

$$J^r_{\pi,\phi}(X) \to J^r_{\pi,\Phi}(X).$$

(4) Assume $\pi = p$ and $\Phi = \{\phi_1, \ldots, \phi_n\}$. Then one trivially checks that for all $i \in \{1, \ldots, n\}$ the composition of the face and degeneration morphisms below is the identity:

$$\mathrm{id} : J^r_{p,\phi_i}(X) \to J^r_{p,\Phi}(X) \to J^r_{p,\phi_i}(X).$$

2.6 Total δ-overconvergence

The notion of δ-overconvergence was introduced in [14] and exploited in [12], cf. [12, Definition 2.5].

Definition 2.28. Assume A is a smooth R_π-algebra. An element $f_\pi \in J^r_{\pi,\Phi}(A)$ is called *totally δ-overconvergent* if it has the following property: for all $\pi'|\pi$ there exists an integer $N \geq 0$ such that $p^N f_\pi \otimes 1$ is in the image of the map

$$\iota_{\pi',\pi} : J^r_{\pi',\Phi}(A) \to J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi'}. \tag{2.14}$$

Let us denote by $J^r_{\pi,\Phi}(A)^\dagger$ the R-algebra of all totally δ-overconvergent elements in $J^r_{\pi,\Phi}(A)$. For every smooth scheme X/R_π an element (arithmetic PDE), $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))$, will be called *totally δ-overconvergent* if for all affine open set $U \subset X$ (equivalently for every affine open set of a given affine open cover of X) the image of f in the ring $\mathcal{O}(J^r_{\pi,\Phi}(U)) = J^r_{\pi,\Phi}(\mathcal{O}(U))$ is totally δ-overconvergent. We denote by $\mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ the ring of all totally δ-overconvergent elements of $\mathcal{O}(J^r_{\pi,\Phi}(X))$. A morphism $J^r_{\pi,\Phi}(X) \to \widehat{\mathbb{A}^1}$ will be called *totally δ-overconvergent* if the corresponding element in $\mathcal{O}(J^r_{\pi,\Phi}(X))$ is totally δ-overconvergent.

Remark 2.29. We caution the reader about the notation $\dagger$. It is common for $\dagger$ superscripts to also denote overconvergence in a difference sense. Specifically, these superscripts are used extensively in the overconvergent Witt vectors or Monsky–Washnitzer algebras of rigid geometry. This memoir is written entirely in the formal setting. There are certainly overlaps between concepts used here and those in rigid geometry, however they remain for now in different realms. We hope this notation causes no confusion. To elucidate, all uses of $\dagger$ are in reference to δ-overconvergence.

Note that, again, the ring $\mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ depends only on r, π, X and on the restriction Φ_π of Φ to R_π.

Using Proposition 2.22 one trivially checks the following two propositions.

Proposition 2.30. *For every smooth scheme X over R_π, every $r \geq 0$, and every map as in (2.7) the ring homomorphisms*

$$\mathcal{O}(J^r_{\pi,\Phi'}(X)) \to \mathcal{O}(J^r_{\pi,\Phi''}(X))$$

induced by the morphisms (2.13) induce ring homomorphisms

$$\mathcal{O}(J^r_{\pi,\Phi'}(X))^\dagger \to \mathcal{O}(J^r_{\pi,\Phi''}(X))^\dagger.$$

We will usually view the above ring homomorphisms as inclusions.

Proposition 2.31. *Assume that $u : \widehat{X} \to \widehat{Y}$ is a morphism between the p-adic completions of two smooth R_π-schemes and let $f : J^r_{\pi,\Phi}(Y) \to \widehat{\mathbb{A}^1}$ be a totally δ-overconvergent morphism. Then the composition*

$$J^r_{\pi,\Phi}(X) \stackrel{J^r(u)}{\longrightarrow} J^r_{\pi,\Phi}(Y) \stackrel{f}{\to} \widehat{\mathbb{A}^1}$$

is totally δ-overconvergent, where $J^r(u)$ is the morphism induced by u via the universal property.

Similarly to [12] we make the following definition.

Definition 2.32. For every $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))$ and every object $S^* = (S^r)$ in $\mathbf{Prol}_{\pi,\Phi}$ the universal property of π-jet spaces yields a map of sets

$$f_{S^*} : X(S^0) \to S^r. \tag{2.15}$$

On the other hand, if $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ then for every object $S^* = (S^r)$ in $\mathbf{Prol}_{\pi,\Phi}$ we can define the map of sets

$$f^{\mathrm{alg}}_{S^*} : X(S^0 \otimes_{R_\pi} R^{\mathrm{alg}}) \to S^r \otimes_{R_\pi} K^{\mathrm{alg}} \tag{2.16}$$

as follows. We may assume $X = \mathrm{Spec} A$ is affine because the construction below allows gluing in the obvious sense. Let $P \in X(S^0 \otimes_{R_\pi} R^{\mathrm{alg}})$. Choose $\pi'|\pi$ such that $P \in X(S^0 \otimes_{R_\pi} R_{\pi'})$ and choose $N \geq 1$ such that $p^N f \otimes 1 \in J^r_{\pi,\Phi}(A) \otimes_{R_\pi} R_{\pi'}$ is the image of some (necessarily unique) element $f_{\pi',N} \in J^r_{\pi',\Phi}(A)$ via the map (2.14). View P as a morphism $P : A \to S^0 \otimes_{R_\pi} R_{\pi'}$. By the universal property of π'-jet spaces we have an induced morphism $J^r(P) : J^r_{\pi',\Phi}(A) \to S^r \otimes_{R_\pi} R_{\pi'}$. Then we define

$$f^{\mathrm{alg}}_{S^*}(P) = p^{-N}(J^r(P))(f_{\pi',N}) \in S^r \otimes_{R_\pi} K_{\pi'} \subset S^r \otimes_{R_\pi} K^{\mathrm{alg}}.$$

The definition is independent of the choice of π' and N due to the injectivity part of Proposition 2.26. On the other hand $f^{\mathrm{alg}}_{S^*}$ effectively depends on Φ (and not only on the restriction Φ_π on K_π). For $S^* = R^*_\pi$ we write $f_{R_\pi} := f_{R^*_\pi}$ and

$$f^{\mathrm{alg}} := f^{\mathrm{alg}}_{R_\pi} := f^{\mathrm{alg}}_{R^*_\pi} : X(R^{\mathrm{alg}}) \to K^{\mathrm{alg}}. \tag{2.17}$$

Proposition 2.33. *Let $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))$ and assume the map f_{S^*} is the zero map for every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ with the property that S^r are integral domains and $\varphi : S^r \to S^{r+1}$ are injective. Then $f = 0$. In particular, if $f \in \mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger$ and the map $f^{\mathrm{alg}}_{S^*}$ is the zero map for every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ as above, then $f = 0$.*

Proof. Take $S^* = (S^r)$, $S^r := \mathcal{O}(J^r_{\pi,\Phi}(U))$ for various affine open sets $U \subset X$; one gets that the image of f in $\mathcal{O}(J^r_{\pi,\Phi}(U))$ is 0, hence $f = 0$. ∎

Remark 2.34. Assume $\phi_1, \dots, \phi_n$ are monomially independent in $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$. It would be interesting to know when/if the ring homomorphism

$$\mathcal{O}(J^r_{\pi,\Phi}(X))^\dagger \to \mathrm{Fun}(X(R^{\mathrm{alg}}), K^{\mathrm{alg}}), \quad f \mapsto f^{\mathrm{alg}} \tag{2.18}$$

is injective. For $n = 1$ this is true; cf. [12, proof of Proposition 4.4]. See also Proposition 3.13 and Proposition 7.38 for related results. Clearly, if we do not assume $\phi_1, \dots, \phi_n$ are monomially independent in $\mathfrak{G}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ then (2.18) is not injective in general: to get an example take X the affine line, $n = 2$, and $\phi_1 = \phi_2$.