Chapter 3

Partial δ-characters

3.1 Definition and the additive group

We start with the following PDE definition extending the ODE case in [6].

Definition 3.1. A *partial δ_π-character of order $\leq r$* of a commutative smooth group scheme G/R_π is a group homomorphism $J^r_{\pi,\Phi}(G) \to \widehat{\mathbb{G}_a}$ in the category of p-adic formal schemes. (So in particular a partial δ_π-character can be identified with an element of $\mathcal{O}(J^r_{\pi,\Phi}(G))$, i.e., with an arithmetic PDE of order $\leq r$.) We denote by

$$\mathbf{X}^r_{\pi,\Phi}(G) := \mathrm{Hom}(J^r_{\pi,\Phi}(G), \widehat{\mathbb{G}_a})$$

the R_π-module of partial δ_π-characters of G of order $\leq r$ which we identify with a submodule of $\mathcal{O}(J^r_{\pi,\Phi}(G))$. For $\pi'|\pi$ we set $\mathbf{X}_{\pi',\Phi}(G) := \mathbf{X}_{\pi',\Phi}(G_{\pi'})$, and we call the elements of the latter *partial $\delta_{\pi'}$-characters* of G. For $n = 1$ partial δ_π-characters will be referred to as *ODE δ_π-characters*. An element of $\mathbf{X}^r_{\pi,\Phi}(G)$ will be said to have order r if it is not in the image of the canonical (injective) map $\varphi : \mathbf{X}^{r-1}_{\pi,\Phi}(G) \to \mathbf{X}^r_{\pi,\Phi}(G)$ induced by φ. We also consider the naturally induced semilinear maps ϕ_i : $\mathbf{X}^{r-1}_{\pi,\Phi}(G) \to \mathbf{X}^r_{\pi,\Phi}(G)$ induced by ϕ_i.

Consider the R_π-module $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$ of totally δ-overconvergent partial δ_π-characters of G. So inside the ring $\mathcal{O}(J^r_{\pi,\Phi}(G))$ we have

$$\mathbf{X}^r_{\pi,\Phi}(G)^\dagger = \mathbf{X}^r_{\pi,\Phi}(G) \cap \mathcal{O}(J^r_{\pi,\Phi}(G))^\dagger.$$

Note that if $\psi \in \mathbf{X}^r_{\pi,\Phi}(G)$ and if $p^N\psi \otimes 1 \in \mathcal{O}(J^r_{\pi,\Phi}(G)) \otimes_{R_\pi} R_{\pi'}$ is the image of some (necessarily unique) $\psi_{\pi'} \in \mathcal{O}(J^r_{\pi',\Phi}(G))$ then $\psi_{\pi'} \in \mathbf{X}^r_{\pi',\Phi}(G)$. In particular, we have the following.

Lemma 3.2. *The image of the natural map*

$$\mathbf{X}^r_{\pi,\Phi}(G)^\dagger \to \mathrm{Fun}(G(R^{\mathrm{alg}}), K^{\mathrm{alg}}), \quad \psi \mapsto \psi^{\mathrm{alg}}$$

is contained in $\mathrm{Hom}_{\mathrm{gr}}(G(R^{\mathrm{alg}}), K^{\mathrm{alg}})$ *where* $\mathrm{Hom}_{\mathrm{gr}}$ *is the* Hom *in the category of abstract groups.*

We also record the following obvious lemma.

Lemma 3.3. *If an element ψ of $\mathbf{X}^r_{\pi,\Phi}(G)$ times a power of p belongs to $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$ then ψ itself belongs to $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$.*

We have the following description of partial δ_π-characters of the additive group $\mathbb{G}_a = \operatorname{Spec} R_\pi[T]$. For $\mu = i_1 \dots i_s \in \mathbb{M}_n$ and $r \geq s$ recall that we write

$$\phi_{\pi,\mu} T := \phi_{\pi,i_1} \dots \phi_{\pi,i_s} T \in \widehat{R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]} = \mathcal{O}(J^r_{\pi,\Phi}(\mathbb{G}_a)).$$

Consider the embedding

$$\mathcal{O}(J^r_{\pi,\Phi}(\mathbb{G}_a)) \subset K_\pi[\![\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!] = K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!]$$

and consider the groups

$$K^r_{\pi,\Phi} T := \sum_{\mu \in \mathbb{M}_n^r} K_\pi \phi_{\pi,\mu} T \subset K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!],$$
$$R^r_{\pi,\Phi} T := \sum_{\mu \in \mathbb{M}_n^r} R_\pi \phi_{\pi,\mu} T \subset K^r_{\pi,\Phi} T.$$

These groups are naturally isomorphic to the groups of symbols $K^r_{\pi,\Phi}$ and $R^r_{\pi,\Phi}$, respectively.

Proposition 3.4. *The following equality holds,*

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a) = (K^r_{\pi,\Phi} T) \cap (R_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]), \tag{3.1}$$

where the intersection is taken inside the ring $K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!]$. In particular

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a) = \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a)^\dagger.$$

Proof. The inclusion $\supset$ in (3.1) is clear. To check the inclusion $\subset$ note that every element in $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a)$ defines an additive polynomial in the ring $K_\pi[\![\phi_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]\!]$ hence, since K_π has characteristic 0, a linear polynomial. ■

Lemma 3.5. *For $\psi := \sum_{\mu \in \mathbb{M}_n^r} \lambda_\mu \phi_\mu T$ in the group in (3.1) the following hold:*

(1) $\lambda_0 \in R_\pi$.

(2) *If $\psi \equiv 0 \bmod T$ in $K_\pi[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}_n^r]$ then $\lambda_\mu = 0$ for all $\mu \in \mathbb{M}_n^{r,+}$.*

(3) *If $n = 1$ then $\lambda_\mu \in R_\pi$ for all $\mu \in \mathbb{M}_n^r$. In other words the intersection in the right-hand side of (3.1) equals $R^r_{\pi,\Phi} T$.*

Proof. Part 1 follows by picking out the coefficient of T.

Part 2 follows by induction on the number of non-zero terms in ψ. For the induction step one orders $\mathbb{M}_n^r$ by letting all members of $\mathbb{M}_n^s \setminus \mathbb{M}_n^{s-1}$ be greater than all members of $\mathbb{M}_n^{s-1}$ for all $s \in \{1, \dots, r\}$ and by taking an arbitrary total order on each set $\mathbb{M}_n^s \setminus \mathbb{M}_n^{s-1}$. Then one picks out the coefficient of $\delta_{\pi,\mu} T$ in ψ where μ is the largest element in $\mathbb{M}_n^r$ with $\delta_{\pi,\mu} T$ appearing in ψ.

Part 3 follows again by induction on the number of non-zero terms in ψ. For the induction step one picks out the coefficient of T^{p^n} in ψ where n is the largest integer such that T^{p^n} appears in ψ. ■

Remark 3.6. Assertion 3 in Lemma 3.5 fails for $n \geq 2$. For instance, for $\pi = p$, one immediately checks that

$$\frac{1}{p}\phi_1\phi_2 T - \frac{1}{p}\phi_2\phi_1 T \in \mathbf{X}^2_{p,\Phi}(\mathbb{G}_a) \setminus (R^r_\Phi T).$$

3.2 Picard–Fuchs symbol

Definition 3.7. Let G have relative dimension 1 over R_π and let ω be an invariant 1-form on G. By an *admissible coordinate* for G we mean an étale coordinate $T \in \mathcal{O}(U)$ on a neighborhood U of the origin of G generating the ideal of the origin of G in $\mathcal{O}(U)$. Let

$$\ell(T) = \ell_\omega(T) = \sum_{m=1}^{\infty} \frac{b_m}{m} T^m \in K_\pi[\![T]\!], \;\; b_m \in R_\pi,$$

be the logarithm of the formal group of G (with respect to T) normalized with respect to ω; so ℓ is the unique series in $K_\pi[\![T]\!]$ without constant term such that $d\ell = \omega$ in $K_\pi[\![T]\!]dT$. (We have $b_1 \in R_\pi^\times$.) Let $e(T) = e_\omega(T) \in K_\pi[\![T]\!]$ be the exponential normalized with respect to ω, i.e., the compositional inverse of $\ell(T)$. Then the series $e(pT)$ belongs to $pR_\pi\widehat{[T]}$ and so defines a morphism of groups in the category of p-adic formal schemes, $\mathcal{E} : \widehat{\mathbb{G}_a} \to \widehat{G}$. For every partial δ_π-character $\psi \in \mathbf{X}^r_{\pi,\Phi}(G)$ the composition

$$\theta(\psi) : J^r_{\pi,\Phi}(\mathbb{G}_a) \xrightarrow{\mathcal{E}^r} J^r_{\pi,\Phi}(G) \xrightarrow{\psi} \widehat{\mathbb{G}_a}$$

is a partial δ_π-character of $\mathbb{G}_a$ so, identified with an element of $\mathcal{O}(J^r_{\pi,\Phi}(\mathbb{G}_a))$, can be written (cf. Proposition 3.4 and Lemma 3.5, Part 1) as

$$\theta(\psi) = \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_{\pi,\mu} T \in \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a) \subset K^r_{\pi,\Phi} T, \;\; \lambda_\mu \in K_\pi, \;\; \lambda_0 \in R_\pi. \tag{3.2}$$

We define the *Picard–Fuchs symbol* (still denoted by $\theta(\psi)$) of ψ (with respect to T and ω) by

$$\theta(\psi) := \sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \in K^r_{\pi,\Phi}.$$

The latter induces a $\mathbb{Q}_p$-linear map

$$\theta(\psi)^{\text{alg}} : K^{\text{alg}} \to K^{\text{alg}}.$$

Remark 3.8. (1) By our very definition, viewing ψ as an element of $R_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}^r_n]\!]$, we have the following equality in $K_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}^r_n]\!]$:

$$\psi = \frac{1}{p}\theta(\psi)\ell(T).$$

(2) For every ψ, writing $\theta(\psi) = \sum_\mu \lambda_\mu \phi_\mu$ we have that

$$\lambda_0 \in pR_\pi.$$

Indeed, by the equality in Part 1 we have that

$$\theta(\psi)\ell(T) \in pR_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^r]\!]$$

and we are done by picking out the coefficient of T.

(3) The map

$$\theta : \mathbf{X}^r_{\pi,\Phi}(G) \to K^r_{\pi,\Phi}, \quad \psi \mapsto \theta(\psi) \tag{3.3}$$

is a group homomorphism. Moreover, for all μ we have

$$\theta(\phi_\mu \psi) = \phi_\mu \theta(\psi).$$

(4) If $\pi = p$ then the action of Σ_n on $J^r_{p,\Phi}(G)$ induces an action of Σ_n on $\mathbf{X}^r_{p,\Phi}(G)$. We also have an obvious action of Σ_n on K^r_Φ and the homomorphism (3.3) is Σ_n-equivariant.

In what follows let $\mathfrak{m} = \mathfrak{m}(R^{\text{alg}})$ be the maximal ideal of R^{alg}, let

$$G(\mathfrak{m}) := \text{Ker}(G(R^{\text{alg}}) \to G(k)),$$

and let $\ell^{\text{alg}} : G(\mathfrak{m}^{\text{alg}}) \to K^{\text{alg}}$ be the map induced by the logarithm series $\ell(T)$.

Corollary 3.9. *Let $\psi \in \mathbf{X}_{\pi,\Phi}(G)^\dagger$ be a totally δ-overconvergent δ_π-character and consider the homomorphism $\psi^{\text{alg}} : G(R^{\text{alg}}) \to K^{\text{alg}}$. The following hold:*

(1) *The restriction $\psi^{\mathfrak{m}}$ of ψ^{alg} to $G(\mathfrak{m})$ fits into a commutative diagram*

$$\begin{array}{ccc} G(\mathfrak{m}) & \xrightarrow{\ell^{\text{alg}}} & K^{\text{alg}} \\ {\scriptstyle \psi^{\mathfrak{m}}}\downarrow & & \downarrow{\scriptstyle \theta(\psi)^{\text{alg}}} \\ K^{\text{alg}} & = & K^{\text{alg}} \end{array} \tag{3.4}$$

(2) *The homomorphism ψ^{alg} can be extended to a (necessarily unique) continuous homomorphism $\psi^{\mathbb{C}_p} : G(\mathbb{C}_p^\circ) \to \mathbb{C}_p$.*

Proof. Part 1 follows directly from Remark 3.8, Part 1. To check Part 2 note that since ψ^{alg} is a homomorphism it is enough to check it can be extended by continuity on a ball in $G(\mathbb{C}_p^\circ)$ around the origin, cf. [12, Section 4.2]. This follows directly, exactly as in [12, proof of Proposition 6.8], from Part 1. ∎

The following is a PDE version of the arithmetic ODE analogue (cf., [6, 8]) of Manin's Theorem of the kernel [27].

Corollary 3.10. *For every* $\psi \in \mathbf{X}_{\pi,\Phi}(G)^{\dagger}$ *there is a natural group isomorphism*

$$(\mathrm{Ker}(\psi^{\mathrm{alg}})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}). \tag{3.5}$$

Proof. The exact sequence

$$0 \to G(\mathfrak{m}) \to G(R^{\mathrm{alg}}) \to G(k)$$

induces an exact sequence

$$0 \to \mathrm{Ker}(\psi^{\mathfrak{m}}) \to \mathrm{Ker}(\psi^{\mathrm{alg}}) \to G(k).$$

Since the group $G(k)$ is torsion we get an induced group isomorphism

$$(\mathrm{Ker}(\psi^{\mathfrak{m}})) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} (\mathrm{Ker}(\psi^{\mathrm{alg}})) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.6}$$

On the other hand recall that the map ℓ^{alg} in diagram (3.4) has a torsion kernel and cokernel; cf. [33, Proposition 3.2 and Theorem 6.4]. So tensoring the diagram (3.4) with $\mathbb{Q}$ the resulting upper horizontal map is an isomorphism. Taking the kernels of the resulting vertical maps we get an induced group isomorphism

$$(\mathrm{Ker}(\psi^{\mathfrak{m}})) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}). \tag{3.7}$$

We are done by considering the composition of the map (3.7) with the inverse of the map (3.6). ∎

The following strengthened version of the above corollary is sometimes useful. Let L be a filtered union of complete subfields of K^{alg}, let $\mathcal{O}$ be the valuation ring of L, and let $\mathfrak{m}(\mathcal{O})$ be the maximal ideal of $\mathcal{O}$. Assume G comes via base change from a smooth group scheme $G_{\mathcal{O}}$ over $\mathcal{O}$ and write $G_{\mathcal{O}}(\mathcal{O}) =: G(\mathcal{O})$.

Corollary 3.11. *For every* $\psi \in \mathbf{X}_{\pi,\Phi}(G)^{\dagger}$ *the isomorphism in Corollary* 3.10 *induces an isomorphism*

$$(\mathrm{Ker}(\psi^{\mathrm{alg}}) \cap G(\mathcal{O})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}) \cap L. \tag{3.8}$$

In particular, if $\mathrm{Ker}(\theta(\psi)^{\mathrm{alg}}) \cap L = 0$ *then*

$$\mathrm{Ker}(\psi^{\mathrm{alg}}) \cap G(\mathcal{O}) = G(\mathcal{O})_{\mathrm{tors}}.$$

Proof. It is enough to prove this for L complete. Let x be in the left-hand side of (3.8), hence in the left-hand side of (3.5). The image x' of x in the right-hand side of (3.5) is obtained as follows. One takes an integer $N \geq 1$ such that $Nx = P \otimes 1$ with $P \in \mathrm{Ker}(\psi^{\mathfrak{m}})$. Then $x' = \frac{1}{N}\ell^{\mathrm{alg}}(P)$. Since L is complete and ℓ has coefficients in $\mathcal{O}$ we get that ℓ^{alg} sends $G_{\mathcal{O}}(\mathfrak{m}(\mathcal{O}))$ into L, so we get that $x' \in L$ hence x' is in the right-hand side of (3.8). Conversely, let y' be in the right-hand side of (3.8). The image y of y' in the left-hand side of (3.5) is obtained as follows. There exists an integer $N \geq 1$ such that $Ny = \ell^{\text{alg}}(P) \otimes 1$ with $P \in G_{\mathcal{O}}(\mathfrak{m}(\mathcal{O}))$. By diagram (3.4) we have $P \in \text{Ker}(\psi^{\text{alg}})$. Then $y = P \otimes \frac{1}{N}$. So y is in the left-hand side of (3.5). ■

We end by providing an easy dimension evaluation. Define:

$$D(n,r) := \#\mathbb{M}_n^r = 1 + n + n^2 + \cdots + n^r. \tag{3.9}$$

The following proposition is trivial to check.

Proposition 3.12. *The map $\mathbf{X}^r_{\pi,\Phi}(G) \to K^r_{\pi,\Phi}$, $\psi \mapsto \theta(\psi)$ is injective. In particular*

$$\text{rank}_{R_\pi} \mathbf{X}^r_{\pi,\Phi}(G) \leq D(n,r). \tag{3.10}$$

3.3 Functions on points

The next proposition shows that, in the case of monomially independent Frobenius automorphisms, polynomial combinations of δ-characters are completely determined by their functions on points.

Proposition 3.13. *Assume $\phi_1, \ldots, \phi_n$ are monomially independent in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$ and denote by $R_\pi[\mathbf{X}^r_{\pi,\Phi}(G)^\dagger]$ the R_π-subalgebra of $\mathcal{O}(J^r_{\pi,\Phi}(G))$ generated by the elements of $\mathbf{X}^r_{\pi,\Phi}(G)^\dagger$. Then the R_π-algebra map*

$$R_\pi[\mathbf{X}^r_{\pi,\Phi}(G)^\dagger] \to \text{Fun}(G(R^{\text{alg}}), K^{\text{alg}}), \quad f \mapsto f^{\text{alg}}$$

is injective. In particular, the R_π-module homomorphism

$$\mathbf{X}^r_{\pi,\Phi}(G)^\dagger \to \text{Hom}_{\text{alg}}(G(R^{\text{alg}}), K^{\text{alg}}), \quad \psi \mapsto \psi^{\text{alg}}$$

is injective.

Proof. Let $\psi_1, \ldots, \psi_N \in \mathbf{X}^r_{\pi,\Phi}(G)^\dagger$, let $F \in R_\pi[y_1, \ldots, y_N]$ be a polynomial in the variables $y_1, \ldots, y_N$, and let

$$f = F(\psi_1, \ldots, \psi_N) \in \mathcal{O}(J^r_{\pi,\Phi}(G)).$$

Assume that the induced map $f^{\text{alg}} : G(R^{\text{alg}}) \to K^{\text{alg}}$ is the zero map. Then the composition of f^{alg} with the induced map $\mathcal{E}^{\text{alg}} : \mathbb{G}_a(R^{\text{alg}}) \to G(R^{\text{alg}})$ is the zero map. Write

$$\theta(\psi_i) = \sum_\mu \lambda_{i,\mu}\phi_\mu, \quad \lambda_{i,\mu} \in K_\mu.$$

Then for every $\lambda \in R^{\text{alg}}$ we have

$$0 = (f^{\text{alg}} \circ \mathcal{E}^{\text{alg}})(\lambda) = F\Big(\sum_\mu \lambda_{1,\mu}\phi_\mu(\lambda), \ldots, \sum_\mu \lambda_{N,\mu}\phi_\mu(\lambda)\Big).$$

Let x_μ be variables indexed by $\mu \in \mathbb{M}_n^r$ and consider the polynomial

$$G(\dots, x_\mu, \dots) := F\Big(\sum_\mu \lambda_{1,\mu} x_\mu, \dots, \sum_\mu \lambda_{N,\mu} x_\mu\Big) \in R_\pi[\dots, x_\mu, \dots].$$

By Lemma 2.6 we get $G = 0$. But clearly f is obtained from G by replacing $x_\mu \mapsto \frac{1}{p}\phi_\mu \ell(T)$. So $f = 0$. ∎