Chapter 4

Multiplicative group

For $\pi \in \Pi$ define

$$N(\pi) := \min\left\{ N \in \mathbb{Z} \;\middle|\; \frac{\pi^n}{n} \in \frac{1}{p^N}\mathbb{Z}_p \;\text{ for all }\; n \geq 1 \right\}. \tag{4.1}$$

In particular, since p is odd we have $N(p) = -1$.

Denote by $\mathbb{G}_m$ the multiplicative group scheme Spec $R_\pi[x, x^{-1}]$ over R_π. Note that for each $i \in \{1, \ldots, n\}$ the series

$$p^{N(\pi)} \log\left(\frac{\phi_{\pi,i} x}{x^p}\right) := p^{N(\pi)} \log\left(1 + \pi \frac{\delta_{\pi,i} x}{x^p}\right) \tag{4.2}$$

(where log is the usual logarithm series) belongs to $R_\pi[x, x^{-1}, \delta_\pi x]\widehat{\ }$ and defines a totally δ-overconvergent δ_π-character $\psi_i \in \mathbf{X}^1_{\pi,\Phi}(\mathbb{G}_m)^\dagger$. Clearly the symbol of ψ_i is given by

$$\theta(\psi_i) = p^{N(\pi)+1}(\phi_i - p).$$

Theorem 4.1. *For all $r \geq 1$ we have*

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)^\dagger = \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$$

and a basis modulo torsion for this R_π-module is given by

$$\{\phi_\mu \psi_i \mid i \in \{1, \ldots, n\}, \mu \in \mathbb{M}_n^{r-1}\}. \tag{4.3}$$

Proof. By Proposition 3.12 the rank of $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ is at most $1 + n + \cdots + n^r$. The symbols of the members of (4.3) are linearly independent so the $n + n^2 + \cdots + n^r$ elements of (4.3) are linearly independent. It is enough to check that $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ does not have rank $1 + n + \cdots + n^r$; indeed this will make (4.3) a basis modulo torsion of $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ and this will also force $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)^\dagger$ and $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ to have the same rank and hence to be equal by Lemma 3.3. Now if $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ has rank $1 + n + \cdots + n^r$, by Proposition 3.12 the map $\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m) \to K^r_{\pi,\Phi}$ tensored with K_π must be an isomorphism. So there must be an element $\psi \in \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ with symbol $\theta(\psi) =: c \in K_\pi$. Taking $T = x - 1$ we immediately get that the logarithm $\ell_{\mathbb{G}_m}(T) = \log(1 + T)$ of the formal group of $\mathbb{G}_m$ times a power of p belongs to $R_\pi[\![T]\!]$ which is a contradiction. ∎

Remark 4.2. The moral of the above theorem is that the PDE story in the case of $\mathbb{G}_m$ can be reduced to the ODE story via face maps. As we shall see in the next section no such reduction is possible in the case of elliptic curves where 'genuinely PDE' δ-characters exist.

The next corollary is a strengthening of [12, Proposition 3.5].

Corollary 4.3. *Let* $\pi \in \Pi$ *and for* $i \in \{1, \ldots, n\}$ *let* ψ_i *be the* δ_π*-character in* (4.2). *Consider the induced homomorphism* $\psi_i^{\mathrm{alg}} : \mathbb{G}_m(R^{\mathrm{alg}}) \to K^{\mathrm{alg}}$. *Then*

$$\mathrm{Ker}(\phi_i^{\mathrm{alg}}) = \mathbb{G}_m(R^{\mathrm{alg}})_{\mathrm{tors}}. \tag{4.4}$$

Proof. This follows directly from Corollary 3.10 in view of the fact that the map

$$\theta(\psi_i)^{\mathrm{alg}} : \mathbb{Q}_p^{\mathrm{alg}} \to \mathbb{Q}_p^{\mathrm{alg}}, \quad \beta \mapsto \phi_i(\beta) - p\beta$$

is injective. ∎