Chapter 5

Elliptic curves

5.1 General case

Throughout this subsection E is an elliptic curve (abelian scheme of dimension 1) over R_π, and we fix an invertible 1-form ω on E. By a *formal group* over a ring we will always understand a formal group law (i.e, a tuple of elements in a formal power series ring). For every family $\Phi := (\phi_1, \dots, \phi_n)$ of Frobenius automorphisms of K^{alg} define

$$N^r_{\pi,\Phi} := \text{Ker}(J^r_{\pi,\Phi}(E) \to \widehat{E_{R_\pi}}).$$

Consider an admissible coordinate T on E. Exactly as in [10, Proposition 4.45] $N^r_{\pi,\Phi}$ is a group object in the category of p-adic formal schemes over R_π whose underlying p-adic formal scheme can be identified with the p-adic completion of the affine space

$$(\mathbb{A}^{n+\dots+n^r}_{R_\pi})\widehat{\ } = \text{Spf } R_\pi[\delta_{\pi,\mu}T \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ }$$

and whose group law is obtained as follows. One considers the formal group law $F(T_1, T_2) \in R_\pi[\![T_1, T_2]\!]$ of E with respect to T and one considers the group law on $N^r_{\pi,\Phi}$ defined by the series $F_\mu := (\delta_{\pi,\mu}F)|_{T=0}$ for $\mu \in \mathbb{M}^r_n$ (which turn out to be restricted power series rather than just formal power series).

Proposition 5.1. *The R_π-module* $\text{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$ *has rank*

$$\text{rank}_{R_\pi}\text{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}) = D(n,r) - 1 = n + \dots + n^r.$$

Proof. Denote by $(N^r_{\pi,\Phi})^{\text{for}}$ the formal group over R_π associated to $N^r_{\pi,\Phi}$ and to the variables $\delta_{\pi,\mu}T$, $\mu \in \mathbb{M}^{r,+}_n$, and denote by $(N^r_{\pi,\Phi})^{\text{for}}_{K_\pi}$ the induced formal group law over K_π which is isomorphic to a power of the additive formal group law $\mathbb{G}^{\text{for}}_{a/K_\pi}$ (since $(N^r_{\pi,\Phi})^{\text{for}}_{K_\pi}$ is commutative over a field of characteristic zero). We have natural injective maps of K_π-vector spaces

$$\text{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}) \otimes_{R_\pi} K_\pi \to \text{Hom}_{\text{for.gr.}}((N^r_{\pi,\Phi})^{\text{for}}_{K_\pi}, \mathbb{G}^{\text{for}}_{a,K_\pi}) \simeq K_\pi^{n+\dots+n^r}.$$

On the other hand we will construct, in what follows, $D(n,r) - 1$ K_π-linearly independent elements in $\text{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$; from this our proposition follows. Recall the logarithm series $\ell(T) = \ell_\omega(T) \in K_\pi[\![T]\!]$ normalized with respect to ω. Recalling the integer $N(\pi)$ in (4.1) we have that, for $\mu \in \mathbb{M}^{r,+}_n$, the series

$$L^\mu_{\pi,\Phi} := (\phi_\mu(\ell(T)))|_{T=0} \in K_\pi[\![\delta_{\pi,\nu}T \mid \nu \in \mathbb{M}^{r,+}_n]\!] \tag{5.1}$$

satisfies

$$\tilde{L}^{\mu}_{\pi,\Phi} := p^{N(\pi)} L^{\mu}_{\pi,\Phi} \in R_{\pi}[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}^{r,+}_n]\widehat{\ }. \tag{5.2}$$

It follows that

$$\tilde{L}^{\mu}_{\pi,\Phi} \in \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}). \tag{5.3}$$

It is trivial to check that the elements $\tilde{L}^{\mu}_{\pi,\Phi}$ are K_π-linearly independent, which ends our proof. ∎

Remark 5.2. Exactly as in [12, Proposition 4.6] we get that for every $\pi'|\pi$ and $\mu \in \mathbb{M}^r_n$ the element $p^{N(\pi')-N(\pi)} \tilde{L}^{\mu}_{\pi,\Phi} \otimes 1$ is the image of $\tilde{L}^{\mu}_{\pi',\Phi}$ via the homomorphism

$$R_{\pi'}[\delta_{\pi',\mu} T \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ } \to R_{\pi}[\delta_{\pi,\nu} T \mid \nu \in \mathbb{M}^{r,+}_n]\widehat{\ } \otimes_{R_\pi} R_{\pi'}.$$

Remark 5.3. Assume $\pi = p$. As in [10, page 124] for all $i_1 \dots i_r \in \mathbb{M}^{r,+}_n$ we have

$$\phi_{i_1\dots i_r} T - T^{p^r} - p(\delta_{i_r} T)^{p^{r-1}} \in (pT, p^2) \subset R[\delta_\nu T \mid \nu \in \mathbb{M}^r_n].$$

Hence, following [10, Proposition 4.41] we get that

$$\tilde{L}^{i_1\dots i_r}_{\pi,\Phi} \equiv (\delta_{i_r} T)^{p^{r-1}} \bmod p \ \text{ in } \ R[\delta_{p,\mu} T \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ }.$$

Remark 5.4. Consider the following standard cohomology sequence (cf. [10, page 191] for the case $n = 1$):

$$0 = \mathrm{Hom}(\widehat{E}, \widehat{\mathbb{G}_a}) \to \mathrm{Hom}(J^r_{\pi,\Phi}(E), \widehat{\mathbb{G}_a}) \to \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a}) \xrightarrow{\partial^r} H^1(\widehat{E}, \mathcal{O}) \tag{5.4}$$

and consider the isomorphism defined by Serre duality,

$$\langle -, \omega \rangle : H^1(\widehat{E}, \mathcal{O}) \to R_\pi.$$

It is useful to recall the explicit construction of the map ∂^r. By Proposition 2.31 there exists an affine open cover $E = \bigcup_i U_i$ and sections $s^r_i : \widehat{U_i} \to \mathrm{pr}_r^{-1}(U_i)$ of the projection $\mathrm{pr}_r : \mathrm{pr}_r^{-1}(U_i) \to \widehat{U_i}$ induced by the projection $\mathrm{pr}_r : J^r_{\pi,\Phi}(E) \to \widehat{E}$. Then for all $L \in \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$ the element $\partial^r(L)$ is defined as the cohomology class

$$[L \circ (s^r_i - s^r_j)] \in H^1(\widehat{E}, \mathcal{O})$$

of the cocycle

$$(L \circ (s^r_i - s^r_j))_{ij}, \quad L \circ (s^r_i - s^r_j) : \widehat{U_i} \cap \widehat{U_j} \to N^r_{\pi,\Phi} \to \widehat{\mathbb{G}_a}.$$

The definition above is independent of the choice of sections s^r_i; such a change would change the cocycle by a coboundary.

Following [10, page 194] and recalling the elements in (5.3), we introduce the following.

Definition 5.5. For $\mu \in \mathbb{M}_n^{r,+}$ we define the *primary arithmetic Kodaira–Spencer class* of E attached to μ by the formula

$$f_\mu := \langle \partial^r(\tilde{L}^\mu_{\pi,\Phi}), \omega \rangle \in R_\pi$$

and consider the vector

$$\mathrm{KS}^r_{\pi,\Phi}(E) = (f_\mu)_{\mu \in \mathbb{M}_n^{r,+}} \in R_\pi^{\mathbb{M}_n^{r,+}}.$$

Remark 5.6. A few comments are important at this point.

(1) The elements f_μ are easily seen to depend only on the pair (E, ω) and not on the choice of T. This follows from the easily-checked fact that our construction can be presented in a coordinate-free manner: instead of the rings $R_\pi[\![T]\!]$, $K_\pi[\![T]\!]$ one may consider the completion A of the local ring of E at the closed point of the identity section and the corresponding completion A_{K_π} for $E \otimes_{R_\pi} K_\pi$. Instead of

$$R_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^r]\!],\ K_\pi[\![\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^r]\!],\ R_\pi[\![T]\!][\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^{r,+}]^{\widehat{}},$$

one may consider certain new types of "π-jet algebras" A^r, $A^r_{K_\pi}$, $\tilde{A}^r$, attached to A respectively, satisfying certain new corresponding universal properties. We will not go here into defining these new types of π-jet algebras. Then $\ell \in A_{K_\pi}$ is defined by the condition that it "vanishes" at the ideal of the zero section and $d\ell = \omega$ in the completed module of Kähler differentials of A_{K_π}, $\phi_\mu \ell$ makes sense as an element of $A^r_{K_\pi}$ while $\phi_\mu \ell_{|T=0}$ makes sense as an element of $\tilde{A}^r$.

(2) We will write $f_\mu(E, \omega)$ instead of f_μ if we want to emphasize the dependence on (E, ω). With notation as in Remark 2.14, we have

$$f_\mu(E, \lambda\omega) = \lambda^{\phi_\mu + 1} f_\mu(E, \omega)$$

for all $\lambda \in R_\pi^\times$; this follows from the fact that if one replaces ω by $\lambda\omega$ in our construction then, since $\ell_{\lambda\omega}(T) = \lambda \ell_\omega(T)$, we have that $L^\mu_{\pi,\Phi}$ gets replaced by $\phi_\mu(\lambda) L^\mu_{\pi,\Phi}$.

(3) It is easy to see that the elements f_μ do not change if r changes, as long as $\mu \in \mathbb{M}_n^{r,+}$; this follows from the fact that changing r amounts to changing the defining cocycle in our construction by a coboundary which does not change the cohomology class. This justifies not including r in our notation for f_μ. In particular, if $\mathrm{KS}^r_{\pi,\Phi}(E) \neq 0$ for some $r \geq 1$ then $\mathrm{KS}^{r'}_{\pi,\Phi}(E) \neq 0$ for all $r' \geq r$.

(4) It is easy to check that the formation of the elements f_μ is compatible with face maps in the sense that for every $\mu' \in \mathbb{M}_{n'}^{r,+}$ we have, in the above notation:

$$f_{\mu'} = f_{\epsilon(\mu')}. \tag{5.5}$$

On the other hand note that in general

$$f_{i\mu} \neq \phi_i f_\mu.$$

(5) For every isogeny $u : E' \to E$ of degree d prime to p of elliptic curves over R_π and every invertible 1-form ω on E, setting $\omega' = u^*\omega$, we have

$$f_\mu(E', \omega') = d \cdot f_\mu(E, \omega).$$

The argument is entirely similar to the one in [10, page 264].

(6) Let us write $f_{\pi,\mu}$ instead of f_μ if we want to emphasize dependence on π. Then for all $\pi'|\pi$ we clearly have

$$f_{\pi',\mu}(E_{\pi'}, \omega_{\pi'}) = p^{N(\pi')-N(\pi)} f_{\pi,\mu}(E, \omega) \in R_{\pi'}$$

where $E_{\pi'} := E \otimes_{R_\pi} R_{\pi'}$ and $\omega_{\pi'}$ is the induced form.

A special role will be played later by the primary arithmetic Kodaira–Spencer classes $f_i, f_{ii}, f_{iii}, \ldots$ We will write

$$f_{i^r} := f_{i\ldots i} \quad \text{with } i \text{ repeated } r \text{ times.}$$

These classes are the images, via the corresponding face maps, of the corresponding classes obtained by replacing Φ by $\{\phi_i\}$ in all our constructions. Note that in [3] and [10] the forms f_{i^r} were denoted by f^r.

Proposition 5.7. *Assume E over R_π has ordinary reduction and assume $f_i = 0$ for some i. Then for all $\mu \in \mathbb{M}_n$ we have $f_\mu = 0$.*

This proposition cannot be proved at this point in the memoir but is an immediate consequence of Theorem 7.39 that will be stated and proved later.

Lemma 5.8. *Assume $\pi = p$. Then for all $\mu \in \mathbb{M}_n^r$ and all $\sigma \in \Sigma_n$ we have*

$$f_{\sigma\mu} = f_\mu.$$

Proof. Let $s_i^r : \widehat{U}_i \to J^r_{p,\Phi}(\mathrm{pr}_r^{-1}(U_i))$ be local sections of the projection $\mathrm{pr}_r : J^r_{p,\Phi}(E) \to \widehat{E}$ as in Remark 5.4, and consider the group automorphism over E,

$$\sigma : J^r_{p,\Phi}(E) \to J^r_{p,\Phi}(E)$$

induced by $\sigma \in \Sigma_n$. Consider the local sections $\sigma \circ s_i^r : \widehat{U}_i \to J^r_{p,\Phi}(\mathrm{pr}_r^{-1}(U_i))$. By the independence of f_μ on the choice of local sections we get

$$\begin{aligned} f_\mu &= \langle [\tilde{L}^\mu_{\pi,\Phi} \circ (s_i^r - s_j^r)], \omega\rangle \\ &= \langle \tilde{L}^\mu_{\pi,\Phi} \circ (\sigma \circ s_i^r - \sigma \circ s_j^r), \omega\rangle \\ &= \langle \tilde{L}^\mu_{\pi,\Phi} \circ \sigma \circ (s_i^r - s_j^r), \omega\rangle \\ &= \langle \tilde{L}^{\sigma\mu}_{\pi,\Phi} \circ (s_i^r - s_j^r), \omega\rangle \\ &= f_{\sigma\mu}. \end{aligned}$$

∎

Remark 5.9. Assume $\pi = p$ and fix an index i. By its very construction $f_i = 0$ if and only if E possesses a Frobenius lift (i.e., a scheme endomorphism reducing modulo p to the p-power Frobenius). Recall that if E has ordinary reduction then E has a Frobenius lift if and only if E is a canonical lift of its reduction [29, Appendix]. On the other hand recall from [10, Corollary 8.89] that if E has supersingular reduction then $f_i \neq 0$. We conclude that for an arbitrary E over $R = R_p$ we have $f_i = 0$ if and only if E has ordinary reduction and is a canonical lift of its reduction.

Consider the K_π-linear space

$$K^{r,+}_{\pi,\Phi} = \left\{ \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu \,\middle|\, \lambda_\mu \in K_\pi \right\} \subset K^r_{\pi,\Phi}$$

and the projection

$$\rho : K^r_{\pi,\Phi} \to K^{r,+}_{\pi,\Phi}, \quad \rho\left(\sum_{\mu \in \mathbb{M}^r_n} \lambda_\mu \phi_\mu \right) = \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu .$$

We may consider the K_π-linear space of relations among the primary arithmetic Kodaira–Spencer classes:

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp := \left\{ \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu \in K^{r,+}_{\pi,\Phi} \,\middle|\, \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu f_\mu = 0 \right\} \tag{5.6}$$

and its R_π-submodule of "integral elements,"

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} := \left\{ \sum_{\mu \in \mathbb{M}^{r,+}_n} \lambda_\mu \phi_\mu \in \mathrm{KS}^r_{\pi,\Phi}(E)^\perp \,\middle|\, \lambda_\mu \in R_\pi \right\}.$$

Finally, recall the *symbol homomorphism*

$$\theta : \mathbf{X}^r_{\pi,\Phi}(E) \to K^r_{\pi,\Phi}, \quad \psi \mapsto \theta(\psi).$$

Theorem 5.10. *The following claims hold.*

(1) *There exists an R_π-module homomorphism P as in (5.7) below such that the composition*

$$\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} \xrightarrow{P} \mathbf{X}^r_{\pi,\Phi}(E)^\dagger \subset \mathbf{X}^r_{\pi,\Phi}(E) \xrightarrow{\theta} K^r_{\pi,\Phi} \xrightarrow{\rho} K^{r,+}_{\pi,\Phi} \tag{5.7}$$

is the multiplication by $p^{N(\pi)+1}$ map. So for $\pi = p$ the composition (5.7) is the inclusion $\mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}} \subset K^{r,+}_{\pi,\Phi}$.

(2) *The map $\rho \circ \theta$ is injective. In particular, if $\theta(\psi) \in K_\pi$ for some $\psi \in \mathbf{X}^r_{\pi,\Phi}(E)$ then $\psi = 0$ and hence $\theta(\psi) = 0$.*

Proof. To prove Part 1 note that if

$$\Lambda := \sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu \in \mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}}$$

and if

$$L_\Lambda := \sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \tilde{L}^{\mu}_{\pi,\Phi} \in \mathrm{Hom}(N^r_{\pi,\Phi}, \widehat{\mathbb{G}_a})$$

then $\partial(L_\Lambda) = 0$ so L_Λ is the restriction of a unique element

$$P(\Lambda) \in \mathrm{Hom}(J^r_{\pi,\Phi}(E), \widehat{\mathbb{G}_a}) = \mathbf{X}^r_{\pi,\Phi}(E). \tag{5.8}$$

Clearly $\Lambda \mapsto P(\Lambda)$ is an R_π-linear map. By an argument entirely similar to the one in the proof of [12, Theorem 6.1] and using Remark 5.2 above it follows that $P(\Lambda)$ is totally δ-overconvergent: $P(\Lambda) \in \mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$. By an argument entirely similar to the one in the proof of [10, Proposition 7.20] one gets that

$$\theta(P(\Lambda))T \equiv p^{N(\pi)+1}\left(\sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu T\right) \bmod T$$

in the ring $K_\pi[\delta_{\pi,\mu} T \mid \mu \in \mathbb{M}_n^r]$. By Lemma 3.5, Part 2, we have

$$\theta(P(\Lambda))T = p^{N(\pi)+1}\left(\sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu T\right) + \lambda_0 T$$

for some $\lambda_0 \in R_\pi$. Hence

$$\rho(\theta(P(\Lambda))) = p^{N(\pi)+1}\left(\sum_{\mu \in \mathbb{M}_n^{r,+}} \lambda_\mu \phi_\mu\right)$$

and Part 1 follows.

Part 2 follows from the observation that if $\theta(\psi) \in K_\pi$ for some $\psi \in \mathbf{X}^r_{\pi,\Phi}(E)$ then by Remark 3.8, Part 1 it easily follows that

$$\psi \in \mathcal{O}(J^2_{\pi,\Phi}(E)) \cap K_\pi[\![T]\!] = \mathcal{O}(\widehat{E})$$

and hence ψ defines a homomorphism $\widehat{E} \to \widehat{\mathbb{G}_a}$; but the only such homomorphism is the zero homomorphism. ∎

Remark 5.11. Note that P in Theorem 5.10 is automatically injective. For all $\Lambda \in \mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}}$ we write $\psi_\Lambda := P(\Lambda)$; hence, by Remark 3.8, Part 2, we have

$$\theta(\psi_\Lambda) = p^{N(\pi)+1}\Lambda + \lambda_0(\Lambda)$$

for some $\lambda_0(\Lambda) \in pR_\pi$. Clearly the map

$$\mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}} \to pR_\pi, \quad \Lambda \mapsto \lambda_0(\Lambda)$$

is an R_π-module homomorphism.

Remark 5.12. The map P in Theorem 5.10 is compatible with the face maps (2.7) in an obvious sense.

Remark 5.13. If $f_i = 0$ for some i then $\phi_i \in \mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}}$ hence

$$\psi_i := \psi_{\phi_i} \in \mathbf{X}^1_{\pi,\Phi}(E)^{\dagger}.$$

Moreover, the symbol of ψ_i is given by

$$\theta(\psi_i) = p^{N(\pi)+1}\phi_i + \lambda_0(\phi_i).$$

Corollary 5.14. *The following claims hold.*

(1) *If* $\mathrm{KS}^r_{\pi,\Phi}(E) \neq 0$ *(in particular if* $\mathrm{KS}^1_{\pi,\Phi}(E) \neq 0$*), then we have* $\mathbf{X}^r_{\pi,\Phi}(E) = \mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ *and the rank of this* R_π*-module equals* $D(n,r) - 2$*.*

(2) *If* $\mathrm{KS}^1_{\pi,\Phi}(E) = 0$ *(equivalently, if* $f_i = 0$ *for all* $i \in \{1,\dots,n\}$*) then the equality* $\mathbf{X}^r_{\pi,\Phi}(E) = \mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ *holds, the rank of this* R_π*-module equals* $D(n,r) - 1$*, and a basis modulo torsion for this* R_π*-module is given by*

$$\{\phi_\mu\psi_i \mid \mu \in \mathbb{M}^{r,+}_n,\ i \in \{1,\dots,n\}\}. \tag{5.9}$$

(3) *The cokernel of the injective homomorphism* P *in Theorem 5.10 is a torsion* R_π*-module.*

Proof. If $\mathrm{KS}^r_{\pi,\Phi}(E) \neq 0$ then the module $\mathrm{KS}^r_{\pi,\Phi}(E)^{\perp}_{\mathrm{int}}$ has rank $D(n,r) - 2$. Since P in Theorem 5.10 is injective the module $\mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ has rank at least $D(n,r) - 2$. On the other hand by Proposition 5.1 and by the exact sequence (5.4) the module $\mathbf{X}^r_{\pi,\Phi}(E)$ has rank at most $D(n,r) - 2$. So the modules $\mathbf{X}^r_{\pi,\Phi}(E)$ and $\mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ have the same rank $D(n,r) - 2$ and hence they are equal by Lemma 3.3. This proves Part 1.

Assume now $\mathrm{KS}^1_{\pi,\Phi}(E) = 0$. The subset (5.9) of $\mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ is linearly independent (because so is the set of symbols of its elements). It follows that $\mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ has rank at least $D(n,r) - 1$. On the other hand by Proposition 5.1 and the sequence (5.4) the module $\mathbf{X}^r_{\pi,\Phi}(E)$ has rank at most $D(n,r) - 1$. So the modules $\mathbf{X}^r_{\pi,\Phi}(E)$ and $\mathbf{X}^r_{\pi,\Phi}(E)^{\dagger}$ have the same rank $D(n,r) - 1$ and hence they are equal by Lemma 3.3, with basis modulo torsion given by (5.9). This proves Part 2.

Part 3 follows from the fact that the source and target of P have the same rank. ∎

As an application to Theorem 5.10 we construct a series of special δ_π-characters of E as follows. Let $\mu, \nu \in \mathbb{M}_n^{r,+}$ be distinct and let ω be an invertible 1-form on E. Recalling the integers $N(\pi)$ in (4.1) set

$$\tilde{f}_\mu := p^{N(\pi)+1} f_\mu, \quad \mu \in \mathbb{M}_n. \tag{5.10}$$

In particular, if $\pi = p$ then $\tilde{f}_\mu = f_\mu$.

Note that

$$f_\nu \phi_\mu - f_\mu \phi_\nu \in \mathrm{KS}^r_{\pi,\Phi}(E)^\perp_{\mathrm{int}}$$

so we may consider the partial δ_π-character

$$\psi_{\mu,\nu} := \psi_{f_\nu \phi_\mu - f_\mu \phi_\nu} \in \mathbf{X}^r_{\pi,\Phi}(E)^\dagger. \tag{5.11}$$

By Theorem 5.10 we have

$$\theta(\psi_{\mu,\nu}) = \tilde{f}_\nu \phi_\mu - \tilde{f}_\mu \phi_\nu + f_{\mu,\nu} \tag{5.12}$$

for some $f_{\mu,\nu} \in pR_\pi$.

Definition 5.15. The above element $f_{\mu,\nu} \in pR_\pi$ is called the *secondary arithmetic Kodaira–Spencer class* attached to μ and ν.

Remark 5.16. Note that $\psi_{\mu,\nu}$ and $f_{\mu,\nu}$ do not change if r changes which justifies r not being included in the notation. Note also that $\psi_{\mu,\nu}$ and $f_{\mu,\nu}$ effectively depend on (E and) ω and if we want to emphasize this dependence we denote them by $\psi_{\mu,\nu}(E,\omega)$ and $f_{\mu,\nu}(E,\omega)$, respectively. Similarly, we write $\tilde{f}_\mu(E,\omega)$ in place of $\tilde{f}_\mu$. Then for all $\lambda \in R_\pi^\times$ we have (using the notation in Remark 2.14):

$$f_{\mu,\nu}(E,\lambda\omega) = \lambda^{\phi_\mu+\phi_\nu} f_{\mu,\nu}(E,\omega).$$

Indeed, by Remark 3.8, Part 1, and Remark 5.6, Part 1, we have the following equalities

$$\begin{aligned}
\psi_{\mu,\nu}(E,\omega) =& \frac{1}{p}(\tilde{f}_\nu(E,\omega)\phi_\mu - \tilde{f}_\mu(E,\omega)\phi_\nu + f_{\mu,\nu}(E,\omega))\ell_\omega(T),\\
\psi_{\mu,\nu}(E,\lambda\omega) =& \frac{1}{p}(\tilde{f}_\nu(E,\lambda\omega)\phi_\mu - \tilde{f}_\mu(E,\lambda\omega)\phi_\nu + f_{\mu,\nu}(E,\lambda\omega))\ell_{\lambda\omega}(T)\\
=& \frac{1}{p}(\lambda^{\phi_\nu+1}\tilde{f}_\nu(E,\omega)\phi_\mu - \lambda^{\phi_\mu+1}\tilde{f}_\mu(E,\omega)\phi_\nu\\
&+ f_{\mu,\nu}(E,\lambda\omega))(\lambda\ell_\omega(T))\\
=& \frac{1}{p}(\lambda^{\phi_\nu+\phi_\mu+1}\tilde{f}_\nu(E,\omega)\phi_\mu - \lambda^{\phi_\mu+\phi_\nu+1}\tilde{f}_\mu(E,\omega)\phi_\nu\\
&+ \lambda f_{\mu,\nu}(E,\lambda\omega))\ell_\omega(T).
\end{aligned}$$

We get

$$\begin{aligned}\psi^* :=&\lambda^{\phi_\mu+\phi_\nu+1}\psi_{\mu,\nu}(E,\omega)-\psi_{\mu,\nu}(E,\lambda\omega)\\ =&\frac{1}{p}(\lambda^{\phi_\mu+\phi_\nu+1}f_{\mu,\nu}(E,\omega)-\lambda f_{\mu,\nu}(E,\lambda\omega))\ell_\omega(T).\end{aligned}$$

Hence

$$\theta(\psi^*)=\frac{1}{p}(\lambda^{\phi_\mu+\phi_\nu+1}f_{\mu,\nu}(E,\omega)-\lambda f_{\mu,\nu}(E,\lambda\omega))\in K_\pi.$$

By Theorem 5.10, Part 2, $\theta(\psi^*)=0$ which ends the proof.

Remark 5.17. For all distinct μ,ν we have

$$f_{\mu,\nu}+f_{\nu,\mu}=0. \tag{5.13}$$

Indeed, switching μ and ν in (5.12) we get

$$\theta(\psi_{\nu,\mu})=\tilde{f}_\mu\phi_\nu-\tilde{f}_\nu\phi_\mu+f_{\nu,\mu}. \tag{5.14}$$

Adding (5.12) and (5.14) we may conclude by Theorem 5.10, Part 2.

Remark 5.18. Fix in what follows the elliptic curve E over R_π and an invertible 1-form ω. Write, as before, $\ell(T)=\ell_\omega(T)=\sum_{m=1}^{\infty}\frac{b_m}{m}T^m$, $b_m\in R_\pi$. Let $\mu,\nu\in\mathbb{M}_n$ be distinct of lengths $r\geq s$, respectively. By Remark 3.8, Part 1, we have that

$$\psi_{\mu,\nu}=\frac{1}{p}(\tilde{f}_\nu\phi_\mu-\tilde{f}_\mu\phi_\nu+f_{\mu,\nu})\ell(T)\in R_\pi[\![\delta_{\pi,\eta}T\mid\eta\in\mathbb{M}_n^r]\!]. \tag{5.15}$$

On the other hand we can write

$$\phi_{\pi,\mu}T=T^{p^r}+G_\mu,\ \ \phi_{\pi,\nu}T=T^{p^s}+G_\nu$$

with G_μ,G_ν in the ideal I_r of $R_\pi[\delta_{\pi,\eta}T\mid\eta\in\mathbb{M}_n^r]$ generated by the set

$$\{\delta_{\pi,\eta}T\mid\eta\in\mathbb{M}_n^{r,+}\}.$$

A direct computation shows

$$\begin{aligned}p\psi_{\mu,\nu}=&\tilde{f}_\nu\Big(\sum_m\frac{\phi_\mu(b_m)}{m}(T^{p^r}+G_\mu)^m\Big)\\ &-\tilde{f}_\mu\Big(\sum_m\frac{\phi_\nu(b_m)}{m}(T^{p^s}+G_\nu)^m\Big)\\ &+f_{\mu,\nu}\Big(\sum_m\frac{b_m}{m}T^m\Big).\end{aligned}$$

Reducing the above equality modulo the ideal I_r we get

$$\tilde{f}_\nu\Big(\sum_m \frac{\phi_\mu(b_m)}{m} T^{p^r m}\Big) - \tilde{f}_\mu\Big(\sum_m \frac{\phi_\nu(b_m)}{m} T^{p^s m}\Big) + f_{\mu,\nu}\Big(\sum_m \frac{b_m}{m} T^m\Big) \in pR_\pi[\![T]\!].$$

For all integers $N \geq 1$, picking out the coefficients of $T^{p^r N}$, we get the following analogue of the integrality conditions of Atkin and Swinnerton-Dyer [1, 34].

Corollary 5.19. *For all integers* $N \geq 1$

$$\tilde{f}_\nu \frac{\phi_\mu(b_N)}{N} - \tilde{f}_\mu \frac{\phi_\nu(b_{p^{r-s}N})}{p^{r-s}N} + f_{\mu,\nu} \frac{b_{p^r N}}{p^r N} \in pR_\pi. \tag{5.16}$$

Remark 5.20. For every isogeny $u : E' \to E$ of degree d prime to p of elliptic curves over R_π and every invertible 1-form ω on E, setting $\omega' = u^*\omega$, we have

$$f_{\mu,\nu}(E', \omega') = d \cdot f_{\mu,\nu}(E, \omega).$$

Indeed, we may identify two admissible coordinates for E and E' (call this parameter T) in which case we identify the images of ω' and ω in $R_\pi[\![T]\!]dT$, and we identify the two series ℓ_ω and $\ell_{\omega'}$ in $R_\pi[\![T]\!]$. As in 5.15 we consider the partial δ_π-characters of E and E' respectively:

$$\psi := \psi_{\mu,\nu} = \frac{1}{p}(\tilde{f}_\nu(E,\omega)\phi_\mu - \tilde{f}_\mu(E,\omega)\phi_\nu + f_{\mu,\nu}(E,\omega))\ell(T), \tag{5.17}$$

$$\psi' := \psi'_{\mu,\nu} = \frac{1}{p}(\tilde{f}_\nu(E',\omega')\phi_\mu - \tilde{f}_\mu(E',\omega')\phi_\nu + f_{\mu,\nu}(E',\omega'))\ell(T). \tag{5.18}$$

Identifying ψ with its image in the space of δ_π-characters of E' and using Remark 5.6, Part 5, we get that

$$\psi' - d \cdot \psi = (f_{\mu,\nu}(E',\omega') - d \cdot f_{\mu,\nu}(E,\omega))\ell_\omega(T).$$

Hence

$$\theta(\psi' - d \cdot \psi) = f_{\mu,\nu}(E',\omega') - d \cdot f_{\mu,\nu}(E,\omega) \in R_\pi.$$

By Theorem 5.10, Part 2, $\theta(\psi' - d \cdot \psi) = 0$ which ends the proof.

Remark 5.21. Let us write $f_{\pi,\mu,\nu}$ and $\psi_{\pi,\mu,\nu}$ instead of $f_{\mu,\nu}$ and $\psi_{\mu,\nu}$ if we want to emphasize dependence on π. Then for all $\pi'|\pi$ we have

$$\psi_{\pi',\mu,\nu} = p^{2N(\pi')-2N(\pi)}\psi_{\pi,\mu,\nu} \in \mathbf{X}^r_{\pi',\Phi}(E), \tag{5.19}$$

$$f_{\pi',\mu,\nu} = p^{2N(\pi')-2N(\pi)} f_{\pi,\mu,\nu} \in R_{\pi'}. \tag{5.20}$$

This follows from Remark 5.6, Part 6 by an argument similar to that in Remark 5.20.

5.2 The case $n = r = 2$

We continue to consider an elliptic curve E over R_π and a 1-form ω. Consider, in what follows, $\Phi = (\phi_1, \phi_2)$. We consider in this subsection the arithmetic Kodaira–Spencer classes of order ≤ 2, and we derive some basic quadratic and cubic relations among them that will play a key role in the next section.

Specializing the construction in the previous section to our case we may consider the partial δ_π-character

$$\psi_{1,2} = \psi_{f_2\phi_1 - f_1\phi_2} \in \mathbf{X}^1_{\pi,\phi_1,\phi_2}(E)^\dagger.$$

Remark 5.22. If $f_1 f_2 \neq 0$ then $\psi_{1,2}$ is a "genuinely partial" object (not expressible in terms of ODE objects via face maps); indeed, in this case, by Theorem 5.10, we have $\mathbf{X}^1_{\pi,\phi_1}(E) = \mathbf{X}^1_{\pi,\phi_2}(E) = 0$. On the other hand ψ^1_{12} can be viewed as an analogue of the transport equation in [17].

By Theorem 5.10 we have

$$\theta(\psi_{1,2}) = \tilde{f}_2\phi_1 - \tilde{f}_1\phi_2 + f_{1,2}.$$

By Remark 5.16 the dependence of $f_{1,2}$ on ω is as follows:

$$f_{1,2}(E, \lambda\omega) = \lambda^{\phi_1+\phi_2} f_{1,2}(E, \omega).$$

Next, for $i \in \{1, 2\}$, we may consider the partial δ_π-characters (induced via face maps by the ODE arithmetic Manin maps in [6]):

$$\psi_{ii,i} := \psi_{\tilde{f}_i\phi_i^2 - \tilde{f}_{ii}\phi_i} \in \mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)^\dagger.$$

By Theorem 5.10 we have

$$\theta(\psi_{ii,i}) = \tilde{f}_i\phi_i^2 - \tilde{f}_{ii}\phi_i + f_{ii,i}.$$

By Remark 5.16 the dependence of $f_{ii,i}$ on ω is as follows:

$$f_{ii,i}(E, \lambda\omega) = \lambda^{\phi_i+\phi_i^2} f_{ii,i}(E, \omega).$$

Finally, we may consider the partial δ_π-character

$$\psi_{11,22} := \psi_{f_{22}\phi_1^2 - f_{11}\phi_2^2} \in \mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)^\dagger.$$

By Theorem 5.10 we have

$$\theta(\psi_{11,22}) = \tilde{f}_{22}\phi_1^2 - \tilde{f}_{11}\phi_2^2 + f_{11,22}.$$

By Remark 5.16 the dependence of $f_{11,22}$ on ω is as follows:

$$f_{11,22}(E,\lambda\omega) = \lambda^{\phi_1^2+\phi_2^2} f_{11,22}(E,\omega).$$

One has the following 6 elements in the module $\mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)$,

$$\psi_{1,2},\ \phi_1\psi_{1,2},\ \phi_2\psi_{1,2},\ \psi_{11,1},\ \psi_{22,2},\ \psi_{11,22}. \tag{5.21}$$

So if $f_1 \neq 0$ or $f_2 \neq 0$, since $\mathbf{X}^2_{\pi,\phi_1,\phi_2}(E)$ has rank $2+2^2-1=5$ (cf. Theorem 5.10), it follows that there must be a non-trivial R_π-linear relation among these 6 elements:

$$\lambda_1\psi_{1,2} + \lambda_2\phi_1\psi_{1,2} + \lambda_3\phi_2\psi_{1,2} + \lambda_4\psi_{11,1} + \lambda_5\psi_{22,2} + \lambda_6\psi_{11,22} = 0, \tag{5.22}$$

for some $\lambda_1,\dots,\lambda_6 \in R_\pi$, not all zero. The existence of such a relation implies the vanishing of all 6×6 minors of the 6×7 matrix Γ of the coefficients of the Picard–Fuchs symbols of the elements in (5.21) with respect to the basis

$$\phi_1^2,\ \phi_2^2,\ \phi_1\phi_2,\ \phi_2\phi_1,\ \phi_1,\phi_2, 1 \tag{5.23}$$

of K^2_{π,ϕ_1,ϕ_2}. One can compute this matrix explicitly. Indeed, denote by $\theta_1,\dots,\theta_6$ the Picard–Fuchs symbols of the elements in (5.21), let $e_1,\dots,e_7$ be the elements in (5.21) and let $\Gamma = (\gamma_{ij})$ be the 6×7 matrix defined by the equalities

$$\theta_i = \sum_{j=1}^{7} \gamma_{ij} e_j,\ \ i = 1,\dots,6.$$

We have the following matrix

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{f}_2 & -\tilde{f}_1 & f_{1,2} \\ \tilde{f}_2^{\phi_1} & 0 & -\tilde{f}_1^{\phi_1} & 0 & f_{1,2}^{\phi_1} & 0 & 0 \\ 0 & -\tilde{f}_1^{\phi_2} & 0 & \tilde{f}_2^{\phi_2} & 0 & f_{1,2}^{\phi_2} & 0 \\ \tilde{f}_1 & 0 & 0 & 0 & -\tilde{f}_{11} & 0 & f_{11,1} \\ 0 & \tilde{f}_2 & 0 & 0 & 0 & -\tilde{f}_{22} & f_{22,2} \\ \tilde{f}_{22} & -\tilde{f}_{11} & 0 & 0 & 0 & 0 & f_{11,22} \end{pmatrix}.$$

The upper left 5×5 minor of the matrix Γ is non-zero if $f_1 f_2 \neq 0$. In particular, the following corollary is proved.

Corollary 5.23. *If $f_1 f_2 \neq 0$ then the first 5 elements in (5.21) are R_π-linearly independent and hence they form a basis up to torsion of $\mathbf{X}^2_{\pi,\Phi}(E)$.*

On the other hand the linear combination of the rows of Γ with coefficients $\lambda_1, \ldots, \lambda_6$ is 0 from which we get the following corollary.

Corollary 5.24. *If* $f_1 f_2 \neq 0$ *then in* (5.22) *we have* $\lambda_2 = \lambda_3 = 0$.

Assume $f_1 f_2 \neq 0$ and denote by $\tilde{\Gamma}$ the 4×5 matrix obtained from Γ by removing the 3rd and 4th columns as well as the 2nd and the 3rd rows. The rows of $\tilde{\Gamma}$ are then linearly dependent, so we get that all 4×4 minors of the matrix $\tilde{\Gamma}$ vanish. The vanishing of the minor obtained by removing the fifth column of $\tilde{\Gamma}$ is tautologically trivial, so it does not yield any information. The vanishings of the rest of the minors of $\tilde{\Gamma}$ is equivalent to one cubic relation (5.24) given in the following corollary.

Corollary 5.25. *If* $f_1 f_2 \neq 0$ *then the following relation holds in* R_π*:*

$$f_{11} f_{22} f_{1,2} + f_2 f_{22} f_{11,1} - f_{11} f_1 f_{22,2} - f_1 f_2 f_{11,22} = 0. \tag{5.24}$$

Proposition 5.26. *Assume* $\pi = p$*. Then the following equalities hold in* R*:*

(1) $f_1 = f_2$, $f_{12} = f_{21}$.

(2) $f_{11,1} = f_{22,2}$.

(3) $f_{1,2} = f_{11,22} = 0$.

Proof. Part 1 follows from Lemma 5.8. Part 2 follows from the compatibility with face maps. In order to check Part 3 consider the compatible actions of $\Sigma_2 = \{e, \sigma\}$ on $\mathbf{X}^1_{p,\Phi}(E)$ and $K^1_{p,\Phi}$. We have

$$\theta(\sigma\psi_{1,2}) = \sigma(\theta(\psi_{1,2})) = \sigma(f_1\phi_1 - f_1\phi_2 + f_{1,2}) = f_1\phi_2 - f_1\phi_1 + f_{1,2}.$$

Hence

$$\theta(\psi_{1,2} + \sigma\psi_{1,2}) = 2f_{1,2} \in K_\pi.$$

By Theorem 5.10, Part 2, it follows that $f_{1,2} = 0$. The equality $f_{11,22} = 0$ follows similarly. ∎

Remark 5.27. Assume that E comes from a curve $E_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ and denote by a_p the trace of Frobenius on $E_{\mathbb{Z}_p} \otimes \mathbb{F}_p$. Also fix an index i. It follows from [8, Theorem 1.10], that if E is not a canonical lift of an ordinary elliptic curve then

$$f_{ii} = a_p f_i, \quad f_{ii,i} = p f_i.$$

Recall that, if in addition $p \geq 5$, then $a_p = 0$ if and only if E has supersingular reduction.

We continue by considering the partial δ_π-character

$$\psi_{12,1} := \psi_{f_1\phi_1\phi_2 - f_{12}\phi_1}.$$

Its symbol is

$$\theta(\psi_{12,1}) = \tilde{f}_1\phi_1\phi_2 - \tilde{f}_{12}\phi_1 + f_{12,1}.$$

This symbol must be a linear combination of the symbols of

$$\psi_{1,2},\ \phi_1\psi_{1,2},\ \phi_2\psi_{1,2},\ \psi_{11,1},\ \psi_{22,2}.$$

Let Γ' be the matrix obtained by replacing the last row in Γ by the row

$$[0\ 0\ \tilde{f}_1\ 0\ -\tilde{f}_{12}\ 0\ f_{12,1}].$$

Then the determinants of the matrices obtained from Γ' by deleting the 5th and the 7th columns respectively must be 0. The vanishing of these determinants yields the following result.

Lemma 5.28. *If $f_1 f_2 \neq 0$ then the following relations hold in R_π:*

$$f_{12,1} f_1^{\phi_1} - f_{11,1} f_2^{\phi_1} = 0, \tag{5.25}$$

$$\tilde{f}_{12}\tilde{f}_1^{\phi_1} - \tilde{f}_{11}\tilde{f}_2^{\phi_1} - \tilde{f}_1 f_{1,2}^{\phi_1} = 0. \tag{5.26}$$

Similarly, by looking at the partial δ_π-character

$$\psi_{21,2} := \psi_{f_2\phi_2\phi_1 - f_{21}\phi_2}$$

we get the following lemma.

Lemma 5.29. *If $f_1 f_2 \neq 0$ then the following relations hold in R_π:*

$$f_{21,2} f_2^{\phi_2} - f_{22,2} f_1^{\phi_2} = 0, \tag{5.27}$$

$$\tilde{f}_{21}\tilde{f}_2^{\phi_2} - \tilde{f}_{22}\tilde{f}_1^{\phi_2} - \tilde{f}_2 f_{2,1}^{\phi_2} = 0. \tag{5.28}$$

Next consider the partial δ_π-character

$$\psi_{12,21} := \psi_{f_{21}\phi_1\phi_2 - f_{12}\phi_2\phi_1}.$$

Its symbol is

$$\theta(\psi_{12,21}) = \tilde{f}_{21}\phi_1\phi_2 - \tilde{f}_{12}\phi_2\phi_1 + f_{12,21}.$$

Set

$$\psi := \tilde{f}_1\tilde{f}_2\psi_{12,21} - \tilde{f}_2\tilde{f}_{21}\psi_{12,1} + \tilde{f}_1\tilde{f}_{12}\psi_{21,2} - \tilde{f}_{12}\tilde{f}_{21}\psi_{1,2}.$$

One trivially checks the following identity

$$\theta(\psi) = f_{12,21}\tilde{f}_1\tilde{f}_2 - \tilde{f}_2\tilde{f}_{21} f_{12,1} + \tilde{f}_1\tilde{f}_{12} f_{21,2} - \tilde{f}_{12}\tilde{f}_{21} f_{1,2}.$$

By Theorem 5.10, Part 1 we get the following.

Lemma 5.30. *If $f_1 f_2 \neq 0$ then the following relation holds in R_π:*

$$f_{12,21} f_1 f_2 - f_2 f_{21} f_{12,1} + f_1 f_{12} f_{21,2} - f_{12} f_{21} f_{1,2} = 0. \tag{5.29}$$

Similarly consider the partial δ_π-characters

$$\begin{gathered}
\tilde{f}_1 \psi_{11,2} - \tilde{f}_2 \psi_{11,1} - \tilde{f}_{11} \psi_{12}, \\
\tilde{f}_1 \psi_{11,12} - \tilde{f}_{12} \psi_{11,1} + \tilde{f}_{11} \psi_{12,1}, \\
\tilde{f}_1 \psi_{12,2} - \tilde{f}_2 \psi_{12,1} - \tilde{f}_{12} \psi_{1,2}, \\
\tilde{f}_2 \psi_{11,21} - \tilde{f}_{21} \psi_{11,2} + \tilde{f}_{11} \psi_{21,2}.
\end{gathered}$$

The symbols of these partial δ_π-characters are equal to the expressions in the left-hand sides of the equalities in the Lemma 5.31 below. By Theorem 5.10, Part 1, since these symbols are in R_π they must vanish. So we have the following lemma.

Lemma 5.31. *If $f_1 f_2 \neq 0$ then the following relations hold in R_π:*

$$f_1 f_{11,2} - f_2 f_{11,1} - f_{11} f_{1,2} = 0, \tag{5.30}$$

$$f_1 f_{11,12} - f_{12} f_{11,1} + f_{11} f_{12,1} = 0, \tag{5.31}$$

$$f_1 f_{12,2} - f_2 f_{12,1} - f_{12} f_{1,2} = 0, \tag{5.32}$$

$$f_2 f_{11,21} - f_{21} f_{11,2} + f_{11} f_{21,2} = 0. \tag{5.33}$$

Moreover, the relations obtained from the above relations by switching the indices 1 and 2 also hold.

Remark 5.32. One can ask if one can "extend" equations (5.24), (5.25), (5.26), (5.27), (5.28), (5.30), (5.31), (5.32), (5.33) by continuity so that these remain true without the condition $f_1 f_2 \neq 0$. We claim this is the case as an immediate consequence of Theorems 7.18, 7.19 and the formulae (7.6) and (7.7) to be stated and proved later. By the way we have the following result; this will be proved after the proof of Proposition 7.38.

Theorem 5.33. *Assume ϕ_1, ϕ_2 are monomially independent in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$. Then there exist $\pi \in \Pi$ and a pair (E, ω) over R_π such that E has ordinary reduction and all classes f_μ, $f_{\mu,\nu}$ with $\mu, \nu \in \mathbb{M}_2^{2,+}$, $\mu \neq \nu$, attached to (E, ω) are non-zero.*