Chapter 6

The relative theory

The theory developed so far over R_π should be viewed as an “absolute” theory and has a “relative” version in which the δ_π-prolongation sequence R_π^* is replaced by an arbitrary object S^* in $\mathbf{Prol}^*_{\pi,\Phi}$. This relative version of the theory is crucial for developing the formalism of *partial δ_π-modular forms* in the next section. In the present section we present a quick discussion of this relative version of the theory.

Again we consider the variables $\delta_{\pi,\mu} y_j$ for $\mu \in \mathbb{M}_n$, $\pi \in \Pi$, $j \in \{1, \dots, N\}$. Let $S^* = (S^r)$ be an object in $\mathbf{Prol}_{\pi,\Phi}$. Fix a positive integer N and consider the ring $S^0[y_1, \dots, y_N]$ and the rings

$$J^r_{\pi,\Phi}(S^0[y_1, \dots, y_N]/S^*) := S^r[\delta_{\pi,\mu} y_j \mid \mu \in \mathbb{M}^r_m, j = 1, \dots, N]\widehat{\ }.$$

The sequence $J^*_{\pi,\Phi}(S^0[y_1, \dots, y_N]/S^*) := (J^r_{\pi,\Phi}(S^0[y_1, \dots, y_N]/S^*))$ has, again, a unique structure of an object in $\mathbf{Prol}^*_{\pi,\Phi}$ such that $\delta_{\pi,i}\delta_{\pi,\mu} y := \delta_{\pi,i\mu} y$ for all $i = 1, \dots, n$; if S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$ then $J^*_{\pi,\Phi}(S^0[y_1, \dots, y_N]/S^*)$ is also an object of $\mathbf{Prol}_{\pi,\Phi}$.

For every object S^* in $\mathbf{Prol}^*_{\pi,\Phi}$ and every S^0-algebra of finite type written as $A := S^0[y_1, \dots, y_N]/I$, we define the ring

$$J^r_{\pi,\Phi}(A/S^*) := J^r_\pi(S^0[y_1, \dots, y_N]/S^*)/(\delta_{\pi,\mu} I \mid m \in \mathbb{M}^r_m).$$

If $S^* = R^*_\pi$ then $J^r_{\pi,\Phi}(A/S^*)$ coincides with the previously defined ring $J^r_{\pi,\Phi}(A)$.

If S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$, A is a smooth S^0-algebra, and $u \colon S^0[T_1, \dots, T_d] \to A$ is an étale morphism of S^0-algebras, then, again, there is a (unique) isomorphism

$$(A \otimes_{S^0} S^r)[\delta_{\pi,\mu} T_j \mid \mu \in \mathbb{M}^{r,+}_m, j = 1, \dots, d]\widehat{\ } \cong J^r_{\pi,\Phi}(A/S^*)$$

sending $\delta_{\pi,\mu} T_j$ into $\delta_{\pi,\mu}(u(T_j))$ for all j and μ. In particular, $J^r_{\pi,\Phi}(A/S^*)$ is Noetherian and flat over R_π so the sequence $J^*_{\pi,\Phi}(A/S^*)$ is an object of $\mathbf{Prol}_{\pi,\Phi}$.

As in Proposition 2.23 we have the following universal property. Assume S^* is an object of $\mathbf{Prol}_{\pi,\Phi}$ and A is a smooth S^0-algebra. For every object T^* of $\mathbf{Prol}_{\pi,\Phi}$ and every S^0-algebra map $u : A \to T^0$ and any morphism $S^* \to T^*$ in $\mathbf{Prol}_{\pi,\Phi}$ there is a unique morphism $J^*_{\pi,\Phi}(A/S^*) \to T^*$ over S^* in $\mathbf{Prol}_{\pi,\Phi}$ compatible with u. (A similar result holds for $\mathbf{Prol}^*_{\pi,\Phi}$.)

As in Definition 2.25 for every object S^* in $\mathbf{Prol}_{\pi,\Phi}$ and every smooth scheme X over S^0 we define the *relative partial π-jet space*

$$J^r_{\pi,\Phi}(X/S^*) = \bigcup \mathrm{Spf}(J^r_{\pi,\Phi}(\mathcal{O}(U_i))/S^*),$$

where $X = \bigcup_i U_i$ is (any) affine open cover. If $S^* = R^*_\pi$, $J^r_{\pi,\Phi}(X/S^*)$ coincides with the previously defined formal scheme $J^r_{\pi,\Phi}(X)$.

Let S^* be an object in $\mathbf{Prol}_{\pi,\Phi}$ and G a commutative smooth group scheme over S^0. We define a *relative partial* δ_π*-character* of order $\leq r$ of G over S^* to be a group homomorphism $J^r_{\pi,\Phi}(G/S^*) \to \widehat{\mathbb{G}_{a,S^r}}$ in the category of p-adic formal schemes. (Here G_{a,S^r} is, of course, the additive group scheme over S^r.) We denote by $\mathbf{X}^r_{\pi,\Phi}(G/S^*) = \mathrm{Hom}(J^r_{\pi,\Phi}(G), \widehat{\mathbb{G}_{a,S^r}})$ the S^r-module of relative partial δ_π-characters of G of order $\leq r$.

For a family $\Phi := (\phi_1, \ldots, \phi_n)$, $\phi_i \in \mathfrak{F}^{(1)}(K^{\mathrm{alg}}/\mathbb{Q}_p)$ of distinct Frobenius automorphisms and for an object S^* in $\mathbf{Prol}_{\pi,\Phi}$ we define the S^r-modules of *symbols* $S^r_{\pi,\Phi}$ to be the free S^r-module with basis $\{\phi_\mu \in \mathbb{M}_\Phi \mid \mu \in \mathbb{M}^r_n\}$. We consider the rings $S^r \otimes \mathbb{Q} = S^r \otimes_{R_\pi} K_\pi$ and the $S^r \otimes \mathbb{Q}$-modules $S^r_{\pi,\Phi} \otimes \mathbb{Q}$; they play the roles, in this relative setting, of K_π and $K^r_{\pi,\Phi}$, respectively.

As in Proposition 3.4 we have

$$\mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_a/S^*) = ((S^r \otimes \mathbb{Q})T) \cap (S^r[\delta_{\pi,\nu}T \mid \nu \in \mathbb{M}^r_n])$$

where the intersection is taken inside the ring $(S^r \otimes \mathbb{Q})[\![\phi_{\pi,\nu}T \mid \nu \in \mathbb{M}^r_n]\!]$.

Let S^* be an object in $\mathbf{Prol}_{\pi,\Phi}$, let G have relative dimension 1 over S^0 and assume we are given an invariant 1-form ω on G/S^0 and an *admissible coordinate* T (defined in the obvious corresponding way) on G/S^0. (Note that ω and T may not exist in general, but they exist locally on $\mathrm{Spec}(S^0)$ in the Zariski topology.) Then, as in Definition 3.7, one can attach to every $\psi \in \mathbf{X}^r_{\pi,\Phi}(G/S^*)$ a *Picard–Fuchs symbol* $\theta(\psi) \in S^r_{\pi,\Phi} \otimes \mathbb{Q}$.

The various results about δ_π-characters obtained in the previous sections have (obvious) relative analogues (over objects S^* in $\mathbf{Prol}_{\pi,\Phi}$ instead of over R^*_π) that are proved using essentially identical arguments. We note, however, that the relative analogues over S^* of Corollary 5.14 and of the results in Section 5.2 need the hypothesis that the rings S^r be integral domains; indeed for Corollary 5.14 we need the concepts of rank and torsion of an S^r-module to be well defined while for the analysis in Section 5.2 we need the fact that linear dependence in torsion-free S^r-modules is expressed via vanishings of corresponding determinants. Rather than stating these relative analogues here we will use them freely in what follows with appropriate references to the corresponding "absolute" results in the previous sections.