Chapter 7

Partial δ-modular forms

7.1 Basic definitions

We start with the standard extension of [9] or [10, Section 8.4.1] to our setting of partial differential equations. In this section, $\pi \in \Pi$. Continue to set $\Phi = (\phi_1, \ldots, \phi_n)$ a fixed choice of Frobenius automorphisms of K^{alg}. Consider the category of triples $(E/S^0, \omega, S^*)$ where S^* is an object in $\mathbf{Prol}_{\pi,\Phi}$, E/S^0 is an elliptic curve, and $\omega \in H^0(E, \Omega_{E/S_0})$ is a basis. A morphism in this category is defined as a map of tuples $(E/S^0, \omega, S^*) \to (E'/T^0, \omega', T^*)$ consisting of a map of prolongation sequences $T^* \to S^*$ and a compatible map $E/S^0 \to E'/T^0$ of curves pulling back ω' to ω.

Definition 7.1. A *partial δ_π-modular function of order at most* $r \geq 0$ is a rule f assigning to each object $(E/S^0, \omega, S^*)$ an element $f(E/S^0, \omega, S^*) \in S^r$, depending only on the isomorphism class of $(E/S^0, \omega, S^*)$, such that f commutes with base change of the prolongation sequence. Specifically, if $u\colon S^* \to T^*$ is a map of prolongation sequences, then $f((E \times_{S^0} T^0)/T^0, u^*\omega, T^0) = u^r f(E/S^0, \omega, S^*)$. We denote by $M^r_{\pi,\Phi}$ the set of all partial δ_π-modular functions of order at most $r \geq 0$; this set has an obvious structure of ring. An element of $M^r_{\pi,\Phi}$ is said to have order r if it is not in the image of the canonical map $M^{r-1}_{\pi,\Phi} \to M^r_{\pi,\Phi}$. The latter map is, by the way, injective as can be seen from the next remark.

Remark 7.2. Consider two variables a_4 and a_6, let $\Delta := 4a_4^3 + 27a_6^2$, and let us consider the R_π-algebra $M_\pi := R_\pi[a_4, a_6, \Delta^{-1}]$ and the affine scheme $B(1) := \mathrm{Spec}(M_\pi)$. (The scheme $B(1)$ has a natural $\mathbb{G}_m$-action and a morphism to the "j-line" $Y(1)$ which is however not a $\mathbb{G}_m$-bundle; later we will consider level $\Gamma_1(N)$-structures, the modular curves $Y_1(N)$, and the corresponding $\mathbb{G}_m$-bundles $B_1(N)$.) For all R_π-algebras S the set $B(1)(S)$ of S-points of $B(1)$ is in a natural bijection with the set of pairs (E, ω) consisting of an elliptic curve E/S and a basis ω for the 1-forms on E/S. Then, as in [9], we have an identification of R_π-algebras

$$M^r_{\pi,\Phi} \simeq J^r_{\pi,\Phi}(M_\pi) = \mathcal{O}(J^r_{\pi,\Phi}(B(1))) \simeq \widehat{R_\pi[\delta_\mu a_4, \delta_\mu a_6, \Delta^{-1} \mid \mu \in \mathbb{M}^r_n]}.$$

In particular, $M^0_{\pi,\Phi} \simeq \widehat{M_\pi}$.

In what follows we discuss weights. In the case $\pi = p$ and $\Phi = \{\phi\}$, weights are taken to be elements of the polynomial ring $\mathbb{Z}[\phi]$ in the "variable" ϕ. In the partial differential setting considered here, we consider weights in the ring of integral symbols, $\mathbb{Z}_\Phi$. As before, if $w = \sum m_\mu \phi_\mu \in \mathbb{Z}_\Phi$, $m_\mu \in \mathbb{Z}$, and if S^* is an object of

$\mathbf{Prol}_{\pi,\Phi}$ and $\lambda \in (S^0)^\times$, we write

$$\lambda^w = \prod_{\mu \in \mathbb{M}_n} (\phi_\mu(\lambda))^{m_\mu} \in (S^r)^\times.$$

We use a similar notation for $\lambda \in S^0$ in case all $\lambda_\mu \geq 0$. We have the formulae $\lambda^{w_1+w_2} = \lambda^{w_1}\lambda^{w_2}$ and $\lambda^{w_1 w_2} = (\lambda^{w_2})^{w_1}$.

Definition 7.3. A partial δ_π-modular function $f \in M^r_{\pi,\Phi}$ is called a *partial δ_π-modular form of weight $w \in \mathbb{Z}^r_\Phi$* provided for all $(E/S^0, \omega, S^*)$ and for each $\lambda \in (S^0)^\times$ we have

$$f(E/S^0, \lambda\omega, S^*) = \lambda^{-w} f(E/S^0, \omega, S^*). \tag{7.1}$$

We denote by $M^r_{\pi,\Phi}(w)$ the R_π-module of partial δ_π-modular forms of weight w.

Remark 7.4. If t is a variable we may consider the ring

$$J^r_{\pi,\Phi}(M_\pi[t, t^{-1}]) \simeq R_\pi[\delta_\mu a_4, \delta_\mu a_6, \delta_\mu t, \Delta^{-1}, t^{-1} \mid \mu \in \mathbb{M}^r_n]\widehat{\ }. \tag{7.2}$$

Then for $f \in M^r_{\pi,\Phi}$ we have that $f \in M^r_{\pi,\Phi}(w)$ if and only if

$$f(\ldots, \delta_\mu(t^4 a_4), \ldots, \delta_\mu(t^6 a_6), \ldots) = t^w f(\ldots, \delta_\mu a_4, \ldots, \delta_\mu a_6, \ldots)$$

in the ring (7.2).

Remark 7.5. The direct sum $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{\pi,\Phi}(w)$ has a natural structure of $\mathbb{Z}_\Phi$-graded R_π-algebra. Moreover, for every i and $f \in M^r_{\pi,\Phi}(w)$ we have a naturally defined form $f^{\phi_i} \in M^r_{\pi,\Phi}(\phi_i w)$. Consequently, we have natural *bracket* $\mathbb{Z}_p$-bilinear maps

$$\{\ ,\ \}_{\pi,i} : M^r_{\pi,\Phi}(w) \times M^r_{\pi,\Phi}(w) \to M^r_{\pi,\Phi}((\phi_i + p)w)$$

defined by

$$\{f, g\}_{\pi,i} := \frac{1}{\pi}(f^{\phi_i} g^p - g^{\phi_i} f^p) = g^p \delta_{\pi,i} f - f^p \delta_{\pi,i} g.$$

7.2 Jet construction

Examples of δ_π-modular forms are provided by primary and secondary arithmetic Kodaira–Spencer classes as we shall explain in what follows. We begin with primary classes where each f_μ for $\mu \in \mathbb{M}_n$ "comes from" a δ_π-modular function (which we denote by $f^{\text{jet}}_{\pi,\mu}$). Indeed, consider a prolongation sequence S^* over R, an elliptic curve E/S^0, and $\omega \in H^1(E, \Omega_{E/S^0})$ a basis. For fixed r and $\mu \in \mathbb{M}^r_n$, replicating the arguments in the construction of f_μ in Remark 5.2 (with jet spaces over R^* replaced by relative jet spaces over S^* as in Chapter 6) and setting η for the class of the corresponding $\partial^r(\tilde{L}^{\mu}_{\pi,\Phi})$ in Definition 5.5 we define $f^{\text{jet}}_{\pi,\mu}(E/S^0,\omega,S^*) = \langle\eta,\omega\rangle \in S^r$. In particular, using the notation in Remark 5.6, Part 2, for every (E,ω) over R_π we have

$$f_\mu(E,\omega) = f^{\text{jet}}_{\pi,\mu}(E/R_\pi,\omega,R^*_\pi). \tag{7.3}$$

Using the corresponding version over S^* of Remark 5.6, Part 2 we get the following.

Theorem 7.6. *The rule $f^{\text{jet}}_{\pi,\mu}$ defines a partial δ_π-modular form of weight $-1-\phi_\mu$.*

These forms are generalizations of the forms f_{jet} constructed in [9, Construction 4.1] or f^r_{jet} in [10, Section 8.4.2]. For $\pi = p$ we write

$$f^{\text{jet}}_{p,\mu} = f^{\text{jet}}_\mu.$$

Remark 7.7. Assume $\pi = p$ and let $E_{p-1} \in \mathbb{Z}_p[a_4,a_6] \subset M^0_{p,\Phi}$ be the polynomial that corresponds to the normalized Eisenstein series of weight $p-1$; we recall that the reduction mod p of E_{p-1} is the Hasse invariant. Then, exactly as in [10, Proposition 8.55] and using Remark 5.3, we get that for every $i_1\dots i_r \in \mathbb{M}^{r,+}_n$ we have

$$f^{\text{jet}}_{i_1\dots i_r} \equiv E_{p-1}^{1+p+\dots+p^{r-2}} \cdot (f^{\text{jet}}_{i_r})^{p^{r-1}} \ \text{mod}\ p \ \text{ in } \ M^r_{p,\Phi}.$$

In particular, for all $\mu \in \mathbb{M}^{r,+}_n$ we have

$$f^{\text{jet}}_\mu \in M^r_{p,\Phi} \setminus pM^r_{p,\Phi}, \ \text{ hence } \ f^{\text{jet}}_\mu \neq 0.$$

Next, for fixed r and $\mu,\nu \in \mathbb{M}^{r,+}_n$ distinct, replicating the arguments in the construction of $f_{\mu,\nu}$ in Definition 5.15 (with jet spaces over R^* replaced by relative jet spaces over S^* as in Chapter 6) we define $f^{\text{jet}}_{\pi,\mu,\nu}(E,\omega,S^*) \in S^r$. In particular, using the notation in Remark 5.16 for every E over R_π we have

$$f_{\mu,\nu}(E,\omega) = f^{\text{jet}}_{\pi,\mu,\nu}(E/R_\pi,\omega,R^*_\pi). \tag{7.4}$$

Using the corresponding version over S^* of Remarks 5.16 and 5.13, we get the following.

Theorem 7.8. *The rule $f^{\text{jet}}_{\pi,\mu,\nu}$, for $\mu,\nu \in \mathbb{M}^{r,+}_n$ distinct, defines a partial δ_π-modular form of weight $-\phi_\nu-\phi_\mu$. Moreover, $f^{\text{jet}}_{\pi,\mu,\nu} + f^{\text{jet}}_{\pi,\nu,\mu} = 0$.*

For $\pi = p$ we write

$$f^{\text{jet}}_{p,\mu,\nu} =: f^{\text{jet}}_{\mu,\nu}.$$

Our next goal is to show the above forms enjoy the special property of being "isogeny covariant" which we now define in our setting.

Definition 7.9. For every weight $w = \sum_{\mu \in \mathbb{M}_n} m_\mu \phi_\mu$, set $\deg(w) := \sum_{\mu \in \mathbb{M}_n} m_\mu$. Let f be a partial δ_π-modular form f of weight $w \in \mathbb{Z}^r_\Phi$ where $\deg(w)$ is even. We say f is *isogeny covariant* provided for every tuple $(E/S^0, \omega, S^*)$ and every isogeny of degree prime to p, $u \colon E' \to E$ over S^0, setting $\omega' = u^*\omega$, we have

$$f(E'/S^0, \omega', S^*) = [\deg(u)]^{-\deg(w)/2} f(E/S^0, \omega, S^*).$$

We denote by $I^r_{\pi,\Phi}(w)$ the R_π-module of isogeny covariant partial δ_π-modular forms of weight w.

Remark 7.10. The direct sum $\bigoplus_{w \in \mathbb{Z}_\Phi} I^r_{\pi,\Phi}(w)$ is a $\mathbb{Z}_\Phi$-graded R_π-subalgebra of $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{\pi,\Phi}(w)$. For every $f \in I^r_{p,\Phi}(w)$ and every i we have $f^{\phi_i} \in I^r_{p,\Phi}(\phi_i w)$ and consequently the brackets in Remark 7.5 induce brackets

$$\{\ ,\ \}_{\pi,i} : I^r_{\pi,\Phi}(w) \times I^r_{\pi,\Phi}(w) \to I^r_{\pi,\Phi}((\phi_i + p)w).$$

Theorem 7.11. *The partial δ_π-modular forms $f^{\mathrm{jet}}_{\pi,\mu}$ and $f^{\mathrm{jet}}_{\pi,\mu,\nu}$ are isogeny covariant.*

Proof. This follows by adapting the arguments in Remark 5.6, Part 5 and Remark 5.20 with R^*_π replaced by an arbitrary prolongation sequence S^*. ∎

Exactly as in [12] the forms $f^{\mathrm{jet}}_{\pi,\mu}$, $f^{\mathrm{jet}}_{\pi,\mu,\nu}$ induce totally δ-overconvergent arithmetic PDEs on $B(1)$ and on certain natural bundles $B_1(N)$ over modular curves $Y_1(N)$. We explain this in what follows.

Definition 7.12. Consider the modular curve $Y_1(N) := X_1(N) \setminus \{\text{cusps}\}$ over R_π where $N \geq 4$, N coprime to p, and let L be the line bundle on $Y_1(N)$ equal to the direct image of $\Omega_{E_{\mathrm{univ}}/Y_1(N)}$ where E_{univ} is the universal elliptic curve over $Y_1(N)$. Let $X \subset Y_1(N)$ be an affine open set, continue to denote by L the restriction of L to X, and consider the natural $\mathbb{G}_m$-bundle

$$B_1(N) := \mathrm{Spec}\Big(\bigoplus_{m \in \mathbb{Z}} L^m\Big) \to X.$$

The main example we have in mind is the case $X = Y_1(N)$.

Recall that if X is such that L is free over $X = \mathrm{Spec}(A)$ with basis ω then we have a natural identification

$$\mathcal{O}(J^r_{\pi,\Phi}(B_1(N))) \simeq J^r_{\pi,\Phi}(A)[x, x^{-1}, \delta_\mu x \mid \mu \in \mathbb{M}^{r,+}_n]\widehat{\ } \tag{7.5}$$

where x is a variable identified with the section ω. We define the R_π-module

$$M^r_{\pi,\Phi,X}(w) := \widehat{\mathcal{O}(X)} \cdot x^w.$$

If X is arbitrary and $X = \bigcup X_i$ is an open cover such that L is trivial on each X_i then we define $M^r_{\pi,\Phi,X}(w)$ to be the submodule of $\mathcal{O}(J^r_{\pi,\Phi}(B_1(N)))$ of all elements whose restriction to every $\mathcal{O}(J^r_{\pi,\Phi}(B_1(N) \times_X X_i))$ lies in $M^r_{\pi,\Phi,X_i}(w)$; this definition is independent of the covering considered.

Recall the scheme $B(1)$ defined in Remark 7.2. Exactly as in [12, Section 5.2] every partial δ_π-modular form

$$f = f^{B(1)} \in M^r_{\pi,\Phi}(w) \subset \mathcal{O}(J^r_{\pi,\Phi}(B(1)))$$

induces an element

$$f^{B_1(N)} \in M^r_{\pi,\Phi,X}(w) \subset \mathcal{O}(J^r_{\pi,\Phi}(B_1(N))).$$

We recall that for $X = \mathrm{Spec}(A)$ such that L has a basis x corresponding to a 1-form ω we define

$$f^{B_1(N)} := f(E_{\mathrm{univ}}, \omega, J^*_{\pi,\Phi}(A)) \cdot x^w;$$

for arbitrary X we glue the elements just defined. Exactly as in [12, Section 5.3] we have the following theorem.

Theorem 7.13. *Let B be either $B(1)$ or $B_1(N)$ with $N \geq 4$. For all $\mu, \nu \in \mathbb{M}^{r,+}_n$ the elements $(f^{\mathrm{jet}}_{\pi,\mu})^B, (f^{\mathrm{jet}}_{\pi,\mu,\nu})^B \in \mathcal{O}(J^r_{\pi,\Phi}(B))$ are totally δ-overconvergent. So there are induced maps*

$$((f^{\mathrm{jet}}_{\pi,\mu})^B)^{\mathrm{alg}}, ((f^{\mathrm{jet}}_{\pi,\mu,\nu})^B)^{\mathrm{alg}} : B(R^{\mathrm{alg}}) \to K^{\mathrm{alg}}.$$

Remark 7.14. For $\pi'|\pi$ and every point $P \in B(R_{\pi'})$, denoting by (E_P, ω_P) the corresponding elliptic curve over $R_{\pi'}$ equipped with the induced 1-form, we have that

$$((f^{\mathrm{jet}}_{\pi,\mu})^B)^{\mathrm{alg}}(P) = p^{-N(\pi')+N(\pi)} f_{\pi',\mu}(E_P/R_{\pi'}, \omega_P, R^*_{\pi'}), \tag{7.6}$$

$$((f^{\mathrm{jet}}_{\pi,\mu,\nu})^B)^{\mathrm{alg}}(P) = p^{-2N(\pi')+2N(\pi)} f_{\pi',\mu,\nu}(E_P/R_{\pi'}, \omega_P, R^*_{\pi'}); \tag{7.7}$$

cf. Remark 5.6, Part 6 and Remark 5.21.

Definition 7.15. For every selection map ϵ with respect to (Φ', Φ'', p) (cf. Definition 2.20) and every $f \in M^r_{p,\Phi'}$ we define $f_\epsilon \in M^r_{p,\Phi''}$ by letting

$$f_\epsilon(E/S^0, \omega, S^*) := f(E/S^0, \omega, S^*_\epsilon)$$

for every prolongation sequence S^*, every elliptic curve E/S^0 and every basis ω for the 1-forms on E. In particular, we get the following special cases:

(1) There is a natural action

$$\Sigma_n \times M^r_{p,\Phi} \to M^r_{p,\Phi}, \quad (\sigma, f) \mapsto \sigma f := f_\sigma.$$

(2) For every $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ there are natural *face* homomorphisms

$$M^r_{p,\phi_{i_1},\ldots,\phi_{i_s}} \to M^r_{p,\Phi}.$$

(3) For every $\phi \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ there is a natural *degeneration* homomorphism

$$M^r_{p,\Phi} \to M^r_{p,\phi}.$$

(4) The composition of the face and degeneration maps below is the identity:

$$\text{id} : M^r_{p,\phi_i} \to M^r_{p,\Phi} \to M^r_{p,\phi_i}.$$

(5) The face and degeneration maps induce maps between the corresponding modules $I^r_{p,\Phi}(w), I^r_{p,\phi_i}(w)$.

On the other hand by Remark 2.27 we have a natural action

$$\Sigma_n \times \mathcal{O}(J^r_{p,\Phi}(B_1(N))) \to \mathcal{O}(J^r_{p,\Phi}(B_1(N))), \quad (\sigma, u) \mapsto \sigma u.$$

The following is trivially checked.

Lemma 7.16. *The Σ_n-actions on $M^r_{p,\Phi} = \mathcal{O}(J^r_{p,\Phi}(B(1)))$ and $\mathcal{O}(J^r_{p,\Phi}(B_1(N)))$ are compatible in the sense that for every $\sigma \in \Sigma_n$ and every $f \in M^r_{p,\Phi}$ we have*

$$\sigma(f^{B_1(N)}) = (\sigma f)^{B_1(N)}.$$

7.3 The case $n = 2, \pi = p$

In this subsection we assume $n = 2, \pi = p$, and we may consider the forms

$$f^{\text{jet}}_\mu, f^{\text{jet}}_{\mu,\nu} \in M^2_{p,\phi_1,\phi_2}, \quad \mu, \nu \in \mathbb{M}^{2,+}_2, \quad \mu \neq \nu. \tag{7.8}$$

The forms (7.8) have weights $-\phi_\mu - 1$ and$-\phi_\mu - \phi_\nu$, respectively.

Remark 5.27 can be amplified as follows.

Remark 7.17. It follows from [10, Proposition 7.20, Corollary 8.84, and Remark 8.85] that for $i \in \{1, 2\}$ we have the following equality in M^2_{p,ϕ_1,ϕ_2}:

$$f^{\text{jet}}_{ii,i} = p(f^{\text{jet}}_i)^{\phi_i}. \tag{7.9}$$

It also follows from [10, Proposition 8.55] that $f^{\text{jet}}_i, f^{\text{jet}}_{ii}$ are non-zero in M^2_{p,ϕ_1,ϕ_2}; this also follows from (7.26) below. Hence by (7.9) it follows that $f^{\text{jet}}_{ii,i}$ are non-zero in M^2_{p,ϕ_1,ϕ_2}. The fact that the rest of the forms f^{jet}_μ are non-zero was proved in Remark 7.7; the fact that the forms $f^{\text{jet}}_{\mu,\nu}$ (for $\mu \neq \nu$) are non-zero is also true but more subtle and will be proved later; cf. Remark 7.31.

Theorem 7.18. *The following relation holds in M^2_{p,ϕ_1,ϕ_2}:*

$$f^{\text{jet}}_{11} f^{\text{jet}}_{22} f^{\text{jet}}_{1,2} + f^{\text{jet}}_2 f^{\text{jet}}_{22} f^{\text{jet}}_{11,1} - f^{\text{jet}}_{11} f^{\text{jet}}_1 f^{\text{jet}}_{22,2} - f^{\text{jet}}_1 f^{\text{jet}}_2 f^{\text{jet}}_{11,22} = 0. \tag{7.10}$$

Proof. In order to check our relation over a prolongation sequence S^* it is enough to check it after base change to a finite étale S^0-algebra. So we may assume E/S^0 has a $\Gamma_1(N)$-level structure with $(N, p) = 1$, $N \geq 4$. By functoriality it is then enough to check (7.1) for S^0 the ring of an affine open set of the modular curve $Y_1(N)$ over R_π and $S^r := J^r_{\pi,\Phi}(S^0)$. Note that f_1^{jet} and f_2^{jet} evaluated at the universal curve over S^0 are non-zero mod p in S^1; cf. [10, Lemma 4.4] (and also follows from Corollary 7.30 below). As in Corollary 5.25 the relation in our theorem holds with (S^r) replaced by (T^r) where $T^0 = S^0$ and $T^r := \widehat{(S^r_{f_1^{\text{jet}} f_2^{\text{jet}}})}$ for $r \geq 1$; this is because T^r are integral domains in which f_1^{jet}, f_2^{jet} are invertible. Note now that the homomorphisms $S^r \to T^r$ are injective; this follows because the homomorphisms $S^r/pS^r \to T^r/pT^r$ are injective as S^r/pS^r are integral domains and $f_1^{\text{jet}} f_2^{\text{jet}}$ is not zero in S^r/pS^r. We conclude that the relation in our theorem holds for (S^r). ■

By an argument similar to the one in the proof of Theorem 7.18, using the corresponding version of Lemmas 5.28, 5.29, 5.30, 5.31 over an arbitrary prolongation sequence, we obtain the following.

Theorem 7.19. *The following relations hold in M^2_{p,ϕ_1,ϕ_2}:*

$$f_{12,1}^{\text{jet}}(f_1^{\text{jet}})^{\phi_1} - f_{11,1}^{\text{jet}}(f_2^{\text{jet}})^{\phi_1} = 0, \qquad (7.11)$$

$$f_{12}^{\text{jet}}(f_1^{\text{jet}})^{\phi_1} - f_{11}^{\text{jet}}(f_2^{\text{jet}})^{\phi_1} - f_1^{\text{jet}}(f_{1,2}^{\text{jet}})^{\phi_1} = 0, \qquad (7.12)$$

$$f_{12,21}^{\text{jet}} f_1^{\text{jet}} f_2^{\text{jet}} - f_2^{\text{jet}} f_{21}^{\text{jet}} f_{12,1}^{\text{jet}} + f_1^{\text{jet}} f_{12}^{\text{jet}} f_{21,2}^{\text{jet}} - f_{12}^{\text{jet}} f_{21}^{\text{jet}} f_{1,2}^{\text{jet}} = 0, \qquad (7.13)$$

$$f_1^{\text{jet}} f_{11,2}^{\text{jet}} - f_2^{\text{jet}} f_{11,1}^{\text{jet}} - f_{11}^{\text{jet}} f_{1,2}^{\text{jet}} = 0, \qquad (7.14)$$

$$f_1^{\text{jet}} f_{11,12}^{\text{jet}} - f_{12}^{\text{jet}} f_{11,1}^{\text{jet}} + f_{11}^{\text{jet}} f_{12,1}^{\text{jet}} = 0, \qquad (7.15)$$

$$f_1^{\text{jet}} f_{12,2}^{\text{jet}} - f_2^{\text{jet}} f_{12,1}^{\text{jet}} - f_{12}^{\text{jet}} f_{1,2}^{\text{jet}} = 0, \qquad (7.16)$$

$$f_2^{\text{jet}} f_{11,21}^{\text{jet}} - f_{21}^{\text{jet}} f_{11,2}^{\text{jet}} + f_{11}^{\text{jet}} f_{21,2}^{\text{jet}} = 0. \qquad (7.17)$$

Moreover, the relations obtained from the above relations by switching the indices 1 *and* 2 *also hold.*

7.4 δ-Serre–Tate expansions

In this subsection we assume $\pi = p$ and n is arbitrary.

For the discussion of formal moduli in this paragraph we refer to [24]; we will use the notation in [10, Section 8.2]. Throughout our discussion we fix an ordinary elliptic curve E_0 over $k = R/pR$, a basis b of the Tate module of $T_p(E_0)$, and a basis $\check{b}$ of the Tate module of the dual $T_p(\check{E}_0)$. Let $S^0_{\text{for}} = R[\![T]\!]$, with T a variable, and consider the Serre–Tate universal deformation space (identified with $\mathrm{Spf}(R[\![T]\!])$) of

E_0/k [24]. Let $E_{\rm for}/S^0_{\rm for}$ be the universal elliptic curve over $R[\![T]\!]$. For all Noetherian complete local rings $(A, \mathfrak{m}(A))$ with residue field k and every elliptic curve E/A lifting E_0/k we let $q(E) = q(E/A) \in 1 + \mathfrak{m}(A)$ be the *Serre–Tate parameter* of E, i.e., the value of the Serre–Tate pairing $q_{E/A} : T_p(E_0) \times T_p(\check{E}_0) \to 1 + \mathfrak{m}(A)$ at the pair $(b, \check{b})$. Then $q(E)$ is the image of $1 + T$ via the classifying map $R[\![T]\!] \to A$ corresponding to E/A. We denote by $\omega_{\rm for}$ the canonical 1-form on $E_{\rm for}$ attached to $\check{b}$; cf. [10, equation (8.67)] and the discussion before it.

Let now

$$S^r_{\rm for} := R[\![T]\!][\delta_{p,\mu} T \mid \mu \in \mathbb{M}^r_n]\widehat{\ }.$$

Clearly $S^*_{\rm for} = (S^r_{\rm for})$ is naturally an object of $\mathbf{Prol}_{p,\Phi}$. We define a ring homomorphism

$$\mathcal{E} = \mathcal{E}_{E_0,b,\check{b}} : M^r_{p,\Phi} \to S^r_{\rm for}$$

by attaching to every $f \in M^r_{p,\Phi}$ its δ-*Serre–Tate expansion* given by

$$\mathcal{E}(f) := f(E_{\rm for}/S^0_{\rm for}, \omega_{\rm for}, S^*_{\rm for}) \in S^r_{\rm for}.$$

Note that we have a natural action

$$\Sigma_n \times S^r_{\rm for} \to S^r_{\rm for}, \quad (\sigma, \delta_{p,i} T) \mapsto \delta_{p,\sigma(i)} T.$$

The following is trivially checked.

Lemma 7.20. *The δ-Serre–Tate expansion map $\mathcal{E}$ is compatible with the Σ_n-actions on $f \in M^r_{p,\Phi}$ and $S^r_{\rm for}$ in the sense that for every $\sigma \in \Sigma_n$ and every $f \in M^r_{p,\Phi}$ we have*

$$\sigma(\mathcal{E}(f)) = \mathcal{E}(\sigma f).$$

As in the arithmetic ODE case [9], one has the following *Serre–Tate expansion principle.*

Theorem 7.21. *For every $w \in \mathbb{Z}_\Phi$ the homomorphism*

$$\mathcal{E} : M^r_{p,\Phi}(w) \to S^r_{\rm for}, \quad f \mapsto \mathcal{E}(f)$$

is injective with torsion-free cokernel.

Proof. Let $f \in M^r_{p,\Phi}(w)$ be such that $\mathcal{E}(f) = 0$ (respectively $\mathcal{E}(f) \in pS^r_{\rm for}$); we want to show that for all S^* and E/S^0 we have $f(E/S^0, \omega, S^*) = 0$ (respectively $f(E/S^0, \omega, S^*) \in pS^r$). As in the proof of Theorem 7.18 it is enough to check this for S^0 the ring of an affine open set of the modular curve $Y_1(N)$ over R, E/S^0 the universal elliptic curve, and $S^r := J^r_{p,\Phi}(S^0)$. We conclude by the injectivity of the homomorphisms $S^r \to S^r_{\rm for}$ (respectively $S^r/pS^r \to S^r_{\rm for}/pS^r_{\rm for}$). ∎

Corollary 7.22. *The ring $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{p,\Phi}(w)$ is an integral domain.*

Remark 7.23. Write

$$f_{i^r}^{\text{jet}} = f_{i\ldots i}^{\text{jet}} \quad \text{with } i \text{ repeated } r \text{ times.}$$

By [10, Propositions 8.22, 8.61, 8.84] plus the equality $\epsilon = 1$ in [10, page 236] we have

$$\mathcal{E}(f_{i^r}^{\text{jet}}) = c_r \Lambda_i^{r-1} \Psi_i, \tag{7.18}$$

where $c_r \in R^\times$,

$$\Lambda_i^{r-1} := \sum_{j=0}^{r-1} p^j \phi_i^{r-1-j},$$

and

$$\Psi_i := \frac{1}{p} \sum_{n \geq 1} (-1)^n \frac{p^n}{n} \left(\frac{\delta_{p,i}(1+T)}{(1+T)^p} \right)^n \in S_{\text{for}}^1 = R[\![T]\!][\delta_{p,i}T]\widehat{\ }.$$

In fact, by the theory over $\mathbb{Z}_p$ in [2, 9] (instead of over R as in [10]) one gets that

$$c_r \in \mathbb{Z}_p^\times. \tag{7.19}$$

Note that one has the following equality in $K[\![T, \delta_{p,i}T, \ldots, \delta_{p,i}^r T]\!]$:

$$\mathcal{E}(f_{i^r}^{\text{jet}}) = c_r \frac{1}{p} (\phi_i^r - p^r) \log(1+T). \tag{7.20}$$

Now recall that the Serre–Tate expansion of E_{p-1} satisfies

$$\mathcal{E}(E_{p-1}) \equiv 1 \bmod p \ \text{ in } \ R[\![T]\!]; \tag{7.21}$$

cf., for instance, [10, Propositions 8.57 and 8.59]. (In loc. cit. $\overline{H}$ denotes the Hasse invariant which is the reduction mod p of E_{p-1}.) Taking $\mathcal{E}$ in the congruence

$$f_{ii}^{\text{jet}} \equiv E_{p-1} \cdot (f_i^{\text{jet}})^p \bmod p \ \text{ in } \ M_{p,\Phi}^2,$$

cf. Remark 7.7, and using Fermat's little theorem we get

$$c_2 \Psi_i \equiv c_1 \Psi_i^p \bmod p \ \text{ in } \ R[\![T]\!]$$

hence

$$c_2 \equiv c_1 \bmod p \ \text{ in } \ \mathbb{Z}_p. \tag{7.22}$$

We claim that we have:

$$c_1 = c_2 = c_3 \tag{7.23}$$

and hence we set, in what follows,

$$c := c_1 = c_2 = c_3. \tag{7.24}$$

To check the equality (7.23) consider a prolongation sequence S^* with S^r integral domains and an arbitrary elliptic curve E/S^0 such that $f_1 \neq 0$. By the relative case of Corollary 5.14 (with $n = 1$) the S^3-module $\mathbf{X}^3_{p,\phi}(E/S^*)$ has rank ≤ 2 so the following δ-characters of E are S^3-linearly dependent:

$$\psi_{11,1},\ \phi_1\psi_{11,1},\ \psi_{111,1}.$$

Therefore the Picard–Fuchs symbols of these δ-characters,

$$f_1\phi^2 - f_{11}\phi + pf_1^{\phi},\ f_1^{\phi}\phi^3 - f_{11}^{\phi}\phi^2 + pf_1^{\phi^2}\phi,\ f_1\phi^3 - f_{111}\phi + f_{111,1}$$

are S^3-linearly dependent. We deduce that the determinant of the matrix

$$\begin{pmatrix} 0 & f_1 & -f_{11} \\ f_1^{\phi} & -f_{11}^{\phi} & pf_1^{\phi^2} \\ f_1 & 0 & -f_{111} \end{pmatrix}$$

vanishes, hence

$$pf_1^{\phi^2}f_1 - f_{11}^{\phi}f_{11} + f_1^{\phi}f_{111} = 0.$$

Since S^* was arbitrary (with S^r integral domains) we get

$$p(f_1^{\mathrm{jet}})^{\phi^2}f_1^{\mathrm{jet}} - (f_{11}^{\mathrm{jet}})^{\phi}f_{11}^{\mathrm{jet}} + (f_1^{\mathrm{jet}})^{\phi}f_{111}^{\mathrm{jet}} = 0.$$

Using the formula (7.18) we get

$$pc_1^2\Psi^{\phi^2}\Psi - c_2^2(\Psi^{\phi} + p\Psi)(\Psi^{\phi^2} + p\Psi^{\phi}) + c_1c_3\Psi^{\phi}(\Psi^{\phi^2} + p\Psi^{\phi} + p^2\Psi) = 0.$$

Identifying the coefficients we get $c_1^2 = c_2^2$ and $c_2^2 = c_3c_1$. So $c_3 = c_1$ and c_2 is either c_1 or $-c_1$. The equality $c_2 = -c_1$ together with the equality (7.22) leads to a congruence $c_1 \equiv -c_1 \bmod p$ which is impossible for p odd. We conclude that $c_1 = c_2 = c_3$.

Theorem 7.24. *For every weight $w \in \mathbb{M}_n^r$ of degree $\deg(w) = -2$ and every $f \in I^r_{p,\Phi}(w)$ we have that $\mathcal{E}(f)$ is a K-linear combination of elements in the set*

$$\{\Psi_i^{\phi_\mu} \mid \mu \in \mathbb{M}_n^{r-1},\ i \in \{1,\dots,n\}\}.$$

Proof. The proof is entirely similar to that of the statement made in [10, paragraph after Proposition 8.30]. Here is a rough guide to the argument. By [10, Proposition 8.22] there exists a prime $l \neq p$ and an endomorphism $a := u_0 \in \mathrm{End}(E_0) \subset \mathbb{Z}_p$ of degree l such that the quotient $\breve{u}_0/u_0 \in \mathbb{Z}_p$ is not a root of unity. By standard Serre–Tate theory u_0 lifts to an isogeny of degree l between $E_{R[\![T]\!]}$ and the curve $E'_{R[\![T]\!]}$

obtained from $E_{R[\![T]\!]}$ by base change via the homomorphism $R[\![T]\!] \to R[\![T]\!]$ given by $T \mapsto (1+T)^a - 1$. This forces the series $F := \mathcal{E}(f)$ to satisfy the equation

$$F(\dots, \delta_{p,\mu}((1+T)^a - 1), \dots) = a \cdot F(\dots, \delta_{p,\mu} T, \dots); \tag{7.25}$$

cf. [10, Proposition 8.30]. We conclude exactly as in [10, Proposition 4.36] that F is a K-linear combination of series Ψ^{ϕ_μ}. ■

By Theorem 7.24 and the Serre–Tate expansion principle in Theorem 7.21 we get the following corollary.

Corollary 7.25. *For every $w \in \mathbb{Z}^r_\Phi$ of degree* $\deg(w) = -2$ *the R-module $I^r_{p,\Phi}(w)$ has rank at most*

$$D(n,r) - 1 = n + \dots + n^r.$$

Definition 7.26. Let $w \in \mathbb{Z}^r_\Phi$ be a weight of degree $\deg(w) = -2$ and $f \in I^r_{p,\Phi}(w)$. Write

$$\mathcal{E}(f) = \sum_{i=1}^{n} \sum_{\mu \in \mathbb{M}_n^{r-1}} \lambda_{i,\mu} \Psi_i^{\phi_\mu}, \quad \lambda_{i,\mu} \in K_\pi;$$

cf. Theorem 7.24. Note that

$$\Psi_i^{\phi_\mu} = \frac{1}{p} \phi_i^\mu (\phi_i - p) \log(1+T).$$

Define the *symbol* $\theta(f) \in K_{\pi,\Phi}$ by

$$\theta(f) = \sum_{i=1}^{n} \sum_{\mu \in \mathbb{M}_n^{r-1}} \lambda_{i,\mu} \phi_\mu (\phi_i - p).$$

Hence the following holds in the ring S^r_{for}:

$$\mathcal{E}(f) = \frac{1}{p} \theta(f) \log(1+T).$$

Remark 7.27. For $n = 2$, using (7.18) and Remarks 7.17 and 7.23, we have:

$$\begin{aligned} \mathcal{E}(f_i^{\text{jet}}) &= c\Psi_i, \\ \mathcal{E}(f_{ii}^{\text{jet}}) &= c(\Psi_i^{\phi_i} + p\Psi_i), \\ \mathcal{E}(f_{ii,i}^{\text{jet}}) &= pc\Psi_i^{\phi_i}. \end{aligned} \tag{7.26}$$

In particular, the symbols of these forms are:

$$\begin{aligned} \theta(f_i^{\text{jet}}) &= c(\phi_i - p), \\ \theta(f_{ii}^{\text{jet}}) &= c(\phi_i^2 - p^2), \\ \theta(f_{ii,i}^{\text{jet}}) &= pc(\phi_i^2 - p^2\phi_i). \end{aligned} \tag{7.27}$$

For the rest of this subsection we assume that $n = 2$.

Theorem 7.28. *We have the following Serre–Tate expansions:*

$$\begin{aligned}
\mathcal{E}(f^{\text{jet}}_{1,2}) &= pc(\Psi_1 - \Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,22}) &= p^2c(\Psi_1^{\phi_1} + p\Psi_1 - \Psi_2^{\phi_2} - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{12,1}) &= pc\Psi_2^{\phi_1},\\
\mathcal{E}(f^{\text{jet}}_{12}) &= c(\Psi_2^{\phi_1} + p\Psi_1),\\
\mathcal{E}(f^{\text{jet}}_{12,21}) &= p^2c(\Psi_2^{\phi_1} + p\Psi_1 - \Psi_1^{\phi_2} - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,2}) &= pc(\Psi_1^{\phi_1} + p\Psi_1 - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,12}) &= p^2c(\Psi_1^{\phi_1} - \Psi_2^{\phi_1}),\\
\mathcal{E}(f^{\text{jet}}_{12,2}) &= pc(\Psi_2^{\phi_1} + p\Psi_1 - p\Psi_2),\\
\mathcal{E}(f^{\text{jet}}_{11,21}) &= p^2c(\Psi_1^{\phi_1} - \Psi_1^{\phi_2} + p\Psi_1 - p\Psi_2).
\end{aligned}$$

Moreover, the relations obtained from the above relations by switching the indices 1 *and* 2 *also hold.*

Proof. We begin by proving the first 2 equalities.

Set $G_1 := \mathcal{E}(f^{\text{jet}}_{1,2})$, $G_2 := \mathcal{E}(f^{\text{jet}}_{11,22})$. Taking $\mathcal{E}$ in the cubic relation of Theorem 7.18 and using the formulae (7.26) we get

$$\begin{aligned}
c^2(\Psi_1^{\phi_1} + p\Psi_1)(\Psi_2^{\phi_2} + p\Psi_2)G_1 + pc^3\Psi_2(\Psi_2^{\phi_2} + p\Psi_2)\Psi_1^{\phi_1}\\
= pc^3(\Psi_1^{\phi_1} + p\Psi_1)\Psi_1\Psi_2^{\phi_2} + c^2\Psi_1\Psi_2G_2.
\end{aligned} \tag{7.28}$$

By Theorem 7.24 we can write

$$\begin{aligned}
G_1 &= \gamma_1\Psi_1 + \gamma_2\Psi_2\\
G_2 &= \sum_\mu \gamma'_\mu\Psi_1^{\phi_\mu} + \sum_\mu \gamma''_\mu\Psi_2^{\phi_\mu}
\end{aligned}$$

with $\gamma_i, \gamma'_\mu, \gamma''_\mu \in K$. Plugging these expressions into the equation (7.28) and using the fact that the set

$$\{\Psi_i^{\phi_\mu} \mid i = 1, 2;\ \mu \in \mathbb{M}_2\}$$

is algebraically independent over K, we see that there is a unique tuple $(\gamma_i, \gamma'_\mu, \gamma''_\mu)$ satisfying the resulting equation, which leads to the desired formulae for $\mathcal{E}(f^{\text{jet}}_{1,2})$ and $\mathcal{E}(f^{\text{jet}}_{11,22})$. Taking $\mathcal{E}$ in (7.11), (7.12), (7.13), (7.14), (7.15), (7.16), (7.17) in this order we get the desired formulae for $\mathcal{E}(f^{\text{jet}}_{12,1})$, $\mathcal{E}(f^{\text{jet}}_{12})$, $\mathcal{E}(f^{\text{jet}}_{12,21})$, $\mathcal{E}(f^{\text{jet}}_{11,2})$, $\mathcal{E}(f^{\text{jet}}_{11,12})$, $\mathcal{E}(f^{\text{jet}}_{12,2})$, $\mathcal{E}(f^{\text{jet}}_{11,21})$, respectively. ∎

Corollary 7.29. *The following equalities hold:*

$$f^{\text{jet}}_{12,1} = p(f^{\text{jet}}_{2})^{\phi_1},$$
$$f^{\text{jet}}_{11,12} = p(f^{\text{jet}}_{1,2})^{\phi_1},$$
$$f^{\text{jet}}_{1,2} f^{\text{jet}}_{12,21} - p^2 (f^{\text{jet}}_{2})^{\phi_1} (f^{\text{jet}}_{1})^{\phi_2} + f^{\text{jet}}_{21,1} f^{\text{jet}}_{12,2} = 0.$$

Moreover, the equalities obtained from the above equalities by switching the indices 1 *and* 2 *also hold.*

Proof. The forms $f^{\text{jet}}_{12,1}$ and $p(f^{\text{jet}}_{2})^{\phi_1}$ have the same weight and the same Serre–Tate expansion so by the Serre–Tate expansion principle they must be equal. The same argument holds for the other equalities. ∎

Corollary 7.30. *Let $\mu, \nu \in \mathbb{M}^{2,+}_2$, $\mu \neq \nu$, of length r, s respectively, with $r \geq s$.*

(1) *The following non-divisibility, respectively divisibility conditions hold:*

$$\begin{aligned} f^{\text{jet}}_{\mu} &\in I^r_{p,\phi_1,\phi_2}(-\phi_\mu - 1) \setminus p I^r_{p,\phi_1,\phi_2}(-\phi_\mu - 1), \\ f^{\text{jet}}_{\mu,\nu} &\in p^s I^r_{p,\phi_1,\phi_2}(-\phi_\mu - \phi_\nu). \end{aligned} \tag{7.29}$$

(2) *The symbols of f^{jet}_{μ}, $f^{\text{jet}}_{\mu,\nu}$ are given by*

$$\begin{aligned} \theta(f^{\text{jet}}_{\mu}) &= c(\phi_\mu - p^r), \\ \theta(f^{\text{jet}}_{\mu,\nu}) &= c(p^s \phi_\mu - p^r \phi_\nu). \end{aligned} \tag{7.30}$$

Proof. Part 1 follows from Theorem 7.28 plus the torsion-freeness part of the Serre–Tate expansion principle. (The first equation in Part 1 also follows from Remark 7.7.) Part 2 follows from a direct computation using our definitions. ∎

Remark 7.31. In particular, we have that the forms f^{jet}_{μ}, $f^{\text{jet}}_{\mu,\nu}$ for $\mu, \nu \in \mathbb{M}^{2,+}_2$, $\mu \neq \nu$ are non-zero in M^2_{p,ϕ_1,ϕ_2}. Note that all these forms, with the exception of the "ODE forms" f^{jet}_{i}, f^{jet}_{ii}, $f^{\text{jet}}_{ii,i}$, are "genuinely PDE" in the sense that they are not sums of products of ODE forms and their images by ϕ_1, ϕ_2 (as one can see by looking at weights).

Remark 7.32. We expect that Corollary 7.30 remains true for $\mathbb{M}^{2,+}_2$ replaced by $\mathbb{M}^{r,+}_n$ for n and r arbitrary. Our method of proof for $\mathbb{M}^{2,+}_2$ was based on "solving" a rather complicated system of quadratic and cubic equations satisfied by the arithmetic Kodaira–Spencer classes; extending this method to the case of $\mathbb{M}^{r,+}_n$ seems tedious. It would be interesting to find another approach to the proof of Corollary 7.30 that easily extends to arbitrary n and r.

Remark 7.33. Let $\sigma \in \Sigma_2$ be the transposition (12). Then by Lemma 7.20 and by the δ-Serre–Tate expansion principle we have:

$$\sigma f^{\text{jet}}_{\mu} = f^{\text{jet}}_{\sigma\mu}, \quad \sigma f^{\text{jet}}_{\mu,\nu} = f^{\text{jet}}_{\sigma\mu,\sigma\nu}.$$

Theorem 7.34. *The form $f_{1,2}^{\text{jet}}$ is a basis modulo torsion of the R-module*

$$I^1_{p,\phi_1,\phi_2}(-\phi_1 - \phi_2)$$

of isogeny covariant δ_p-modular forms of order 1 *and weight $-\phi_1 - \phi_2$.*

Proof. Assume this R-module has rank ≥ 2 and seek a contradiction. Let f be an element of this module that is R-linearly independent of $\psi_{p,1,2}$. By Theorem 7.24 we have $\mathcal{E}(f) = \gamma_1\Psi_1 + \gamma_2\Psi_2$ for some $\gamma_1, \gamma_2 \in K$. By the Serre–Tate expansion principle (Theorem 7.21) $\mathcal{E}(f)$ is K-linearly independent of $\mathcal{E}(f_{1,2}^{\text{jet}})$. By Theorem 7.28 the latter equals $p\Psi_1 - p\Psi_2$. Hence $\gamma_1 + \gamma_2 \neq 0$. Consider the degeneration morphism $d : M^1_{p,\phi_1,\phi_2} \to M^1_{p,\phi}$ where $\phi \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$ and let Ψ be the series corresponding to f. Then $d(f) \in I^1_{p,\phi}(-2\phi)$. By [10, Theorem 8.83, Part 2] we have $I^1_{p,\phi}(-2\phi) = 0$; hence $f = 0$. But on the other hand $\mathcal{E}(f) = (\gamma_1 + \gamma_2)\Psi \neq 0$, a contradiction. ∎

7.5 δ-period maps

In this subsection we revert to the case of an arbitrary value of n. For the next result let us consider a weight $w \in \mathbb{Z}^r_\Phi$ together with the set $P^r(w)$ of all polynomials $F(\ldots, y_{\eta,\mu}, \ldots, y_{\eta,\mu,\nu}, \ldots)$ with R-coefficients in the variables $y_{\eta,\mu}, y_{\eta,\mu,\nu}$, $\eta, \mu, \nu \in \mathbb{M}_n$, that are homogeneous of weight w when $y_{\eta,\mu}, y_{\eta,\mu,\nu}$ are given weights $-\phi_\eta - \phi_{\eta\mu}$ and $-\phi_{\eta\mu} - \phi_{\eta\nu}$, respectively. Moreover, we denote by $\text{KSI}^r_{p,\Phi}(w)$ the R-submodule of $M^r_{p,\Phi}$ of all elements f of the form

$$f = F(\ldots, \phi_\eta f_\mu^{\text{jet}}, \ldots, \phi_\eta f_{\mu,\nu}^{\text{jet}}, \ldots), \quad F \in P^r(w). \tag{7.31}$$

Clearly the elements of $\text{KSI}^r_{p,\Phi}(w)$ have weight w and are isogeny covariant, i.e.,

$$\text{KSI}^r_{p,\Phi}(w) \subset I^r_{p,\Phi}(w). \tag{7.32}$$

Alternatively $\text{KSI}^r_{p,\Phi}(w)$ is the R-span of all the products of the form

$$\prod_\mu (f_\mu^{\text{jet}})^{w_\mu} \cdot \prod_{\mu,\nu} (f_{\mu,\nu}^{\text{jet}})^{w_{\mu,\nu}} \tag{7.33}$$

where μ, ν run through $\mathbb{M}_n$, $w_\mu, w_{\mu,\nu} \in \mathbb{Z}_\Phi$ are ≥ 0 and

$$\sum_\mu w_\mu(1 + \phi_\mu) + \sum_{\mu,\nu} w_{\mu,\nu}(\phi_\mu + \phi_\nu) = -w.$$

One should view the ring

$$\text{KSI}^r_{p,\Phi} := \bigoplus_{w \in \mathbb{Z}^r_\Phi} \text{KSI}^r_{p,\Phi}(w) \subset \bigoplus_{w \in \mathbb{Z}^r_\Phi} I^r_{p,\Phi}(w)$$

as the "Φ-stable R-subalgebra generated by the arithmetic Kodaira–Spencer classes"; whence our notation. By the way we do not know if the inclusion (7.32) is an equality (or an equality modulo torsion).

Definition 7.35. Let $B = B_1(N)$, $N \geq 4$, $\pi = p$ and assume as before that the reduction mod p of $X \subset Y_1(N)$ is non-empty. Fix an order $r \geq 1$ and weight $w \in \mathbb{Z}_\Phi^r$ and consider a basis

$$f_{(0)}, \dots, f_{(N_w)}$$

of the R-module $\mathrm{KSI}^r_{p,\Phi}(w)$ (so $N_w + 1$ is the rank of this module). Consider the map

$$\mathfrak{P}^B_w : B(R^{\mathrm{alg}}) \to \mathbb{A}^{N_w}(K^{\mathrm{alg}}) = (K^{\mathrm{alg}})^{N_w}$$

defined by

$$\mathfrak{P}^B_w(P) := (((f_{(0)})^B)^{\mathrm{alg}}(P), \dots, ((f_{(N_w)})^B)^{\mathrm{alg}}(P))$$

and the induced map to the set of points of the projective space:

$$\mathfrak{p}^B_w : B(R^{\mathrm{alg}})^{\mathrm{ss}}_w := B(R^{\mathrm{alg}}) \setminus ((\mathfrak{P}^B_w)^{-1}(0)) \to \mathbb{P}^{N_w}(K^{\mathrm{alg}}) = \mathbb{P}^{N_w}(R^{\mathrm{alg}}).$$

The "ss" superscript stands for "semistable" (in analogy with geometric invariant theory; here instead of group actions we have an action of the Hecke correspondences and isogeny covariant forms are viewed as analogues of invariant sections of line bundles in geometric invariant theory). Assuming, for a moment, that the universal elliptic curve over X possesses a global invertible relative 1-form ω we get an induced section $\sigma : X \to B$ of the projection $B \to X$ and hence an induced map

$$\mathfrak{p}_w := \mathfrak{p}^X_w : X(R^{\mathrm{alg}})^{\mathrm{ss}}_w := \sigma^{-1}(B(R^{\mathrm{alg}})^{\mathrm{ss}}_w) \xrightarrow{\sigma} B(R^{\mathrm{alg}})^{\mathrm{ss}}_w \xrightarrow{\mathfrak{P}^B_w} \mathbb{P}^{N_w}(R^{\mathrm{alg}}). \quad (7.34)$$

The map (7.34) does not depend on the choice of ω (due to the fact that $f_{(i)}$ have the same weight) and hence this map is well defined for any X (not only for X such that an ω as above exists) and only depends on X and w (up to a projective transformation). The map (7.34) will be referred to as the δ-*period map* (of weight w) and the set $X(R^{\mathrm{alg}})^{\mathrm{ss}}_w$ will be referred to as the set of *semistable* points (relative to w).

Note that for $w, w' \in \mathbb{M}_n$ the composition

$$X(R^{\mathrm{alg}})^{\mathrm{ss}}_w \cap X(R^{\mathrm{alg}})^{\mathrm{ss}}_{w'} \xrightarrow{\mathfrak{p}_w \times \mathfrak{p}_{w'}} \mathbb{P}^{N_w}(R^{\mathrm{alg}}) \times \mathbb{P}^{N_{w'}}(R^{\mathrm{alg}}) \xrightarrow{\mathrm{Segre}} \mathbb{P}^{N_w N_{w'} + N_w + N_{w'}}(R^{\mathrm{alg}})$$

is obtained by composing the map

$$\mathfrak{p}_{w+w'} : X(R^{\mathrm{alg}})^{\mathrm{ss}}_w \to \mathbb{P}^{N_{w+w'}}(R^{\mathrm{alg}})$$

with a projection. And similarly $\mathfrak{p}_w$ is obtained from any of the maps $\mathfrak{p}_{\phi_i w}$ by composing with a projection followed by the ϕ_i map.

Theorem 7.36. *The δ-period map*

$$\mathfrak{p}_w : X(R^{\text{alg}})^{\text{ss}}_w \to \mathbb{P}^{N_w}(R^{\text{alg}})$$

is constant on prime to p isogeny classes in the following sense: for every two points $P, Q \in X(R^{\text{alg}})^{\text{ss}}_w$ if there exists an isogeny of degree prime to p between the elliptic curves over R^{alg} corresponding to P and Q then

$$\mathfrak{p}_w(P) = \mathfrak{p}_w(Q).$$

Proof. By the density of prime to p ordinary isogeny classes [19] we may assume X is such that the universal elliptic curve over X possesses an invertible 1-form ω. Assume P and Q are as in the statement of the theorem and let $u : E_P \to E_Q$ be an isogeny of degree prime to p between the corresponding elliptic curves over R^{alg}. Let ω_P and ω_Q be the 1-forms on E_P and E_Q induced by ω, respectively. We may view both elliptic curves and the isogeny as being defined over some R_π. For each i let $f^X_{(i)}$ be the composition

$$f^X_{(i)} : J^r_{\pi,\Phi}(X) \xrightarrow{J^r(\sigma)} J^r_{\pi,\Phi}(B) \xrightarrow{f^B_{(i)}} \widehat{\mathbb{A}^1}.$$

Write for simplicity $f_{(i)}(P) := ((f_{(i)})^X)^{\text{alg}}(P)$ for $P \in X(R^{\text{alg}})$. By isogeny covariance, the weight condition, and the equalities (7.3), (7.4), (7.6), (7.7) we get that

$$f_{(i)}(P) = \lambda \cdot f_{(i)}(Q), \quad i \in \{0, \dots, N_w\}$$

for some $\lambda \in R^\times_\pi$ depending on $E_P, E_Q, \omega_P, \omega_Q, u$ but not on i. This implies that $\mathfrak{p}_w(P) = \mathfrak{p}_w(Q)$. ■

Example 7.37. Assume $n = r = 2$. Then one can explicitly describe the algebra $\text{KSI}^2_{p,\Phi} \otimes_R K$ as follows. Consider the ring of polynomials

$$\mathcal{P} := K[\Psi_i, \Psi_i^{\phi_j} \mid i, j \in \{1, 2\}],$$

where $\Psi_i^{\phi_j}$ are viewed as variables and view this ring as graded by giving the variables the degree 1. We denote by $\mathcal{P}(i)$ the graded piece of degree $i \in \mathbb{Z}_{\geq 0}$, we set

$$t_0 := \frac{\Psi_2}{\Psi_1}, \quad t_{ij} := \frac{\Psi_i^{\phi_j}}{\Psi_1}, \quad i, j \in \{1, 2\},$$

and we consider the field of "homogeneous fractions of degree 0":

$$\mathcal{F} := \left\{ \frac{F}{G} \;\middle|\; F, G \in \mathcal{P}(i),\ i \geq 1,\ G \neq 0 \right\} = K(t_0, t_{11}, t_{12}, t_{21}, t_{22});$$

this is a field of rational functions over K in five variables. Consider new variables Λ^{ϕ_μ} for $\mu \in \mathbb{M}_2^2$ and the algebra of polynomials

$$\mathcal{P}[\Lambda^{\phi_\mu} \mid \mu \in \mathbb{M}_2^2].$$

Inside the latter consider the K-subalgebra $\mathcal{KSI}^2$ generated by all the elements of the form

$$\mathcal{E}(f_\mu^{\text{jet}})\Lambda^{1+\phi_\mu}, \quad \mathcal{E}(f_{\mu,\nu}^{\text{jet}})\Lambda^{\phi_\mu+\phi_\nu},$$

which have been explicitly computed in Remark 7.27 and Theorem 7.28. We then write

$$\mathcal{KSI}^2 = \bigoplus_{w \in \mathbb{Z}_\Phi^2} \mathcal{KSI}^2(w)\Lambda^w, \quad \mathcal{KSI}^2(w) \subset \mathcal{P}\left(\frac{\deg(w)}{2}\right)$$

(which makes sense since $\mathcal{KSI}^2(w) = 0$ for $\deg(w)$ odd). On the other hand, by the Serre–Tate expansion principle we get isomorphisms of K-vector spaces

$$\text{KSI}_{p,\Phi}^2 \otimes_R K \simeq \mathcal{KSI}^2(w).$$

One may consider the subfield $\mathcal{F}_w \subset \mathcal{F}$ generated by the set

$$\left\{\frac{F}{G} \;\middle|\; F, G \in \mathcal{KSI}^2(w),\ m \geq 0,\ G \neq 0\right\}.$$

This field could be intuitively interpreted as the "field of rational functions on the image of the δ-period map $\mathfrak{p}_w$." If $\mathcal{KSI}^2(w) \neq 0$ and $\mathcal{KSI}^2(w') \neq 0$ then, clearly, $\mathcal{F}_w$ and $\mathcal{F}_{w'}$ are subfields of $\mathcal{F}_{w+w'}$.

As a special case of the above discussion let $w = -(1 + \phi_1 + \phi_2 + \phi_1^2)$. Then $\text{KSI}_{p,\Phi}^r(w_1)$ is spanned by

$$f_1^{\text{jet}} f_{11,2}^{\text{jet}}, \ f_2^{\text{jet}} f_{11,1}^{\text{jet}}, \ f_{11}^{\text{jet}} f_{1,2}^{\text{jet}},$$

and has rank 2 with the "only relation" (7.14). The fact that this module has rank 2 follows by looking at the Serre–Tate expansions of the generators; cf. Theorem 7.28. Note that

$$\tau := \frac{\Psi_1(\Psi_1^{\phi_1} + p\Psi_1 - p\Psi_2)}{\Psi_2\Psi_1^{\phi_1}} = \frac{t_{11} + p - pt_0}{t_0 t_{11}} \in \mathcal{F}_w. \tag{7.35}$$

Similarly, let $w' = -(1 + \phi_1 + \phi_2 + \phi_1\phi_2)$. Then $\text{KSI}_{p,\Phi}^r(w')$ is spanned by

$$f_1^{\text{jet}} f_{12,2}^{\text{jet}}, \ f_2^{\text{jet}} f_{12,1}^{\text{jet}}, \ f_{12}^{\text{jet}} f_{1,2}^{\text{jet}},$$

and has rank 2 with the "only relation" (7.16). Note that

$$\tau' := \frac{\Psi_1(\Psi_2^{\phi_1} + p\Psi_1 - p\Psi_2)}{\Psi_2\Psi_2^{\phi_1}} = \frac{t_{21} + p - pt_0}{t_0 t_{21}} \in \mathcal{F}_{w'}. \tag{7.36}$$

One has a similar discussion for the weights w'', w''' obtained by switching the indices 1 and 2 in the weights w, w', and we have corresponding elements

$$\tau'' := \frac{\Psi_2(\Psi_2^{\phi_2} + p\Psi_2 - p\Psi_1)}{\Psi_1\Psi_2^{\phi_2}} = \frac{t_0(t_{22} + pt_0 - p)}{t_{22}} \in \mathcal{F}_{w''}, \tag{7.37}$$

$$\tau''' := \frac{\Psi_2(\Psi_1^{\phi_2} + p\Psi_2 - p\Psi_1)}{\Psi_1\Psi_1^{\phi_2}} = \frac{t_0(t_{12} + pt_0 - p)}{t_{12}} \in \mathcal{F}_{w'''}. \tag{7.38}$$

It is trivial to check that

$$K(t_0, \tau, \tau', \tau'', \tau''') = K(t_0, t_{11}, t_{12}, t_{21}, t_{22}).$$

So in particular the field $K(\tau, \tau', \tau'', \tau''')$ is a field of rational functions in 4 variables. The union $\bigcup_w \mathcal{F}_w$ for all w's of order 2 is a subfield of $\mathcal{F}$, so it is finitely generated, hence equal to one of the fields $\mathcal{F}_{w_0}$. The field $\mathcal{F}_{w_0}$ has then transcendence degree 4 or 5 over K. It would be interesting to compute this field and in particular to compute its transcendence degree over K. The heuristic we are employing is that the order 2 partial p-jet space of $Y_1(N)$ has relative dimension $7 = |\mathbb{M}_2^2|$ over K and the field $\mathcal{F}_{w_0}$ (which has transcendence degree 4 or 5 over K) plays the role of "field of rational functions" of the quotient "of order 2" of $Y_1(N)$ by the action of the Hecke correspondences. The difference between the dimensions (which is either 3 or 2) should play the role of "dimension of the fibers" of the "order 2" projection from $Y_1(N)$ to this "quotient". All of this can be made rigorous in a "partial δ-geometry" which is a PDE analogue of the ODE δ-geometry in [10]; we will not pursue this in the present work. Suffices to say that, for every $P \in X(R^{\text{alg}})^{\text{ss}}_{w_0}$, we get in this way, a "large" system of arithmetic differential equations of order 2 satisfied by the points $P' \in X(R^{\text{alg}})^{\text{ss}}_{w_0}$ in the prime to p isogeny class of P. Roughly speaking these equations have the form

$$f_{(i)}(P)f_{(j)}(P') - f_{(j)}(P)f_{(i)}(P') = 0,$$

where we are using the notation in the proof of Theorem 7.36. Note that the analogous ODE δ-period maps of minimum order (implicit in [10, Section 8.6]) have order 3 rather than 2; this is an instance of the principle, already encountered in this memoir in the case of δ-characters, that replacing ODEs by PDEs reduces the order of the interesting arithmetic differential equations.

We end by discussing the maps on points on the ordinary locus.

Proposition 7.38. *Let $B = B_1(N)$, $N \geq 4$. Assume the reduction mod p of X is contained in the ordinary locus of the modular curve. Then the following hold:*

(1) *For all distinct $\mu, \nu \in \mathbb{M}_n$ and all $\eta \in \mathbb{M}_n$ the maps*

$$((\phi_\eta f^{\text{jet}}_\mu)^B)^{\text{alg}}, ((\phi_\eta f^{\text{jet}}_{\mu,\nu})^B)^{\text{alg}} : B(R^{\text{alg}}) \to K^{\text{alg}}$$

(cf. Theorem 7.13) extend to continuous maps

$$((\phi_\eta f_\mu^{\text{jet}})^B)^{\mathbb{C}_p}, ((\phi_\eta f_{\mu,\nu}^{\text{jet}})^B)^{\mathbb{C}_p} : B(\mathbb{C}_p^\circ) \to \mathbb{C}_p.$$

(2) *Assume Φ is monomially independent in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$. Then for every $w \in \mathbb{Z}_\Phi$ the R-module homomorphism*

$$\text{KSI}_{p,\Phi}^r(w) \to \text{Fun}(B(R^{\text{alg}}), K^{\text{alg}}), \quad f \mapsto (f^B)^{\text{alg}}$$

is injective.

Proof. Part 1 follows exactly as in [12, Proposition 5.17]. Here is a guide to the argument. Let $\text{pr} : B \to X$ be the projection and let $P \in B(\mathbb{C}_p^\circ)$. Since $\text{pr}(P) \in X(\mathbb{C}_p^\circ)$ corresponds to an elliptic curve with ordinary reduction E_0 we have at our disposal the δ-Serre–Tate expansion at E_0. Similar to [12, equation (5.29)] one gets that there exists a p-adic ball $\mathbb{B} \subset B(\mathbb{C}_p^\circ)$ containing P and a nowhere vanishing analytic map $u : \mathbb{B} \to \mathbb{C}_p$ with the following property: for all μ and ν, $((f_\mu^{\text{jet}})^B)^{\text{alg}}$ and $((f_{\mu,\nu}^{\text{jet}})^B)^{\text{alg}}$ may be expressed on $\mathbb{B} \cap B(R^{\text{alg}})$ as

$$((f_\mu^{\text{jet}})^B)^{\text{alg}}(P) = u(P)^{-1-\phi_\mu} \cdot \theta(f_\mu^{\text{jet}})^{\text{alg}}[\log(q(E_P))], \tag{7.39}$$

$$((f_{\mu,\nu}^{\text{jet}})^B)^{\text{alg}}(P) = u(P)^{-\phi_\mu-\phi_\nu} \cdot \theta(f_{\mu,\nu}^{\text{jet}})^{\text{alg}}[\log(q(E_P))], \tag{7.40}$$

where log is the usual logarithm defined on the open ball of $\mathbb{C}_p$ of radius 1, E_P is the elliptic curve corresponding to P, and $\theta(f_\mu^{\text{jet}})^{\text{alg}}$, $\theta(f_{\mu,\nu}^{\text{jet}})^{\text{alg}}$ are, as usual, the induced maps $K^{\text{alg}} \to K^{\text{alg}}$. Cf. loc. cit. for details on this representation. Extension by continuity then follows. A similar argument works for the above maps composed with ϕ_η.

By the way, if P corresponds to a pair (E_P, ω_P) then by the construction in loc. cit. we have

$$u(P) = 1 \text{ if } \omega_P \text{ is induced by } \omega_{\text{for}} \text{ via the classifying map.} \tag{7.41}$$

To check Part 2 let f be as in (7.31) and assume $(f^B)^{\text{alg}} = 0$. Then, by the homogeneity of F, the map $R^{\text{alg}} \to K^{\text{alg}}$ given by

$$\lambda \mapsto F(\ldots, \theta(\phi_\eta f_\mu^{\text{jet}})^{\text{alg}}(\lambda), \ldots, \theta(\phi_\eta f_{\mu,\nu}^{\text{jet}})^{\text{alg}}(\lambda), \ldots)$$

must vanish on the additive group $p^N R^{\text{alg}}$ for some N hence, again, by the homogeneity of F, this map vanishes on R^{alg}. Now we proceed as in the proof of Proposition 3.13. Indeed, write $\theta(\phi_\eta f_{\eta,\mu}) = \sum_\epsilon \lambda_{\eta,\mu,\epsilon}\phi_\epsilon$ and $\theta(\phi_\eta f_{\mu,\nu}) = \sum_\epsilon \lambda_{\eta,\mu,\nu,\epsilon}\phi_\epsilon$. Let x_ϵ be variables indexed by $\epsilon \in \mathbb{M}_n$. By Lemma 2.6 we get that the following polynomial vanishes:

$$F(\ldots, \sum_\epsilon \lambda_{\eta,\mu,\epsilon} x_\epsilon, \ldots, \sum_\epsilon \lambda_{\eta,\mu,\nu,\epsilon} x_\epsilon, \ldots) = 0. \tag{7.42}$$

But f is obtained from the left-hand side of (7.42) by replacing $x_\epsilon \mapsto \frac{1}{p}\phi_\epsilon T$. Hence $f = 0$. ∎

At this point we are ready to give the proof of Theorem 5.33.

Proof of Theorem 5.33. By Remark 7.31 the forms corresponding to the classes in Theorem 5.33 are all non-zero. Let f be the product of all these forms. By Corollary 7.22 $f \neq 0$. By Proposition 7.38, Part 2, for $B = B_1(N)$, $N \geq 4$, we get that $(f^B)^{\text{alg}} \neq 0$, so there exists $\pi \in \Pi$ and a point $P \in B(R_\pi)$ such that the pair (E_P, ω_P) over R_π corresponding to P satisfies $(f^B)^{\text{alg}}(P) \neq 0$. We conclude by (7.6) and (7.7). ∎

Here is a characterization of ordinary elliptic curves with vanishing arithmetic Kodaira–Spencer classes; it is an improvement (and generalization) of [12, Proposition 5.10].

Proposition 7.39. *Let E/R_π be an elliptic curve with ordinary reduction and ω a basis for its* 1*-forms. The following are equivalent:*

(1) $f_{\pi,i}(E, \omega) = 0$ *for some* $i \in \{1, \dots, n\}$.

(2) $f_{\pi,\mu}(E, \omega) = f_{\pi,\mu,\nu}(E, \omega) = 0$ *for all* $\mu, \nu \in \mathbb{M}_n$.

(3) *The Serre–Tate parameter $q(E)$ is a root of unity.*

Proof. Assume condition (1) holds. Let $P \in B(R_\pi)$ represent (E, ω). By formula (7.6) we have $((f_i^{\text{jet}})^B)^{\text{alg}}(P) = 0$ for $B = B_1(N)$, $N \geq 4$. By (7.39) (with $\mu = i$), since the map $\theta(f_i^{\text{jet}})^{\text{alg}} : R^{\text{alg}} \to K^{\text{alg}}$, $\beta \mapsto \phi_i(\beta) - p\beta$ is injective, it follows that $\log(q(E)) = 0$. Hence $q(E)$ is a root of unity, i.e. condition (3) holds. Similarly, condition (3) implies condition (2) due to (7.6) and (7.39), (7.40) (applied to arbitrary μ, ν). Finally, condition (2) trivially implies condition (1). ∎

7.6 Theorem of the kernel and Reciprocity theorem

Recall from the Introduction the following pairing.

Definition 7.40. Let $\mu, \nu \in \mathbb{M}_2^2$ have length $r, s \in \{1, 2\}$, respectively. Define the $\mathbb{Q}_p$-bilinear map

$$\langle\ ,\ \rangle_{\mu,\nu} : K^{\text{alg}} \times K^{\text{alg}} \to K^{\text{alg}}$$

by the formula

$$\langle \alpha, \beta \rangle_{\mu,\nu} = \beta^{\phi_\nu}\alpha^{\phi_\mu} - \beta^{\phi_\mu}\alpha^{\phi_\nu} + p^s(\alpha\beta^{\phi_\mu} - \beta\alpha^{\phi_\mu}) + p^r(\beta\alpha^{\phi_\nu} - \alpha\beta^{\phi_\nu}). \quad (7.43)$$

Note that the above expression is antisymmetric in α, β:

$$\langle \alpha, \beta \rangle_{\mu,\nu} = -\langle \beta, \alpha \rangle_{\mu,\nu}$$

and also in μ, ν:

$$\langle \alpha, \beta \rangle_{\mu,\nu} = -\langle \alpha, \beta \rangle_{\nu,\mu}.$$

Consider in what follows the following data. Fix $\pi \in \Pi$ and an elliptic curve E over R_π with ordinary reduction E_0. Choose bases $b, \check{b}$ of $T_p(E_0), T_p(\check{E}_0)$ as in the beginning of Section 7.4. Now set $S^0_{\text{for}} = R[\![T]\!] \to R_\pi$ the classifying homomorphism given by the Serre–Tate theory. Choose the 1-form ω on E induced by the canonical form ω_{for} on the universal elliptic curve $E_{\text{for}}/S^0_{\text{for}}$ via the classifying homomorphism and set $q(E) \in R_\pi$ the Serre–Tate parameter of E, i.e., the image of $1+T$ via the classifying homomorphism. Denote also by $\beta := \beta(E) := \log(q(E)) \in K_\pi$, where $\log\colon 1 + \pi R_\pi \to K_\pi$ is the usual logarithm. When $n = 2$, set $\Phi = \{\phi_1, \phi_2\}$, $\mu \neq \nu \in \mathbb{M}_2^{2,+}$ of length r, s respectively, with $r \geq s$. Consider $\psi_{\mu,\nu} := \psi_{\pi,\mu,\nu}(E,\omega) \in \mathbf{X}^2_{\pi,\Phi}(E)$ the δ_π-character attached to (E,ω) and $c \in \mathbb{Z}_p^\times$ the constant in (7.24). Recall this constant depends only on p.

Proposition 7.41. *The Picard–Fuchs symbol of $\psi_{\mu,\nu}$ is given by the following formula:*

$$\theta(\psi_{\mu,\nu}) = p^{N(\pi)+1} c[(\beta^{\phi_\nu} - p^s\beta)\phi_\mu - (\beta^{\phi_\mu} - p^r\beta)\phi_\nu + (p^s\beta^{\phi_\mu} - p^r\beta^{\phi_\nu})].$$

Proof. A direct computation using (in the following order) (5.12), (5.10), (7.6), (7.7), (7.39), (7.40), (7.41), (7.30). ∎

In view of Definition 7.40, Proposition 7.41 yields then the formula

$$\theta(\psi_{\mu,\nu})^{\text{alg}}(\alpha) = p^{2N(\pi)+1} c\langle \alpha, \beta \rangle_{\mu,\nu}. \tag{7.44}$$

Hence

$$\operatorname{Ker}(\theta(\psi_{\mu,\nu})^{\text{alg}}) = \{\alpha \in K^{\text{alg}} \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}.$$

The above is a $\mathbb{Q}_p$-linear space; this space contains β which is non-zero if $q(E)$ is not a root of unity. So by Corollary 3.10 if $q(E)$ is not a root of unity then the group $\operatorname{Ker}(\psi^{\text{alg}}_{\mu,\nu})$ in *not torsion*. More generally by Corollary 3.10 we get the following.

Theorem 7.42. *(Theorem of the kernel) We have a natural group isomorphism*

$$\operatorname{Ker}(\psi^{\text{alg}}_{\mu,\nu}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \{\alpha \in K^{\text{alg}} \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}.$$

Remark 7.43. In fact, in view of Corollary 3.11 a stronger result holds as follows. Let L be a filtered union of complete subfields of K^{alg} and let $\mathcal{O}$ be the valuation ring of L. Assume E comes via base change from an elliptic curve over $\mathcal{O}$. Then

$$(\operatorname{Ker}(\psi^{\text{alg}}_{\mu,\nu}) \cap E(\mathcal{O})) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \{\alpha \in L \mid \langle \alpha, \beta \rangle_{\mu,\nu} = 0\}. \tag{7.45}$$

In order to state our next result fix an ordinary elliptic curve E_0 over k and bases $b, \check{b}$ of $T_p(E_0), T_p(\check{E}_0)$ as in the beginning of Section 7.4. Let $\pi \in \Pi$ and let

$$\alpha, \beta \in K_\pi, \quad |\alpha|, |\beta| < p^{-\frac{1}{p-1}}. \tag{7.46}$$

By [33, Theorem 6.4] there exists $q \in 1 + \pi R_\pi$ such that $\log q = \beta$. Then the homomorphism $R[\![T]\!] \to R_\pi$, $T \mapsto q - 1$, defines an elliptic curve E_β over R_π with logarithm of the Serre–Tate parameter satisfying $\log(q(E_\beta)) = \beta$. Let $\ell^{\text{alg}}_{E_\beta} : E_\beta(\pi R_\pi) \to K_\pi$ be the logarithm of E_β and let ω_β be the 1-form on E_β induced from the canonical 1-form ω_{for} defined by $\check{b}$ on the universal deformation $E_{\text{for}}/R[\![T]\!]$ of E_0. Again, by [33, Theorem 6.4] there exists a point $P_{\alpha,\beta} \in E_\beta(\pi R_\pi)$ such that $\ell^{\text{alg}}_{E_\beta}(P_{\alpha,\beta}) = \alpha$. Since the roles of α and β can be interchanged we also have at our disposal an elliptic curve E_α over R_π, a 1-form ω_α on E_α, and a point $P_{\beta,\alpha} \in E_\alpha(\pi R_\pi)$. Let $\mu, \nu \in \mathbb{M}^2_n$ be distinct and let $\psi_{\mu,\nu,\beta}$ and $\psi_{\mu,\nu,\alpha}$ be the corresponding δ_π-characters attached to (E_β, ω_β) and $(E_\alpha, \omega_\alpha)$ over R_π, respectively. Then formula (7.44), the antisymmetry of $\langle\ ,\ \rangle_{\mu,\nu}$, and the commutative diagram (3.4) imply the following theorem.

Theorem 7.44. *(Reciprocity theorem) For every α, β as in (7.46) and every distinct $\mu, \nu \in \mathbb{M}^2_2$ we have*

$$\psi^{\text{alg}}_{\mu,\nu,\beta}(P_{\alpha,\beta}) = \psi^{\text{alg}}_{\nu,\mu,\alpha}(P_{\beta,\alpha}).$$

Remark 7.45. (1) The Reciprocity theorem works because of the antisymmetry of $\langle\ ,\ \rangle_{\mu,\nu}$. However, we feel that this antisymmetry comes as a surprise and should not be expected a priori; the only "explanation" we could give is the explicit computation of the constants involved in the expression of our bilinear map.

(2) Note that by antisymmetry in μ, ν we get

$$\psi^{\text{alg}}_{\mu,\nu,\alpha}(P_{\alpha,\alpha}) = 0. \tag{7.47}$$

In case $\pi = p$ and $\alpha \in pR$, (7.47) can also be derived as follows. Recall from [6, Section (4.1)] that if $\pi = p$ and $\alpha \in pR$ then the point $P_{\alpha,\alpha}$ belongs to the group $\bigcap_{m=1}^{\infty} p^m E(R)$ of infinitely p-divisible points of $E(R)$. On the other hand for every partial δ_p-character ψ of E_α the homomorphism $\psi^{\text{alg}} : E_\alpha(R^{\text{alg}}) \to K^{\text{alg}}$ sends $E_\alpha(R)$ into R. Since $\bigcap_{m=1}^{\infty} p^m R = 0$ we get $\psi^{\text{alg}}(P_{\alpha,\alpha}) = 0$. We expect that (7.47) can be derived along similar lines in the general case when π is arbitrary and α is arbitrary, satisfying $|\alpha| < p^{-\frac{1}{p-1}}$.

Example 7.46. Here is an illustration of the Theorem of the kernel; the example below can be easily generalized.

Let $\ell \leq p - 1$ be a prime and consider the field $K^{(l)}$ in (2.2) and the notation of the paragraph containing that equation; in particular recall the elements π_m, ζ_{l^m}, and the automorphisms $\phi^{(\gamma)} \in \mathfrak{F}^{(1)}(K^{\text{alg}}/\mathbb{Q}_p)$. Set $\phi_1 := \phi^{(0)}$ and $\phi_2 := \phi^{(1)}$; hence ϕ_1

and ϕ_2 are Frobenius automorphisms whose restriction to K_{π_m} satisfy $\phi_1\pi_m = \pi_m$, $\phi_2\pi_m = \zeta_{l^m}\pi_m$. Let $\beta = \pi_1$ (hence $|\beta| < p^{-\frac{1}{p-1}}$), let $\mu, \nu \in \mathbb{M}_2^{2,+}$ be distinct of length 2 (a similar computation holds for length 1) and consider the $\mathbb{Q}_p$-bilinear map $\langle\ ,\ \rangle_{\mu,\nu} : K^{\text{alg}} \times K^{\text{alg}} \to K^{\text{alg}}$ in (7.43).

Claim 1. For every $\alpha \in K^{(l)}$ we have $\langle \alpha, \beta\rangle_{\mu,\nu} = 0$ if and only if there exists $\lambda \in \mathbb{Q}_p$ such that $\alpha = \lambda\beta$.

The "if" part is clear. To check the "only if" part note that we may assume $\alpha \in R_{\pi_m}$ for some m, so we may write

$$\alpha = \sum_{i=0}^{l^m-1} \alpha_i \pi_m^i, \quad \alpha_i \in R.$$

Let ϕ be the restriction of ϕ_1, ϕ_2 to R. Picking out the coefficient of π_m^i in the right-hand side of the equality (7.43) we get from the equality $\langle \alpha, \beta\rangle_{\mu,\nu} = 0$ that

$$\alpha_i^{\phi}(\zeta_l - \zeta_{l^m}^i - p^2 + p^2\zeta_{l^m}^i) + \alpha_i(p^2 - p^2\zeta_l) = 0, \quad i \in \{0, \ldots, l^m - 1\}.$$

If i is such that $\alpha_i \neq 0$, since $|\phi(\alpha_i)| = |\alpha_i|$, it follows that p divides $\zeta_l - \zeta_{l^m}^i$ in R which forces $i = l^{m-1}$. This in turn implies $\phi(\alpha_{l^{m-1}}) = \alpha_{l^{m-1}}$, hence $\alpha_{l^{m-1}} \in \mathbb{Z}_p$ and our Claim 1 is proved.

Consider now the data $E_0, b, \check{b}$ in the paragraph before Theorem 7.44 and consider the elliptic curve E_β over R_{π_1} whose Serre–Tate parameter has logarithm equal to β. Consider, as in that paragraph, the partial δ_{π_1}-character $\psi_{11,22,\beta}$. Then by Claim 1 above and by the strengthening in Remark 7.43 of our Theorem of the kernel, the following claims hold.

Claim 2. The group $(\text{Ker}(\psi^{\text{alg}}_{\mu,\nu,\beta}) \cap E_\beta(K^{(l)})) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a one-dimensional $\mathbb{Q}_p$-linear space with basis any point $P_{\beta,\beta}$ whose elliptic logarithm is β.

On the other hand, if instead of the elliptic curve $E_\beta = E_{\pi_1}$ over R_{π_1} above we consider an elliptic curve E_γ over R with $\gamma \in pR$ then we get the following.

Claim 3. The group $(\text{Ker}(\psi^{\text{alg}}_{\mu,\nu,\gamma}) \cap E_\gamma(K)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is naturally isomorphic to K, hence this group is an infinite-dimensional $\mathbb{Q}_p$-linear space.

Indeed, in this case, one has $\langle \alpha, \gamma\rangle_{\mu,\nu} = 0$ for every $\alpha \in K$.

The above is a "genuine PDE" example of an explicit computation for the kernel of a δ-character, in a tamely ramified situation. For the ODE case one has a complete description of the kernels of δ-characters if one restricts to the unramified situation; cf. [8, Introduction].

7.7 Crystalline construction

For background here we refer to [10, Section 8.4.3]. Let S^* be an object of $\mathbf{Prol}_{p,\Phi}$ and E/S^0 an elliptic curve. Let $H^1_{DR}(E/S^0)$ be the de Rham S^0-module. By crystalline theory for every $i \in \{1,\dots,n\}$ and every $r \geq 0$ we have ϕ_i-linear maps

$$H^1_{DR}(E/S^0) \otimes_{S^0} S^r \xrightarrow{\phi_i} H^1_{DR}(E/S^0) \otimes_{S^0} S^{r+1}.$$

Also, one has the de Rham pairing

$$\langle\, , \,\rangle_{DR} : H^1_{DR}(E/S^0) \otimes_{S^0} S^r \times H^1_{DR}(E/S^0) \otimes_{S^0} S^r \to S^r.$$

Definition 7.47. For every basis ω of 1-forms on E/S^0 and every distinct $\mu, \nu \in \mathbb{M}_n^{r,+}$ we set

$$f^{\text{crys}}_{\mu}(E/S^0, \omega, S^*) = \frac{1}{p}\langle \phi_\mu \omega, \omega\rangle_{DR} \in S^r,$$

$$f^{\text{crys}}_{\mu,\nu}(E/S^0, \omega, S^*) = \frac{1}{p}\langle \phi_\mu \omega, \phi_\nu \omega\rangle_{DR} \in S^r.$$

It is trivial to check that the following proposition holds.

Proposition 7.48. *The rules f^{crys}_{μ} and $f^{\text{crys}}_{\mu,\nu}$ define isogeny covariant partial δ_p-modular forms of order $\leq r$ and weights $-1-\phi_\mu$ and $-\phi_\mu - \phi_\nu$, respectively.*

On the other hand we have the next proposition.

Proposition 7.49. *Let $\mu, \nu \in \mathbb{M}_n^r$, $\mu \neq \nu$, of length r, s respectively, with $r \geq s$. The symbols of f^{crys}_{μ}, $f^{\text{crys}}_{\mu,\nu}$ are given by*

$$\begin{aligned} \theta(f^{\text{crys}}_{\mu}) &= \phi_\mu - p^r, \\ \theta(f^{\text{crys}}_{\mu,\nu}) &= p^s\phi_\mu - p^r\phi_\nu. \end{aligned} \tag{7.48}$$

In particular, the Serre–Tate expansions $\mathcal{E}(f^{\text{crys}}_{\mu})$ are R-linearly independent and not divisible by p in S^r_{for}.

Proof. This follows exactly as in the proof of [10, Proposition 8.61]. ∎

Remark 7.50. Following the lead of the ODE case we expect that the forms f^{crys}_{μ} and $f^{\text{crys}}_{\mu,\nu}$ coincide up to a multiplicative constant in $\mathbb{Z}_p^\times$ with the forms f^{jet}_{μ} and $f^{\text{jet}}_{\mu,\nu}$. Cf. also Remark 7.59 for more on this. In any case, by [10, Corollary 8.84] (adapted to the theory over $\mathbb{Z}_p$ instead of over R, as explained in Remark 7.23) we have the next corollary.

Corollary 7.51. *Assume $n = 1$. Then for all μ we have $f^{\text{jet}}_{\mu} \in \mathbb{Z}_p^\times \cdot f^{\text{crys}}_{\mu}$.*

For $n = 2$ and with $c \in \mathbb{Z}_p^\times$ as in (7.24) our theory yields the following result.

Corollary 7.52. *Let $\mu, \nu \in \mathbb{M}_2^{2,+}$. Then we have*

$$f_\mu^{\text{jet}} = c \cdot f_\mu^{\text{crys}},$$
$$f_{\mu,\nu}^{\text{jet}} = c \cdot f_{\mu,\nu}^{\text{crys}}.$$

Proof. By Corollary 7.30 and Proposition 7.49 the forms in the left-hand sides of our equations have the same Serre–Tate expansion as the forms in the right-hand sides, respectively. Since these forms have the same corresponding weights we conclude by the Serre–Tate expansion principle, cf. Theorem 7.21. ∎

7.8 Forms on the ordinary locus

Definition 7.53. A *δ_p-modular function of order $\leq r$ on the ordinary locus* is a rule f assigning to each object $(E/S^0, \omega, S^*)$ with E/S^0 ordinary an element $f(E/S^0, \omega, S^*)$ of the ring S^r, depending only on the isomorphism class of $(E/S^0, \omega, S^*)$, such that f commutes with base change of the prolongation sequence. We denote by $M^r_{p,\Phi,\text{ord}}$ the set of all δ_p-modular function of order $\leq r$ on the ordinary locus; it has a structure of R-algebra.

In [10] such f's were called *ordinary*, but we want to avoid here the term *ordinary* so no confusion arises with the use of this word in relation to ODEs/PDEs.

Remark 7.54. The general theory developed in the preceding subsections can be developed in this context as follows.

(1) The set $M^r_{p,\Phi,\text{ord}}$ has an obvious structure of ring. As in [9] we have a natural ring isomorphism

$$M^r_{p,\Phi,\text{ord}} \simeq J^r_{\pi,\Phi}(M_{p,\Phi}[E_{p-1}^{-1}]) = R[\delta_\mu a_4, \delta_\mu a_6, \Delta^{-1}, E_{p-1}^{-1} \mid \mu \in \mathbb{M}_n^r]\widehat{\ },$$

where $E_{p-1} \in \mathbb{Z}_p[a_4, a_6]$ corresponds to the Eisenstein series of weight $p-1$.

(2) As in Definition 7.3 one defines what it means for an element $f \in M^r_{p,\Phi,\text{ord}}$ to have *weight* $w \in \mathbb{Z}^r_\Phi$ by requiring that the condition in that definition be satisfied only for ordinary elliptic curves. We denote by $M^r_{p,\Phi,\text{ord}}(w)$ the R-submodule of $M^r_{p,\Phi,\text{ord}}$ of weight w.

(3) As in Definition 7.9, first one defines what it means for an element $f \in M^r_{p,\Phi,\text{ord}}(w)$ to be *isogeny covariant* by requiring that the condition in that definition be satisfied only for ordinary elliptic curves. We denote by $I^r_{p,\Phi,\text{ord}}(w)$ the submodule of all isogeny covariant elements of $M^r_{p,\Phi,\text{ord}}(w)$. The direct sum $\bigoplus_{w \in \mathbb{Z}_\Phi} I^r_{p,\Phi,\text{ord}}(w)$ is a Z_Φ-graded R-subalgebra of the R-algebra $\bigoplus_{w \in \mathbb{Z}_\Phi} M^r_{p,\Phi,\text{ord}}(w)$. For every $f \in I^r_{p,\Phi,\text{ord}}(w)$ and every i we have $f^{\phi_i} \in I^r_{p,\Phi,\text{ord}}(\phi_i w)$.

(4) As in Theorem 7.21 for every $w \in \mathbb{Z}_\Phi$ there is a natural *Serre–Tate expansion* homomorphism $\mathcal{E} : M^r_{p,\Phi,\mathrm{ord}}(w) \to S^r_{\mathrm{for}}$, $f \mapsto \mathcal{E}(f)$, which is injective with torsion-free cokernel.

(5) As in Theorem 7.24 for every weight w of degree $\deg(w) = -2$ and every $f \in I^r_{p,\Phi,\mathrm{ord}}(w)$ we have that $\mathcal{E}(f)$ is a K-linear combination of elements in the set

$$\{\Psi_i^{\phi_\mu} \mid \mu \in \mathbb{M}_n^{r-1},\ i \in \{1,\dots,n\}\}.$$

In particular, $I^r_{p,\Phi,\mathrm{ord}}(w)$ has rank $\leq D(n,r) - 1$.

(6) For every weight w of degree $\deg(w) = 0$ and every $f \in I^r_{p,\Phi,\mathrm{ord}}(w)$ we have that $\mathcal{E}(f) \in K$. (To prove this one proceeds as in the proof of Theorem 7.24 by noting that, in this case, the right-hand side of (7.25) reduces to $F(\dots,\delta_{p,\mu}T,\dots)$ which forces $F \in K$.) In particular, by the Serre–Tate expansion principle (4) above, the R-module $I^r_{p,\Phi,\mathrm{ord}}(w)$ has rank 1.

(7) There are natural R-module homomorphisms $M^r_{p,\Phi}(w) \to M^r_{p,\Phi,\mathrm{ord}}(w)$ and $I^r_{p,\Phi}(w) \to I^r_{p,\Phi,\mathrm{ord}}(w)$ that are injective with torsion-free cokernel.

Recall the following result due to Barcau; cf. [3, Theorem 5.1, Corollary 5.1, Proposition 5.2] and [10, Theorem 8.83].

Theorem 7.55. *Assume $n = 1$, $\Phi = \{\phi\}$. There exist elements $f^\partial \in I^1_{p,\phi,\mathrm{ord}}(\phi - 1)$ and $f_\partial \in I^1_{p,\phi,\mathrm{ord}}(1 - \phi)$ such that*

(1) *f^∂ and f_∂ are bases modulo torsion for these R-modules, respectively;*

(2) *$f^\partial \cdot f_\partial = 1$ in $M^1_{p,\phi,\mathrm{ord}}$;*

(3) *$\mathcal{E}(f^\partial) = \mathcal{E}(f_\partial) = 1$;*

(4) *$f^\partial \equiv E_{p-1}$ and $f_\partial \equiv E_{p-1}^{-1}$ mod p in $M^1_{p,\phi,\mathrm{ord}}$;*

(5) *$I^1_{p,\phi}(\phi - 1) = I^1_{p,\phi}(1 - \phi) = 0$ hence $f^\partial, f_\partial \notin M^1_{p,\phi}$.*

Part 5 says intuitively that $f^\partial_\phi, f_\partial$ are "genuinely singular along the supersingular locus."

Proof. We recall the idea of the argument using references to [10]. For any triple $(E/S^0, \omega, S^*)$ with E/S^0 ordinary we define

$$f^\partial(E/S^0,\omega,S^*) := \frac{\langle \phi u, \omega\rangle_{DR}}{\phi(\langle u,\omega\rangle_{DR})} \in S^r$$

where $u \in H^1_{DR}(E/S^0)$ is a basis of the unit root subspace of $H^1_{DR}(E/S^0)$; cf. [10, page 269]. We also define

$$f_\partial(E/S^0,\omega,S^*) := \frac{\phi(\langle u,\omega\rangle_{DR})}{\langle \phi u, \omega\rangle_{DR}} \in S^r.$$

One readily checks that these formulae define elements of

$$I^1_{p,\phi,\mathrm{ord}}(\phi - 1),\ I^1_{p,\phi,\mathrm{ord}}(1 - \phi),$$

respectively. For the computation of their Serre–Tate expansion (Part 2 in the theorem) we refer to [10, Proposition 8.59]. Then these elements being non-zero are bases modulo torsion of the corresponding modules by Remark 7.54, Part 6, hence Part 1 of the theorem follows. Part 3 is obvious. Part 4 follows from [10, Theorem 8.83, Part 3]. ∎

Definition 7.56. Let n be arbitrary and denote by f_i^∂ and $f_{i,\partial}$ the images of f^∂ and f_∂ via the face maps

$$M^1_{p,\phi,\mathrm{ord}} \simeq M^1_{p,\phi_i,\mathrm{ord}} \to M^1_{p,\Phi,\mathrm{ord}}.$$

Corollary 7.57. *The following claims hold for every $i \in \{1, \dots, n\}$:*

(1) *f_i^∂ and $f_{i,\partial}$ are bases modulo torsion for the R-modules, $I^1_{p,\Phi,\mathrm{ord}}(\phi_i - 1)$ and $I^1_{p,\Phi,\mathrm{ord}}(1 - \phi_i)$, respectively;*

(2) *$f_i^\partial \cdot f_{i,\partial} = 1$ in $M^1_{p,\Phi,\mathrm{ord}}$;*

(3) *$\mathcal{E}(f_i^\partial) = \mathcal{E}(f_{i,\partial}) = 1$;*

(4) *$f_i^\partial \equiv E_{p-1}$ and $f_{i,\partial} \equiv E_{p-1}^{-1}$ mod p in $M^1_{p,\Phi,\mathrm{ord}}$;*

(5) *$f_i^\partial, f_{i,\partial} \notin M^1_{p,\Phi}$.*

Proof. Parts 1 to 4 follow from Parts 1 to 4 of Theorem 7.55. Part 5 follows from Part 5 of Theorem 7.55 by using the fact that the images of f_i^∂ and $f_{i,\partial}$ via the degeneration map $M^1_{p,\Phi} \to M^1_{p,\phi_i}$ are f^∂ and f_∂, respectively. ∎

For the next result let us consider, for every $r \geq 1$, the unique group homomorphism

$$\{w \in \mathbb{Z}^r_\Phi \mid \deg(w) = 0\} \to (M^r_{p,\Phi,\mathrm{ord}})^\times, \quad w \mapsto f_{(w)}, \tag{7.49}$$

satisfying

$$f_{(\phi_{i_1 \dots i_s} - 1)} := f_{i_1}^\partial \cdot (f_{i_2}^\partial)^{\phi_{i_1}} \cdot (f_{i_3}^\partial)^{\phi_{i_1 i_2}} \cdots (f_{i_s}^\partial)^{\phi_{i_1 i_2 i_3 \dots i_{s-1}}}, \quad s \in \{1, \dots, r\}. \tag{7.50}$$

Note that, by Corollary 7.57, Part 4, we have the following congruences in $M^s_{p,\Phi,\mathrm{ord}}$:

$$f_{(\phi_{i_1 \dots i_s} - 1)} \equiv E_{p-1}^{1+p+p^2+\dots+p^{s-1}} \bmod p. \tag{7.51}$$

Corollary 7.58. *For every $r \geq 1$ the following claims hold.*

(1) *For every $w \in \mathbb{Z}^r_\Phi$ of degree $\deg(w) = 0$ the form $f_{(w)}$ is a basis of the R-module $I^r_{p,\Phi,\mathrm{ord}}(w)$.*

(2) *For every* $v \in \mathbb{Z}^r_\Phi$ *of degree* $\deg(v) = -2$ *the* R*-module* $I^r_{p,\Phi,\mathrm{ord}}(v)$ *has rank* $D(n,r) - 1$ *and a basis modulo torsion is given by the set*

$$\{f_{(v+\phi_\mu+1)} f_\mu^{\mathrm{crys}} \mid \mu \in \mathbb{M}_n^{r,+}\}.$$

Proof. To check Part 1 note that by Remark 7.54, Part 6, $I^r_{p,\Phi,\mathrm{ord}}(w)$ has rank 1. We are done by noting that $f_{(w)}$ belongs to this module and is not divisible by p in this module.

To check Part 2 note that the forms $f_{(v+\phi_\mu+1)} f_\mu^{\mathrm{crys}}$ have weight v hence belong to $I^r_{p,\Phi,\mathrm{ord}}(v)$. Since the latter module has rank $\leq D(n,r) - 1$ (cf. Remark 7.54, Part 5) it is enough to check that the forms $f_{(v+\phi_\mu+1)} f_\mu^{\mathrm{crys}}$ are R-linearly independent. For this it is enough to check that their Serre–Tate expansions are R-linearly independent. However, by Corollary 7.57, Part 3, we have

$$\mathcal{E}(f_{(v+\phi_\mu+1)} f_\mu^{\mathrm{crys}}) = \mathcal{E}(f_\mu^{\mathrm{crys}}),$$

and we may conclude by Proposition 7.49. ∎

Remark 7.59. Following the lead from the ODE case (cf. [3] or [10, Theorem 8.83, Part 2]) it is natural to ask if for all distinct $\mu, \nu \in \mathbb{M}_n^r$ we have

$$\mathrm{rank}_R I^r_{p,\Phi}(-1 - \phi_\mu) = 1, \tag{7.52}$$

$$\mathrm{rank}_R I^r_{p,\Phi}(-\phi_\mu - \phi_\nu) = 1, \tag{7.53}$$

$$I^r_{p,\Phi}(-2) = I^r_{p,\Phi}(-2\phi_\mu) = 0. \tag{7.54}$$

By loc. cit. the above equations hold if $n = 1$. For $n = 2$ the "simplest case" ($r = 1$, $\mu = 1$, $\nu = 2$) of (7.53) holds; cf. Theorem 7.34. Note that if the conditions (7.52) and (7.53) hold in general then there exist constants $\lambda_\mu \in \mathbb{Z}_p$ and $\lambda_{\mu,\nu} \in \mathbb{Q}_p$ such that

$$f_\mu^{\mathrm{jet}} = \lambda_\mu \cdot f_\mu^{\mathrm{crys}},$$
$$f_{\mu,\nu}^{\mathrm{jet}} = \lambda_{\mu,\nu} \cdot f_{\mu,\nu}^{\mathrm{crys}}.$$

Indeed, by Theorems 7.8 and 7.11 and by Proposition 7.48 the left-hand sides and the right-hand sides of the above equations belong to the same R-modules of rank one, respectively. On the other hand the forms f_μ^{crys} are not divisible by p while the forms $f_{\mu,\nu}^{\mathrm{crys}}$ are non-zero (cf. Proposition 7.49). Moreover, again under the assumption that conditions (7.52) and (7.53) hold, since by Remark 7.7 the forms f_μ^{jet} are not divisible by p, we get $\lambda_\mu \in \mathbb{Z}_p^\times$. Nevertheless, even under the assumption that conditions (7.52) and (7.53) hold, we cannot conclude that $\lambda_{\mu,\nu} \neq 0$ (let alone that $\lambda_{\mu,\nu} \in \mathbb{Z}_p^\times$, as in Corollary 7.52). We recall that the proof of Corollary 7.52 involved "solving a system of quadratic and cubic equations" satisfied by the f^{jet} forms as in the proof of Theorem 7.28. So even if one can prove conditions (7.52) and (7.53) one still cannot go around solving our system of quadratic and cubic equations if one wants to prove the non-vanishing of the forms $f^{\text{jet}}_{\mu,\nu}$ for distinct μ, ν as in Corollary 7.52.

Finally, one may hope to prove conditions (7.52) and (7.53) along the lines of [3] or [10, Theorem 8.83]. However, even proving condition (7.52) for $n = 2, \mu = 1$ along these lines does not seem to work in an obvious way. Indeed, by Corollary 7.58 we have that $I^1_{p,\phi_1,\phi_2,\text{ord}}(-1-\phi_1)$ has a basis modulo torsion consisting of f_1 and $f_{1,\partial} f_2^{\partial} f_2$. So in order to prove that $I^1_{p,\phi_1,\phi_2}(-1-\phi_1)$ has rank 1 we need to show that the form $f_{1,\partial} f_2^{\partial} f_2 \in M^1_{p,\Phi,\text{ord}}$ does not belong to $M^1_{p,\Phi}$. The argument in loc. cit. for this type of statement was to show that the image of the corresponding form in $M^1_{p,\Phi,\text{ord}} \otimes_R k$ does not belong to $M^1_{p,\Phi} \otimes_R k$. But in our case the image of $f_{1,\partial} f_2^{\partial} f_2$ in $M^1_{p,\Phi,\text{ord}} \otimes_R k$ equals the image of f_2 which *does* belong to $M^1_{p,\Phi} \otimes_R k$.

On the other hand, as an application of the theory we get a whole series of identities between our forms f^{jet} and f^{∂}; here is an example.

Corollary 7.60. *The following formula holds in $I^1_{p,\phi_1,\phi_2,\text{ord}}(-\phi_1-\phi_2)$:*

$$f^{\text{jet}}_{1,2} = p(f^{\text{jet}}_1 f_{2,\partial} - f^{\text{jet}}_2 f_{1,\partial}).$$

Proof. The two sides of the formula have the same weight equal to $-\phi_1 - \phi_2$ and the same Serre–Tate expansions (cf. Remark 7.27 and Theorem 7.28). So they must be equal by the Serre–Tate expansion principle (Remark 7.54, Part 4). ■

Remark 7.61. The formula in Corollary 7.60 is interesting in that $f_{1,\partial}$, $f_{2,\partial}$ in the right-hand side do not belong to $M^1_{p,\Phi}$ (they are "genuinely singular along the supersingular locus", cf. Corollary 7.57, Part 4) while the left-hand side does belong to $M^1_{p,\Phi}$; so, intuitively, the "singularities" in the right-hand side "cancel each other out." In view of Corollary 7.58 one has similar formulae (exhibiting similar "cancellations of singularities") for every $f \in I^r_{p,\phi_1,\phi_2}(w)$ with w of degree $\deg(w) = -2$.

Finally we address the total δ-overconvergence aspect, there by strengthening the results in [12].

Theorem 7.62. *Let $B = B_1(N)$ be the natural bundle over an open set $X \subset Y_1(N)$ of the modular curve $Y_1(N)$ over R, for $N \geq 4$, N coprime to p, and assume the reduction mod p of X is contained in the ordinary locus of the reduction mod p of $Y_1(N)$. Then the following hold:*

(1) *For every weight w of degree* $\deg(w) = 0$ *and every $f \in I^r_{p,\Phi,\text{ord}}(w)$ the element $f^B \in \mathcal{O}(J^r_{p,\Phi}(B))$ is totally δ -overconvergent.*

(2) *For every f as in (1) the map $(f^B)^{\text{alg}} : B(R^{\text{alg}}) \to K^{\text{alg}}$ extends to a continuous map $(f^B)^{\mathbb{C}_p} : B(\mathbb{C}_p^{\circ}) \to \mathbb{C}_p$.*

Proof. By Corollary 7.58 we may assume f is either $f_{i,\partial}$ or f_i^{∂}. Assume $f = f_i^{\partial}$; the other case is similar. Note that since f_i^{∂} is induced via a face map from the form f^{∂}

we are reduced to check Part 1 in case $n = 1$; but this was proved in [12, Corollary 5.12]. For Part 2 we proceed exactly as in the proof of Proposition 7.38; note that in our case here, since $\mathcal{E}(f_i^{\partial}) = 1$, we will simply have

$$((f_i^{\partial})^B)^{\text{alg}}(P) = u(P)^{\phi_i - 1} \tag{7.55}$$

for u as in that proof. ∎

Remark 7.63. The maps $(f^B)^{\mathbb{C}_p}$ in Part 2 of Theorem 7.62 satisfy a compatibility property with respect to isogenies at points with coordinates belonging to R because of the isogeny covariance of f^{∂}, f_{∂}; however we do not know if the maps $(f^B)^{\mathbb{C}_p}$ continue to satisfy a compatibility property with respect to isogenies at points with coordinates *not* belonging to R. For this to hold it would be sufficient to "naturally extend" the crystalline definition of f^{∂}, f_{∂} in the proof of Theorem 7.55 to the ramified case.

7.9 Finite covers defined by δ-modular forms

Throughout the discussion below fix an element $\pi \in \Pi$ and $\Phi = (\phi_1, \dots, \phi_n)$ with $n \geq 1$. We recall the forms $f_{\pi,i}^{\text{jet}}$ for $i \in \{1, \dots, n\}$; cf. Theorem 7.6. For an affine open set $X = \text{Spec}(A) \subset Y_1(N)$ with non-empty reduction mod π denote by E_X the corresponding universal elliptic curve over X. Assume there is a basis ω for the 1-forms on E_X/X (which can be achieved by shrinking X) and consider the unique elements

$$\check{f}_{\pi,i} \in J^1_{\pi,\phi_i}(A) \setminus \pi J^1_{\pi,\phi_i}(A)$$

such that there exist (necessarily unique) integers $n_i \geq 0$ with

$$\pi^{n_i} \cdot \check{f}_{\pi,i} = f_{\pi,i}^{\text{jet}}(E_X/X, \omega, J^*_{\pi,\phi_i}(A)) \in J^1_{\pi,\phi_i}(A).$$

We continue to denote by $\check{f}_{\pi,i}$ the images of these elements in $J^1_{\pi,\Phi}(A)$. Our main result here is the following theorem.

Theorem 7.64. *There exists an affine open set* $X = \text{Spec}(A) \subset X_1(N)$ *of the modular curve* $X_1(N)$ *over* R_π, *with non-empty reduction mod* π, *and a basis* ω *for the* 1-*forms on* E_X/X *such that the ring homomorphism*

$$\widehat{A} \to J^1_{\pi,\Phi}(A)/(\check{f}_{\pi,1}, \dots, \check{f}_{\pi,n})$$

is a finite algebra map.

If the map above is an isomorphism (which happens for instance if $\pi = p$ as one can easily see from the proof below) then one can view the arithmetic differential equations $\check{f}_{\pi,1}, \dots, \check{f}_{\pi,n}$ as defining an 'arithmetic flow' on X; in the more general case when the map in the theorem is merely a finite algebra map one should view $\check{f}_{\pi,1}, \dots, \check{f}_{\pi,n}$ as defining a structure slightly more general than that of an 'arithmetic flow.'

Proof. Since the source and the target of the map in the theorem are p-adically complete rings it is enough to show that there exists $X = \mathrm{Spec}(A)$ such that the map

$$A/\pi A \to (J^1_{\pi,\Phi}(A)/(\check{f}_{\pi,1}, \dots, \check{f}_{\pi,n}))/(\pi)$$

is a finite algebra map. Start with an arbitrary X as in the paragraph before our theorem. Replacing X by an affine open set we may assume there is an étale map $R_\pi[y] \to A$, so we have identifications

$$J^1_{\pi,\phi_i}(A) = A[\delta_{\pi,i}y]\widehat{\ }, \quad J^1_{\pi,\Phi}(A) = A[\delta_{\pi,1}y, \dots, \delta_{\pi,n}y]\widehat{\ }.$$

The theorem will be proved if we show that for every $i \in \{1, \dots, n\}$ the image $\overline{f}_{\pi,i}$ of $\check{f}_{\pi,i}$ in the ring $(A/\pi A)[\delta_{\pi,i}y]$ is not contained in the ring $A/\pi A$. Note that by definition $\overline{f}_{\pi,i} \neq 0$ for all i. Assume for some i we have $\overline{f}_{\pi,i} \in A/\pi A$ and seek a contradiction. Consider the natural map

$$\mathcal{E}_\pi : J^1_{\pi,\phi_i}(A) \to R_\pi[\![T]\!][\delta_{\pi,i}T]\widehat{\ }$$

defined similarly to the Serre–Tate expansion map. Since the reduction mod π of this map,

$$\overline{\mathcal{E}_\pi} : J^1_{\pi,\phi_i}(A)/(\pi) = (A/\pi A)[\delta_{\pi,i}y] \to k[\![T]\!][\delta_{\pi,i}T]$$

is injective it follows that the image $\overline{E_\pi}(\overline{f}_{\pi,i}) \in k[\![T]\!][\delta_{\pi,i}T]$ of $\overline{f}_{\pi,i}$ is non-zero and is contained in $k[\![T]\!]$. Let z be a variable and consider the $k[\![T]\!]$-algebra isomorphism

$$\sigma : k[\![T]\!][z] \to k[\![T]\!][\delta_{\pi,i}T], \quad \sigma(z) = \frac{\delta_{\pi,i}(1+T)}{(1+T)^p}.$$

We have

$$0 \neq \sigma^{-1}(\overline{\mathcal{E}_\pi}(\overline{f}_{\pi,i})) \in k[\![T]\!] \subset k[\![T]\!][z]. \tag{7.56}$$

By the compatibility of $\mathcal{E}_\pi$ with the Serre–Tate expansion map and in view of Remark 7.27 it follows that there exists an integer $N \in \mathbb{Z}$ such that

$$\mathcal{E}(\check{f}_{\pi,i}) = u(T)^{-1-\phi_i} \cdot \pi^N \sum_{m \geq 1} (-1)^{m+1} \frac{\pi^m}{m} \left(\frac{\delta_{\pi,i}(1+T)}{(1+T)^p} \right)^m$$

for some invertible series $u(T) \in R_\pi[\![T]\!]^\times$. Letting $\lambda_m \in k$ be the image of

$$(-1)^{m+1} \frac{\pi^{N+n}}{m} \in R_\pi$$

we have that

$$\sigma^{-1}(\overline{\mathcal{E}_{\pi}}(\overline{f}_{\pi,i})) = u(T)^{-1-p} \cdot \sum_{m=1}^{d} \lambda_m z^m \in k[\![T]\!][z] \tag{7.57}$$

for some $d \geq 1$ which by (7.56) implies

$$0 \neq \sum_{m=1}^{d} \lambda_m z^m \in k[\![T]\!] \subset k[\![T]\!][z],$$

a contradiction. ■

Remark 7.65. The integer N and the polynomial $S(z) = \sum_{i=1}^{d} \lambda_i z^i$ in the above proof can be computed explicitly. Indeed, let e be the ramification index of R_π over R. Then $-N$ is the minimum of the π-adic valuations of the numbers $\frac{\pi^m}{m}$ hence writing $m = p^\kappa s$ with $s \in \mathbb{Z} \setminus p\mathbb{Z}$ we have

$$-N = \min\{p^\kappa s - \kappa e \mid \kappa \geq 0,\ s \geq 1\} = \min\{p^\kappa - \kappa e \mid \kappa \geq 0\}.$$

On the other hand the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = p^x - ex$ has a unique minimum at

$$\theta := \frac{\log e - \log \log p}{\log p}.$$

So if $e < \log p$ then f is strictly increasing on $\mathbb{Z}_{\geq 0}$ which implies that $N = -1$ and $S(z) = z$. On the other hand if $e > \log p$ and $\kappa_0 \in \mathbb{Z}_{\geq 0}$ is such that $\theta \in [\kappa_0, \kappa_0 + 1]$ then the restriction of f to $\mathbb{Z}_{\geq 0}$ attains its minimum either at κ_0 or $\kappa_0 + 1$ or at both (the last case occurring if and only if $e = p^{\kappa_0+1} - p^{\kappa_0}$). Consequently we have that $S(z)$ is either $\lambda z^{p^{\kappa_0}}$ or $\lambda z^{p^{\kappa_0+1}}$ or $\lambda z^{p^{\kappa_0}} + \lambda' z^{p^{\kappa_0+1}}$ for some $\lambda, \lambda' \in k^\times$ (the last case occurring if and only if $e = p^{\kappa_0+1} - p^{\kappa_0}$).

7.10 Application to modular parameterizations

We present in what follows an application of Theorem 7.64 along the lines of [13, Theorem 1.3]. In this section, we prove Theorem 1.1 as well as the enhanced version of Strassman's theorem. We need some notation first.

We fix again $\pi \in \Pi$ and $n \geq 2$. Consider an elliptic curve E over R_π and a surjective morphism of R_π-schemes

$$\Theta : X_1(N) \to E$$

where $X_1(N)$ is the complete modular curve over R_π. In particular, one can take E to come from an elliptic curve over $\mathbb{Q}$ and Θ to be induced by a new form of weight 2 as in the Eichler–Shimura theory. We denote by

$$\Theta_{R_\pi} : X_1(N)(R_\pi) \to E(R_\pi)$$

the induced map on R_π-points. Similarly, for an open set $X = \mathrm{Spec}(A) \subset X_1(N)$ and a function $g \in \widehat{A}$ we denote by

$$g_{R_\pi} : X(R_\pi) \to R_\pi$$

the induced map on R_π-points. For such a g we consider the sets of zeros of g:

$$Z(g) = \{P \in X(R_\pi) \mid g_{R_\pi}(P) = 0\} \subset X(R_\pi).$$

Definition 7.66. A point $P \in X_1(N)(R_\pi)$ is called a *quasi-canonical lift* if the corresponding elliptic curve is ordinary with Serre–Tate parameter a root of unity. For an open set $X \subset X_1(N)$ we denote by $\mathrm{QCL}(X(R_\pi))$ the set of all points in $X(R_\pi)$ that are quasi-canonical lifts.

For every δ_π-character $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ denote, as usual, by

$$\psi_{R_\pi} : E(R_\pi) \to R_\pi$$

the induced group homomorphism. For instance, if $n = 2$ one can take $\psi = \psi_{1,2}$ from Section 5.2. For $n \geq 3$ one can take ψ to be any R_π-linear combination of images of $\psi_{1,2}$ via the different face maps.

Theorem 7.67. *There exists an affine open set $X = \mathrm{Spec}(A) \subset X_1(N)$ with non-empty reduction mod π such that for every δ_π-character $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ there exists a monic polynomial $G \in \widehat{A}[t]$ with the property that for every $P \in E(R_\pi)$ the following holds:*

$$\mathrm{QCL}(X(R_\pi)) \cap \Theta^{-1}_{R_\pi}(\mathrm{Ker}(\psi_{R_\pi}) + P) \subset Z(G(\psi_{R_\pi}(P))).$$

Proof. Take X as in Theorem 7.64. Then for every $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ consider the composition

$$\Theta^\sharp : J^1_{\pi,\Phi}(X_1(N)) \xrightarrow{J^1(\Theta)} J^1_{\pi,\Phi}(E) \xrightarrow{\psi} \widehat{\mathbb{G}_{a,R_\pi}},$$

which we identify with an element (still denoted by)

$$\Theta^\sharp \in \mathcal{O}(J^1_{\pi,\Phi}(X_1(N))) \subset \mathcal{O}(J^1_{\pi,\Phi}(X)) = J^1_{\pi,\Phi}(A).$$

By Theorem 7.64 the image of $\Theta^\sharp$ in the ring

$$J^1_{\pi,\Phi}(A)/(\check{f}_{\pi,1}, \ldots, \check{f}_{\pi,n})$$

is integral over $\widehat{A}$ hence there exists a monic polynomial

$$G(t) = t^s + g_1 t^{s-1} + \cdots + g_s \in \widehat{A}[t], \quad g_1, \ldots, g_s \in \widehat{A}$$

and there exist $h_1, \dots, h_n \in J^1_{\pi,\Phi}(A)$ such that

$$(\Theta^\sharp)^s + g_1 \cdot (\Theta^\sharp)^{s-1} + \cdots + g_s = h_1 \cdot \check{f}_{\pi,1} + \cdots + h_n \cdot \check{f}_{\pi,n}$$

in the ring $J^1_{\pi,\Phi}(A)$. Let us denote by g_{i,R_π}, $\check{f}_{\pi,i,R_\pi}$, and h_{j,R_π}, respectively, the functions $X(R_\pi) \to R_\pi$ induced by g_i, $\check{f}_{\pi,i}$, and h_j. Then for all $Q \in X(R_\pi)$ we get an equality

$$(\psi_{R_\pi}(\Theta_{R_\pi}(Q)))^s + g_{1,R_\pi}(Q) \cdot (\psi_{R_\pi}(\Theta_{R_\pi}(Q)))^{s-1} + \cdots + g_{s,R_\pi}(Q)$$
$$= h_{1,R_\pi}(Q) \cdot \check{f}_{\pi,1,R_\pi}(Q) + \cdots + h_{n,R_\pi}(Q) \cdot \check{f}_{\pi,n,R_\pi}(Q).$$

By Proposition 7.39 we have

$$\check{f}_{\pi,i,R_\pi}(Q) = 0 \text{ for } i \in \{1, \dots, n\}, \quad Q \in \mathrm{QCL}(X(R_\pi)).$$

Hence for all $P \in E(R_\pi)$ and all $Q \in \mathrm{QCL}(X(R_\pi)) \cap \Theta^{-1}_{R_\pi}(\mathrm{Ker}(\psi_{R_\pi}) + P)$ we have the equality $\Theta_{R_\pi}(Q) = \psi_{R_\pi}(P)$ hence

$$(\psi_{R_\pi}(P))^s + g_{1,R_\pi}(Q) \cdot (\psi_{R_\pi}(P))^{s-1} + \cdots + g_{s,R_\pi}(Q) = 0,$$

which implies $Q \in Z(G(\psi_{R_\pi}(P)))$. ∎

We conclude with a finiteness result; cf. Corollary 7.69 below. We need the following variant of Strassman's theorem [32, page 306]. The classic case is that of the affine line over a not necessarily discrete valuation ring (DVR). We need here the case of an arbitrary smooth curve over a complete discrete valuation ring; the fact that the valuation ring is discrete greatly simplifies the proof.

Lemma 7.68 (Strassman's theorem for curves over a DVR). *Let V be a complete DVR with maximal ideal generated by $\pi \in V$. Fix X/V a smooth affine curve with connected closed fiber and let $\widehat{\mathcal{O}(X)}$ be the π-adic completion of $\mathcal{O}(X)$. Then every non-zero $g \in \widehat{\mathcal{O}(X)}$ has finitely many zeros in $\widehat{X}(V) = X(V)$.*

Proof. Set $A = \mathcal{O}(X)$ and for $g \in \widehat{A}$ denote by $Z(g)$ the set of zeros of g in $\widehat{X}(V) = X(V)$. Without loss of generality, assume g is not in $\pi\widehat{A}$ as π is a regular element. Note the usual bijection $Z(g) \cong \mathrm{Hom}_V(\widehat{A}/(g), V)$. For each map φ in this set denote by P_φ the kernel of φ and by M_φ the kernel of the composition

$$\overline{\varphi}\colon \widehat{A}/(g) \xrightarrow{\varphi} V \to k := V/\pi V.$$

Claim. *The map $\mathrm{Hom}_V(\widehat{A}/(g), V) \to \mathrm{Spec}(\widehat{A}/(g))$, $\varphi \mapsto P_\varphi$ is injective.*

Indeed, if $P_{\varphi_1} = P_{\varphi_2}$ and $\iota : R_\pi \to \widehat{A}/(g)$ is the natural map then for all $x \in \widehat{A}$ we have

$$\varphi_1(x - \iota(\varphi_1(x))) = \varphi_1(x) - \varphi_1(x) = 0$$

hence $x - \iota(\varphi_1(x)) \in P_{\varphi_1} = P_{\varphi_2}$ so

$$0 = \varphi_2(x - \iota(\varphi_1(x))) = \varphi_2(x) - \varphi_1(x)$$

and our claim is proved.

Similarly, we have that the map $\mathrm{Hom}_V(\widehat{A}/(g), k) \to \mathrm{Spec}(\widehat{A}/(g, \pi))$ is injective. Since $A/\pi A$ is regular and connected, it is an integral domain of dimension 1. It follows that $\widehat{A}/(g, \pi)$ is an Artin ring, and therefore it has a finite spectrum. In particular, the set $\mathrm{Hom}_V(\widehat{A}/(g), k)$ is finite.

Consider the natural map

$$\rho\colon \mathrm{Hom}_V(\widehat{A}/(g), V) \to \mathrm{Hom}_V(\widehat{A}/(g), k). \tag{7.58}$$

As the target of this map was shown to be finite the lemma follows by showing that the fibers of (7.58) are also finite. Fix a V-homomorphism $\varphi_0\colon \widehat{A}/(g) \to V$ and let $\varphi\colon \widehat{A}/(g) \to V$ be any V-homomorphism such that φ and φ_0 induce the same map $\overline{\varphi} = \overline{\varphi_0}\colon \widehat{A}/(g) \to k$ hence $M_\varphi = M_{\varphi_0}$. In particular, P_φ is contained in M_{φ_0}. But there are only finitely many prime ideals of $\widehat{A}/(g)$ contained in M_{φ_0} because the ring $(\widehat{A}/(g))_{M_{\varphi_0}}$ is local and Noetherian of dimension 1. We conclude by the claim above that there are only finitely many V-homomorphisms $\varphi : \widehat{A}/(g) \to V$ for which $\overline{\varphi} = \overline{\varphi_0}$ which proves the finiteness of the fibers of the map (7.58). ∎

Corollary 7.69. *There exists an open set $X \subset X_1(N)$ with non-empty reduction mod π such that for every δ_π-character $\psi \in \mathbf{X}^1_{\pi,\Phi}(E)$ there exists a finite set $\Sigma \subset R_\pi$ with the following property. For all $P \in E(R_\pi)$ if $\psi_\pi(P) \notin \Sigma$ then the set*

$$\mathrm{QCL}(X(R_\pi)) \cap \Theta^{-1}_{R_\pi}(\mathrm{Ker}(\psi_{R_\pi}) + P)$$

is finite.

Proof. Let G be the polynomial in Theorem 7.67. Then, in view of Lemma 7.68 applied to $V = R_\pi$, it is enough to take Σ the set of all roots of G. ∎

Remark 7.70. The above result is a ramified version of [13, Theorem 1.3] which dealt with the case $\pi = p$ and with the set of canonical lifts in $X(R)$. The group $\mathrm{Ker}(\psi_{R_\pi})$ in Corollary 7.69 contains the torsion of $E(R_\pi)$ but is generally bigger than the torsion. For $n = 2$ and $\psi = \psi_{1,2}$ this group was explicitly described in our arithmetic "Theorem of the kernel," cf. Theorem 7.42. As remarked before that theorem, for every E that is ordinary but not a quasi-canonical lift the group $\mathrm{Ker}(\psi_{R_\pi})$ is not torsion.

7.11 δ-Serre operators

We conclude now by applying, as in the ODE case, δ-Serre operators and δ-Euler operators to produce systems of differential equations satisfied by our partial δ-modular forms. In this subsection we assume n is arbitrary, $\pi = p$, and we consider an affine open subset $X \subset Y_1(N)$ with non-empty reduction mod p, where $N \geq 4$ is coprime to p. We let $B := B_1(N) = \mathrm{Spec}(\oplus_{m\in\mathbb{Z}} L^m) \to X$ be as in Definition 7.12. Recall from [10, Section 8.3.2 and Remark 3.58] the classic *Serre operator*

$$\partial \colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m$$

(which has the property $\partial(L^{\otimes m}) \subset L^{\otimes m+2}$) and the *Euler operator*

$$\mathcal{D} \colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m, \quad \mathcal{D}\Big(\sum \alpha_m\Big) := m\alpha_m, \quad \alpha_m \in L^{\otimes m}$$

(which has the property that $\mathcal{D}(L^{\otimes m}) \subset L^{\otimes m}$). The above operators induce R-derivations of the algebra $\mathcal{O}(B_1(N))$. Hence, if $\mu \in \mathbb{M}_n^r$ we may consider the induced derivations

$$\partial_\mu, \mathcal{D}_\mu \colon \mathcal{O}(J^r_{p,\Phi}(B_1(N))) \to \mathcal{O}(J^r_{p,\Phi}(B_1(N)));$$

cf. Proposition 2.24. Note that under the identification (7.5) we have $\mathcal{D} = x\frac{d}{dx}$ and hence, for $w = \sum a_\nu \phi_\nu$ and $f \in \mathcal{O}(J^r_{p,\Phi}(X))$ we have

$$\mathcal{D}_\mu(f \cdot x^w) = p^r \cdot a_\mu \cdot f \cdot x^w.$$

Proposition 7.71. *Assume we are given a derivation* $D \colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m$ *so that there is* $c \geq 0$ *with* $\partial(L^m) \subset L^{m+c}$ *for all* $m \geq 0$*. Then, for all* $\mu \in \mathbb{M}_n^r$ *and all* $w \in \mathbb{Z}^r_\Phi$*, the induced derivation (cf. Proposition 2.24)*

$$D_\mu \colon \mathcal{O}(J^r_{p,\Phi}(B_1(N))) \to \mathcal{O}(J^r_{p,\Phi}(B_1(N)))$$

satisfies

$$D_\mu(M^r_{p,\Phi,X}(w)) \subset M^r_{p,\Phi,X}(w + c\phi_\mu).$$

Proof. Similar to [10, Proposition 3.56]. ∎

Recall from [10, Section 8.3.3] that if X has reduction mod p contained in the ordinary locus then we may consider the *Ramanujan form*

$$P \in L^2. \tag{7.59}$$

Then, as in [10, equation (8.94)], we consider the derivation

$$\partial^* := \partial + P\mathcal{D} \colon \bigoplus_{m\in\mathbb{Z}} L^m \to \bigoplus_{m\in\mathbb{Z}} L^m$$

which has the property that $\partial^*(L^m) \subset L^{m+2}$ for all $m \in \mathbb{Z}$.

On the other hand, as in [10, Section 8.3.5] we may consider the *canonical derivation*

$$\partial^{\text{can}} := (1+T)\frac{d}{dT} : R[\![T]\!] \to R[\![T]\!].$$

As in Proposition 2.24 one shows that for every $\mu \in \mathbb{M}^r_\mu$ there is a unique derivation

$$\partial^{\text{can}}_\mu : S^r_{\text{for}} \to S^r_{\text{for}}$$

satisfying

(1) $\partial^{\text{can}}_\mu \phi_\mu F = p^r \cdot \phi_\mu \partial^{\text{can}} F$ for all $F \in R[\![T]\!]$;

(2) $\partial^{\text{can}}_\mu \phi_\nu F = 0$ for all $F \in R[\![T]\!]$ and all $\nu \in \mathbb{M}^r_n \setminus \{\mu\}$.

It is given by

$$F \mapsto \partial^{\text{can}}_\mu F := (1+T^{\phi_\mu})\frac{\partial F}{\partial \delta_\mu T}.$$

Finally, recall the injective homomorphism $\mathcal{E} : M^r_{p,\Phi,\text{ord}}(w) \to S^r_{\text{for}}$ from Remark 7.54, Part 4. With the notation above we have the following.

Proposition 7.72. *For every $\mu \in \mathbb{M}^r_n$ and every $w \in \mathbb{Z}^r_\Phi$ we have an equality*

$$\mathcal{E} \circ \partial^*_\mu = \partial^{\text{can}}_\mu \circ \mathcal{E}$$

of maps $M^r_{\pi,\Phi,X}(w) \to S^r_{\text{for}}$. Equivalently, for $w = \sum a_\nu \phi_\nu$ and $f \in M^r_{p,\Phi,X}(w)$ we have

$$\mathcal{E}(\partial^*_\mu f) = \mathcal{E}(\partial_\mu f + a_\mu p^r P^{\phi_\mu} f) = (1+T^{\phi_\mu})\frac{\partial(\mathcal{E}(f))}{\partial \delta_\mu T}.$$

Proof. Similar to [10, Proposition 8.42]. The value of ϵ in loc. cit. is 1 in view of the comment after [10, equation (8.52)]. ∎

Remark 7.73. Note that for $\mu = i \in \{1,\ldots,n\}$ we easily compute

$$\partial^{\text{can}}_i \Psi_i = (1+T^{\phi_i})\frac{\partial \Psi_i}{\partial \delta_i T} = (1+T^{\phi_i})\frac{\partial}{\partial \delta_i T}\left\{\frac{1}{p}(\phi_i - p)\log(1+T)\right\} = 1. \quad (7.60)$$

The following corollary follows from the ODE case in [10, Proposition 8.64].

Corollary 7.74. *The following equality holds:*

$$\partial^*_i f^{\text{jet}}_i = \partial_i f^{\text{jet}}_i - pP^{\phi_i} f^{\text{jet}}_i = c f^{\partial}_i. \quad (7.61)$$

Proof. For convenience we repeat the argument. The form $\partial^*_i f^{\text{jet}}_i$ has weight $\phi_i - 1$ (cf. Proposition 7.71) and Serre–Tate expansion c (cf. Proposition 7.72 and Remarks 7.26 and 7.73) while the form $c f^{\partial}_i$ has the same weight and Serre–Tate expansion (cf. Theorem 7.55); hence the two forms must coincide by the Serre–Tate expansion principle. ∎

One can get new results along these lines in the PDE case; here is an example.

Corollary 7.75. *For $n = 2$ the following equalities hold:*

$$\partial_1^* f_{1,2}^{\text{jet}} = \partial_1 f_{1,2}^{\text{jet}} - pP^{\phi_1} f_{1,2}^{\text{jet}} = cpf_1^{\partial} f_{2,\partial},$$
$$\partial_2^* f_{1,2}^{\text{jet}} = \partial_2 f_{1,2}^{\text{jet}} - pP^{\phi_2} f_{1,2}^{\text{jet}} = -cpf_2^{\partial} f_{1,\partial},$$
$$(\partial_1 f_{1,2}^{\text{jet}} - pP^{\phi_1} f_{1,2}^{\text{jet}})(\partial_2 f_{1,2}^{\text{jet}} - pP^{\phi_2} f_{1,2}^{\text{jet}}) + c^2 p^2 = 0.$$

Proof. The form $\partial_1^* f_{1,2}^{\text{jet}}$ has weight $\phi_1 - \phi_2$ (cf. Proposition 7.71) and Serre–Tate expansion cp (cf. Proposition 7.72 and Remarks 7.26 and 7.73) while the other form $cpf_1^{\partial} f_{2,\partial}$ has the same weight and Serre–Tate expansion (cf. Theorem 7.55); hence the two forms must coincide by the Serre–Tate expansion principle which proves the first equality. The second equality is proved similarly. The third equality follows by multiplying the first two equalities. ■