Appendix

Partial π-jet spaces with relations

Throughout this memoir, commutation relations between Frobenius lifts on particular fields K_π played no role; and similarly, no role was given to the inverses of our Frobenius lifts. In short, we assumed no dependence among the Frobenius lifts chosen. The aim of the Appendix is to briefly discuss a more general theoretical framework in which commutation relations and inversion of Frobenius lifts are "built into" our jet spaces. We will also provide some simple computations illustrating the complexity of this more general framework.

A.1 Main definitions

We now discuss technical structures on monoids which help to describe relationships among Frobenius lifts and their inverses. At the most basic, for every homomorphism $\rho : \mathbb{M} \to \mathbb{M}'$ of monoids with identity one defines its *kernel* $\mathbb{K}_\rho$ to be the set of all pairs $(\mu, \nu) \in \mathbb{M} \times \mathbb{M}$ such that $\rho(\mu) = \rho(\nu)$.

Fix in what follows $\pi \in \Pi$, R_π, K_π as in Subsection 2.3, and a family

$$\Phi = (\phi_1, \ldots, \phi_n)$$

of distinct Frobenius elements in $\mathfrak{G}(K^{\text{alg}}/\mathbb{Q}_p)$. Furthermore, fix an integer n^* so that $0 \le n^* \le n$ and set

$$\Phi^* = (\phi_{n+1}, \ldots, \phi_{n+n^*}) := (\phi_1^{-1}, \ldots, \phi_{n^*}^{-1}).$$

By convention, for $n^* = 0$ we take $\Phi^* = \emptyset$. For all $i \in \{1, \ldots, n^*\}$ we write $i^* = n + i$. From now on we set

$$\mathbb{M} := \mathbb{M}_{n+n^*}, \quad \mathbb{M}^r := \mathbb{M}^r_{n+n^*}.$$

We have a canonical monoid homomorphism

$$\text{can}_\pi : \mathbb{M} \to \mathfrak{G}(K_\pi/\mathbb{Q}_p)$$

defined by

$$\text{can}_\pi(i) = \phi_{\pi,i}, \quad i \in \{1, \ldots, n + n^*\}.$$

Finally, fix a homomorphism of monoids with identity

$$\rho : \mathbb{M} \to \mathbb{M}'$$

into a monoid with identity $\mathbb{M}'$ such that $(ii^*, 0)$ and $(i^*i, 0)$ belong to the kernel $\mathbb{K}_\rho$ for all $i \in \{1, \dots, n^*\}$ and assume ρ is *compatible* with π in the sense that

$$\mathbb{K}_\rho \subset \mathbb{K}_{\mathrm{can}_\pi}.$$

We set

$$\mathbb{K}_\rho^r := (\mathbb{M}^r \times \mathbb{M}^r) \cap \mathbb{K}_\rho.$$

We will also fix a subset $\mathbb{M}_\rho \subset \mathbb{M}$ with following properties (such a set always exists):

(1) For all $\mu \in \mathbb{M}$ there exists a unique element $\overline{\mu} \in \mathbb{M}_\rho$ such that $\rho(\mu) = \rho(\overline{\mu})$.

(2) For all $\mu \in \mathbb{M}$ we have $|\mu| \geq |\overline{\mu}|$, where $|\ |$ denotes the length of a word.

Example A.1. We will consider later the following special cases:

(i) (Invertible case) Take $n = n^*$ and $\rho : \mathbb{M} = \mathbb{M}_{2n} \to G$ where G is a group. In case G is the free group on n generators and ρ sends $1, \dots, n$ into the generators one can take M_ρ to be the set of all words $\mu \in \mathbb{M}$ such that no sequence of 2 consecutive letters in μ is of the form ii^* or i^*i.

(ii) (Abelian case) Take $n^* = 0$ and

$$\rho : \mathbb{M} := \mathbb{M}_n \to \mathbb{M}_{\mathrm{ab}} := \mathbb{M}_{n,\mathrm{ab}} := \mathbb{Z}_{\geq 0}^n$$

the canonical homomorphism $\rho(i) = (0, \dots, 1, \dots, 0)$ with 1 on the i-th position. We assume that $\phi_1, \dots, \phi_n$ commute on K_π so ρ is compatible with π. In this case we can take $\mathbb{M}_\rho \subset \mathbb{M}$ to consist of all words of the form $1^{i_1} \dots n^{i_n}$. We will identify $\mathbb{M}_\rho$ with $\mathbb{M}_{\mathrm{ab}}$ via the bijection induced by ρ.

As π is fixed throughout, we will sometimes drop π from the notation $\delta_{\pi,i}, \phi_{\pi,i}$; this should not create confusion as the meaning of δ_i, ϕ_i will be clear every time from the context.

Definition A.2. Define the category $\mathbf{Prol} = \mathbf{Prol}_{\pi,\Phi,\Phi^*,\rho}$ as follows. An object of this category is a countable family of p-adically complete Noetherian flat R_π-algebras $S^* = (S^r)_{r \geq 0}$ equipped with the following data:

(1) R_π-algebra homomorphisms $\varphi\colon S^r \to S^{r+1}$;

(2) π-derivations $\delta_i\colon S^r \to S^{r+1}$ with attached Frobenius lifts mod π denoted by $\phi_i\colon S^r \to S^{r+1}$ for $1 \leq i \leq n$;

(3) ring homomorphisms $\phi_j\colon S^r \to S^{r+1}$ for $n + 1 \leq j \leq n + n^*$.

For all $i \in \{1, \dots, n + n^*\}$ we write

$$\epsilon_i := \begin{cases} \delta_i & \text{if } 1 \leq i \leq n, \\ \phi_i & \text{if } n + 1 \leq i \leq n + n^*. \end{cases}$$

For $\mu := i_1 \dots i_l \in \mathbb{M}$ and $r \geq 0$ we write

$$\epsilon_\mu = \epsilon_{i_1} \dots \epsilon_{i_l} \ : S^r \to S^{r+l},$$
$$\phi_\mu = \phi_{i_1} \dots \phi_{i_l} : S^r \to S^{r+l}.$$

We require that ϕ_i on S^r be compatible with the ϕ_i on R_π for $i \in \{1, \dots, n+n^*\}$, and we have

$$\begin{aligned} \phi_i \circ \varphi = \varphi \circ \phi_i \quad & \text{for all } \ i \in \{1, \dots, n+n^*\}, \\ \phi_\mu = \phi_\nu \quad & \text{on } S^r \text{ for all } \ r \geq 0, \ (\mu, \nu) \in \mathbb{K}_\rho. \end{aligned}$$

Morphisms in this category are defined in the obvious way.

Note that since $(i i^*, 0), (i^* i, 0) \in \mathbb{K}_\rho^2$ we always have in the above definition that

$$\phi_i(\phi_{i^*}(x)) = \phi_{i^*}(\phi_i(x)) = x$$

for all $x \in S^r$ with $r \geq 0$ and all $i \in \{1, \dots, n^*\}$. So the homomorphisms ϕ_{i^*} play the role of "inverses" of ϕ_i for $i \in \{1, \dots, n^*\}$.

We next consider variables $\epsilon_\mu y_s$ for $\mu \in \mathbb{M}$ and $s \in \{1, \dots, N\}$ and consider the rings

$$J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]) := R_\pi[\epsilon_\mu y_s \mid \mu \in \mathbb{M}^r, \ s \in \{1, \dots, N\}]\widehat{\ }.$$

We define ring homomorphisms

$$\phi_i : J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]) \to J^{r+1}_{\pi,\Phi,\Phi^*}(R_\pi[N]), \ \ i \in \{1, \dots, n+n^*\},$$

extending ϕ_i on R_π by letting

$$\phi_i(\epsilon_\mu y_s) := \begin{cases} (\epsilon_\mu y)^p + \pi \epsilon_{i\mu} & \text{if } 1 \leq i \leq n, \\ \epsilon_{i\mu} y_s & \text{if } n \leq i \leq n+n^*. \end{cases}$$

We then have π-derivations

$$\delta_i : J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]) \to J^{r+1}_{\pi,\Phi,\Phi^*}(R_\pi[N]), \ \ i \in \{1, \dots, n\},$$

with

$$\delta_i(\epsilon_\mu y_s) = \epsilon_{i\mu} y_s.$$

For each pair $(\mu, \nu) \in \mathbb{K}_\rho^r$ define the N-tuple

$$\Delta_{\mu,\nu} := \phi_\mu y - \phi_\nu y.$$

For an ideal I in a ring A we denote by $I : p^\infty$ the ideal of all $a \in A$ for which there exists an integer $m(a) \geq 0$ such that $p^{m(a)} a \in I$. If A is a $\mathbb{Z}_{(p)}$-algebra then

$$A/(I : p^\infty) \simeq (A/I)/\text{tors}.$$

Let $I \subset R_\pi[N]$ be an ideal. For $r \geq 1$ define

$$I_r := (\phi_\eta I, \Delta_{\mu,\nu} \mid \eta \in \mathbb{M}^r,\ (\mu,\nu) \in \mathbb{K}^r_\rho) : p^\infty \subset J^r_{\pi,\Phi,\Phi^*}(R_\pi[N]).$$

Definition A.3. For every finitely generated R_π-algebra $A := R_\pi[N]/I$ define the *π-ρ-jet algebra* of A by the formula

$$J^r_{\pi,\Phi,\Phi^*,\rho}(A) := J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])/I_r.$$

If $m = 0$ (hence $\Phi^* = \emptyset$) we write $J^*_{\pi,\Phi,\emptyset,\rho}(A) =: J^*_{\pi,\Phi,\rho}(A)$.

Clearly the rings $J^r_{\pi,\Phi,\Phi^*,\rho}(A)$ are torsion free as groups, so they are flat over R_π. They are also Noetherian and p-adically complete. We claim they inherit π-derivations and ring homomorphisms from those on $J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])$. This follows from the fact that for all $i \in \{1, \dots, n+n^*\}$ we have that the components of the tuple $\phi_i(\Delta_{\mu,\nu})$ belong to the ideal $(\Delta_{\mu,\nu}, \Delta_{i\mu,i\nu})$ and hence

$$\delta_i I_r \subset I_r. \tag{A.1}$$

We claim $J^*_\rho(A) := J^*_{\pi,\Phi,\Phi^*,\rho}(A) := (J^r_{\pi,\Phi,\Phi^*,\rho}(A))$ is an object of **Prol**. Indeed, for all $(\mu,\nu) \in \mathbb{K}^r_\rho$ we have $\phi_\mu a_s = \phi_\nu a_s$ where a_s is the image of y_s. So $\phi_\mu a = \phi_\nu a$ for all $a \in A$. In particular,

$$\phi_\mu\phi_\eta a = \phi_{\mu\nu} a = \phi_{\nu\eta} a = \phi_\nu\phi_\eta a$$

for all $a \in A$ and all $\eta \in \mathbb{M}^r$. Using the fact that π is a non-zero divisor in $J^r_\rho(A)$ we get that $\phi_\mu\epsilon_\eta a = \phi_\nu\epsilon_\eta a$ for $a \in A$. By p-adic continuity we get that $\phi_\mu b = \phi_\nu b$ for all $b \in J^r_\rho(A)$.

Remark A.4. (1) The object $J^*_{\pi,\Phi,\Phi^*,\rho}(A)$ has the obvious universal property: for every object S^* in **Prol** and every R_π-algebra homomorphism $u : A \to S^0$ there is a unique morphism $J^*_{\pi,\Phi,\Phi^*,\rho}(A) \to S^*$ in **Prol** compatible with u.

(2) One has the following compatibility with fractions. For every object $S^* = (S^r)$ in **Prol** and every $f \in S^0 \setminus \pi S^0$ the sequence $(\widehat{S^r_f})$ has a natural structure of object in **Prol**. To check this it is enough to check that for every $r \geq 0$ and every $i \in \{1, \dots, n+n^*\}$ we have that $\phi_i(f)$ is invertible in $\widehat{S^{r+1}_f}$. If $i \in \{1, \dots, n\}$ the element $\phi_i(f) = f^p + \pi\delta_i f$ has inverse

$$\frac{1}{f^p}\Big(1 - \pi\frac{\delta_i f}{f^p} + \pi^2\Big(\frac{\delta_i f}{f^p}\Big)^2 + \cdots\Big).$$

If $i \in \{1, \dots, n^*\}$ we have

$$f = \phi_{i^*}(\phi_i(f)) = \phi_{i^*}(f)^p + \phi_i^{-1}(\pi)\cdot\phi_{i^*}\delta_i f$$

hence the element $\phi_{i*}(f)$ has inverse

$$\frac{\phi_{i*}(f)^{p-1}}{f}\Big(1+\phi_i^{-1}(\pi)\cdot\frac{\phi_{i*}\delta_i f}{f}+\phi_i^{-1}(\pi)^2\cdot\Big(\frac{\phi_{i*}\delta_i f}{f}\Big)^2+\cdots\Big).$$

Using the above compatibility we get that for every finitely generated R_π-algebra A, $f\in A\setminus\pi A$, and every $r\geq 0$ one has natural isomorphisms

$$J^r_{\pi,\Phi,\rho}(A_f)\simeq((J^r_{\pi,\Phi,\rho}(A))_f)\widehat{\ }. \tag{A.2}$$

In view of (A.2) the functors $A\mapsto J^r_{\pi,\Phi,\Phi^*\rho}(A)$ can be globalized to give functors $X\mapsto J^r_{\pi,\Phi,\Phi^*,\rho}(X)$ from (not necessarily smooth) schemes of finite type over R_π to p-adic formal schemes. In case $\Phi^*=\emptyset$ we write $J^r_{\pi,\Phi,\Phi^*,\rho}(X)=:J^r_{\pi,\Phi,\rho}(X)$.

(3) If $n^*=0$ and ρ is injective, then for A smooth over R_π we have that the ring $J^r_{\pi,\Phi,\emptyset,\rho}(A)$ coincides with the ring $J^r_{\pi,\Phi}(A)$ previously defined in the body of the memoir; this is because the ring $J^r_{\pi,\Phi}(A)$ is, in this case, torsion free. So for X smooth over R_π we have that $J^r_{\pi,\Phi,\rho}(X)$ coincides with the formal scheme $J^r_{\pi,\Phi}(X)$ defined in the body of the memoir.

(4) For every $f\in J^r_{\pi,\Phi,\Phi^*,\rho}(A)$ and $X:=\mathrm{Spec}(A)$ we have an induced map $f_{R_\pi}:X(R_\pi)\to R_\pi$.

A.2 Structure over $\mathbb{Q}$

The π-ρ-jet algebras mod π have a complicated structure as we shall presently see. However, we have the following theorem about the behavior of these algebras over $\mathbb{Q}$. We need the following trivial fact.

Lemma A.5. *Let A be a flat R_π-algebra, P a prime ideal in A not containing π, $A':=A\otimes_{R_\pi}K_\pi$, and $P':=PA'$. Then $A_P\simeq A'_{P'}$.*

In what follows for a local ring B we denote by B^{for} the completion of B with respect to its maximal ideal. For a finitely generated R_π-algebra A we simply denote by $J^r_\rho(A)$ the algebra $J^r_{\pi,\Phi,\Phi^*,\rho}(A)$. For $u:A\to R_\pi$ an R_π-algebra map with kernel P we denote by P_r the kernel of the surjective homomorphism $J^r_\rho(A)\to R_\pi$ induced by u via the universal property of π-jet algebras; we refer to P_r as the r-th *prolongation* of P. We continue to denote by $P\in X(R_\pi)$ the point of $X:=\mathrm{Spec}(A)$ defined by $u:A\to R_\pi$.

Theorem A.6. *Let A be a finitely generated R_π-algebra and $A\to R_\pi$ be an R_π-algebra map with kernel P. Write $A=R_\pi[y]/I$ where y is an N-tuple of variables such that $P=(y)/I$ and let P_r be the prolongation of P. Then there is a canonical isomorphism*

$$(J^r_\rho(A)_{P_r})^{\mathrm{for}}\simeq\frac{K_\pi[\![\phi_{\overline{\mu}}y\mid\overline{\mu}\in\mathbb{M}^r_\rho]\!]}{(\phi_{\overline{\mu}}I\mid\overline{\mu}\in\mathbb{M}^r_\rho)}.$$

Moreover, if the image of some $F \in J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])$ *in* $K_\pi[\![\phi_\mu y \mid \mu \in \mathbb{M}^r]\!]$ *belongs to the ideal* $(\Delta_{\mu,\nu} \mid (\mu,\nu) \in \mathbb{K}^r_\rho)$ *and if* f *is the image of* F *in* $J^r_\rho(A)$ *then* $f_{R_\pi}(P) = 0$.

Proof. In what follows (if not otherwise stated) η runs through $\mathbb{M}^r_n$ and (μ,ν) run through $\mathbb{K}^r_\rho$. In particular, $P_r = (\delta_\eta y)/I_r$. By Lemma A.5 we have

$$(J^r_\rho(A))_{(\delta_\eta y)} \simeq (J^r_\rho(A) \otimes_{R_\pi} K_\pi)_{(\delta_\eta y)}. \tag{A.3}$$

Note that

$$\begin{aligned} J^r_\rho(A) \otimes_{R_\pi} K_\pi &\simeq (\widehat{R_\pi[\delta_\eta y]} \otimes_{R_\pi} K_\pi)/((\delta_\eta I, \Delta_{\mu,\nu}) : p^\infty) \\ &\simeq (\widehat{R_\pi[\delta_\eta y]} \otimes_{R_\pi} K_\pi)/(\phi_\eta I, \Delta_{\mu,\nu}). \end{aligned} \tag{A.4}$$

Also, one easily checks that

$$((R_\pi[\delta_\eta y]\widehat{\ } \otimes_{R_\pi} K_\pi)_{(\delta_\eta y)})^{\text{for}} \simeq K_\pi[\![\delta_\eta y]\!] \simeq K_\pi[\![\phi_\eta y]\!]. \tag{A.5}$$

Finally note that since $I \subset (y)$, we have that

$$\phi_\mu f - \phi_\nu f \in (\Delta_{\mu,\nu}). \tag{A.6}$$

Using (A.5) and (A.6) we compute:

$$\begin{aligned} &((((\widehat{R_\pi[\delta_\eta y]}) \otimes_{R_\pi} K_\pi)/(\phi_\eta I, \Delta_{\mu,\nu}))_{(\delta_\eta y)})^{\text{for}} \\ &\quad \simeq (((\widehat{R_\pi[\delta_\eta y]}) \otimes_{R_\pi} K_\pi)_{(\delta_\eta y)})^{\text{for}}/(\phi_\eta I, \Delta_{\mu,\nu}) \\ &\quad \simeq K_\pi[\![\phi_\eta y]\!]/(\phi_\eta I, \Delta_{\mu,\nu}) \\ &\quad \simeq K_\pi[\![\phi_{\overline{\eta}} y : \overline{\eta} \in \mathbb{M}^r_\rho]\!]/(\phi_{\overline{\eta}} I \mid \overline{\eta} \in \mathbb{M}^r_\rho). \end{aligned} \tag{A.7}$$

We conclude the first assertion of the theorem by combining the isomorphisms (A.3), (A.4), (A.7). To check the second assertion of the theorem note that if the image of $F \in J^r_{\pi,\Phi,\Phi^*}(R_\pi[N])$ in $K_\pi[\![\phi_\mu y \mid \mu \in \mathbb{M}^r]\!]$ belongs to the ideal $(\Delta_{\mu,\nu} \mid (\mu,\nu) \in \mathbb{K}^r_\rho)$ then the image of F in $\frac{K_\pi[\![\phi_{\overline{\mu}} y \mid \overline{\mu} \in \mathbb{M}^r_\rho]\!]}{(\phi_{\overline{\mu}} I \mid \overline{\mu} \in \mathbb{M}^r_\rho)}$ is 0 hence the image of f in $J^r_\rho(A)_{P_r}$ is zero. So the image of f via the homomorphism $J^r_\rho(A)_{P_r} \to K_\pi$ is zero. Hence the image of f via the homomorphism $J^r_\rho(A) \to R_\pi$ is zero, hence $f_{R_\pi}(P) = 0$. ∎

Corollary A.7. *Under the assumptions of Theorem A.6 if* A *is smooth over* R_π *then the rings* $J^r_\rho(A)_{P_r}$ *are regular and the canonical homomorphisms*

$$J^r_\rho(A)_{P_r} \to J^{r+1}_\rho(A)_{P_{r+1}}$$

are injective.

Proof. If the R_π-algebra A is smooth the (y)-adic completion of $A \otimes_{R_\pi} K_\pi$ is isomorphic to $K_\pi[\![T]\!]$ for some tuple of variables T and hence

$$\frac{K_\pi[\![\phi_\mu y \mid \mu \in \mathbb{M}_\rho^r]\!]}{(\phi_\mu I \mid \mu \in \mathbb{M}_\rho^r)} \simeq K_\pi[\![\phi_\mu T \mid \mu \in \mathbb{M}_\rho^r]\!].$$

The latter power series ring is a regular local ring hence so is the ring $J_\rho^r(A)_{P_r}$. Hence the canonical homomorphisms in the theorem are injective because the corresponding homomorphisms between the power series rings are injective. ■

Remark A.8. It would be interesting to know if for A smooth over R_π the rings $J_\rho^r(A)$ themselves are regular. If these rings even just domains then this would also imply that the homomorphisms

$$J_\rho^r(A) \to J_\rho^{r+1}(A)$$

are injective.

A.3 Invertible π-jets

Assume $n = n^*$ and $\rho : \mathbb{M} = \mathbb{M}_{2n} \to G$ with G a group; cf. Example A.1 (i). We refer to the algebra $J_{\pi,\Phi,\Phi^*,\rho}^r(A)$ as the *invertible π-jet algebra* of order r of A attached to ρ. The reduction modulo π of this algebra has a complicated structure as we shall illustrate in what follows.

First note that since $(ii^*, 0), (i^*i, 0) \in \mathbb{K}_\rho^2$ for $i \in \{1, \ldots, n\}$ the following elements belong to the ideals I_r for all $r \geq 2$, $s \in \{1, \ldots, N\}$, and $i \in \{1, \ldots, n\}$:

$$\begin{aligned}\Delta_{ii^*,0,s} &= \phi_i\phi_{i^*}y_s - y_s = (\phi_{i^*}y_s)^p + \pi\delta_i\phi_{i^*}y_s - y_s,\\ \Delta_{i^*i,0,s} &= \phi_{i^*}\phi_i y_s - y_s = (\phi_{i^*}y_s)^p + \phi_i^{-1}\pi \cdot \phi_{i^*}\delta_i y_s - y_s.\end{aligned}$$

Take now $N = 1$, $n = 2$, $y = y_1$ (so we drop the index s), $I = 0$, and $\rho = \mathrm{can}_\pi : \mathbb{M} = \mathbb{M}_4 \to F_2$ the natural homomorphism to the free group on 2 generators. Consider the "linear relations:"

$$F_i := \delta_i\phi_{i^*}y - \frac{\phi_i^{-1}\pi}{\pi}\phi_{i^*}\delta_i y.$$

Then we have

$$\begin{aligned}J^2_{\pi,\Phi,\Phi^*,\mathrm{can}_\pi}(R_\pi[y]) &= J^2_{\pi,\Phi,\Phi^*}(R_\pi[y])/((\Delta_{13,0}, \Delta_{31,0}, \Delta_{24,0}, \Delta_{42,0}) : p^\infty)\\ &= (J^2_{\pi,\Phi,\Phi^*}(R_\pi[y])/(\Delta_{13,0}, \Delta_{24,0}, F_1, F_2))/\mathrm{tors}.\end{aligned}$$

A.4 Abelian π-jets

Assume $n^* = 0$ and $\rho : \mathbb{M} := \mathbb{M}_n \to \mathbb{M}_{\text{ab}} := \mathbb{M}_{n,\text{ab}} := \mathbb{Z}^n_{\geq 0}$ is the canonical homomorphism. We identify $\mathbb{M}_{\text{ab}}$ with the subset $\mathbb{M}_\rho$ of $\mathbb{M}$ consisting of words of the form $1^{i_1} \dots n^{i_n}$; cf. Example A.1, (ii). We assume that $\phi_1, \dots, \phi_n$ commute on K_π so ρ is compatible with π. The algebra $J^r_{\pi,\Phi,\emptyset,\text{ab}}(A) = J^r_{\pi,\Phi,\text{ab}}(A)$ is referred to as the *abelian π-jet algebra* of order r. The reduction modulo π of this algebra also has a complicated structure and some comments on this will be made in what follows.

Assume for simplicity $N = 1$ $y = y_1$, $I = 0$. We begin with the following observation.

Lemma A.9. *For $\mu = i_1, \dots, i_r$ and variable y we have*

$$\phi_\mu(y) \equiv y^{p^r} + \pi(\delta_{i_1}\pi)(\delta_{i_2}\pi)^p \cdots (\delta_{i_{r-1}}(\pi)^{p^{r-2}})(\delta_{i_r} y)^{p^{r-1}} \bmod \pi^2.$$

Proof. We proceed by induction on r. The case $r = 1$ holds by definition. For the inductive step note that, for any fixed Frobenius lift ϕ mod π with associated π-derivation δ, any $m \geq 0$ and any $F \in R_\pi[\delta_\mu y | \mu \in \mathbb{M}]$ we have

$$\phi(F^{p^m}) = (F^p + \pi\delta F)^{p^m} \equiv F^{p^{m+1}} + p^m \pi F^{p(p^m-1)} \delta F \bmod \pi^2. \tag{A.8}$$

Now assume $r \geq 2$ and set $\mu' = i_2 \dots i_r$ and $\phi = \phi_{i_1}$ so $\mu = i_1\mu'$. By induction

$$\phi_{\mu'}(y) \equiv y^{p^r} + \pi(\delta_{i_2}\pi)(\delta_{i_3}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-3}}(\delta_{i_r} y)^{p^{r-2}} \bmod \pi^2.$$

Repeatedly using (A.8) we have

$$\begin{aligned}
&\phi(\pi(\delta_{i_2}\pi)(\delta_{i_3}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-3}}(\delta_{i_r} y)^{p^{r-2}}) \\
&\equiv (\pi^p + \pi\delta\pi)((\delta_{i_2}\pi)^p + \pi\delta\delta_{i_2}\pi) \cdots ((\delta_{i_r} y)^{p^{r-1}} + \pi p^{r-2}\delta\delta_{i_r} y(\delta_{i_r} y)^{p(p^{r-2}-1)}) \\
&\equiv (\pi\delta\pi)(\delta_{i_2}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-2}}(\delta_{i_r} y)^{p^{r-1}} \bmod \pi^2.
\end{aligned}$$

The result follows. ∎

For each $\mu = i_1 \dots i_r$ with $r \geq 2$ set

$$F_\mu := (\delta_{i_1}\pi)(\delta_{i_2}\pi)^p \cdots (\delta_{i_{r-1}}\pi)^{p^{r-2}}(\delta_{i_r} y)^{p^{r-1}}.$$

Note in particular that F_μ has order 1. Note also that if $(\mu, \nu) \in \mathbb{K}^r_\rho$ then μ and ν must have, in particular, the *same* length and the difference $F_\mu - F_\nu$ is a p^{r-1}-th power of a linear polynomial in the variables $\delta_i y$.

Proposition A.10. *One has a surjective ring homomorphism*

$$\frac{k[y, \delta_1 y, \dots, \delta_n y]}{(\{F_\mu - F_\nu : (\mu, \nu) \in \mathbb{K}^r_\rho\})}[\delta_\mu y : \mu \in \mathbb{M}^r \setminus \mathbb{M}^1] \to J^r_{\pi,\Phi,\text{ab}}(R_\pi[y])/(\pi).$$

For $n = r = 2$ the above homomorphism is an isomorphism, i.e.,

$$J^2_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi) \simeq \frac{k[y, \delta_1 y, \delta_2 y]}{(\delta_1\pi \cdot (\delta_2 y)^p - \delta_2\pi \cdot (\delta_1 y)^p)}[\delta_1^2 y, \delta_1\delta_2 y, \delta_2\delta_1 y, \delta_2^2 y].$$

Proof. Recall that $J^r_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])$ is obtained as $J^r_{\pi,\Phi,\Phi^*}(R_\pi[y])$ divided by the ideal $I_r = (\{\Delta_{\mu,\nu}\}) : p^\infty$, where the generators run over all $(\mu,\nu) \in \mathbb{K}^r_\rho$. From Lemma A.9, $\pi(F_\mu - F_\nu) \equiv \Delta_{\mu,\nu} \bmod \pi^2$, and so $F_\mu - F_\nu \in (I_r, \pi)$. Then the first part of the proposition follows. Assume now $n = r = 2$. Since π is a prime element in the ring $J^2_{\pi,\Phi}(R_\pi[y])$ and does not divide $F_{12,21}$ in this ring it follows that

$$I_2 := (\Delta_{12,21}) : p^\infty = (F_{12,21})$$

which implies the second part of the proposition. ∎

Remark A.11. In particular, for $n = r = 2$ the reduced ring

$$(J^2_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi))_{\mathrm{red}}$$

is isomorphic to a polynomial ring in 6 variables. However, note that, contrary to what one might expect, there is no equality (or even relation) in this ring between the images of the variables $\delta_1\delta_2 y$ and $\delta_2\delta_1 y$; instead we have an identification between the images of $(\delta_1\pi)^{1/p} \cdot \delta_2 y$ and $(\delta_2\pi)^{1/p} \cdot \delta_1 y$. As we see, a relation between the images of the variables $\delta_1\delta_2 y$ and $\delta_2\delta_1 y$ pops up in the ring $J^3_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi)$; cf. Proposition A.12.

To tackle the case $n = 2$ and $r = 3$ note that since $p \geq 3$,

$$\begin{aligned}
\phi_1\phi_2 y &= \phi_1(y^p + \pi\delta_2 y) \\
&= (y^p + \pi\delta_1 y)^p + \phi_1(\pi)((\delta_2 y)^p + \pi\delta_1\delta_2 y) \\
&\equiv y^{p^2} + p\pi y^{p(p-1)}\delta_1 y + \pi\delta_1\pi \cdot (\delta_2 y)^p + \pi^2\delta_1\pi \cdot \delta_1\delta_2 y \mod \pi^3.
\end{aligned}$$

hence

$$\check{\Delta}_{12,21} := \frac{1}{\pi}\Delta_{12,21} = \frac{\phi_1\phi_2 y - \phi_2\phi_1 y}{\pi} \equiv A + pB + \pi C \mod \pi^2$$

where

$$\begin{aligned}
A &= \delta_1\pi \cdot (\delta_2 y)^p - \delta_2\pi \cdot (\delta_1 y)^p \\
B &= y^{p(p-1)}(\delta_1 y - \delta_2 y) \\
C &= \delta_1\pi \cdot \delta_1\delta_2 y - \delta_2\pi \cdot \delta_2\delta_1 y.
\end{aligned}$$

Let $i \in \{1, 2\}$. Using the fact that for any element z in a δ_π-ring we have

$$\delta_\pi z^p \equiv 0 \mod \pi$$

and assuming for simplicity that $R_\pi \neq R$ (so $p/\pi \in \pi R_\pi$) we get that

$$\begin{aligned} \delta_i(A) &\equiv \delta_i\delta_1\pi \cdot (\delta_2 y)^{p^2} - \delta_i\delta_2\pi \cdot (\delta_1 y)^{p^2} && \bmod \pi, \\ \delta_i(pB) &\equiv 0 && \bmod \pi, \\ \delta_i(\pi C) &\equiv \delta_i\pi \cdot C^p && \bmod \pi. \end{aligned}$$

Hence

$$\begin{aligned} \delta_i(\check{\Delta}_{12,21}) \equiv& \delta_i(A + pB + \pi C) && \bmod \pi \\ \equiv& \delta_i(A) + \delta_i(pB) + \delta_i(\pi C) && \bmod \pi \\ \equiv& \delta_i\delta_1\pi \cdot (\delta_2 y)^{p^2} - \delta_i\delta_2\pi \cdot (\delta_1 y)^{p^2} \\ & + \delta_i\pi \cdot (\delta_1\pi \cdot \delta_1\delta_2 y - \delta_2\pi \cdot \delta_2\delta_1 y)^p && \bmod \pi. \end{aligned}$$

On the other hand, by (A.1) and since $\check{\Delta}_{12,21} \in (\Delta_{12,21}) : p^\infty$ it follows that

$$\delta_i(\check{\Delta}_{12,21}) \in I_3 = (\Delta_{\mu,\nu} \mid (\mu,\nu) \in \mathbb{K}^3_\rho) : p^\infty.$$

In particular, we have proved the following.

Proposition A.12. *Assume that $R_\pi \neq R$. The image of the element*

$$\delta_i\pi \cdot (\delta_1\pi \cdot \delta_1\delta_2 y - \delta_2\pi \cdot \delta_2\delta_1 y)^p + \delta_i\delta_1\pi \cdot (\delta_2 y)^{p^2} - \delta_i\delta_2\pi \cdot (\delta_1 y)^{p^2} \in J^3_{\pi,\Phi}(R_\pi[y])$$

in the ring $J^3_{\pi,\Phi,\mathrm{ab}}(R_\pi[y])/(\pi)$ is 0.

Note that the image of the above element in $J^3_{\pi,\Phi}(R_\pi[y])/(\pi)$ is a p-th power. A similar (but slightly more complicated) formula holds in case $R_\pi = R$.

A.5 $\delta_{\pi\text{-}\rho}$-characters

We conclude with a discussion of characters for π-ρ-jets. Specifically, partial characters restrict to π-ρ-characters, but we will show this restriction map is not injective in the abelian case. It is a question whether the restriction map is surjective.

Definition A.13. Fix G a smooth commutative group scheme over R_π. The R_π-module

$$\mathbf{X}^r_{\pi,\Phi,\Phi^*,\rho}(G) := \mathrm{Hom}(J^r_{\pi,\Phi,\Phi^*,\rho}(G), \widehat{\mathbb{G}_a}).$$

will be called the *module of $\delta_{\pi\text{-}\rho}$-characters* of G of order $\leq r$. Let

$$\rho_{\mathrm{free}} : \mathbb{M}_{n+n^*} \to F_n$$

be the unique homomorphism of monoids with identity into the free group F_n on $\{1,\ldots,n\}$ that is the identity on $\mathbb{M}_n$ and sends i^* into i^{-1} for all $i \in \{1,\ldots,n\}$.

Note that $\mathbb{K}_{\rho_{\text{free}}} \subset \mathbb{K}_{\rho}$, so we have an induced closed immersion $J^r_{\pi,\Phi,\Phi^*,\rho}(G) \to J^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G)$. The latter induces a restriction R_π-module homomorphism

$$\mathbf{X}^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G) \xrightarrow{\text{res}} \mathbf{X}^r_{\pi,\Phi,\Phi^*\rho}(G).$$

For the further development of the theory it is important to understand the behavior of this canonical restriction. Note that in case $\Phi^* = \emptyset$ we have that $J^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G)$ and $\mathbf{X}^r_{\pi,\Phi,\Phi^*,\rho_{\text{free}}}(G)$ identify with $J^r_{\pi,\Phi}(G)$ and $\mathbf{X}^r_{\pi,\Phi}(G)$ as defined in the body of the memoir.

Of particular interest are the *abelian δ_π-characters* of a commutative smooth group scheme G, defined as the $\delta_{\pi\text{-ab}}$-characters, i.e., the elements of the R_π-module

$$\mathbf{X}^r_{\pi,\Phi,\text{ab}}(G) := \text{Hom}(J^r_{\pi,\Phi,\text{ab}}(G), \widehat{\mathbb{G}_a}).$$

We therefore have a naturally induced restriction homomorphism

$$\mathbf{X}^r_{\pi,\Phi}(G) \to \mathbf{X}^r_{\pi,\Phi,\text{ab}}(G). \tag{A.9}$$

Define the K_π-module of *abelian symbols* $K^r_{\pi,\Phi,\text{ab}}$ as the free K_π-module with basis $\mathbb{M}^r_{\text{ab}}$ and define $R^r_{\pi,\Phi,\text{ab}}$ similarly. We would like to briefly look into the abelian δ_π-characters of $\mathbb{G}_a$ and $\mathbb{G}_m$.

Write $\mathbb{G}_a = \text{Spec}\, R_\pi[T]$. Recall from Corollary A.7 that we have an injective homomorphism

$$\mathcal{O}(J^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a))_{P_r} \to K_\pi[\![\phi_\mu T : \mu \in \mathbb{M}^r_{\text{ab}}]\!] \tag{A.10}$$

where $P_r = (\delta_\mu T : \mu \in \mathbb{M}^r)$. We may define

$$K^r_{\pi,\Phi,\text{ab}}T = \sum_{\mu \in \mathbb{M}^r_{\text{ab}}} K_\pi \phi_\mu T \subset K_\pi[\![\phi_\mu T : \mu \in \mathbb{M}^r_{\text{ab}}]\!]$$

and similarly for $R^r_{\pi,\Phi,\text{ab}}T$ which again are naturally isomorphic to the groups of abelian symbols. As in the non-abelian case, the image of $\mathbf{X}^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a)$ via the homomorphism (A.10) is contained in $K^r_{\pi,\Phi,\text{ab}}T$, so we get an induced homomorphism

$$\mathbf{X}^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a) \to K^r_{\pi,\Phi,\text{ab}}T, \ \ \psi \mapsto \psi(T). \tag{A.11}$$

If G has relative dimension 1 with invariant 1-form ω, the standard theory provides again a map of p-formal schemes $\mathcal{E} : \widehat{\mathbb{G}_a} \to \widehat{G}$. This again yields for each $\psi \in \mathbf{X}^r_{\pi,\Phi,\text{ab}}(G)$ a composition $\psi \circ \mathcal{E}^r \in \mathbf{X}^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a)$,

$$\psi \circ \mathcal{E}^r : J^r_{\pi,\Phi,\text{ab}}(\mathbb{G}_a) \xrightarrow{\mathcal{E}^r} J^r_{\pi,\Phi,\text{ab}}(G) \xrightarrow{\psi} \widehat{\mathbb{G}_a}.$$

We get an induced homomorphism

$$\mathbf{X}^r_{\pi,\Phi,\text{ab}}(G) \to K^r_{\pi,\Phi,\text{ab}}T, \ \ \psi \mapsto (\psi \circ \mathcal{E}^r)(T). \tag{A.12}$$

Writing $(\psi \circ \mathcal{E})(T) = \sum \lambda_\mu \phi_\mu T$ we define

$$\theta(\psi) := \sum \lambda_\mu \phi_\mu \in K^r_{\pi,\Phi,\mathrm{ab}}$$

to be the *abelian Picard–Fuchs symbol* of ψ.

Write now $\mathbb{G}_m = \operatorname{Spec} R_\pi[x, x^{-1}]$. Let $n = 2$ and note that $\phi_{21}(x)$ is invertible in the ring

$$J^2_{\pi,\Phi}(\mathbb{G}_m) = R_\pi[x, x^{-1}, \delta_1 x, \delta_2 x, \delta_1^2 x, \delta_1\delta_2 x, \delta_2\delta_1 x, \delta_2^2 x]\widehat{}\,.$$

We can now show the restriction map (A.9) can fail to be injective.

Proposition A.14. *Let N be the smallest integer so that $\psi := \frac{\pi^N}{p} \log\left(\frac{\phi_{12}(x)}{\phi_{21}(x)}\right)$ belongs to the ring $J^2_{\pi,\Phi}(\mathbb{G}_m)$. Then the restriction of $\psi \in \mathbf{X}^r_{\pi,\Phi}(\mathbb{G}_m)$ to $\mathbf{X}^r_{\pi,\Phi,\mathrm{ab}}(\mathbb{G}_m)$ vanishes.*

Proof. We can write

$$\frac{\phi_{12}(x)}{\phi_{21}(x)} = 1 + \left(\frac{\phi_{12}(x)}{\phi_{21}(x)} - 1\right) = 1 + \frac{\phi_{12}x - \phi_{21}x}{\phi_{21}x} = 1 + \pi \frac{\Delta_{12,21}(x)}{\phi_{21}x}$$

which yields

$$\frac{\pi^N}{p} \log\left(\frac{\phi_{12}(x)}{\phi_{21}(x)}\right) = \frac{\pi^N}{p} \sum (-1)^n \frac{\pi^n}{n} \left(\frac{\Delta_{12,21}(x)}{\phi_{21}x}\right)^n.$$

Clearly the latter series is in the ideal generated by $\Delta_{12,21}$ hence lies in the ideal I_2 and is therefore zero in $\mathcal{O}(J^2_{\pi,\Phi,\mathrm{ab}}(\mathbb{G}_m))$. ∎

Remark A.15. The previous discussion offers a glimpse into what the theory of abelian δ_π-characters should look like; the first non-trivial steps would have to tackle the case of elliptic curves which will not be pursued at this time.