To the memory of Jacques Tits

Preface

Building theory is a manifestation of the interaction between geometry and groups described by Felix Klein in his Erlangen program (as early as in 1872): a geometry (affine, Euclidean, projective, ...) is the study of those properties of figures that are invariant under the action of a group of transformations; and one of the best ways to study a group is to make it act on a geometric object. Buildings were introduced by Jacques Tits in the 1950s to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups (in particular, of the exceptional groups) and for the construction and study of semi-simple groups over general fields.

These first buildings are now known as Tits buildings and are among the buildings of spherical type. Such a building is a simplicial complex and, for a good geometric realization, its apartments are Euclidean spheres, and its associated Weyl group is a finite Coxeter group (associated with a root system).

Later, in the 1960s, François Bruhat, Nagayoshi Iwahori and Hideya Matsumoto (among others) found similar properties for the semi-simple groups over non-archimedean local fields. Then François Bruhat and Jacques Tits introduced a new kind of buildings. They are now known as Bruhat–Tits buildings and are among the (discrete) buildings of affine type. Such a building is a simplicial complex, its associated Weyl group is an infinite Coxeter group (the affine Weyl group of a root system) and, for a good geometric realization, its apartments are Euclidean affine spaces. Later on, new types of (combinatorial) buildings were introduced with more general kinds of apartments and more general types of Coxeter groups. Most of them are suitable to study Kac–Moody groups. In another direction F. Bruhat and J. Tits introduced a slight modification of their buildings, adapted to the study of reductive groups over fields endowed with a non-discrete real valuation. Such a building is no longer a simplicial complex, its Weyl group is no longer a Coxeter group, but, for a good geometric realization, its apartments are still Euclidean affine spaces.

In this book, we are interested in the buildings that are useful for studying reductive groups over general fields or real valued fields, i.e., in the Tits or Bruhat–Tits buildings. So we develop a general theory of Euclidean buildings, where the apartments are Euclidean spaces. This clearly includes the Bruhat–Tits buildings of reductive groups over non-archimedean local fields, or more generally over fields endowed with a (discrete or not) real valuation. On the other hand, this theory includes also the Tits buildings; we just have to change the geometric realization: for the apartments we replace their realization as Euclidean sphere by their realization as the ambient Euclidean vector space; a simplex in the sphere is replaced by the simplicial cone (with vertex 0) it generates.

This book is intended as the first part of a two-volume project. We give an abstract definition of Euclidean buildings (as certain metric spaces of "negative curvature," analogous to the Riemannian symmetric spaces of non-compact type), study their properties and give some applications. We indicate only a number of examples of Tits or Bruhat–Tits buildings associated to some reductive algebraic groups over a field $\mathcal{K}$ (actually split or classical). The second volume (with the title "Euclidean buildings, buildings for reductive groups") will recall some details on the algebraic structure of reductive algebraic groups and explain the construction of their associated Tits buildings (for a general field $\mathcal{K}$) or their associated Bruhat–Tits buildings (when the field $\mathcal{K}$ is endowed with a non-trivial real valuation).

Needless to say, this book comes entirely from Jacques Tits' ideas and works. It is now impossible to thank him directly; nevertheless, I am very much indebted to him. To tell it once again, maybe the best is to say that the present book is a (long) introduction to his works (see also [Rou09b] for some complements).

I would like to thank my university and the CNRS for their support, offered without too heavy administrative obligations (such as repetitive applications). I also thank the anonymous referees for their valuable comments and suggestions, as well as Andrei Iacob and Simon Winter for making a nice book out of my poorly written text, and Apostolos Damialis and Theresa Haney of EMS Press for their editorial and publishing work.

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Finally, I certainly do not thank Covid-19; it caused the death of many people, including some well-known mathematicians, although I have to admit that the associated lockdowns accelerated the completion of this book.