Introduction

As buildings were introduced by Jacques Tits in the 1950s, the best way to start is certainly to quote him in [Tits80]: “Originally the theory of buildings was aimed primarily at understanding the exceptional Lie groups from a geometrical point of view. The starting point was the observation that it is possible to associate with each complex analytic semi-simple group a certain well defined geometry, in such a way that the basic properties of the geometries thus obtained and their mutual relationships can be easily read from the Dynkin diagrams of the corresponding groups.” The model was the n-dimensional projective space $\mathbb{P}_n(\mathbb{C})$ and the action of the projective group $\mathrm{PGL}_{n+1}(\mathbb{C})$ on it: the linear subspaces of $\mathbb{P}_n(\mathbb{C})$ are in a one-to-one correspondence with their stabilizers in $\mathrm{PGL}_{n+1}(\mathbb{C})$ and the incidence properties between these subspaces are easily translated into properties between their stabilizers. And one of the goals was to construct (over any field) groups, analogues of exceptional semi-simple groups, as automorphism groups of these buildings.

Actually the construction of (split) semi-simple groups over general fields corresponding to each type of semi-simple Lie group was made by Claude Chevalley with algebraic means around 1955 [Che55]. So the theory of buildings was essentially first developed to study these groups. The first detailed and complete account appeared in 1974 [Tits74]; therein an abstract definition of buildings is given. Meanwhile the real estate terminology for these mathematical objects had been suggested by Nicolas Bourbaki (compare [Tits61] or [Tits65] with [Bou-Lie, §IV.1, Exers. 15 to 24]). Thus, now to any semi-simple algebraic group over any field there is associated a building (called a Tits building) on which the group acts. This building is an abstract simplicial complex with a covering by subcomplexes called apartments, and the geometric realization of such an apartment is a Euclidean sphere with a finite (Weyl) group of isometries. Accordingly, these buildings are said to be of spherical type. They encode geometrically the most part of the algebraic structure of (possibly non-split) reductive groups determined by Armand Borel and Jacques Tits [BorT65].

François Bruhat [Bru64], Nagayoshi Iwahori and Hideya Matsumoto [IwaM65] discovered interesting properties of the semi-simple groups over non-archimedean local fields (see also [GolI63]). Then F. Bruhat and J. Tits elaborated on this and constructed buildings with an action of these groups [BruT67]. The theory of the (now called) Bruhat–Tits buildings grew up along several publications [BruT72], [BruT84a] and [BruT87a], with some details for classical groups given in the papers [BruT84b] and [BruT87b]. The Bruhat–Tits buildings provide, in particular, a good classification of maximal compact subgroups of a given group. One can now associate such a building to any semi-simple (or reductive) group over a field endowed with a non-archimedean real valuation (at least if the group is quasi-split or if the valuation is discrete, complete and the residue field perfect, see Example 4.4.6 (3)). If the valuation is discrete, these buildings are still simplicial complexes and (if moreover the group is almost simple) the geometrical realization of any apartment is a finite-dimensional Euclidean affine space with an infinite (Weyl) group of affine isometries, generated by orthogonal reflections with respect to some hyperplanes called walls. When the valuation is not discrete, one cannot consider a simplicial structure, so the Bruhat–Tits buildings (identified with their geometric realizations) are metric spaces with a covering by subspaces called apartments. Each such apartment is a finite-dimensional Euclidean affine space with an infinite (Weyl) group of affine isometries, generated by orthogonal reflections with respect to some hyperplanes called walls. However, now the simplices are replaced by facets which are filters of subsets of these apartments. In [BruT72] is given the abstract construction of the (Bruhat–Tits) building associated to a group endowed with an (abstract) structure of valued root datum. These buildings are said to be of affine type. They look much like the Riemannian symmetric spaces of non-compact semi-simple Lie groups, and are their true analogues when the field $\mathbb{R}$ or $\mathbb{C}$ is replaced by a non-archimedean local field.

The Weyl group W of any of the above buildings is a Coxeter group (at least in the discrete case), more precisely a group generated by orthogonal reflections in a Euclidean space; these reflections are linear (and the group is finite) for buildings of spherical type, and they are affine for buildings of affine type. But there are more general buildings with more general Coxeter groups as Weyl groups. In them the apartments are no longer Euclidean spaces; they may be, e.g., hyperbolic spaces or without a simply defined geometric structure. These buildings are simplicial complexes and they may even be defined as a set (of chambers) with a "distance" taking values in the Weyl group W. Many of them are associated with Kac–Moody groups (of indefinite type). A construction without a group was given by Mark Ronan and Jacques Tits [RonT87]. There are also generalizations of the non-discrete affine buildings, where $\mathbb{R}$ is replaced by a non-archimedean ordered commutative group Λ. They are known as Λ-buildings, see [Benn94] and [HébIL20]. We shall not deal in this book with all these more general buildings. The interested readers may consult [AbrB08] or [Ron89] and, for Kac–Moody groups, [Tits87], [Tits89] or [Rém02].

The aim of this book is to introduce and investigate the buildings used to study reductive groups. Accordingly, we shall deal only with Euclidean buildings (i.e., buildings of spherical or affine type), essentially the Tits buildings and Bruhat–Tits buildings introduced above. We want to include in our study the Bruhat–Tits buildings associated to reductive groups over a field endowed with a real (potentially non-discrete) valuation. Hence, we cannot view buildings as simplicial complexes. Instead, we consider buildings as metric spaces: in the more classical "discrete" case, where a simplicial complex version exists, we consider its geometric realization (potentially slightly modified).

Thus, for us a Euclidean building is a metric space endowed with a covering by subspaces, called apartments, that are isometric to a given finite-dimensional Euclidean space $\mathbb{A}$. This model apartment $\mathbb{A}$ is itself endowed with a group $\mathbb{W}$ of isometries generated by orthogonal reflections with respect to a $\mathbb{W}$-stable set $\mathscr{M}$ of hyperplanes called walls; a requirement is that there are finitely many parallel classes of such walls. Therefore, these buildings look much like the Riemannian symmetric spaces of non-compact type, where the apartments are the maximal flats (see §3.1.18.3). The fact preventing such a building to be a simplicial complex is that $\mathscr{M}$ may be a non-discrete set of hyperplanes. Actually, in the discrete case the facets of $\mathbb{A}$ are its subsets cut out by the walls: they provide a partition of $\mathbb{A}$ and the top-dimensional ones, the chambers, are the connected components of $\mathbb{A} \setminus \bigcup_{M \in \mathscr{M}} M$. When $\mathscr{M}$ is not discrete, the facets are no longer subsets of apartments, but filters in these apartments. This situation appeared naturally for Bruhat–Tits buildings associated to reductive groups over fields with a non-discrete real valuation [BruT72].

Here we want to give an abstract definition of these (possibly non-discrete) Euclidean buildings. In [BruT72] one can find only a construction of such buildings that starts from a valued root datum in a group. In [Tits86] J. Tits gave already an abstract definition. But his definition emphasizes the role of sectors against that of facets; it is the definition of $\mathbb{R}$-buildings adopted in this book (§2.4.4). On the contrary, we keep here the definition introduced in the eponymous article [Rou09a], which is closer to the classical definition of [Tits74] or [Bou-Lie, §IV.1, Exer. 15]. The equivalence with Tits' definition (under some additional hypothesis) is stated in [Rou09a], as a simple corollary of previous results of Anne Parreau [Parr00a].

We are lead to consider the following abstract definition.

Definition. A Euclidean building is a triple $(\mathscr{I}, \mathcal{F}, \mathscr{A})$ where $\mathscr{I}$ is a metric space, $\mathcal{F}$ is a poset of filters in $\mathscr{I}$ called facets, and $\mathscr{A}$ is a covering of $\mathscr{I}$ by subsets called apartments. This datum has to satisfy:

(I0) Each apartment A endowed with the restriction of the metric and the set of facets $F \in \mathcal{F}$ contained in A is a Euclidean apartment (in particular, a Euclidean affine space).

(I1) Any two facets are contained in a common apartment.

(I2) Two apartments containing the same two facets F, F' are isometric by an isomorphism fixing (pointwise) the closures $\overline{F}$ and $\overline{F'}$.

We gave above only an idea of what a Euclidean apartment and the set of facets inside it are. For details and the definition of isomorphisms, see Chapter 1. The above definition is stated precisely in Definition 2.1.2. Apart from the technical aspects of

(I0) and the definition of facets, the axioms (I1) and (I2) are precisely the main ingredients of the definition of buildings as simplicial complexes, see, e.g., [Tits74, §3.1] and [AbrB08, Def. 4.1]. Actually, in the literature, there are several definitions of (Euclidean) buildings: the initial one by J. Tits, our definition (above and in Definition 2.1.2), and several variants, with slight differences and each adapted to a specific context. We compare them in §2.2.4.

The Bruhat–Tits buildings are clearly examples of Euclidean buildings (defined as above). But the Tits buildings associated to semi-simple groups over any field [Tits74] admit also geometrical realizations (called vectorial buildings) as discrete Euclidean buildings. Then an apartment is a Euclidean vector space, the Weyl group is a finite group of linear isometries, and the facets are vectorial cones. The associated spherical building is simply the unit sphere in it centered at the common origin of all apartments; it is the classical geometrical realization of the simplicial complex.

The easiest examples of affine Euclidean buildings are the trees (discrete without leaves, or $\mathbb{R}$-trees). They provide a clear illustration of the notions we introduce, in particular for the properties of the systems of apartments, see, e.g., Example 3.1.6.1 (c).

As already stated, our goal is to describe and analyze the Euclidean buildings useful for studying reductive groups, in general or over a field endowed with a real valuation. However, we postpone, to the second volume of this book, the construction and precise study of the Tits buildings or Bruhat–Tits buildings associated to these groups. Nevertheless, these main examples are already explained quickly in the present volume. In §2.2.3 and Example 4.3.13 (2) the readers will find a description (without proof) of the Tits building of a reductive group over a general field and of how it fits with our definitions. The details for SL_2 are in Example 4.3.13 (1) and specific classical examples are provided in §6.1 and §6.3. In Example 4.4.6 (3) we present (without proof) the valued root datum associated to a reductive group endowed with a real valuation: this is the abstract algebraic structure used by F. Bruhat and J. Tits to build their Bruhat–Tits buildings in [BruT72] (this construction is explained in §5.4). The details for SL_2 are provided in Example 4.4.6 (1), Example 5.2.5 and Exercise 5.4.12 (then the buildings are trees). And specific classical examples are studied in detail in §6.2 and §6.4.

The abstract developments of the theory of Euclidean buildings we shall present have actually very important applications in the theory of reductive groups over general fields or over local fields. This is clear from the above discussion and from Chapter 6, where the cases of many classical groups are presented in detail. In Part 2 of this book, we shall give more applications, in particular a building-theoretic proof of the main structure results for non-split reductive groups, which are due to A. Borel and J. Tits.

Among the main features and possible applications of Euclidean buildings, one may choose three aspects (explained in the chapters of this book):

From our point of view, Euclidean buildings are metric spaces and, as proved by F. Bruhat and J. Tits, they are CAT(0) spaces, the abstract version of "spaces of negative curvature" introduced by M. Gromov. Hence, one can develop, for these Euclidean buildings, the general theory of CAT(0) spaces (actually with some simplifications). In particular, one can construct at the infinity of an affine building a spherical building (with its Tits metric) and also a boundary that is a union of affine buildings of smaller rank. When the building is locally finite (e.g., a Bruhat–Tits building over a local field), this results in two compactifications: the visual compactification and the polyhedral compactification (see Chapter 3).

From the CAT(0) property one can deduce the Bruhat–Tits fixed point theorem: any bounded group of automorphisms of a Euclidean building has a fixed point (provided that the building is complete, which always holds in the discrete case). For the Bruhat–Tits building of a reductive group over a local field, this enables one to classify the conjugacy classes of maximal compact subgroups. Actually the same proof, applied to symmetric spaces of non-compact type, establishes the conjugacy of all maximal compact subgroups in a semi-simple Lie group (see §4.1).

To a pair (G, K) consisting of a topological group G and a compact, open subgroup K, one can associate the algebra $\mathscr{H}$ of complex functions on G that are bi-invariant under K. It is known as a Hecke algebra and is an important tool for the study of some representations of G. Here we consider the case where G acts strongly transitively on a discrete Euclidean building and K is the stabilizer of either a vertex or a chamber. This leads to two interesting Hecke algebras, and we are able to prove the results that are classical in the particular case of reductive groups over local fields. Our arguments are purely building-theoretic, essentially based on the study of the image of a line segment under a retraction; this image is called a Hecke path (see Chapter 7).

Thus, one will find in this book a detailed study of Euclidean buildings and of the groups acting nicely on them. We try to give more complete proofs in the case of discrete buildings; for dense buildings we allow ourselves to refer to the literature in the proofs. Also, we do not prove completely all the consequences of our hypotheses for Euclidean buildings (or CAT(0) spaces) whenever they are easier to explain for Tits buildings or Bruhat–Tits buildings.

In spite of its length, this book does not touch upon all interesting properties of Euclidean buildings. In particular, we do not present their classification, referring instead [Wei03] and [Wei09], where a rather different language is used. For further developments, details or applications the interested readers may consult [AbrB08], [Brow89] (more or less contained in the preceding reference), [BruT72], [Garr97], [KalP23], [Ron89], [Tits74], [TitsW02] and all articles cited in this introduction, or many articles by F. Bruhat and/or J. Tits.

The building blocks of a building are the apartments. They are defined as affine Euclidean spaces endowed with a structure (some facets in them) derived from a (Weyl) group W of isometries that is generated by orthogonal reflections. This theory is explained in Chapter 1, including the very closely related theories of Coxeter groups or root systems; there, some references to the literature are used. But the rest of the book is essentially self-contained (at least for discrete buildings).

Our general definition of Euclidean buildings is given in Chapter 2. We explain the particular properties of discrete buildings and $\mathbb{R}$-buildings. The negative curvature aspects of the metric appear in §2.3: these buildings are important examples of CAT(0) spaces. An important tool is the retraction centered at a chamber (§2.1.7).

The properties at infinity of Euclidean buildings are presented in Chapter 3; they depend heavily on the chosen system of apartments. We investigate there the notions of sector-friendly and chimney-friendly buildings (or systems of apartments); they will be shown to be equivalent at the end of the book, in Theorem 7.1.18. In this situation one may define retractions centered at a sector germ (§3.1.3) or at a full chimney germ (§3.3.2). We build the spherical building at infinity and the polyhedral compactification.

In Chapter 4 we study the groups G acting on Euclidean buildings. We obtain the well-known Bruhat–Tits fixed point theorem and then concentrate on strongly transitive actions. We then explain the equivalence between the notion of a thick, discrete (combinatorial) building with a strongly transitive action of a group G and the notion of a saturated Tits system (with Euclidean Weyl group) in G. The stronger condition of Moufang provides in G a root group datum (RGD) system (or sometimes a root datum), even in the non-discrete case. If the building is affine, we get moreover a valuation of this system (or datum).

Conversely, in Chapter 5, we consider a group endowed with an RGD system or root datum (plus possibly a valuation of it) and we build an associated vectorial building (plus possibly an affine building) on which the group acts strongly transitively (Theorem 5.4.10). This abstract construction is mainly due to F. Bruhat and J. Tits [BruT72]. It is the main result of this chapter (and perhaps of this book).

Chapter 6 is devoted to some precise examples; there were only a few of them in the preceding chapters. We present almost all cases of classical semi-simple groups (over general, resp., ultrametric fields) and their associated Tits buildings (resp., Bruhat–Tits buildings). In §6.1 (resp., §6.2) we deal with the general linear group over a (skew or commutative) field, possibly infinite dimensional over its center (resp., with moreover a real valuation); this is largely inspired by [BruT84b] and [Parr00a]. In §6.3 (resp., §6.4) we deal with the isometry groups (orthogonal, symplectic or hermitian) over a commutative field (resp., with moreover a real valuation). More general cases (e.g., on skew fields) of this last situation are treated in [BruT87b]. We finish with the exotic example of the dense, affine building one gets by taking the asymptotic cone of a Riemannian symmetric space of non-compact type. We follow there Bruce Kleiner and Bernhard Leeb [KleL97].

Chapter 7 offers a personal selection of applications of Euclidean buildings. Other possibilities are explained, e.g., in [AbrB08], [Ji06], [RohS95], [Ron92] or [Tits75]. We essentially use the different kinds of retractions (centered at a chamber, a sector germ or a full chimney germ) and study the shape of the image (under such a retraction) of a line segment: we call it a Hecke path. This enables us to prove that a sector-friendly building is chimney-friendly: Theorem 7.1.18. We can also define various Hecke algebras and determine their structures. The last section is devoted to the metric properties of buildings that are connected with the discreteness properties of the latter (even if they are not discrete buildings).

This book is intended to be largely self-contained. The targeted audience includes anyone interested in these buildings, as a tool for studying some groups or as non-classical metric spaces with interesting properties, or … But some prerequisites are certainly useful, even if not compulsory. For example, in Chapter 1 we present without proofs the needed results about Coxeter groups and root systems. A previous elementary knowledge of these theories may be useful. It may also be interesting to look first at [Rou09a]: it is a good introduction to this book, with the same definitions. Another useful option is to read (perhaps quickly) some introductory article such as [Brow91], [Eve14], [Ron92, Part I], [Rou09b] or [Scha95]. Without such preparatory reading, the readers are advised not to read this book in linear order. For instance, they may look at the example of the general linear group in §6.1 or §6.2 simultaneously with the general theory of buildings in Chapter 2. Depending on their taste or for a first reading, they may also concentrate on the discrete buildings, skipping the difficult aspects of dense buildings.

A word about the internal references:

- Subsections and theorem-like structures that are directly subordinate to a *section* (denoted always by a two-level counter such as §1.4) are assigned a three-level counter, like §1.4.1 or Lemma 1.4.2.
- Theorem-like structures that are part of a *subsection* receive a four-level counter, like Lemma 1.4.1.1 (in §1.4.1).
- All other theorem-like structures remain without counter; when referenced, they carry the four-level number of their superordinate enunciation, like Nota Bene 1.2.11.2 (following Proposition 1.2.11.2).

- Figures and tables are given another three-level counter whose first two levels indicate the associated section and whose third level is a letter to distinguish the counter from those of the other structures, e.g., Figure 1.2.A (in §1.2).

In Appendix A.1 one will find the way a specific type of mathematical object will be denoted by a specific type of symbol; this should facilitate the reading.