Chapter 1

Introduction

A classical question in differential geometry is the following one, posed in [140,144]:

(KW-1) *Given a manifold M and a function $K\colon M \to \mathbb{R}$, find a Riemannian metric $\hat{g}$ on M such that the scalar curvature $R_{\hat{g}}$ of $\hat{g}$ coincides with K.*

In general, the answer to the above problem is negative. For example, in dimension $n = 2$, the scalar curvature is twice the Gaussian curvature. Therefore, if M is compact without boundary, by the Gauss–Bonnet theorem,

$$\int_M R_g \, dV_g = 4\pi\chi(M).$$

In particular, if there is a solution to the prescribed curvature problem, somewhere K has to possess the sign of $\chi(M)$.

Another version of the above problem is the restriction to a fixed conformal class:

(KW-2) *Given a Riemannian metric g on M, try to solve* (KW-1) *with a metric $\tilde{g}$ in the* conformal class *of g, i.e. $\tilde{g}$ is of the form $\tilde{g}(x) = \Lambda(x)g(x)$, where Λ is a smooth positive function on M.*

Main focus of these notes. We will assume that (M^n, g) is *closed*, meaning – unless specified – *compact without boundary and of dimension* $n \geq 3$. We mainly deal with the conformal version (KW-2) of the above problem. In fact, as we will see in Chapter 3, (KW-1) is solvable under rather mild requirements on the prescribed curvature function K.

There are both conceptual and technical differences for the case $n = 2$. We will make several comments here and there in this regard.

Remark 1.1. We point out the following facts:

(i) There exists a version of the problem between (KW-1) and (KW-2): this consists in prescribing the scalar curvature combining conformal changes and pull-backs of the metric via diffeomorphisms on M. In the terminology of Kazdan–Warner, such changes of metric are referred to as *conformal equivalences*, but *not* in our notation!

(ii) The following are interesting particular cases of (KW-2) (conformal case):

- Yamabe's problem [218]: on (M^n, g) closed with $n \geq 3$, find a conformal metric with constant scalar curvature. By [15, 193, 211] this is always solvable. For $n = 2$ this is the classical *uniformization problem*, initiated by Klein, Koebe, and Poincaré.

- Nirenberg's problem: on $(M^n, g) = (\mathbb{S}^n, g_{\mathbb{S}^n})$, prescribe conformally a non-constant function. This is a well-studied problem, originally posed for $n = 2$, and we will give an account in the notes of several results on this research line.

The books [16, 65, 97, 130, 131, 209] are particularly useful references for the purposes of these notes, and contain detailed presentations of some of the topics covered here.

1.1 Problem (KW-1)

Even though we will mainly focus on (KW-2), we will illustrate the solvability of the first version of the Kazdan–Warner problem since its proof is rather self-contained, apart from the fact that it requires the resolution of Yamabe's problem. This will not be discussed in detail here, even though we will present the main ingredients needed to prove it. We begin by introducing the following definition.

Definition 1.2. A closed manifold M of dimension $n \geq 3$ is said to be of *type I* if it admits a metric of non-negative and non-identically zero scalar curvature. It is said to be of *type II* if it admits a metric of non-negative scalar curvature, but any such metric has scalar curvature identically equal to zero. The manifold M is said to be of *type III* if the scalar curvature of any metric on M has to be negative somewhere.

An important fact to note is that if M^n is closed and of dimension $n \geq 3$, then by a result in [13] there always exists a metric $\bar{g}$ with $R_{\bar{g}} < 0$. Indeed, there even exists $\bar{g}$ with $\mathrm{Ric}_{\bar{g}} < 0$ [110, 160]. However, not all manifolds carry a metric with positive scalar curvature: understanding when it happens is a hard problem, with important contributions such as [116, 197, 206]; see e.g. the recent survey [50].

The flexibility in the choice of metric for problem (KW-1) yields the following general result, by which it is possible to prescribe a large class of functions.

Theorem A ([141–143]). *Suppose M^n is closed, with $n \geq 3$. If M is of type I, then (KW-1) is solvable for every $K \in C^\infty(M)$. If M is of type II, then (KW-1) is solvable for every $K \in C^\infty(M)$, which is either negative somewhere or identically zero. If M is of type III, then (KW-1) is solvable for every $K \in C^\infty(M)$ that is negative somewhere.*

Concerning the first statement of the latter theorem, Kazdan and Warner assumed the existence of metrics with positive and constant scalar curvature. Indeed, at the time this was not known to hold in general, but was later proved in [15, 193, 211] for every conformal class, and hence Theorem A can be stated without any extra assumptions.

Remark 1.3. We note the following:

(i) For $n = 2$ and M closed, (KW-1) is solvable provided somewhere K attains the sign of $\chi(M)$.

(ii) If M^n, $n \geq 3$ is non-compact and diffeomorphic to an open set U of a compact manifold $\bar{M}$, then (KW-1) is solvable for every $K \in C^\infty(M)$.

The proof of Theorem A relies on tools that are of functional-analytic nature, and exploits the invertibility of the operator that assigns to each conformal factor the scalar curvature of the metric it induces. The idea is then, starting from a metric with constant scalar curvature, to get any other function close to that constant in L^p to it by composing with a proper diffeomorphism on M. In positive curvature, this argument needs a special spectral property of the linearized equation for the scalar curvature, guaranteed by the above-mentioned results by Trudinger, Aubin, and Schoen.

1.2 Problem (KW-2)

Recall that for this version of the problem one tries to prescribe a given function on M as the scalar curvature via conformal deformations of a background metric g. As will be shown in Chapter 2, for $\tilde{g}(x) = u(x)^{\frac{4}{n-2}} g(x)$, $n \geq 3$, the scalar curvature R_g transforms as

$$-c_n \Delta_g u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}, \quad c_n = \frac{4(n-1)}{n-2}.$$

Therefore, (KW-2) becomes equivalent to finding a positive solution u to

$$-c_n \Delta_g u + R_g u = K u^{\frac{n+2}{n-2}} \quad \text{on } M. \tag{E_K}$$

The linear operator L_g on the left-hand side is the so-called *conformal Laplacian*, and it transforms naturally under conformal changes of metric; see Section 2.2. In two dimensions, the problem instead becomes

$$-\Delta_g w + \frac{1}{2} R_g = \frac{1}{2} R_{\tilde{g}} e^{2w}, \quad \tilde{g} = e^{2w} g.$$

Problem (KW-2) is more rigid than (KW-1), so there are more obstructions to existence, as shown by the following two examples.

Example 1. If R_g has a given sign for $n \geq 3$, then it is impossible to reverse it using conformal deformations; see Chapter 3. This aspect is somehow more similar to the two-dimensional case.

Example 2. There is a well-known obstruction due to Kazdan–Warner on the standard sphere. Suppose u solves (E_K) on $(\mathbb{S}^n, g_{\mathbb{S}^n}) \subseteq \mathbb{R}^{n+1}$, $n \geq 3$, and let $y^1, \ldots, y^{n+1}$ be the Euclidean coordinates restricted to $\mathbb{S}^n$. Then one has the identity (see Section A.2)

$$\int_{\mathbb{S}^n} \langle \nabla_{g_{\mathbb{S}^n}} K, \nabla_{g_{\mathbb{S}^n}} y^i \rangle u^{\frac{2n}{n-2}} \, dV_{g_{\mathbb{S}^n}} = 0 \quad \text{for every } i = 1, \ldots, n+1.$$

In dimension $n = 2$, the above identity becomes

$$\int_{\mathbb{S}^2} \langle \nabla_{\mathbb{S}^2} K, \nabla_{\mathbb{S}^2} y^i \rangle e^{2w} \, dV_{g_{\mathbb{S}^2}} = 0 \quad \text{for every } i = 1, 2, 3.$$

As a consequence, equation (E_K) is not solvable on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ if K is monotone with respect to any of the Euclidean variables, for example when it is affine. More obstructions to existence are given in [36, 166, 208]; see Chapter 9.

The above non-existence result on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is due to the presence of the *Möbius group*. This is a non-compact group of conformal maps on $\mathbb{S}^n$, which is obtained by composing the stereographic projection, Euclidean dilations, and then the inverse stereographic map. In this respect, the global maximum of K acts like an *attractor* for the conformal volume; see Chapter 4.

For reasons of brevity we will limit ourselves to the case of curvatures with constant sign: for the changing-sign case we refer the interested reader to the non-exhaustive list [16, 19, 35, 84, 100, 133, 162, 186, 189].

1.3 Variational analysis of (KW-2)

Each conformal class of metrics carries naturally scalar curvatures with a given sign, which is determined by two quantities introduced in Chapter 3. The first is the principal eigenvalue of the conformal Laplacian, $\lambda_1(L_g)$, whose sign is a conformal invariant. The second is the *Yamabe quotient*, defined on conformal factors as

$$Q_g(u) := \frac{\int_M (c_n |\nabla_g u|^2 + R_g u^2) \, dV_g}{\left(\int_M u^{2^*} \, dV_g\right)^{\frac{2}{2^*}}}, \quad 2^* := \frac{2n}{n-2}.$$

Its infimum, the *Yamabe constant*, is conformally invariant and denoted by $\mathcal{Y}(M, [g])$. It turns out that $\mathcal{Y}(M, [g])$ and $\lambda_1(L_g)$ have the same sign.

In negative curvature it is possible to prove existence of solutions for (KW-2) rather directly using a variational approach. In fact, consider the functional

$$I_K(u) := \frac{1}{2}\int_M (c_n|\nabla_g u|_g^2 + R_g u^2)\, dV_g - \frac{1}{2^*}\int_M K|u|^{2^*}\, dV_g,$$

defined on the space $H^1(M)$ of functions u that are of class $L^2(M)$, together with their gradients. If K is everywhere negative on M, then I_K admits a global minimizer, which is then a solution of (E_K).

For positive curvature the problem is more challenging, as the above Example 2 shows. The energy I_K is in this case unbounded from below, so it is convenient to introduce a modified functional in the form of a Sobolev-type quotient, namely

$$J_K(u) := \frac{\int_M (c_n|\nabla_g u|^2 + R_g u^2)\, dV_g}{\left(\int_M K|u|^{2^*}\, dV_g\right)^{\frac{2}{2^*}}}.$$

Notice that if $\lambda_1(L_g) > 0$, the quadratic form in the numerator is equivalent to the squared norm of $H^1(M)$. The advantage of this formulation is that J_K is bounded from below by the Sobolev embedding of $H^1(M)$ into $L^{2^*}(M)$, and its critical points give rise to solutions of (E_K), after a proper dilation by a positive constant.

However, the latter embedding being non-compact, minimizing sequences may not converge. This phenomenon can be clearly illustrated on the round sphere, especially in relation to the Kazdan–Warner obstruction.

When $K \equiv 1$, J_K on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ reduces to

$$J_{K\equiv 1}(u) := \frac{\int_{\mathbb{S}^n} (c_n|\nabla_g u|^2 + n(n-1)u^2)\, dV_{g_{\mathbb{S}^n}}}{\left(\int_{\mathbb{S}^n} |u|^{2^*}\, dV_{g_{\mathbb{S}^n}}\right)^{\frac{2}{2^*}}}. \tag{1.1}$$

It turns out that on the round sphere the infimum $\mathcal{Y}(\mathbb{S}^n, [g_{\mathbb{S}^n}])$ of the above quotient is equal to $n(n-1)|\mathbb{S}^n|^{\frac{2}{n}}$, and that minimizers can be explicitly classified due to some independent work by Aubin and Talenti; see Chapter 4. Minimizers turn out to be of the form

$$\varphi_{p,\lambda} = \left(\frac{2\lambda}{\lambda^2 + 1 - (\lambda^2 - 1)\cos d_{\mathbb{S}^n}(p,x)}\right)^{\frac{n-2}{2}}, \quad p \in \mathbb{S}^n,\ \lambda > 0. \tag{1.2}$$

Such functions, called *bubbles*, arise as conformal factors of Möbius maps from $\mathbb{S}^n$ to itself: for $\lambda = 1$, $\varphi_{p,\lambda}$ is identically equal to 1, while for λ large the conformal volume density $\varphi_{p,\lambda}^{2^*}$ associated to the corresponding Möbius map concentrates distributionally as a Dirac mass, precisely as $|\mathbb{S}^n|\delta_p$. It also turns out that both the numerator and the denominator in (1.1) are independent of p and λ. It is then possible to prove that J_K on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ has no minimizer unless K is identically constant; see Proposition 4.3.

Extending a result by Moser [178] for the two-dimensional sphere, Escobar and Schoen were able to prove an existence theorem for symmetric functions, stated here in a particular form.

Theorem B ([100]). *Let $n = 3$ and let $K\colon \mathbb{S}^3 \to \mathbb{R}$ be positive and antipodally symmetric. Then* (E_K) *is solvable.*

The strategy relies on working in the class of antipodally symmetric functions, where loss of compactness may only occur with the formation of multiple bubbles at couples of antipodal points. The minimal blow-up energy in a symmetric situation can be computed explicitly as $2^{1-\frac{2}{2^*}} \mathcal{Y}(\mathbb{S}^3, [g_{\mathbb{S}^3}])$; see Chapter 4. Escobar and Schoen proved that for λ large one has

$$J_K(\varphi_{p,\lambda} + \varphi_{-p,\lambda}) < 2^{1-\frac{2}{2^*}} \mathcal{Y}(\mathbb{S}^3, [g_{\mathbb{S}^3}]),$$

showing that minimizing sequences within the symmetric class must stay compact.

1.4 Sub-critical approximation and blow-up analysis

For a given function K not necessarily symmetric, in view of the above-mentioned non-existence of minimizers, one may wonder whether there could be critical points of saddle type for the functional J_K. Such critical points are usually found via min-max schemes or Morse-theoretical tools. When a lack of compactness occurs, as in the present case, these topological tools yield in general *Palais–Smale sequences*, namely sequences of functions along which J_K converges and the gradient of J_K tends to zero. Such sequences were analyzed for a slightly different setting in [207], where it was proved that they *split* into their weak limit and a given number of bubbles as in (1.2) with λ large.

Since there is no information in general about the location or the concentration rates of such bubbles, a useful tool for studying the problem without any symmetry requirement is the sub-critical approximation of (E_K), in the spirit of the well-known paper [192] by Sacks and Uhlenbeck.

For τ positive and small and L_g the conformal Laplacian, we introduce the problem

$$L_g u = K u^{\frac{n+2}{n-2}-\tau}. \qquad (\mathrm{E}_{K,\tau})$$

Its solutions are critical points of the modified Euler–Lagrange functional

$$J_{K,\tau}(u) := \frac{\int_M (c_n |\nabla_g u|^2 + R_g u^2)\, dV_g}{\left(\int_M K u^{2^*-\tau}\, dV_g\right)^{\frac{2}{2^*-\tau}}}.$$

This time, by the compactness of the Sobolev embedding $H^1(M) \to L^{2^*-\tau}(M)$, it is easy to minimize $J_{K,\tau}$, and to find (positive) solutions of ($E_{K,\tau}$). Furthermore, by a result in [112], for τ positive there is an upper bound (depending on τ) on all positive solutions of this equation.

On the other hand, as $\tau \to 0$, the latter might sometimes blow up. Indeed, this must necessarily happen in situations where the Kazdan–Warner obstruction holds, since there cannot be non-trivial limits in this case. The hope, however, is that some classes of solutions might anyway stay bounded for suitable K, in addition to those that are blowing up.

To understand the diverging behavior one can use the so-called *blow-up analysis* method, which consists in rescaling solutions using the dilation covariance (of either (E_K) or ($E_{K,\tau}$)) in order to obtain bounded positive functions satisfying

$$-c_n \Delta u = K(\bar{x}) u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n,$$

where $\bar{x}$ is a blow-up point. Such limiting profiles were classified in [47], and for $K(\bar{x}) = c_n n(n-2)$ are of type

$$U_{\lambda,x_0}(x) := \lambda^{\frac{n-2}{2}} U(\lambda(x - x_0)), \quad \lambda > 0,\ x_0 \in \mathbb{R}^n,$$

where

$$U(x) := \frac{1}{(1 + |x|^2)^{\frac{n-2}{2}}}.$$

If the value of $K(\bar{x})$ is different, it is sufficient to multiply the above expression by a suitable factor, due to the double homogeneity of the equation. Since in the *blow-down* procedure for solutions of ($E_{K,\tau}$) one can obtain entire solutions in $\mathbb{R}^n$ which are upper bounded by, say, the constant 1, it means that original solutions with large L^∞-norm have the shape of $U_{\lambda,0}(x)$ in suitable coordinates x for some large λ.

The above analysis helps in proving that blow-ups for equation ($E_{K,\tau}$) are sometimes *isolated simple*, meaning that near each blow-up point there is formation of at most one bubbling profile. This is a rather delicate property that depends on the dimension, on the underlying manifold, and on the structure of the function K, especially on its Taylor-type expansion near its critical points.

There are several results in this direction, some of which are recalled in Chapter 6. The properties that are most relevant for our purposes are the following ones. We consider a sequence $\tau_j \searrow 0$ and a sequence $(u_j)_j$ of positive solutions of (E_{K,τ_j}) (i.e. of ($E_{K,\tau}$) with $\tau = \tau_j$) with K satisfying

$$K > 0 \text{ is a Morse function such that } \Delta_g K(x) \neq 0 \text{ whenever } \nabla_g K(x) = 0. \quad \text{(ND)}$$

Then the following properties hold:

(i) If $n = 3$ and K satisfies (ND), there is at most one blow-up point, which is isolated simple. If (M^3, g) is not conformally equivalent to $(\mathbb{S}^3, g_{\mathbb{S}^3})$, then there cannot be blow-ups of solutions.

(ii) If $n = 4$ and K satisfies (ND), blow-ups are finitely many and isolated simple.

(iii) If $n \geq 5$ and K satisfies (ND), blow-ups are finitely many and isolated simple provided that the sequence $(u_j)_j$ is uniformly bounded in the norm $H^1(M)$ with zero weak limit.

In each of these three cases, blow-ups for $(u_j)_j$ may only occur at critical points of K with negative Laplacian. More comments for higher dimensions are given in Chapter 10.

The above results are complemented by Theorem 7.1, proven in Chapter 7, in which solutions with isolated simple blow-ups, possibly multiple in dimension $n \geq 4$, are constructed via a refined implicit function argument. One can also characterize the limit values of J_{K,τ_j} on such solutions for j tending to infinity, as well as their Morse index.

In dimension $n = 4$ there are indeed restrictions on the location of blow-up points (see Remark 7.2), while in dimension $n \geq 5$ blow-up may occur at any chosen set among critical points of K with negative Laplacian. If in dimension $n \geq 5$ one removes the above extra assumptions on the Sobolev norm and the weak limit of solutions, then the blow-up behavior might be more involved; see e.g. [62,63]. Exclusion of blow-ups on three-dimensional manifolds differing from the sphere is proved via the *positive mass theorem* by Schoen–Yau; see [198].

1.5 Some general existence results

The above characterizations of blowing-up solutions allow one to derive some general existence results, not requiring any symmetry on K. For example, on $\mathbb{S}^3$ the following theorem holds true, in the spirit of one by Chang and Yang [59] in two dimensions (see also [124]).

Theorem C ([24,57]). *Let $K\colon \mathbb{S}^n \to \mathbb{R}$ be a positive Morse function satisfying* (ND). *Let $(x_i)_i$ be the (finitely many) critical points of K, with Morse index k_i. Suppose that*

$$\sum_{x_i \text{ s.t. } \Delta K(x_i)<0} (-1)^{k_i} \neq (-1)^n. \tag{$*$}$$

If $n = 3$, then (E_K) *is solvable.*

The theorem is proved via a degree-counting formula. By the above results on blow-up analysis, a dichotomy for solutions of ($E_{K,\tau}$) can be shown, namely that they either stay uniformly bounded as $\tau \to 0$ or they develop simple blow-ups at a single point. For τ small, however, it can be shown via a homotopy argument (deforming the function K to 1 in an affine way) and a rigidity result from [34, 112] that the total Leray–Schauder degree of solutions is equal to 1. Subtracting by excision the contribution to the degree from blowing-up solutions, one obtains from the above condition ($*$) that the uniformly bounded solutions of ($E_{K,\tau}$) contribute non-trivially to the degree formula, proving that they cannot be an empty set.

Remark 1.4. For $n = 4$, a more complicated formula than ($*$) appears due to the presence of multiple blow-ups: more details are given in Remark 6.16. An existence result on $\mathbb{S}^n$ under condition ($*$) in any dimension was proved in [5, 61] provided K satisfies a *pinching condition*, that is,

$$\frac{\sup_{\mathbb{S}^n} K}{\inf_{\mathbb{S}^n} K} < p_n,$$

for some constant $p_n > 1$ close enough to 1. In this case we can use a finite-dimensional reduction of the problem related to one presented in Chapter 7, but using degree-theoretical arguments for the reduced problem.

Another recent result requiring pinching conditions is the following one.

Theorem D ([166]). *Suppose $n \geq 5$ and that (M^n, g) is a closed Einstein manifold of positive scalar curvature. Assume $K: M \to \mathbb{R}$ is Morse, satisfies (ND), and that one of the following two conditions is fulfilled:*

(i) $\frac{\sup_M K}{\inf_M K} < 2^{\frac{1}{n-2}}$ *and* ($*$) *holds;*

(ii) $\frac{\sup_M K}{\inf_M K} < (\frac{3}{2})^{\frac{1}{n-2}}$ *and K has at least two critical points with negative Laplacian.*

Then (E_K) *is solvable.*

Remark 1.5. Some comments are in order:

(i) The second pinching condition in Theorem D is clearly stronger than the first one, but in order for ($*$) to hold one needs at least two critical points with negative Laplacian. Notice that, even under the strongest pinching conditions, the inequality in ($*$) is sufficient but not necessary for existence; see also [174]. Under hypothesis (i) the theorem is proven in [61] for $K = 1 + \varepsilon \widetilde{K}$ and in [70] for $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

(ii) The result is false in dimensions $n = 3, 4$.

(iii) As we will see in Chapter 9, assuming that only one critical point of K has negative Laplacian is not sufficient in general to guarantee existence of solutions, even if one imposes stronger pinching conditions.

The latter result extends a theorem from [70] which is valid for the sphere under assumption (i). Recall that Einstein means that $\mathrm{Ric}_g = \Lambda g$ for some $\Lambda \in \mathbb{R}$. Also in this case, the rigidity results from [112] and [34] apply, giving that the total degree of the solutions of ($\mathrm{E}_{K,\tau}$) is equal to 1.

Differently from the three-dimensional case though, blow-ups may now occur at multiple critical points of K. However, the blow-up analysis presented before still allows one to compute their contribution to the degree *within some given levels* of $J_{K,\tau}$ that include single- and double-bubbling solutions. We can then conclude the proof by applying the classical *Poincaré–Hopf theorem*.

In Chapter 9 we give examples of arbitrarily pinched curvature functions with only one critical point of negative Laplacian and arbitrary Morse structure such that (E_K) is not solvable, as well as non-existence examples in dimensions $n = 3$ or 4 for functions with two critical points of negative Laplacian. These show the sharpness of the assumptions in Theorem D.

We conclude the notes with some perspectives and a set of open problems. These concern extensions of the above-mentioned blow-up analysis, the possible use of Morse homology, higher-order and fully non-linear versions of the Kazdan–Warner problem, complete manifolds, manifolds with boundary, and the role of conformal geometry in the study of Einstein's constraint equations.