To Elisa and Bianca

Preface

These notes originate from a Nachdiplomvorlesung held by the author at ETH during the spring term of 2021. The main focus of the course was the classical problem initiated by Kazdan and Warner in the 1970s, about prescribing a given function on a closed manifold M as the scalar curvature of some Riemannian metric. An interesting case of the problem is when one looks for such a metric within a given conformal class, which makes the problem equivalent to solving an elliptic PDE involving the critical Sobolev exponent. This feature results in a loss of compactness, which is typical of scaling-invariant problems. The issue can be tackled by combining several techniques, like exploiting the variational structure of the problem, the use of blow-up analysis, and asymptotic expansions, as well as employing topological tools such as min-max, Morse, or Leray–Schauder degree theory. The aim of the course was to give a general overview of the problem, to present some key results and techniques, and to cover some of the progress on this topic, ranging from the original contributions by Kazdan–Warner, to the most recent results on the subject.

The notes are structured as follows. In the introduction we present the two main versions of the Kazdan–Warner problem, illustrate the variational strategy and the blow-up analysis technique, and state the main results we will focus on. Chapter 2 collects some preliminary material: basic notions in differential geometry, useful functional spaces on compact manifolds, and some elliptic regularity theory, both at linear and non-linear levels. In Chapter 3 we will present the covariance property and some spectral features of the *conformal Laplacian* and the *Yamabe quotient*, which characterize conformal classes of metrics. An exact expression for the extremal value of the quotient will be given for the round sphere, and the resolution of the Yamabe conjecture will be discussed. We will then show the solvability of the Kazdan–Warner problem in its most general form, without any restriction to a conformal class.

We will then turn to the conformally constrained Kazdan–Warner problem in dimension $n \geq 3$, which will indeed be our main focus. As we remarked above, this version of the problem will amount to solving an elliptic equation with a non-linearity of *critical type* on the underlying manifold. In Chapter 4 we will introduce the variational setting for treating the conformal Kazdan–Warner problem. This will lead to full solvability in the case of non-positive Yamabe class by a global minimization procedure. We will show that this approach is not possible in positive Yamabe class, but that in some symmetric situations this property might be recovered, illustrating a result by Escobar and Schoen.

In Chapter 5 a classification theorem for the Yamabe equation on $\mathbb{R}^n$ will be proved, using a variant of the classical moving plane method. We will then turn to sub-critical equations, for which a Liouville-type theorem will also be shown, useful to prove a priori estimates. These results will be crucial to study the phenomenon of loss of compactness, typical of geometric non-linear equations. In Chapter 6 we will perform a blow-up analysis on solutions of sub-critical equations with large L^∞-norm on compact manifolds: it will be shown that in dimensions 3 and 4, blow-ups are *isolated simple*, and that this condition persists in higher dimensions for sequences of solutions with zero weak limit and uniformly bounded Sobolev norm. In Chapter 7 a converse result will be proven, and solutions developing isolated simple blow-ups for sub-critical approximations of the prescribed curvature problem will be constructed using a finite-dimensional reduction. These techniques can also be used in general dimension to perform blow-up analysis of particular classes of solutions.

In the eighth chapter we then present an application of the previous two, combined with some topological tools. In low dimension we present a general existence theory for the scalar curvature prescription problem, and in arbitrary dimension also under suitable pinching conditions on the curvature function. Chapter 9 is then devoted to some results about non-existence, showing the sharpness of the assumptions in the theorems presented in Chapter 8. Such results are built over a classical obstruction due to Kazdan and Warner for the sphere, which is extended with the aid of blow-up analysis. Finally, in Chapter 10 we present some related research directions and open questions.

Many contributions from the literature on this topic, which is vast, cannot unfortunately be covered in these notes. We hope anyway to give an opportunity for the interested reader to get a general view of the subject and to create occasions for further exploration. Some of the most technical parts of some proofs are omitted for brevity, but a precise account of the sources with full details will be given.

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