Chapter 1

General overview

1.1 Historical context

At the dawn of the 20th century, the debate was still raging on the existence of atoms, and Einstein's PhD thesis "A New Determination of Molecular Dimensions" [22] aimed to support the atomic theory. This was the second of his five celebrated 1905 contributions and still constitutes his most cited work. The main part was devoted to the hydrodynamic derivation of a formula for the effective viscosity of a fluid with a dilute suspension of rigid particles: the so-called Einstein formula in fluid mechanics, which is the focus of the present memoir. In the same work, Einstein also derived a relation between the diffusion constant for suspended particles and their mobility: the so-called Einstein relation in kinetic theory. He then applied these two relations to sugar dissolved in water: using available empirical data, he deduced an estimate of the Avogadro number and of the size of sugar molecules (after eliminating a calculation error [23]). We refer to [61] for an inspiring account of this seminal work. As discussed by Perrin in his extensive report [58] at the first Solvay conference in 1911 in Brussels, these discoveries were confirmed by further experiments and shown to agree with other methods to determine the Avogadro number, which sealed the triumph of the atomic theory.

We briefly describe Einstein's argument to estimate the effective viscosity of a dilute suspension. Viscosity of a fluid is usually measured by shear-flow experiments: a cylindrical vessel is filled with the fluid, a rotating spindle is immersed in it, and one measures the torque needed to make it rotate at constant angular speed. Assume now that the fluid contains a suspension of small rigid spherical particles and consider their influence on the measured viscosity. As particles are rigid, they act as obstacles and hinder the fluid flow, thus effectively increasing the measured viscosity. A first challenging question concerns the dynamics of the particles: do they reach a statistical steady state? If this is the case and if one indeed measures a constant-in-time effective viscosity, then the latter depends on the steady state, hence possibly on the speed of the spindle itself, which corresponds to possible non-Newtonian behaviors [31, Section 7]. Einstein's main idea in [22] was that, in the low-density regime, for spherical particles, the first-order effective change in viscosity should only depend on the volume fraction of the particles and not on their distribution. In particular, this universality would relegate non-Newtonian effects to higher-order corrections. More precisely, in 3D, given a fluid with isotropic viscosity Id and given suspended spherical particles with small volume fraction $\varphi \ll 1$, Einstein's formula for the effective

viscosity takes the form

$$\bar{\mathbf{B}} = \mathrm{Id}\left(1 + \frac{5}{2}\varphi + o(\varphi)\right). \tag{1.1}$$

Heuristically, the argument is as follows: at low density, particles are scarce and typically well separated, hence their interactions are negligible to leading order. The first-order effect on the viscosity should thus be proportional to the volume fraction and correspond to the energy dissipation of a single isolated particle in the fluid. The latter can be computed explicitly for spherical particles and leads to the celebrated $\frac{5}{2}$ factor in (1.1); we refer to Section 2.6 below, where this classical calculation is reproduced.

This type of low-density expansion was not new in the physics community at the time: it was very much in line with other work on the micromechanics of heterogeneous media in the late 19th century. Einstein's formula is indeed comparable to the Clausius–Mossotti formula for the effective dielectric constant [10, 51, 52], to Maxwell's formula for the effective conductivity in electrostatics [49], or to the Lorentz–Lorenz formula for the effective refractive index in optics [46, 47]; we refer to [48] for an account of the historical context.

Einstein's formula triggered a lot of long-lasting activity in fluid mechanics: the large-scale rheology of suspensions was soon considered as a topic in its own right [24, 41, 42]. Various works have aimed at understanding to what extent Einstein's formula is robust and accurate. Robustness has been addressed in particular by establishing corresponding formulas for particles of different shapes, as e.g. the explicit formulas by Jeffery [40] for suspensions of ellipsoids (see also [35, 44]). Accuracy is a more subtle issue and essentially amounts to capturing the next-order term in the low-density expansion. While particle interactions are neglected at first order, the next-order correction consists of including the effects of pairwise interactions. Due to their long-range nature, the sum of pairwise contributions is not summable and some renormalization is therefore needed. This was first achieved by Batchelor and Green [7], and we refer to [1, 34, 56] for other formal renormalization ideas. A related, yet different, topic concerns the sedimentation of suspended particles under gravity and the computation of their effective settling speed, which happens to require a similar renormalization: the above-mentioned contribution by Batchelor and Green [7] was indeed inspired by Batchelor's work [6] on sedimentation. Interestingly, the renormalization of higher-order corrections to the effective viscosity had remained an open problem in the physics community.

We also refer to [2, 55, 63] for the asymptotic analysis of the effective viscosity for dilute *periodic* arrays of suspended particles and, in a more mathematical spirit, we mention the pioneering work by Sánchez-Palencia et al. [45, 60] using formal two-scale expansions for locally periodic suspensions.

1.2 Mathematical reformulation and objectives

As described above, Einstein's formal derivation of (1.1) in [22] relies on the following two implicit hypotheses:

(E1) *Scale separation.* There is a scale separation between the "microscopic" particle size and the "macroscopic" observation scale. Therefore, the suspension behaves on the observation scale like an "effective" fluid with some effective viscosity tensor $\overline{\mathbf{B}}$ that can then be measured by shear-flow experiments.

(E2) *Particle interactions are negligible.* In the low-density regime, particles are typically well separated and therefore, to leading order, they do not interact and can be treated as being isolated.

We briefly discuss the validity of these two working hypotheses and then turn to describing the literature and our objectives in the present memoir.

1.2.1 Einstein's hypothesis (E1): Scale separation

This first hypothesis concerns the definition of a notion of effective viscosity for suspensions when the particle size $O(\varepsilon)$ is much smaller than the observation scale $O(1)$. Consider a shear-flow experiment to measure the viscosity, say using a rotational viscometer. Let D denote the fluid domain in this device and let $\{x_{\varepsilon,n}^t\}_n \subset D$ stand for positions of suspended particles at time t, which evolve over time in the fluid flow. If inertia is neglected, the dynamics is greatly simplified: given particle positions at a given time, the fluid velocity satisfies steady Stokes equations, which determine instantaneous particle velocities. In this context, the emergence of an effective viscosity can be split into two parts:

— *Steady-state microstructure.* As the measured effective viscosity is expected not to depend on time, it implicitly requires particle positions to reach a statistical steady state in the long run. Focussing on a portion of the fluid in the bulk, we may consider without much loss of generality that the statistical ensemble is stationary (henceforth, "stationarity" stands for statistical spatial homogeneity). In other words, the point set $\{x_{\varepsilon,n}^t\}_n$ can be approximately replaced by a random point set $\{\varepsilon x_n : \varepsilon x_n \in D\}$ that is the ε-rescaling of some stationary random point process $\mathcal{P} = \{x_n\}_n$. The law of this steady state may depend itself on the prescribed shear flow in the viscometer, which leads to possible non-Newtonian effects [31, Section 7].

— *Steady homogenization problem.* Given a statistical ensemble of particle positions, under an ergodicity assumption, the steady Stokes equations for the fluid velocity are expected to homogenize on the macroscopic observation scale, in the sense that it can be replaced by effective steady Stokes equations with some effective viscosity tensor $\overline{\mathbf{B}}$, which amounts to an averaged effect of suspended particles.

While the rigorous analysis of the steady-state flow-induced microstructure remains a fully open problem at this time, the steady homogenization problem, in contrast, has been extensively studied under various assumptions in our recent series of articles [13, 18, 19, 21] and is by now very well understood. Given a statistical ensemble of particle positions, this provides a rigorous definition of the effective viscosity together with a homogenization result. More precisely, considering the system at the particle scale, we denote by $\mathcal{I} = \bigcup_n I_n$ the random ensemble of particles (not necessarily spherical), centered at the points of a point process $\mathcal{P} = \{x_n\}_n$, say in the d-dimensional Euclidean space $\mathbb{R}^d$ for generality. The effective viscosity tensor $\bar{\mathbf{B}}$ is defined as a quadratic form on the set $\mathbb{M}^{\mathrm{sym}}_0 \subset \mathbb{R}^{d\times d}$ of trace-free symmetric matrices,

$$E : \bar{\mathbf{B}}E := \mathbb{E}\big[\big|\mathrm{D}(\psi_E) + E\big|^2\big] = |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_E)\big|^2\big], \quad E \in \mathbb{M}^{\mathrm{sym}}_0, \tag{1.2}$$

where $\mathrm{D}(\psi_E)$ is the unique stationary symmetric gradient solution, with bounded second moment and vanishing expectation, of the corrector problem

$$\begin{cases} -\triangle\psi_E + \nabla\Sigma_E = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{div}(\psi_E) = 0, & \text{in } \mathbb{R}^d \setminus \mathcal{I}, \\ \mathrm{D}(\psi_E + Ex) = 0, & \text{in } \mathcal{I}, \\ \int_{\partial I_n} \sigma_E \nu = 0, & \forall n, \\ \int_{\partial I_n} \Theta(x - x_n)\cdot\sigma_E\nu = 0, & \forall n,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{1.3}$$

in terms of the associated Cauchy stress tensor

$$\sigma_E := \sigma(\psi_E + Ex, \Sigma_E) := 2\mathrm{D}(\psi_E + Ex) - \Sigma_E\,\mathrm{Id}, \tag{1.4}$$

where $\mathbb{M}^{\mathrm{skew}} \subset \mathbb{R}^{d\times d}$ is the set of skew-symmetric matrices. Throughout this work, we assume for simplicity that the plain fluid has isotropic viscosity Id. The so-called corrector ψ_E can be viewed as the correction of the linear straining flow

$$x \mapsto Ex$$

in presence of rigid suspended particles $\{I_n\}_n$. The last two boundary conditions in (1.3) correspond to the balance of forces and torques on each particle. Note that, if $\mathcal{I}$ contains an unbounded chain of touching particles, then the rigidity constraint

$$\mathrm{D}(\psi_E + Ex)|_{\mathcal{I}} = 0$$

entails that the field ψ_E would grow linearly along this chain, which would prevent $\mathrm{D}(\psi_E)$ from having vanishing expectation: it shows that this corrector problem can only be well posed provided that some suitable non-clustering assumption is made. Different sets of sufficient assumptions are recalled in Section 2.1 below and we refer to our previous work [13, 18, 19, 21] for a detailed account.

1.2.2 Einstein's hypothesis (E2): Interactions are negligible

As it appears from (1.3), the corrector ψ_E depends nonlocally and nonlinearly on the set $\mathcal{I}$ of particles via boundary conditions: this corresponds to the multibody nature of hydrodynamic interactions. Einstein's second hypothesis can be reinterpreted as claiming that ψ_E can be approximated around each inclusion I_n by the unique decaying solution $\psi_E^{\{n\}}$ of the single-particle corrector problem

$$\begin{cases} -\triangle \psi_E^{\{n\}} + \nabla \Sigma_E^{\{n\}} = 0, & \text{in } \mathbb{R}^d \setminus I_n, \\ \operatorname{div}\big(\psi_E^{\{n\}}\big) = 0, & \text{in } \mathbb{R}^d \setminus I_n, \\ \mathrm{D}\big(\psi_E^{\{n\}} + Ex\big) = 0, & \text{in } I_n, \\ \int_{\partial I_n} \sigma\big(\psi_E^{\{n\}} + Ex, \Sigma_E^{\{n\}}\big)\nu = 0, & \\ \int_{\partial I_n} \Theta(x - x_n) \cdot \sigma\big(\psi_E^{\{n\}} + Ex, \Sigma_E^{\{n\}}\big)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{1.5}$$

This amounts to neglecting the effect of other particles on ψ_E around I_n, thus precisely neglecting the multibody nature of the problem. To give a more precise statement, consider the Voronoi tessellation $\{V_n\}_n$ associated with the set of particles $\{I_n\}_n$, that is,

$$V_n := \big\{x : \operatorname{dist}(x, I_n) < \inf_{m:m\neq n} \operatorname{dist}(x, I_m)\big\}.$$

The relevant approximation of ψ_E then takes the form

$$\mathrm{D}(\psi_E) \approx \Psi_E^{\mathrm{Einstein}} := \sum_n \mathrm{D}\big(\psi_E^{\{n\}}\big)\mathbb{1}_{V_n}. \tag{1.6}$$

Inserting this into the definition (1.2) of the effective viscosity yields, after straightforward calculations,

$$\begin{aligned} E : \bar{\mathbf{B}}E &= |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_E)\big|^2\big] \\ &\approx |E|^2 + \mathbb{E}\big[\big|\Psi_E^{\mathrm{Einstein}}\big|^2\big] \\ &= |E|^2 + \sum_n \mathbb{E}\left[\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n} \big|\mathrm{D}\big(\psi_E^{\{n\}}\big)\big|^2\right], \end{aligned}$$

and thus, replacing single-particle energies $\int_{V_n} |\mathrm{D}(\psi_E^{\{n\}})|^2$ in Voronoi cells by corresponding whole-space energies,

$$E : \bar{\mathbf{B}}E \approx |E|^2 + \sum_n \mathbb{E}\left[\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{\mathbb{R}^d} \big|\mathrm{D}\big(\psi_E^{\{n\}}\big)\big|^2\right]. \tag{1.7}$$

In case of spherical particles, single-particle problems can be solved explicitly and we get

$$E : \bar{\mathbf{B}}E = |E|^2\left(1 + \frac{d+2}{2}\varphi + o(\varphi)\right), \tag{1.8}$$

in terms of the particle volume fraction

$$\varphi := \varphi(\mathcal{I}) := \lim_{R\uparrow\infty} R^{-d}|\mathcal{I} \cap RQ|, \tag{1.9}$$

where $Q := [-\frac{1}{2}, \frac{1}{2}]^d$ stands for the unit cube. We refer to Section 2.6 for the detailed computation, and we note that in 3D we recover Einstein's celebrated $\frac{5}{2}$ factor, cf. (1.1). Corrections to Einstein's formula are naturally obtained by further taking into account particle interactions. As we shall see, in the low-density regime, this is naturally captured in form of a cluster expansion: the next-order correction, known in the physics literature as the Batchelor–Green correction [7], involves the sum of two-particle contributions, and so on.

1.2.3 Objectives

In this memoir, we focus on the rigorous analysis of Einstein's hypothesis (E2): we start from the relevant notion of effective viscosity (1.2) as defined by homogenization theory and we study its asymptotic behavior at low density, aiming to justify Einstein's formula (1.8) and to describe all higher-order corrections.

The early works [32, 45, 60] focussed on Einstein's formula for locally periodic dilute arrays of particles. It was extended in [33, 54] to the dilute disordered setting under the simplifying assumption that the minimal interparticle distance is large enough (that is, $\ell(\mathcal{P}) \gg 1$ with the notation (1.13) below). The next-order Batchelor–Green correction was captured in [27, 29] in the same setting. The uniform separation assumption is particularly convenient as it allows to exploit the reflection method and rigorously neglect many-particle interactions, e.g. [36–39, 54], but it is physically quite restrictive and unsatisfactory. More recently, it was replaced in [28] by some weaker non-concentration condition in the proof of Einstein's formula, however still requiring some control on the minimal interparticle distance. In this context, we shall address the following two main points:

— We shall justify Einstein's formula under the weakest assumptions under which homogenization is known to hold, in particular covering the case of the general subcritical percolation condition in [21]. At the same time, we aim at optimal error estimates: the error $o(\varphi)$ in (1.1) was often claimed to be $O(\varphi^2)$ in the physics literature, but we shall see that it actually strongly depends on the structure of the random ensemble of particles.

— We shall describe higher-order corrections to Einstein's formula in form of a cluster expansion. Due to the long-range nature of hydrodynamic interactions, renormalizations are needed to make sense of cluster contributions. In the physics literature, formal renormalizations were actually still missing beyond the second-order Batchelor–Green correction. On the rigorous side, even the justification of the latter was restricted to some specific regimes [26, 27, 29].

In terms of techniques, previous results on the topic mostly relied on a deterministic analysis, more precisely on various forms of the reflection method. In the present memoir, we rather take inspiration from our work [15] on the Clausius–Mossotti conductivity formula based on the triad consisting of: (1) finite-volume approximation; (2) cluster expansion; (3) uniform $\ell^1 - \ell^2$ energy estimates. Substantially refining on this analysis, we go far beyond [15] (and beyond [26], which also builds on the cluster expansion of [15]) as we manage to cover general dilute regimes (beyond the case of explicit dilution by random deletion in [15]). We further describe explicitly for the first time the renormalization of cluster coefficients to all orders.

1.3 Cluster expansion formalism

While Einstein's formula (1.8) is obtained by considering dilute particles as being isolated, next-order corrections amount to taking into account many-particle interactions and the multibody structure of the corrector field ψ_E. At low density, particles are scarce, hence have weak interactions, and one might want to consider contributions of finite subsets of particles only. As in [15], taking inspiration from statistical mechanics, see e.g. [62, Chapter 19], this is naturally expressed by means of cluster expansions, which provide natural asymptotic series at low density. We recall the formalism, discuss the accuracy of cluster expansions, and describe the key difficulty to apply it to the effective viscosity problem: the long-range nature of hydrodynamic interactions.

1.3.1 Cluster expansions of multibody quantities

We recall the cluster expansion formalism in the compact form that we introduced in [15]. As particles are indexed by natural numbers, we denote by $P(\mathbb{N})$ the set of subsets of the index set $\mathbb{N}$ and we consider the space $M(\mathbb{N})$ of set functions from $P(\mathbb{N})$ to a given vector space V. Starting from the corrector problem (1.3), for any index subset $H \in P(\mathbb{N})$, we may consider[1] the associated corrector ψ_E^H obtained by replacing the full set $\mathcal{I}$ of particles by its corresponding subset $\mathcal{I}^H := \bigcup_{n\in H} I_n$ in the corrector equation (1.3). The map

$$\psi_E^\# : H \mapsto \psi_E^H$$

is then viewed as an element of $M(\mathbb{N})$, where $\psi_E^\varnothing \equiv 0$ and where $\psi_E^{\mathbb{N}} \equiv \psi_E$ is the original corrector defined in (1.3).

[1] The corrector problem (1.3) is, in fact, not well posed in general for a given deterministic infinite subset H of particles. In the sequel, we shall rather consider finite-volume approximations of the corrector problem, for which well-posedness is always trivial. We skip this detail at the level of the present discussion.

In this setting, for all $n \in \mathbb{N}$, we introduce a *difference operator* $\delta^{\{n\}} : M(\mathbb{N}) \to M(\mathbb{N})$, defined for all $\Phi \in M(\mathbb{N})$ by

$$\delta^{\{n\}}\Phi^H := \delta^{\{n\}}\Phi^{H\cup\#} := \Phi^{H\cup\{n\}} - \Phi^H, \quad H \subset \mathbb{N},$$

which provides a natural measure of the sensitivity of Φ with respect to the index n (it plays the role of a discrete derivative). Note that for all $n \neq m$,

$$(\delta^{\{n\}})^2 = -\delta^{\{n\}}, \quad \delta^{\{n\}}\delta^{\{m\}} = \delta^{\{m\}}\delta^{\{n\}}.$$

For any finite $F \subset \mathbb{N}$, we also define the higher-order difference operator

$$\delta^F := \prod_{n\in F} \delta^{\{n\}},$$

which acts as follows: for all $\Phi \in M(\mathbb{N})$,

$$\delta^F\Phi^H = \sum_{G\subset F} (-1)^{|F\setminus G|}\Phi^{G\cup H}, \quad H \subset \mathbb{N}. \tag{1.10}$$

We take the natural convention $\delta^\varnothing\Phi^H := \Phi^H$. These difference operators are the building blocks to construct the so-called *cluster expansions*, e.g. [62, Chapter 19]: to order k, the cluster expansion of $\Phi \in M(\mathbb{N})$ takes the form

$$\Phi^{\mathbb{N}} \sim \Phi^\varnothing + \sum_n \delta^{\{n\}}\Phi^\varnothing + \frac{1}{2!}\sum_{n_1,n_2}^{\neq} \delta^{\{n_1,n_2\}}\Phi^\varnothing + \cdots + \frac{1}{k!}\sum_{n_1,\ldots,n_k}^{\neq} \delta^{\{n_1,\ldots,n_k\}}\Phi^\varnothing,$$

where we use the shorthand notation $\sum^{\neq}_{n_1,\ldots,n_j}$ for sums over j-tuples $(n_1,\ldots,n_j)$ of distinct indices. This can be rewritten in the more compact form

$$\Phi^{\mathbb{N}} \sim \sum_{j=0}^{k}\sum_{\sharp F=j} \delta^F\Phi^\varnothing, \tag{1.11}$$

where $\sum_{\sharp F=j}$ stands for the sum over all sets F of j distinct indices. This expansion is particularly relevant in the low-density regime when particles are very scarce: the 0th-order term corresponds to the situation without any particle, the 1st-order term corresponds to contributions of isolated particles, the 2nd-order term to contributions of pairs of particles, etc. Formally, it can be viewed as a Taylor expansion at $\varnothing$ with respect to the difference operator δ, where under suitable assumptions higher-order terms will be shown to be indeed of higher order at low density. Note that, if $\Phi \in M(\mathbb{N})$ only depends on indices in a finite subset $K \subset \mathbb{N}$ in the sense that $\Phi^H = \Phi^{H\cap K}$ for all $H \subset \mathbb{N}$, then the expansion (1.11) is always a finite sum and is actually *equal* to $\Phi^{\mathbb{N}}$ provided that $k \geq \sharp K$.

1.3.2 Multi-point intensities

The general estimation of the terms in the cluster expansion (1.11) naturally leads to the notion of multi-point intensities, which appear as refined measures of diluteness and seem new to the literature. Given an ergodic stationary point process $\mathcal{P} = \{x_n\}_n$, we start by recalling the standard notion of *intensity* of the point process (or *one-point intensity* in our terminology below),

$$\lambda(\mathcal{P}) := \lambda_1(\mathcal{P}) := \mathbb{E}\big[\sharp(\mathcal{P} \cap Q)\big].$$

By the ergodic theorem, we have almost surely

$$\lambda(\mathcal{P}) = \lim_{R\uparrow\infty} R^{-d}\sharp\{n : x_n \in RQ\}. \tag{1.12}$$

In particular, provided that random shapes satisfy $|I_n^\circ| \simeq 1$ almost surely for all n, this relates to the particle volume fraction (1.9) via

$$\varphi(\mathcal{I}) \simeq \lambda(\mathcal{P}),$$

so that the low-density regime $\varphi(\mathcal{I}) \ll 1$ is equivalently characterized by the condition $\lambda(\mathcal{P}) \ll 1$. Yet, as we consider nonlinear functions of the point process (like the effective viscosity $\overline{\mathbf{B}}$), this linear notion of diluteness is not strong enough and we need to introduce refined notions of "multi-point intensities".

For that purpose, we start by introducing a notation for the *minimal distance* of the point process $\mathcal{P}$,

$$\ell := \ell(\mathcal{P}) := \inf_{n\neq m} |x_n - x_m|_\infty, \tag{1.13}$$

which is almost surely a deterministic characteristic length of $\mathcal{P}$. The point process is called hardcore if $\ell(\mathcal{P}) > 0$, which is the case of all the processes considered in this memoir, cf. (H_ρ) below. For all $j \geq 1$, provided $\ell = \ell(\mathcal{P}) > 0$, we then define the *$j$-point intensity*

$$\lambda_j(\mathcal{P}) := \sup_{z_1,\ldots,z_j} \mathbb{E}\left[\sum_{n_1,\ldots,n_j}^{\neq} \ell^{-d}\mathbb{1}_{Q_\ell(z_1)}(x_{n_1})\cdots\ell^{-d}\mathbb{1}_{Q_\ell(z_j)}(x_{n_j})\right], \tag{1.14}$$

where $Q_r(z) := z + rQ$ stands for the cube of sidelength r centered at z. Note that, by definition (1.13), each cube $Q_\ell(z)$ contains at most one point of $\mathcal{P}$. This definition corresponds to the maximum expected number of j-tuples of points of $\mathcal{P}$ that lie in the ℓ-neighborhood of an element of $(\mathbb{R}^d)^j$, properly normalized by ℓ. Alternatively, recalling that the *j-point density* f_j associated with $\mathcal{P}$ is the nonnegative function defined by the following relation,

$$\mathbb{E}\left[\sum_{n_1,\ldots,n_j}^{\neq} \zeta(x_{n_1},\ldots,x_{n_j})\right] = \int_{(\mathbb{R}^d)^j} \zeta f_j \quad \text{for all } \zeta \in C_c^\infty\big((\mathbb{R}^d)^j\big), \tag{1.15}$$

the definition (1.14) of j-point intensity can be reformulated as

$$\lambda_j(\mathcal{P}) = \sup_{z_1,\dots,z_j} \fint_{Q_\ell(z_1)\times\cdots\times Q_\ell(z_j)} f_j. \tag{1.16}$$

In the case $\ell(\mathcal{P}) = 0$, this definition is naturally extended to $\lambda_j(\mathcal{P}) = \|f_j\|_{\mathrm{L}^\infty((\mathbb{R}^d)^j)}$ for completeness. In view of upcoming arguments, it is convenient to further introduce the following quantities,

$$\underline{\lambda}_j(\mathcal{P}) := \min_{\sum_i j_i = j} \prod_i \lambda_{j_i}(\mathcal{P}) \le \overline{\lambda}_j(\mathcal{P}) := \max_{\sum_i j_i = j} \prod_i \lambda_{j_i}(\mathcal{P}). \tag{1.17}$$

For a Poisson point process, these quantities are, in fact, all equivalent since independence yields $\lambda_j(\mathcal{P}) = \lambda(\mathcal{P})^j$ for all $j \ge 1$, hence $\underline{\lambda}_j(\mathcal{P}) = \overline{\lambda}_j(\mathcal{P}) = \lambda(\mathcal{P})^j$. For a hardcore Poisson point process, we similarly find $\lambda_j(\mathcal{P}) \simeq_j \lambda(\mathcal{P})^j$. In other words, the one-point intensity $\lambda(\mathcal{P})$ is enough to fully describe low-density regimes in those cases. However, multi-point intensities are nontrivial in general: for any $\beta \in [0,1]$, one can construct examples of point processes with $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^{1+\beta}$ (see last paragraph of Section 5.1). For instance, given $e \in \mathbb{R}^d$, the point process

$$\mathcal{P}_e := \mathcal{P} \cup (\mathcal{P} + e)$$

consists of pairs of points $\{x_n, x_n + e\}$ and thus satisfies $\lambda_2(\mathcal{P}_e) \simeq \ell(\mathcal{P}_e)^{-d}\lambda(\mathcal{P}_e)$, hence $\lambda_2(\mathcal{P}_e) \simeq \lambda(\mathcal{P}_e)$ provided that $\mathcal{P}_e$ is hardcore. This indicates that heuristically the condition $\lambda_2(\mathcal{P}) \ll \lambda(\mathcal{P})$ can be understood as the scarcity of clusters in $\mathcal{P}$, and more generally the condition $\lambda_{k+1}(\mathcal{P}) \ll \lambda_k(\mathcal{P})$ as the scarcity of k-point clusters. The following lemma states some general properties.

Lemma 1.1 (Multi-point intensities). *Let $\mathcal{P} = \{x_n\}_n$ be an ergodic stationary random point process.*

(i) *For all $j \ge 1$, we have*

$$\lambda_{j+1}(\mathcal{P}) \le \ell(\mathcal{P})^{-d}\lambda_j(\mathcal{P}).$$

(ii) *If $\mathcal{P}$ is strongly mixing, then for all $j \ge 1$ we have*

$$\lambda(\mathcal{P})^j = \underline{\lambda}_j(\mathcal{P}) \le \overline{\lambda}_j(\mathcal{P}) = \lambda_j(\mathcal{P}).$$

(The same holds for all $j \le n$ under the mixing assumption (Mix^n_ω) introduced in Section 4.3 provided the rate ω decays at infinity.)

(iii) *Given $j \ge 2$ and $\theta \in [0,1]$, for any nonnegative function $\phi \in C^\infty_c((\mathbb{R}^d)^j)$ that satisfies*

$$\begin{cases} \max_i |z_i - z_i'|_\infty \le \theta\ell(\mathcal{P}) \\ \min_{j\neq i} |z_i - z_j|_\infty \ge \ell(\mathcal{P}) \end{cases} \implies \phi(z_1,\dots,z_j) \le C\phi(z_1',\dots,z_j'),$$

we have

$$\int_{(\mathbb{R}^d)^k} \phi f_k \le C\theta^{-dk}\lambda_k(\mathcal{P}) \int_{(\mathbb{R}^d)^k} \phi.$$

Proof. As each cube $Q_\ell(z)$ contains at most one point of the point process $\mathcal{P}$, we find that $\sum_n \ell^{-d} \mathbb{1}_{Q_\ell(z)}(x_n) \le \ell^{-d}$, so that item (i) readily follows from definition (1.14).

We turn to the proof of (ii). Given $j \ge 1$, for any partition $0 = k_1 < k_2 < \cdots < k_l = j$, setting $j_i := k_{i+1} - k_i$, the strong mixing of the point process implies

$$⨍_{Q_\ell(z_1)\times\cdots\times Q_\ell(z_j)} f_j - \prod_{i=1}^{l} \Big(⨍_{Q_\ell(z_{k_i+1})\times\cdots\times Q_\ell(z_{k_{i+1}})} f_{j_i} \Big) \to 0,$$

as $\min_{i\neq i'} \operatorname{dist}(Z_i, Z_{i'}) \to \infty$, where we have set for shortness

$$Z_i := \{z_{k_i+1}, \ldots, z_{k_{i+1}}\}.$$

In view of (1.16), using stationarity, this proves the estimate $\lambda_j(\mathcal{P}) \ge \prod_{i=1}^l \lambda_{j_i}(\mathcal{P})$, from which the claim (ii) easily follows.

Finally, item (iii) is a direct consequence of definition (1.16) of multi-point intensities, further using that the j-point density satisfies $f_j(x_1, \ldots, x_j) = 0$ whenever there are some indices $i \neq i'$ with $|x_i - x_{i'}| < \ell(\mathcal{P})$. ∎

1.3.3 Scaling of cluster expansions

With the above definitions, we may now determine the scaling of the terms in the cluster expansion (1.11) and show the relevance of multi-point intensities in this context. For that purpose, by way of illustration, we place ourselves in the elementary setting of short-range interactions, which will serve as a guideline in the sequel. More precisely, consider a set function $\Phi : P(\mathbb{N}) \to \mathbb{R}$ of the form

$$\Phi^H := \mathbb{E}\Big[g\Big(\sum_{n\in H} h(x_n) \Big) \Big], \tag{1.18}$$

for some $h : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

(a) h is *short-range*, in the sense that $\int_{\mathbb{R}^d} (\sup_{B(z)} |h|)\, dz < \infty$;

(b) g is *smooth*, in the sense that $g \in C_b^\infty(\mathbb{R})$,

and set $\Phi^\varnothing = g(0)$. The cluster expansion of $\Phi^{\mathbb{N}}$, cf. (1.11), then takes the form

$$\Phi^{\mathbb{N}} \sim \sum_{j=0}^{\infty} \frac{1}{j!} \bar{\Phi}^j, \quad \text{where } \bar{\Phi}^j := j! \sum_{\sharp F = j} \delta^F \Phi^\varnothing. \tag{1.19}$$

Although cluster coefficients $\{\bar{\Phi}^j\}_j$ are defined by infinite series, these series are always summable in this short-range setting and we show that they are naturally estimated by multi-point intensities. In particular, the second-order coefficient $\bar{\Phi}^2$ is of

order $O(\lambda_2(\mathcal{P}))$, which contradicts in general the bound $O(\lambda(\mathcal{P})^2) = O(\varphi^2)$ that one could have naively expected. Our main goal in this memoir is precisely to establish corresponding expansions and estimates for the effective viscosity (1.2).

Lemma 1.2 (Cluster expansions in the short-range setting). *Let $\mathcal{P} = \{x_n\}_n$ be an ergodic stationary point process on $\mathbb{R}^d$ with $0 < \ell(\mathcal{P}) \lesssim 1$, let Φ be a set function of the form (1.18) satisfying the short-range and smoothness assumptions* (a) and (b) *above, and let $\{\bar\Phi^j\}_j$ be the associated cluster coefficients (1.19). Then we have for all $k \geq 1$,*

$$\left| \Phi^{\mathbb{N}} - \sum_{j=0}^{k} \frac{1}{j!} \bar\Phi^j \right| \lesssim_{k,g,h} \lambda_{k+1}(\mathcal{P}), \quad |\bar\Phi^k| \lesssim_{k,g,h} \lambda_k(\mathcal{P}), \tag{1.20}$$

in terms of multi-point intensities $\{\lambda_j(\mathcal{P})\}_j$, cf. (1.14).

Proof. Given a sequence $Y := \{y_n\}_n \subset \mathbb{R}$ (that will be chosen as $y_n = h(x_n)$ below), define a set function $\Psi_Y : P(\mathbb{N}) \to \mathbb{R}$ by

$$\Psi_Y^H := g\Big(\sum_{n \in H} y_n \Big), \quad H \subset \mathbb{N}.$$

By definition of difference operators, cf. (1.10), we find, in the spirit of Taylor's remainder formulas,

$$\delta^{\{n_1,\ldots,n_k\}} \Psi_Y^{\varnothing} = \int_0^{y_{n_1}} \cdots \int_0^{y_{n_k}} g^{(k)}(t_1 + \cdots + t_k)\, dt_1 \cdots dt_k,$$

$$\Psi_Y^{\mathbb{N}} - \sum_{j=0}^{k} \sum_{\sharp F = j} \delta^F \Psi_Y^{\varnothing}$$

$$= \sum_{n_1 < \cdots < n_{k+1}} \int_0^{y_{n_1}} \cdots \int_0^{y_{n_{k+1}}} g^{(k+1)}\Big(t_1 + \cdots + t_{k+1} + \sum_{n < n_1} y_n \Big)\, dt_1 \cdots dt_{k+1}.$$

These identities yield in particular

$$|\delta^{\{n_1,\ldots,n_k\}} \Psi_Y^{\varnothing}| \leq \|g^{(k)}\|_{\mathrm{L}^\infty(\mathbb{R})} \prod_{j=1}^{k} |y_{n_j}|,$$

$$\left| \Psi_Y^{\mathbb{N}} - \sum_{j=0}^{k} \sum_{\sharp F = j} \delta^F \Psi_Y^{\varnothing} \right| \leq \|g^{(k+1)}\|_{\mathrm{L}^\infty(\mathbb{R})} \sum_{\sharp F = k+1} \prod_{n \in F} |y_n|.$$

Setting $Y := \{h(x_n)\}_n$, noting that (1.18) reads $\Phi^H = \mathbb{E}[\Psi_Y^H]$, inserting the definition (1.19) of cluster coefficients, and recalling the definition (1.15) of multi-point

density functions, this yields

$$|\bar\Phi^k| \le \|g^{(k)}\|_{\mathrm{L}^\infty(\mathbb{R})} \int_{(\mathbb{R}^d)^k} |h|^{\otimes k} f_k,$$
$$\left|\Phi^{\mathbb{N}} - \sum_{j=0}^{k} \frac{1}{j!}\bar\Phi^j\right| \le \frac{1}{(k+1)!}\|g^{(k+1)}\|_{\mathrm{L}^\infty(\mathbb{R})} \int_{(\mathbb{R}^d)^{k+1}} |h|^{\otimes(k+1)} f_{k+1}.$$

By definition (1.16) of multi-point intensities, the conclusion follows. ∎

1.3.4 Effective viscosity: long-range issues and renormalization

We now apply the above cluster expansion formalism to the effective viscosity (1.2). For a finite subset $H \subset \mathbb{N}$, recall the notation ψ_E^H for the solution of the corrector problem (1.3) with the set $\mathcal{I}$ of particles replaced by its subset $\mathcal{I}^H = \bigcup_{n\in H} I_n$ (this corrector problem is trivially well posed in $\dot H^1(\mathbb{R}^d)^d$ when H is finite). We then define a symmetric linear map $\bar{\mathbf{B}}^H$ on $\mathbb{M}_0^{\mathrm{sym}}$ by

$$E : \bar{\mathbf{B}}^H E := \mathbb{E}\big[\big|\mathrm{D}(\psi_E^H)(0) + E\big|^2\big], \quad E \in \mathbb{M}_0^{\mathrm{sym}}.$$

(This indeed makes sense for a given measurable enumeration of the point process.) In these terms, the formal cluster expansion of the effective viscosity (1.2) takes the form

$$\bar{\mathbf{B}} \sim \sum_{j=0}^{\infty} \frac{1}{j!}\bar{\mathbf{B}}^j, \quad \text{where } \bar{\mathbf{B}}^j := j! \sum_{\sharp F = j} \delta^F \bar{\mathbf{B}}^\varnothing. \tag{1.21}$$

Note that $\bar{\mathbf{B}}^0 = \bar{\mathbf{B}}^\varnothing = \mathrm{Id}$ is the plain fluid viscosity. In contrast with the short-range setting of Lemma 1.2 above, however, series defining cluster coefficients $\{\bar{\mathbf{B}}^j\}_{j\ge1}$ are not summable due to the long-range nature of hydrodynamic interactions. Indeed, the first coefficient $\bar{\mathbf{B}}^1$ takes the form

$$\begin{aligned} E : \bar{\mathbf{B}}^1 E &= \sum_n E : \delta^{\{n\}}\bar{\mathbf{B}}^\varnothing E \\ &= \sum_n \mathbb{E}\big[\big|\mathrm{D}(\psi_E^{\{n\}})(0)\big|^2 + 2E : \mathrm{D}(\psi_E^{\{n\}})(0)\big]. \end{aligned} \tag{1.22}$$

As $\psi_E^{\{n\}}$ satisfies the single-particle problem (1.5), it is easily checked to have borderline decay $|\mathrm{D}(\psi_E^{\{n\}})(x)| \simeq \langle x - x_n\rangle^{-d}$, which entails that the above series is not absolutely convergent,

$$\sum_n \mathbb{E}\big[\big|\mathrm{D}(\psi_E^{\{n\}})(0)\big|\big] = \infty.$$

The same borderline divergence is observed for all cluster coefficients $\{\bar{\mathbf{B}}^j\}_{j\ge1}$. In order to make sense of them, suitable renormalization procedures are thus required and constitute the major difficulty of the problem.

To first order, the needed renormalization happens to be trivial: by definition of the intensity of the point process, identity (1.22) can be equivalently rewritten as follows (say, in case of deterministic particle shapes),

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \int_{\mathbb{R}^d} \left(\left| \mathrm{D}(\psi_E^\circ) \right|^2 + 2E : \mathrm{D}(\psi_E^\circ) \right),$$

where ψ_E° stands for the solution of the single-particle problem (1.5) with a particle centered at the origin. Here, we observe that in any finite-volume approximation the linear term $\int_{\mathbb{R}^d} E : \mathrm{D}(\psi_E^\circ)$ would be given a vanishing value as the integral of a gradient. Removing this linear term, we are left with the following summable integral,

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \int_{\mathbb{R}^d} \left| \mathrm{D}(\psi_E^\circ) \right|^2, \tag{1.23}$$

which coincides with Einstein's formula (1.7). In contrast, higher-order renormalizations are not obtained by such simple cancellations. In the physics literature, the difficulty was recognized by Batchelor and Green [7], who managed to provide a heuristic renormalization for the second-order term $\bar{\mathbf{B}}^2$. The systematic renormalization of higher-order terms is more involved and has remained an open problem so far even on the heuristic level in physics. The present memoir is precisely devoted to the systematic treatment of this difficulty: we provide suitable renormalizations of cluster coefficients and in turn justify the expansion (1.21) to all orders. In the end, we prove essentially the same estimates on the cluster expansion as in the short-range setting (1.20), up to (sharp) logarithmic corrections that are persisting manifestations of the long-range nature of interactions, cf. (1.26) below.

1.4 Main results

This section is devoted to a brief, informal account of the main results of this memoir, with precise references to the relevant chapters. We refer to the conclusion in Chapter 5 for a detailed recap of all our results. We start with the main assumptions on the ensemble of rigid particles.

1.4.1 Main assumptions

Given an underlying probability space $(\Omega, \mathbb{P})$, let $\mathcal{P} = \{x_n\}_n$ be a random point process on $\mathbb{R}^d$, consider an associated collection of random shapes $\{I_n^\circ\}_n$, where each I_n° is a random simply connected open subset of the unit ball B, centered at the origin in the sense of $\int_{I_n^\circ} y \, dy = 0$, and then define the corresponding inclusions

$$I_n := x_n + I_n^\circ.$$

Note that random shapes are not required to be independent of the point process $\mathcal{P}$. We then consider the random set

$$\mathcal{I} := \bigcup_n I_n,$$

which we assume to satisfy the following conditions. Note that the disjointness and ρ-regularity conditions below entail that the point process $\mathcal{P}$ is hardcore with $\ell(\mathcal{P}) \gtrsim \rho$, cf. (1.13).

Assumption ($\mathbf{H}_\rho$) – General conditions with parameter $\rho > 0$.

• Stationarity and ergodicity: *The point process $\mathcal{P} = \{x_n\}_n$ and the associated random set $\mathcal{I}$ are stationary and ergodic.*[2]

• Disjointness: *There holds $I_n \cap I_m = \varnothing$ almost surely for all $n \neq m$.*

• ρ-Regularity: *Random shapes $\{I_n^\circ\}_n$ almost surely satisfy interior and exterior ball conditions with radius ρ.*

Next, we define the effective viscosity tensor $\overline{\mathbf{B}}$ associated with the suspension $\mathcal{I}$ as the quadratic form on $\mathbb{M}^{\mathrm{sym}}_0$ given in (1.2). We emphasize that the corrector problem (1.3) only makes sense provided that all particles $\{I_n\}_n$ are well enough separated. If this separation is uniform, the pressure $\Sigma_E \mathbb{1}_{\mathbb{R}^d \setminus \mathcal{I}}$ can also be uniquely constructed as a stationary field with finite second moment and vanishing expectation, cf. [18, Proposition 2.1]. When particles are not well separated, the corrector problem should rather be considered via its variational formulation and the effective viscosity is then defined as the minimum value

$$\begin{aligned} E : \overline{\mathbf{B}} E = \inf \big\{ \mathbb{E}\big[\big|\mathrm{D}(\psi) + E\big|^2\big] : \psi \in \mathrm{L}^2\big(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d\big), \nabla\psi \text{ stationary}, \\ \mathrm{div}(\psi) = 0, \big(\mathrm{D}(\psi) + E\big)\big|_{\mathcal{I}} = 0, \mathbb{E}\big[\mathrm{D}(\psi)\big] = 0 \big\}. \end{aligned} \tag{1.24}$$

In general, nothing prevents this infimum from being infinite: as explained after (1.3) above, the issue originates from the possible existence of unbounded chains of touching particles. This will be excluded by means of further geometric assumptions, cf. ($\mathrm{H}^{\mathrm{unif}}_\rho$), ($\mathrm{H}^{\mathrm{mom}}_{\rho,\underline{\kappa}}$), or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) below. Even if the infimum is finite, nothing ensures in general that $\overline{\mathbf{B}}$ defines the effective viscosity in the sense of homogenization theory: we view this as a separate question, which is extensively discussed in different settings in our previous work [13, 18, 19, 21] and will not be further discussed here. We are now in the position to describe our main results.

[2] More precisely, stationarity means that the laws of the translated point process $x + \mathcal{P}$ and of the translated random set $x + \mathcal{I}$ are independent of the shift $x \in \mathbb{R}^d$. Ergodicity then means that a measurable function of $\mathcal{P}$ or $\mathcal{I}$ is almost surely unchanged for $\mathcal{P}$ or $\mathcal{I}$ replaced by $x + \mathcal{P}$ or $x + \mathcal{I}$ for any $x \in \mathbb{R}^d$ only if it is almost surely constant. Note that shifts $x \in \mathbb{R}^d$ can be replaced by discrete shifts $x \in \mathbb{Z}^d$, and periodic point sets can be considered as a particular case, for which the expectation is replaced by the average over a period.

1.4.2 First-order expansion: Einstein's formula

Assuming that particles are uniformly separated by a positive distance, cf. ($\mathrm{H}_\rho^{\mathrm{unif}}$) below, we prove in the general ergodic stationary setting,

$$|\bar{\mathbf{B}} - \mathrm{Id} - \bar{\mathbf{B}}^1| \lesssim \lambda_2(\mathcal{P}) \log\left(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})(\ell(\mathcal{P})+1)^d}\right), \quad |\bar{\mathbf{B}}^1| \simeq \lambda(\mathcal{P}), \tag{1.25}$$

where $\bar{\mathbf{B}}^1$ is given by the renormalized cluster formula (1.7) and takes the explicit form of Einstein's formula $\bar{\mathbf{B}}^1 = \frac{d+2}{2}\varphi\,\mathrm{Id}$ in case of spherical particles. This error estimate is new and optimal, and the stochastic assumption of mere ergodicity is minimal. In particular, we find that Einstein's formula provides a leading-order approximation, in the sense of $|\bar{\mathbf{B}} - \mathrm{Id} - \bar{\mathbf{B}}^1| \ll |\bar{\mathbf{B}}^1|$, provided that the condition $\lambda_2(\mathcal{P}) \ll \lambda(\mathcal{P})$ holds, which heuristically amounts to the scarcity of particle clusters and follows for instance from some local independence of $\mathcal{P}$.

Yet, the uniform separation assumption ($\mathrm{H}_\rho^{\mathrm{unif}}$) is not satisfactory from the physical point of view. At the price of weakening the error estimate (1.25), we may relax this assumption as we did for the homogenization result in [13,21]: either we assume moment bounds on the interparticle distance ($\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}}$) (see also [28]), or we consider a subcritical percolation condition ($\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}}$) (in which case particles are allowed to touch provided they do not cluster). We refer to Theorem 2.1 in Chapter 2 for a detailed statement.

1.4.3 Higher-order cluster corrections

For the higher-order analysis, we focus for simplicity on the setting of the uniform separation assumption ($\mathrm{H}_\rho^{\mathrm{unif}}$). Under a slight strengthening of ergodicity, the formal cluster expansion is well defined, up to suitable renormalization of cluster coefficients (1.21), and it essentially[3] satisfies for all $k \geq 1$,

$$\left|\bar{\mathbf{B}} - \sum_{j=0}^{k} \bar{\mathbf{B}}^j\right| \lesssim \lambda_{k+1}(\mathcal{P})\left|\log \lambda(\mathcal{P})\right|^k, \quad |\bar{\mathbf{B}}^k| \lesssim \lambda_k(\mathcal{P})\left|\log \lambda(\mathcal{P})\right|^{k-1}. \tag{1.26}$$

These estimates coincide remarkably with the corresponding result (1.20) in the short-range setting, to the exception of logarithmic corrections that are precisely the manifestations of the long-range nature of hydrodynamic interactions. The result is new for any $k \geq 2$ and logarithmic corrections are expected to be optimal (optimality is proved for $k = 2$, cf. Theorem 4.4). We also believe that the slightly strengthened ergodicity assumption is necessary for the result to hold. We refer to Theorem 5.2 in Chapter 5

[3] The correct estimate is in general slightly more complicated than what is stated here; cf. Theorem 5.2 in Chapter 5.

for a detailed statement. In particular, our analysis justifies the Batchelor–Green formula for the second-order term $\overline{\mathbf{B}}^2$, cf. Proposition 4.9 (see also Corollary 5.6), and we develop a systematic renormalization scheme for all higher-order cluster coefficients by means of diagrammatic expansions, cf. Section 4.4.

We emphasize that the above result (1.26) holds without any structural assumption on the dilution process (which we call the "model-free" setting). If we make the dilution more specific, considering for instance a random deletion procedure (as in [15]) or dilation, then the full cluster expansion can be further shown to define an absolutely converging series. We refer to Theorem 5.4 for a precise analyticity statement. All previous results on the second-order expansion [26, 27, 29] were, in fact, essentially restricted to such specific settings.

1.5 Roadmap to the main results

The rest of the memoir is divided into four chapters. Chapter 2 is dedicated to the proof of Einstein's formula. Chapter 3 studies the cluster expansion of finite-volume approximations $\{\overline{\mathbf{B}}_L\}_L$ of the effective viscosity $\overline{\mathbf{B}}$. In Chapter 4, we deal with the issue of systematic renormalization of cluster coefficients, which leads us to justifying the cluster expansion of $\overline{\mathbf{B}}$. Our different results are combined and summarized in Chapter 5. We briefly describe below our approach for each step.

1.5.1 Einstein's formula: first-order expansion—Chapter 2

We develop a new, purely variational approach to Einstein's formula (1.25); a short self-contained proof is given in Chapter 2. It amounts to constructing competitors for the variational problem (1.24) and to controlling their energy difference by means of elliptic regularity. The variational nature of the argument allows us to avoid uniform particle separation assumptions and to cover in particular the case of colliding particles under a general non-clustering assumption. It also allows to avoid the need for fine pressure estimates, which is a crucial point as such estimates would be problematic in case of colliding particles.

1.5.2 Cluster expansion of the effective viscosity—Chapter 3

While coefficients in the formal cluster expansion of the effective viscosity $\overline{\mathbf{B}}$ are given by infinite series that are not summable due to the long-range nature of hydrodynamic interactions, cf. Section 1.3.4, we start by considering finite-volume approximations $\{\overline{\mathbf{B}}_L\}_{L\geq 1}$ obtained by periodization of the variational problem (1.24). Chapter 3 provides a detailed analysis of the cluster expansion of $\overline{\mathbf{B}}_L$ for fixed L.

- First, we give explicit formulas for the coefficients $\{\overline{\mathbf{B}}_L^j\}_j$ of the cluster expansion, as well as an explicit estimate for the remainder $R_L^{k+1} := \overline{\mathbf{B}}_L - \sum_{j=0}^k \frac{1}{j!}\overline{\mathbf{B}}_L^j$, in terms of correctors associated with finite subsets of particles; see Theorem 3.1. The argument is essentially combinatorial. Note that the proof of remainder estimates further makes key use of the rigidity of the particles.
- Second, we prove that the cluster coefficients $\{\overline{\mathbf{B}}_L^j\}_j$ and the remainder R_L^{k+1} are bounded uniformly in L. The idea of the proof is as follows: if infinite-volume cluster formulas are given by infinite series that are not summable, they can in fact be viewed as complicated (non-explicit) combinations of Calderón–Zygmund kernels. As the effective viscosity is an L^2-based quantity, we may expect to estimate cluster formulas by means of suitable energy estimates, carefully avoiding taking absolute values of any Calderón–Zygmund kernel. Taking inspiration from our previous work [15], this is achieved by means of a hierarchy of so-called interpolating $\ell^1 - \ell^2$ energy estimates (also crucially used in [20,30]). As a corollary, uniform estimates allow us to define infinite-volume cluster coefficients in the limit $\{\overline{\mathbf{B}}^j\}_j := \lim_{L\uparrow\infty}\{\overline{\mathbf{B}}_L^j\}_j$. Yet, being based on energy arguments, these estimates do not display the desired dependence (1.26) on multi-point intensities $\{\lambda_j\}_j$.
- Third, we prove corresponding cluster estimates that have the same dependence on multi-point intensities as in the short-range setting, but display a logarithmic divergence in the large-volume limit. This is obtained by proceeding as for the short-range setting of Lemma 1.2, and the logarithmic divergence follows from estimating hydrodynamic interactions too roughly.

It remains to show that the dependence on multi-point intensities is actually kept in the large-volume limit (at the price of logarithmic corrections).

1.5.3 Renormalization of cluster formulas—Chapter 4

In order to prove the relevant infinite-volume cluster estimates (1.26), we need a better understanding of cluster formulas and of the underlying compensations that make them well defined in the large-volume limit. A first route proceeds by assuming an algebraic convergence rate for the finite-volume approximations $\{\overline{\mathbf{B}}_L\}_L$ of the effective viscosity: this is known to hold under quantitative α-mixing condition whose rate is then transmitted (suboptimally) to cluster coefficients $\{\overline{\mathbf{B}}_L^j\}_j$, which allows in turn to keep the desired dependence on multi-point intensities in the cluster estimates while removing the logarithmic divergence. This implicit renormalization argument is particularly robust (see also [15]), but it does not provide any understanding of underlying cancellations and leaves several questions open.

Next, further assuming for simplicity that particle shapes are independent of particle positions, we show that an explicit renormalization of cluster formulas can be

developed: taking advantage of several explicit cancellations, cluster formulas can be transformed into summable integral formulas. This renormalization is trivial for $\overline{\mathbf{B}}^1$, cf. (1.23), and the required cancellations are already more involved for $\overline{\mathbf{B}}^2$, as formally understood by Batchelor and Green [8]. At higher orders, renormalizations rely on a suitable diagrammatic decomposition of cluster formulas to make cancellations manifest. Finally, the direct analysis of renormalized formulas allows to recover the desired cluster estimates (1.26) and to show that logarithmic corrections in those bounds are actually optimal in general.

Notation

- For vector fields u, u' and matrix fields T, T', we set $(\nabla u)_{ij} = \nabla_j u_i$, $\mathrm{div}(T)_i = \nabla_j T_{ij}$, $T : T' = T_{ij} T'_{ij}$, $(u \otimes u')_{ij} = u_i u'_j$, where we systematically use Einstein's summation convention on repeated indices. We also denote by $(\mathrm{D}(u))_{ij} = \frac{1}{2}(\nabla_j u_i + \nabla_i u_j)$ the symmetrized gradient. For a velocity field u and associated pressure field P, we define the associated Cauchy stress tensor, cf. (1.4),

$$\sigma(u, P) := 2\mathrm{D}(u) - P\,\mathrm{Id}\,.$$

- We denote by $\mathbb{M}^{\mathrm{sym}}_0 \subset \mathbb{R}^{d\times d}$ the subset of symmetric trace-free matrices, and by $\mathbb{M}^{\mathrm{skew}}$ the subset of skew-symmetric matrices.
- We use the notation $\lesssim$ (resp. $\gtrsim$) for $\le C\times$ (resp. $\ge \frac{1}{C}\times$) with a constant C that depends only on the dimension d and on the parameters appearing in the different assumptions when applicable. Note that the value of the constant C is allowed to change from one line to another. We add subscripts to C, $\lesssim$, or $\gtrsim$ to indicate the dependence on other parameters. We write $a \simeq b$ when both $a \lesssim b$ and $a \gtrsim b$ hold. In addition, we write $\ll$ (resp. $\gg$) for $\le \frac{1}{C}\times$ (resp. $\ge C\times$) for some sufficiently large constant C.
- The ball centered at x of radius r in $\mathbb{R}^d$ is denoted by $B_r(x)$, and we set $B(x) = B_1(x)$, $B_r = B_r(0)$, and $B = B_1(0)$. We denote by $Q_r(x) = x + [-\frac{r}{2}, \frac{r}{2})^d$ the cube of sidelength r centered at x, and we set $Q(x) = Q_1(x)$, $Q_r = Q_r(0)$, and $Q = Q_1(0)$.
- For $x \in \mathbb{R}^d$, we denote by $|x|$ its Euclidean norm and by $|x|_\infty$ its supremum norm. We also set $\langle x\rangle = (1 + |x|^2)^{1/2}$, and similarly $\langle\nabla\rangle = (1 - \triangle)^{1/2}$.
- We use the shorthand notation $\sum^{\neq}_{n_1,\dots,n_j}$ for sums over j-tuples $(n_1, \dots, n_j)$ of distinct indices. We also use the notation $\sum_{\sharp F = j}$ for the sum over all subsets F of j distinct indices.