Chapter 2

Einstein's formula: First-order expansion

2.1 Main result

Assumption (H_ρ) first needs to be complemented with suitable geometric assumptions on the ensemble of particles to ensure that the effective viscosity (1.24) is finite. This can either be performed by means of conditions on interparticle distances,

$$\rho_n := \frac{1}{2} \min_{m:m\neq n} \operatorname{dist}(I_n, I_m), \tag{2.1}$$

or in terms of conditions on the size of clusters of close particles. This has been the subject of our recent series of articles [13, 18, 21], where the finiteness of the effective viscosity and the validity of a homogenization result are obtained under any of the following three types of assumptions:

— interparticle distances are uniformly bounded below, cf. [18];

— interparticle distances satisfy suitable reciprocal moment bounds, cf. [13, Theorem 1 for $d > 2$ and Remark 3.4 for $d = 2$];

— diameters of clusters of close particles satisfy suitable moment bounds in a subcritical percolation perspective, cf. [21].

These are formulated more precisely in Assumptions $(\mathrm{H}^{\mathrm{unif}}_\rho)$, $(\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa})$, and $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$ below, respectively.

Assumption $(\mathrm{H}^{\mathrm{unif}}_\rho)$ – Uniform separation with parameter $\rho > 0$. *Particles are uniformly separated with minimal distance $\rho_n > \rho$, that is, we have almost surely*

$$(I_n + \rho B) \cap (I_m + \rho B) = \varnothing \quad \textit{for all } n \neq m.$$

Assumption $(\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa})$ – Moment condition with parameters $\rho > 0$, $\kappa > 1$.

• ρ-Uniform non-degeneracy of contact points: *Pairs of "ρ-close" particles can be "ρ-locally" included in pairs of disjoint spheres with "ρ-uniformly" bounded radius. For "ρ-close" particles, instead of (2.1), we then define ρ_n as (half of) the distance between locally covering spheres. For a more precise statement of this geometric condition, we refer to* [13, Assumption (H'_δ)]. *Note that this condition is trivially satisfied in case of spherical particles.*

• Reciprocal moment bound: *There exists $\mathcal{K}_\kappa < \infty$ such that*

$$\limsup_{R\uparrow\infty} \Bigg(\frac{1}{\sharp\{n : I_n \subset Q_R\}} \sum_{n:I_n\subset Q_R} \mu(\rho_n)^\kappa \Bigg)^{\frac{1}{\kappa}} \le \mathcal{K}_\kappa,$$

in terms of

$$\mu(t) := \begin{cases} t^{-\frac12(5-d)} & \text{if } d < 5, \\ \log(2+\frac1t) & \text{if } d = 5, \\ 1 & \text{if } d > 5. \end{cases} \tag{2.2}$$

Note that this condition is trivially satisfied for any $\kappa > 1$ in case $d > 5$.

Assumption ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) – Cluster condition with parameters $\rho > 0$, $\kappa > 1$. *Let $\{K_{q,\rho}\}_q$ be the family of connected components of the fattened set $\mathcal{I} + \rho B$, and consider the corresponding clusters*

$$J_{q,\rho} := \bigcup_{I_n \subset K_{q,\rho}} I_n.$$

Given $p_0 \gg 1$ large enough (related to the existence of correctors in [21, Proposition 2]*), there exists $\mathcal{K}_\kappa < \infty$ such that*

$$\limsup_{R\uparrow\infty} \bigg(\frac{1}{\sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\}} \sum_{q : J_{q,\rho}\cap Q_R \neq\varnothing} \mathrm{diam}(J_{q,\rho})^{\kappa p_0 + d} \bigg)^{\frac1\kappa} \le \mathcal{K}_\kappa. \tag{2.3}$$

Assumptions ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$) and ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) are always weaker than ($\mathrm{H}^{\mathrm{unif}}_{\rho}$). While ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$) only allows for particle contacts in dimension $d > 5$ and is in particular incompatible with the 3D steady-state behavior of the two-particle density as computed in [8], Assumption ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) is of a different nature and allows for particle contacts in any dimension, but implicitly requires some strong mixing condition to ensure the validity of moment bounds on cluster diameters, cf. [21]. In [13, 18, 21], we show that these assumptions ensure the finiteness of the effective viscosity (1.24) and the well-posedness of the corrector problem. In case of ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$) or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$), the validity of the homogenization result requires further strengthened conditions.

The following theorem states the validity of Einstein's formula under each of the above assumptions. The proof is split between Sections 2.2, 2.3, 2.4, and 2.5 below.

Theorem 2.1 (Einstein's formula). *Under Assumption (H_ρ), provided that either ($\mathrm{H}^{\mathrm{unif}}_{\rho}$), ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$), or ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) holds for some $\rho > 0$ and $\kappa > 1$, we have*

$$\begin{aligned} \big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big| &\lesssim_\rho \lambda_2(\mathcal{P}) \log\bigg(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\bigg) \\ &\quad + \begin{cases} 0 & \text{in case of } (\mathrm{H}^{\mathrm{unif}}_{\rho}), \\ \mathcal{K}_\kappa \lambda_2(\mathcal{P})^{1-\frac1\kappa} \lambda(\mathcal{P})^{\frac1\kappa} & \text{in case of } (\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}) \text{ or } (\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}), \end{cases} \end{aligned} \tag{2.4}$$

where $\bar{\mathbf{B}}^1$ satisfies

$$|\bar{\mathbf{B}}^1| \simeq \lambda(\mathcal{P}),$$

and is defined for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ *by*

$$E : \bar{\mathbf{B}}^1 E := \sum_n \mathbb{E}\Big[\frac{\mathbb{1}_{0 \in I_n}}{|I_n|} \int_{\mathbb{R}^d} \big|\mathrm{D}(\psi_E^{\{n\}})\big|^2\Big], \tag{2.5}$$

where $\psi_E^{\{n\}}$ *is the unique decaying solution of the single-particle problem* (1.5)*. In particular, the estimate* $|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)| = o(\lambda(\mathcal{P}))$ *holds provided the point process* $\mathcal{P}$ *satisfies* $\lambda_2(\mathcal{P}) = o(\lambda(\mathcal{P}))$.

As outlined in Section 2.2, our proof is variational and amounts to proving lower and upper bounds on $\bar{\mathbf{B}}$ that match with $\mathrm{Id} + \bar{\mathbf{B}}^1$ to the required accuracy. (For the case of the Clausius–Mossotti conductivity formula, we refer to [14], where we provide a PDE version of the present variational argument.) This new approach is very robust: it yields the first optimal error estimate and allows to cover the most general setting regarding particle separation assumptions. We briefly emphasize these two points:

— *Optimality:* In case of $(\mathrm{H}_\rho^{\mathrm{unif}})$, the error estimate (2.4) for Einstein's formula is new and sharp. As will be seen in Theorem 4.4, it indeed coincides with the general scaling of the next term $\bar{\mathbf{B}}^2$ in the cluster expansion: the logarithmic correction is related to the long-range nature of hydrodynamic interactions and cannot be avoided in general, thus contrasting with the short-range setting (1.20).[1] In case of $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ or $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$, the error estimate (2.4) displays a further algebraic loss, which is also new and expected to be optimal in general. If for some exponent $\gamma \geq 1$ the moment bounds in $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ or $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$ hold with constant $\mathcal{K}_\kappa \lesssim \kappa^\gamma$ for all $\kappa \geq 1$,[2] then the error estimate (2.4) could be upgraded to

$$\lambda_2(\mathcal{P}) \log^{1\vee\gamma}\Big(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})}\Big)$$

after optimizing in κ.

— *Particle separation:* Most works on the topic [26–29, 33, 54] have focussed so far on the simplest setting of $(\mathrm{H}_\rho^{\mathrm{unif}})$ in case when diluteness further holds in the strong form of $\ell(\mathcal{P}) \gg 1$. The only exception is the recent independent work [28], where this last condition is relaxed and where the case of $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$ is also covered. More generally, our approach allows to further cover essentially any situation for which the effective viscosity (1.24) can be proved to be finite. Applied to $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$, it allows us to treat for the first time a 3D setting where particles are allowed to touch.

[1] In some special cases, however, for instance in the statistically isotropic setting, the logarithmic correction can be removed, cf. Theorem 4.4 (i).

[2] For $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$, this would amount to having stretched exponential moment bounds. For $(\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}})$, this holds for some point processes such as the random parking measure with γ large enough, cf. [21].

Next, we further simplify formula (2.5) for the first-order cluster coefficient $\bar{\mathbf{B}}^1$ in the case when particle shapes are independent: we recover the formula obtained in [33], as well as Einstein's explicit formula (1.8) in case of spherical particles. The proof is postponed to Section 2.6.

Proposition 2.2 (First-order coefficient). *On top of Assumption (H$_\rho$), further assume*

(Indep) Independent shapes: *Random shapes $\{I_n^\circ\}_n$ are iid copies of a given random subset I°, independently of the point process $\mathcal{P}$.*

Then, the first-order coefficient $\bar{\mathbf{B}}^1$ defined in (2.5) can be written as

$$\bar{\mathbf{B}}^1 = \lambda(\mathcal{P})\hat{\mathbf{B}}^1, \quad E : \hat{\mathbf{B}}^1 E = \mathbb{E}\Big[\int_{\mathbb{R}^d} \big|\mathrm{D}(\psi_E^\circ)\big|^2 \Big], \tag{2.6}$$

in terms of the unique decaying solution of the single-particle problem

$$\begin{cases} -\triangle \psi_E^\circ + \nabla \Sigma_E^\circ = 0, & \text{in } \mathbb{R}^d \setminus I^\circ, \\ \operatorname{div}(\psi_E^\circ) = 0, & \text{in } \mathbb{R}^d \setminus I^\circ, \\ \mathrm{D}(\psi_E^\circ) + E = 0, & \text{in } I^\circ, \\ \int_{\partial I^\circ} \sigma_E^\circ \nu = 0, & \\ \int_{\partial I^\circ} \Theta x \cdot \sigma_E^\circ \nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{2.7}$$

In case of spherical particles, $I_n = B(x_n, r_n)$, with iid random radii $\{r_n\}_n$, this reduces to Einstein's celebrated formula

$$\bar{\mathbf{B}}^1 = \frac{d+2}{2} \varphi \,\mathrm{Id}, \tag{2.8}$$

where the volume fraction is in this case $\varphi = \lambda(\mathcal{P})\mathbb{E}[|I_0|]$.

2.2 Variational approach

This section is devoted to setting up our variational approach to prove Theorem 2.1, which is partly inspired by the theory of optimal bounds in homogenization; see e.g. [50, Chapters 13 and 23]. The new main ingredients are the use of Voronoi tessellations and of elliptic regularity. Let $E \in \mathbb{M}_0^{\mathrm{sym}}$ be fixed. In the spirit of the heuristic approximation (1.6) for the corrector, we start by defining single-particle problems in the neighborhood of each particle. For a random set $\mathcal{I}$ satisfying (H$_\rho$), we define the associated Voronoi tessellation $\{V_n\}_n$ as follows,

$$V_n := \big\{ x \in \mathbb{R}^d : \operatorname{dist}(x, I_n) < \operatorname{dist}(x, I_m) \ \forall m \neq n \big\}.$$

By definition, these Voronoi cells pave the whole space $\mathbb{R}^d$ and each V_n contains exactly one inclusion I_n. We then consider the single-particle problems in V_n, with either homogeneous Dirichlet or Neumann boundary conditions on ∂V_n,

$$\mathcal{E}_{n,D} := \inf\left\{\int_{V_n} \left|\mathrm{D}(\psi)\right|^2 : \psi \in H_0^1(V_n)^d,\ \mathrm{div}(\psi) = 0,\ \left(\mathrm{D}(\psi) + E\right)\big|_{I_n} = 0\right\}, \tag{2.9}$$

$$\mathcal{E}_{n,N} := \inf\left\{\int_{V_n} \left|\mathrm{D}(\psi)\right|^2 : \psi \in H^1(V_n)^d,\ \mathrm{div}(\psi) = 0,\ \left(\mathrm{D}(\psi) + E\right)\big|_{I_n} = 0\right\}. \tag{2.10}$$

Provided that $\mathcal{E}_{n,D} < \infty$ (which is the case as soon as $I_n \Subset V_n$), the Dirichlet problem (2.9) is well posed and we denote by $\psi_{n,D}$ its unique minimizer. The Neumann problem (2.10), on the other hand, is always well posed and one has the deterministic uniform bound $\mathcal{E}_{n,N} \lesssim 1$. We denote by $\psi_{n,N}$ the corresponding minimizer, which is only defined up to a rigid motion and can be fixed for instance by choosing $\int_V \psi_{n,N} = 0$ and $\int_V \nabla\psi_{n,N} \in \mathbb{M}_0^{\mathrm{sym}}$. Next, we define the single-particle problem on the whole space via

$$\mathcal{E}_{n,\infty} := \inf\left\{\int_{\mathbb{R}^d} \left|\mathrm{D}(\psi)\right|^2 : \psi \in H^1(\mathbb{R}^d)^d,\ \mathrm{div}(\psi) = 0,\ \left(\mathrm{D}(\psi) + E\right)\big|_{I_n} = 0\right\}. \tag{2.11}$$

Note that the unique minimizer $\psi_{n,\infty}$ of this variational problem coincides with the solution $\psi_E^{\{n\}}$ of (1.5). In case of ($\mathrm{H}_{\rho,\kappa}^{\mathrm{perc}}$), as we only control clusters of close particles, we naturally merge Voronoi cells that intersect the same cluster: more precisely, we consider the Voronoi cell associated with each cluster $J_{q,\rho}$,

$$W_q := \bigcup_{n:I_n \subset J_{q,\rho}} V_n,$$

and we then partition the whole space as

$$\mathbb{R}^d = \Big(\bigcup_{n\in\mathcal{S}} V_n\Big) \cup \Big(\bigcup_{q\in\mathcal{S}'} W_q\Big),$$

where $\mathcal{S} := \{n : \rho_n \geq \rho\}$ is the set of indices of well-separated particles and where $\mathcal{S}'$ is the set of indices q such that the cluster $J_{q,\rho}$ is made of at least two particles. For $n \in \mathcal{S}$ we shall consider the single-inclusion problems $\mathcal{E}_{n,D}, \mathcal{E}_{n,N}, \mathcal{E}_{n,\infty}$ as above, while for $q \in \mathcal{S}'$ it will suffice to consider the single-cluster problem with Dirichlet conditions,

$$\mathcal{F}_{q,D} := \inf\left\{\int_{W_q} \left|\mathrm{D}(\psi)\right|^2 : \psi \in H_0^1(W_q)^d,\ \mathrm{div}(\psi) = 0,\ \left(\mathrm{D}(\psi) + E\right)\big|_{J_{q,\rho}} = 0\right\}. \tag{2.12}$$

The upcoming lemma shows that the error in the first-order expansion $\bar{\mathbf{B}} \sim \mathrm{Id} + \bar{\mathbf{B}}^1$ can be controlled using single-particle problems (as well as single-cluster problems in case of $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$). This provides a drastic reduction of complexity since $\overline{\mathbf{B}}$ itself involves the corrector ψ_E with the full set of particles. The proof is postponed to Section 2.4 below.

Lemma 2.3. *Under the assumptions of Theorem 2.1, using the notation (2.9)–(2.12), we have*

$$\big|E:\big(\overline{\mathbf{B}}-(\mathrm{Id}+\overline{\mathbf{B}}^1)\big)E\big| \lesssim \begin{cases} \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}(\mathcal{E}_{n,D}-\mathcal{E}_{n,N})\Big] & \text{in case of } (\mathrm{H}^{\mathrm{unif}}_{\rho}) \text{ or } (\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}),\\ \mathbb{E}\Big[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}(\mathcal{E}_{n,D}-\mathcal{E}_{n,N})\Big] & \\ \quad +\mathbb{E}\Big[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathcal{F}_{q,D}\Big] & \text{in case of } (\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}), \end{cases} \tag{2.13}$$

where $\overline{\mathbf{B}}^1$ is defined in (2.5).

It remains to control the right-hand side in the above preliminary error estimate (2.13), which amounts to comparing the single-particle problems with Dirichlet or Neumann boundary conditions on Voronoi cells. The proof is postponed to Section 2.5 below.

Lemma 2.4. *For all n, we have almost surely*

$$0\le \mathcal{E}_{n,D}-\mathcal{E}_{n,N}\lesssim \mu(\rho_n)\mathbb{1}_{\rho_n<1}+\rho_n^{-d}\mathbb{1}_{\rho_n\ge 1}, \tag{2.14}$$

where we recall that ρ_n stands for (half of) the interparticle distance, cf. (2.1), and that the weight μ is defined in (2.2). In addition, there is $p_0<\infty$ such that for all q we have almost surely

$$\mathcal{F}_{q,D}\lesssim \mathrm{diam}(J_{q,\rho})^{p_0}. \tag{2.15}$$

With these two lemmas at hand, combining the estimates, we may now quickly conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. Combining Lemmas 2.3 and 2.4, we get

$$\big|\overline{\mathbf{B}}-(\mathrm{Id}+\overline{\mathbf{B}}^1)\big| \lesssim \begin{cases} \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\big(\mu(\rho_n)\mathbb{1}_{\rho_n<1}+\rho_n^{-d}\mathbb{1}_{\rho_n\ge 1}\big)\Big] & \text{in case of } (\mathrm{H}^{\mathrm{unif}}_{\rho}) \text{ or } (\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}),\\ \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\rho_n^{-d}\mathbb{1}_{\rho_n\ge\rho}\Big] & \\ \quad +\mathbb{E}\Big[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathrm{diam}(J_{q,\rho})^{p_0}\Big] & \text{in case of } (\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}), \end{cases} \tag{2.16}$$

and it remains to estimate these expectations. We split the proof into two steps.

Step 1. Proof that, if $g \in \mathrm{L}^\infty(\mathbb{R}^+)$ is non-increasing with $g(r) \downarrow 0$ as $r \uparrow \infty$, then

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|} g(\rho_n)\bigg] \lesssim \lambda_2(\mathcal{P})\|g\|_{\mathrm{L}^\infty(\mathbb{R}^+)} + \int_{(\frac12\ell-1)_+}^\infty |g'(r)|\Big(\big(\lambda_2(\mathcal{P})\langle r\rangle^d\big)\wedge\lambda(\mathcal{P})\Big)\,dr. \tag{2.17}$$

To start with, we rewrite the left-hand side as

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|} g(\rho_n)\bigg] = \int_0^\infty g(r)\,d\Lambda(r), \tag{2.18}$$

where the positive measure Λ on $\mathbb{R}^+$ is defined by its distribution function

$$\Lambda\big([0,r]\big) := \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n\le r}\bigg] = \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\exists m\ne n:\frac12\operatorname{dist}(I_m,I_n)\le r}\bigg]. \tag{2.19}$$

As $I_n \subset B(x_n)$ for all n, we can estimate the latter as

$$\Lambda\big([0,r]\big) \le \mathbb{E}\Big[\sum_n \mathbb{1}_{|x_n|\le1}\mathbb{1}_{\exists m\ne n:|x_m-x_n|\le2(r+1)}\Big].$$

Recalling that $\ell = \ell(\mathcal{P})$ is the minimal distance (1.13), we deduce that $\Lambda([0,r]) = 0$ for all $r \le \frac12\ell - 1$. Moreover, we can bound, on the one hand,

$$\Lambda\big([0,r]\big) \le \mathbb{E}\Big[\sum_n \mathbb{1}_{|x_n|\le1}\Big] = \lambda(\mathcal{P})|B|,$$

and on the other hand, in terms of the two-point density and intensity, for $r \ge \frac12\ell - 1$,

$$\begin{aligned}
\Lambda\big([0,r]\big) &\le \mathbb{E}\Big[\sum_{n\ne m} \mathbb{1}_{|x_n|\le1}\mathbb{1}_{|x_m-x_n|\le2(r+1)}\Big]\\
&= \iint_{B\times B_{2(r+1)}} f_2(x,x+y)\,dx\,dy\\
&= \big(2(r+1)\big)^{-d}\iint_{B_{2(r+1)}\times B_{2(r+1)}} f_2(x,x+y)\,dx\,dy\\
&\lesssim \lambda_2(\mathcal{P})\langle r\rangle^d.
\end{aligned}$$

Combining these estimates yields

$$\Lambda\big([0,r]\big) \lesssim \big(\lambda_2(\mathcal{P})\langle r\rangle^d\big)\wedge\lambda(\mathcal{P}). \tag{2.20}$$

Under our assumptions on g, an integration by parts yields

$$\int_0^\infty g(r)\,d\Lambda(r) = -g(0)\Lambda\big(\{0\}\big) + \int_0^\infty |g'(r)|\Lambda\big([0,r]\big)\,dr,$$

and the conclusion follows in combination with (2.18) and (2.20).

Step 2. Conclusion. In case of $(\mathrm{H}_\rho^{\mathrm{unif}})$, as we have $\rho_n \ge \rho$ for all n, the contributions of $\rho_n < \rho$ can be removed in (2.16). Applying (2.17) with $g(r) = \langle r\rangle^{-d}$, we are then led to

$$\begin{aligned}&\big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big|\\ &\lesssim \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\langle\rho_n\rangle^{-d}\Big] \lesssim \lambda_2(\mathscr{P}) + \int_{(\frac12\ell-1)_+}^\infty \langle r\rangle^{-d-1}\Big(\big(\lambda_2(\mathscr{P})\langle r\rangle^d\big)\wedge\lambda(\mathscr{P})\Big)\,dr,\end{aligned}$$

and the conclusion (2.4) follows after estimating this integral.

Next, in case of $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$, repeating the same computation as above for the contributions of $\rho_n \ge 1$ in (2.16), and separating the contributions of $\rho_n \le 1$, we find

$$\begin{aligned}&\big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big|\\ &\lesssim \lambda_2(\mathscr{P})\log\Bigg(2 + \frac{\lambda(\mathscr{P})}{\lambda_2(\mathscr{P})\big(\ell(\mathscr{P})+1\big)^d}\Bigg) + \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)\mathbb{1}_{\rho_n<1}\Big],\end{aligned}$$

and it remains to estimate the last term. By Hölder's inequality, we can write for any $\kappa \ge 1$,

$$\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)\mathbb{1}_{\rho_n<1}\Big] \le \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n<1}\Big]^{1-\frac1\kappa}\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)^\kappa\Big]^{\frac1\kappa}.$$

On the one hand, (2.20) yields

$$\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathbb{1}_{\rho_n<1}\Big] = \Lambda\big([0,1]\big) \lesssim \lambda_2(\mathscr{P}).$$

On the other hand, by the ergodic theorem, using (1.12) and the reciprocal moment condition in $(\mathrm{H}_{\rho,\kappa}^{\mathrm{mom}})$, we find

$$\begin{aligned}&\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mu(\rho_n)^\kappa\Big]\\ &= \lim_{R\uparrow\infty} R^{-d}\sum_n \frac{|I_n\cap Q_R|}{|I_n|}\mu(\rho_n)^\kappa\\ &\le \limsup_{R\uparrow\infty}\frac{\sharp\{n : I_n\cap Q_R\ne\varnothing\}}{R^d}\frac{1}{\sharp\{n : I_n\cap Q_R\ne\varnothing\}}\sum_{n:I_n\cap Q_R\ne\varnothing}\mu(\rho_n)^\kappa\\ &\le \lambda(\mathscr{P})(\mathcal{K}_\kappa)^\kappa,\end{aligned}$$

and the conclusion (2.4) follows.

Finally, in case of $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$, repeating again the same computation for the contributions of $\rho_n \geq \rho$ in (2.16), we find

$$\begin{aligned}&\big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big| \\ &\quad \lesssim \lambda_2(\mathcal{P}) \log\Big(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\Big) + \mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|} \operatorname{diam}(J_{q,\rho})^{p_0}\bigg],\end{aligned}$$

and it remains to estimate the last term. By Hölder's inequality, we can write for any $\kappa \geq 1$,

$$\begin{aligned}&\mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|} \operatorname{diam}(J_{q,\rho})^{p_0}\bigg] \\ &\qquad \leq \mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\bigg]^{1-\frac{1}{\kappa}} \mathbb{E}\bigg[\sum_{q} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|} \operatorname{diam}(J_{q,\rho})^{\kappa p_0}\bigg]^{\frac{1}{\kappa}}.\end{aligned}$$

On the one hand, by definition of $\mathcal{S}'$ and by definition (2.19) of Λ, we get from (2.20),

$$\mathbb{E}\bigg[\sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\bigg] \leq \mathbb{E}\bigg[\sum_{n} \frac{\mathbb{1}_{0\in I_n}}{|I_n|} \mathbb{1}_{\rho_n\leq\rho}\bigg] = \Lambda\big([0,\rho]\big) \lesssim \lambda_2(\mathcal{P}).$$

On the other hand, by the ergodic theorem, using (1.12), the condition (2.3) in $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$, and the fact that there are less clusters than particles, we find

$$\begin{aligned}&\mathbb{E}\bigg[\sum_{q} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|} \operatorname{diam}(J_{q,\rho})^{\kappa p_0}\bigg] = \lim_{R\uparrow\infty} R^{-d} \sum_{q} \frac{|J_{q,\rho}\cap Q_R|}{|J_{q,\rho}|} \operatorname{diam}(J_{q,\rho})^{\kappa p_0} \\ &\leq \limsup_{R\uparrow\infty} \frac{\sharp\{q : J_{q,\rho}\cap Q_R \neq \varnothing\}}{R^d} \frac{1}{\sharp\{q : J_{q,\rho}\cap Q_R \neq \varnothing\}} \sum_{q: J_{q,\rho}\cap Q_R\neq\varnothing} \operatorname{diam}(J_{q,\rho})^{\kappa p_0} \\ &\leq \limsup_{R\uparrow\infty} \frac{\sharp\{n : I_n\cap Q_R \neq \varnothing\}}{R^d} \frac{1}{\sharp\{q : J_{q,\rho}\cap Q_R \neq \varnothing\}} \sum_{q: J_{q,\rho}\cap Q_R\neq\varnothing} \operatorname{diam}(J_{q,\rho})^{\kappa p_0} \\ &\leq \lambda(\mathcal{P})(\mathcal{K}_\kappa)^\kappa,\end{aligned}$$

and the conclusion (2.4) follows. ■

2.3 Preliminary lemmas

Before turning to the proof of Lemmas 2.3 and 2.4, which are key to Theorem 2.1 as explained above, we start with a few preliminary PDE and probabilistic lemmas. We first prove the following trace estimates at particle boundaries.

Lemma 2.5 (Trace estimates). *For all n, we have the following estimates.*

(i) *For any $\psi \in H^1(I_n)$, we have*

$$\inf_{\kappa\in\mathbb{R}^d,\,\Theta\in\mathbb{M}^{\mathrm{skew}}} \int_{\partial I_n} \big|\psi - \big(\kappa + \Theta(x-x_n)\big)\big|^2 \lesssim \int_{I_n} \big|\mathrm{D}(\psi)\big|^2.$$

(ii) *For any $\psi \in H^1(I_n + \rho B)$ satisfying the following relations, and for some $E \in \mathbb{M}_0^{\mathrm{sym}}$,*

$$\begin{cases} -\triangle\psi + \nabla\Sigma = 0, & \text{in } (I_n+\rho B)\setminus I_n,\\ \mathrm{div}(\psi) = 0, & \text{in } (I_n+\rho B)\setminus I_n,\\ \mathrm{D}(\psi) + E = 0, & \text{in } I_n, \end{cases}$$

we have

$$\inf_{c\in\mathbb{R}} \int_{\partial I_n} \big|\sigma(\psi,\Sigma) - c\,\mathrm{Id}\big|^2 \lesssim \int_{I_n+\rho B} \big|\mathrm{D}(\psi)\big|^2,$$

where we recall that multiplicative constants may implicitly depend on ρ.

Proof. We split the proof into two steps.

Step 1. Proof of (i). We appeal to a trace estimate in form of

$$\int_{\partial I_n} \big|\psi - \big(\kappa + \Theta(x-x_n)\big)\big|^2 \lesssim \int_{I_n} \big|\langle\nabla\rangle^{\frac12}\big(\psi - \big(\kappa + \Theta(x-x_n)\big)\big)\big|^2,$$

and the conclusion then follows from Poincaré's and Korn's inequalities.

Step 2. Proof of (ii). By definition of the Cauchy stress tensor, a trace estimate yields

$$\int_{\partial I_n} \big|\sigma(\psi,\Sigma) - c\,\mathrm{Id}\big|^2 \lesssim \int_{(I_n+\frac12\rho B)\setminus I_n} \big|\langle\nabla\rangle^{\frac12}\nabla\psi\big|^2 + \big|\langle\nabla\rangle^{\frac12}(\Sigma - c)\big|^2. \qquad (2.21)$$

By the local regularity theory for the steady Stokes equation near a boundary, e.g. [25, Theorems IV.5.1–IV.5.3], we have for all $m \ge 0$, for all constants $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\begin{aligned} &\|\nabla\psi\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)}\\ &\quad\lesssim \|\psi|_{I_n} - \kappa\|_{H^{m+\frac32}(\partial I_n)} + \|\psi-\kappa\|_{H^1((I_n+\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{\mathrm{L}^2((I_n+\rho B)\setminus I_n)}. \end{aligned}$$

Choosing $c := \fint_{(I_n+\rho B)\setminus I_n} \Sigma$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma - c\,\mathrm{Id}\|_{\mathrm{L}^2((I_n+\rho B)\setminus I_n)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2((I_n+\rho B)\setminus I_n)},$$

so that the above reduces to

$$\begin{aligned} &\|\nabla\psi\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac12\rho B)\setminus I_n)}\\ &\quad\lesssim \|\psi|_{I_n} - \kappa\|_{H^{m+\frac32}(\partial I_n)} + \|\psi-\kappa\|_{H^1((I_n+\rho B)\setminus I_n)}. \end{aligned}$$

As ψ is affine in I_n, we have

$$\|\psi|_{I_n} - \kappa\|_{H^{m+\frac{3}{2}}(\partial I_n)} \lesssim \|\psi - \kappa\|_{H^{m+2}(I_n)} = \|\psi - \kappa\|_{H^1(I_n)},$$

and the above then becomes

$$\|\nabla\psi\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} \lesssim \|\psi - \kappa\|_{H^1(I_n+\rho B)}.$$

Further, choosing $\kappa := \fint_{I_n+\rho B} \psi$ and applying Poincaré's inequality, we deduce

$$\|\nabla\psi\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} + \|\Sigma - c\,\mathrm{Id}\|_{H^{m+1}((I_n+\frac{1}{2}\rho B)\setminus I_n)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2(I_n+\rho B)}.$$

In particular, combined with (2.21), this leads us to

$$\inf_{c\in\mathbb{R}} \int_{\partial I_n} \left|\sigma(\psi,\Sigma) - c\,\mathrm{Id}\right|^2 \lesssim \int_{I_n+\rho B} |\nabla\psi|^2.$$

Noting that $\sigma(\psi,\Sigma)$ and the equations satisfied by (ψ,Σ) are unchanged if a rigid motion is added to ψ, the conclusion now follows from Korn's inequality. ∎

Next, we recall the following standard elliptic regularity estimate for solutions of the free steady Stokes equation.

Lemma 2.6 (Mean-value property). *Given $r > 0$, if (ψ,Σ) is a weak solution of the free Stokes equation in B_r,*

$$-\triangle\psi + \nabla\Sigma = 0, \quad \mathrm{div}(\psi) = 0, \quad \text{in } B_r,$$

then it satisfies

$$\left|\mathrm{D}(\psi)(0)\right|^2 \lesssim \fint_{B_r} \left|\mathrm{D}(\psi)\right|^2.$$

Proof. By scaling, it suffices to consider $r = 1$. For $m > \frac{d}{2}$, the Sobolev embedding yields

$$\left|\mathrm{D}(\psi)(0)\right| \lesssim \left\|\mathrm{D}(\psi)\right\|_{H^m(\frac{1}{2}B)}, \tag{2.22}$$

and it remains to estimate this Sobolev norm. By the local regularity theory for the steady Stokes equation, e.g. [25, Theorem IV.4.1], we find for all $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\|\nabla\psi\|_{H^m(\frac{1}{2}B)} + \|\Sigma - c\|_{H^m(\frac{1}{2}B)} \lesssim \|\psi - \kappa\|_{H^1(B)} + \|\Sigma - c\|_{\mathrm{L}^2(B)}.$$

Choosing $c = \fint_B \Sigma$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma - c\|_{\mathrm{L}^2(B)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2(B)}.$$

Inserting this into the above and applying Poincaré's inequality for the choice $\kappa = \fint_B \psi$, we deduce

$$\|\nabla\psi\|_{H^m(\frac{1}{2}B)} \lesssim \|\nabla\psi\|_{\mathrm{L}^2(B)}.$$

For any $\Theta \in \mathbb{M}^{\mathrm{skew}}$, this entails

$$\|\mathrm{D}(\psi)\|_{H^m(\frac{1}{2}B)} \le \|\nabla(\psi - \Theta x)\|_{H^m(\frac{1}{2}B)} \lesssim \|\nabla(\psi - \Theta x)\|_{\mathrm{L}^2(B)},$$

hence, by Korn's inequality,

$$\|\mathrm{D}(\psi)\|_{H^m(\frac{1}{2}B)} \lesssim \|\mathrm{D}(\psi)\|_{\mathrm{L}^2(B)}.$$

Inserting this into (2.22), the conclusion follows. ■

Finally, the following lemma provides a useful property of Voronoi tessellations. Although it could be obtained as a direct consequence of Palm theory, we include a more elementary proof by means of an approximation argument.

Lemma 2.7 (Property of Voronoi tessellations). *Under Assumption* (H_ρ)*, for all stationary random fields ζ with $\mathbb{E}[|\zeta|] < \infty$, we have*

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\zeta\bigg]. \tag{2.23}$$

In case of ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$)*, we can alternatively decompose*

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\zeta + \sum_{q\in\mathcal{S}'} \frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\int_{W_q}\zeta\bigg]. \tag{2.24}$$

Remark 2.8. In [14], we refer to Lemma 2.7 above[3] in the following slightly different form: given a general stationary ergodic point process $\widetilde{\mathcal{P}} = \{\tilde{x}_n\}_n$, considering spherical inclusions $\widetilde{I}_n = B(\tilde{x}_n)$, denoting by $\{\widetilde{\rho}_n\}_n$ the associated interparticle distances, by $\{\widetilde{V}_n\}_n$ the associated Voronoi cells, and defining

$$\widetilde{\mathcal{S}} := \{n : \widetilde{\rho}_n \ge 1\}$$

(which can possibly be empty), we have for all stationary random fields ζ with $\mathbb{E}[|\zeta|] < \infty$,

$$\mathbb{E}[\zeta] = \mathbb{E}\bigg[\sum_{n\in\widetilde{\mathcal{S}}} \frac{\mathbb{1}_{0\in B(\tilde{x}_n)}}{|B|}\int_{\widetilde{V}_n}\zeta\bigg] + \mathbb{E}\big[\zeta\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{n\in\widetilde{\mathcal{S}}}\widetilde{V}_n}\big]. \tag{2.25}$$

Indeed, given that the restricted inclusion process $\{B(\tilde{x}_n) : n \in \widetilde{\mathcal{S}}\}$ satisfies (H_ρ),

[3] Though with the erroneous reference "Lemma 2.5".

denoting the associated Voronoi cells by $\{\widetilde{V}_n' : n \in \widetilde{\mathcal{S}}\}$, we can apply (2.23) in the form

$$\begin{aligned}\mathbb{E}[\zeta] &= \mathbb{E}\big[\zeta \mathbb{1}_{\bigcup_{m\in\widetilde{\mathcal{S}}}\widetilde{V}_m}\big] + \mathbb{E}\big[\zeta \mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{m\in\widetilde{\mathcal{S}}}\widetilde{V}_m}\big] \\ &= \mathbb{E}\bigg[\sum_{n\in\widetilde{\mathcal{S}}} \frac{\mathbb{1}_{0\in B(\tilde{x}_n)}}{|B|}\int_{\widetilde{V}_n'} \zeta \mathbb{1}_{\bigcup_{m\in\widetilde{\mathcal{S}}}\widetilde{V}_m}\bigg] + \mathbb{E}\big[\zeta \mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{m\in\widetilde{\mathcal{S}}}\widetilde{V}_m}\big],\end{aligned}$$

and identity (2.25) immediately follows since the trivial inclusion

$$\{\tilde{x}_n : n \in \widetilde{\mathcal{S}}\} \subset \widetilde{\mathcal{P}}$$

implies $\widetilde{V}_n \subset \widetilde{V}_n'$ for all n.

Proof. By the monotone convergence theorem, it is enough to prove the result for any bounded nonnegative random field $0 \le \zeta \le M$ with any fixed $M > 0$. Let such a ζ be fixed. We split the proof into two steps.

Step 1. Proof of (2.23) and (2.24) under the additional assumption that almost surely

$$\sup_n \operatorname{diam}(V_n) < \infty. \tag{2.26}$$

In that case, let $K \ge 1$ be such that $\operatorname{diam}(V_n) \le K$ almost surely for all n. We consider (2.23) and (2.24) separately, and split the proof into two further substeps.

Substep 1.1. Proof of (2.23) under assumption (2.26). By the ergodic theorem, we have almost surely

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\zeta\bigg] = \lim_{R\uparrow\infty} R^{-d}\sum_n \frac{|I_n\cap Q_R|}{|I_n|}\int_{V_n}\zeta.$$

As $\zeta \ge 0$ and as assumption (2.26) entails $V_n \subset B_K(x_n)$ for all n, we easily get the two-sided estimate

$$\int_{Q_{R-CK}}\zeta \le \sum_n \frac{|I_n\cap Q_R|}{|I_n|}\int_{V_n}\zeta \le \int_{Q_{R+CK}}\zeta,$$

and the claim (2.23) then follows from the ergodic theorem.

Substep 1.2. Proof of (2.24) under assumption (2.26). By the ergodic theorem, we have almost surely

$$\begin{aligned}&\mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\zeta + \sum_{q\in\mathcal{S}'}\frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\int_{W_q}\zeta\bigg] \\ &\quad= \lim_{R\uparrow\infty} R^{-d}\bigg(\sum_{n\in\mathcal{S}}\frac{|I_n\cap Q_R|}{|I_n|}\int_{V_n}\zeta + \sum_{q\in\mathcal{S}'}\frac{|J_{q,\rho}\cap Q_R|}{|J_{q,\rho}|}\int_{W_q}\zeta\bigg).\end{aligned}$$

As $0 \le \zeta \le M$ and as assumption (2.26) entails $V_n \subset B_K(x_n)$ for all n, we get the two-sided estimate

$$\begin{aligned}
&\int_{Q_{R-CK}} \zeta - M \sum_{q: J_{q,\rho} \cap (Q_{R+1} \setminus Q_R) \neq \varnothing} |W_q| \\
&\quad \le \sum_{n \in \mathcal{S}} \frac{|I_n \cap Q_R|}{|I_n|} \int_{V_n} \zeta + \sum_{q \in \mathcal{S}'} \frac{|J_{q,\rho} \cap Q_R|}{|J_{q,\rho}|} \int_{W_q} \zeta \\
&\quad \le \int_{Q_{R+CK}} \zeta + M \sum_{q: J_{q,\rho} \cap (Q_{R+1} \setminus Q_R) \neq \varnothing} |W_q|.
\end{aligned}$$

By the ergodic theorem, in order to prove (2.24), it thus remains to show almost surely

$$\lim_{R \uparrow \infty} R^{-d} \sum_{q: J_{q,\rho} \cap (Q_{R+1} \setminus Q_R) \neq \varnothing} |W_q| = 0,$$

which would follow provided that we show almost surely

$$\limsup_{R \uparrow \infty} R^{-d} \sum_{q: J_{q,\rho} \cap Q_R \neq \varnothing} |W_q| < \infty. \tag{2.27}$$

As $|W_q| \lesssim (\operatorname{diam}(J_{q,\rho}) + K)^d$, we can estimate

$$\begin{aligned}
R^{-d} \sum_{q: J_{q,\rho} \cap Q_R \neq \varnothing} |W_q| &\lesssim K^d \big(R^{-d} \,\sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\} \big) \\
&\quad \times \bigg(\frac{1}{\sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\}} \sum_{q: J_{q,\rho} \cap Q_R \neq \varnothing} \operatorname{diam}(J_{q,\rho})^d \bigg).
\end{aligned}$$

To bound the first factor, we simply note that

$$\begin{aligned}
R^{-d} \,\sharp\{q : J_{q,\rho} \cap Q_R \neq \varnothing\} &\le R^{-d} \,\sharp\{n : I_n \cap Q_R \neq \varnothing\} \\
&\le R^{-d} \,\sharp\{n : x_n \in Q_{R+1}\} \xrightarrow{R \uparrow \infty} \lambda(\mathcal{P}).
\end{aligned}$$

Appealing to the condition (2.3) in ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$) to estimate the second factor, the claim (2.27) follows.

Step 2. Relaxing assumption (2.26). It remains to consider the case when

$$\sup_n \operatorname{diam}(V_n) = \infty,$$

and we proceed by approximation. Consider a point process

$$\mathcal{P}' = \{x'_n\}_n$$

independent of $\mathcal{P}$ and of $\mathcal{I}$ such that almost surely

$$\min_{n\neq m}|x'_n - x'_m| \geq \frac{1}{2}, \qquad \min_{m:m\neq n}|x'_n - x'_m| \leq 1 \quad \text{for all } n.$$

For instance, $\mathcal{P}'$ can be chosen as the random parking process of parameter $\frac{1}{4}$, cf. [57]. Now, for any integer $\alpha \geq 1$, we define the "enriched" point process $\mathcal{P}_\alpha$ as follows,

$$\mathcal{P}_\alpha := \mathcal{P} \cup \left\{2^{2\alpha}x'_n : \mathrm{dist}(2^{2\alpha}x'_n, \mathcal{P}) \geq 2^{2\alpha+3}\right\},$$

as well as the corresponding random set

$$\mathcal{I}_\alpha := \mathcal{I} \cup \bigcup_{n:\mathrm{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}} B(2^{2\alpha}x'_n).$$

Denote by $V_\alpha(x_n)$ the Voronoi cell associated with I_n in $\mathcal{I}_\alpha$, and by $V_\alpha(x'_n)$ the Voronoi cell associated with $B(2^{2\alpha}x'_n)$. By construction, it can be checked that for all n,

$$V_n \cap B_{2^{2\alpha+2}}(x_n) \subset V_\alpha(x_n) \subset V_n \cap B_{2^{2\alpha+4}}(x_n),$$

which entails that $V_\alpha(x_n) \uparrow V_n$ increasingly as $\alpha \uparrow \infty$ (over integers). In addition, $\mathcal{P}_\alpha, \mathcal{I}_\alpha$ satisfy (H_ρ), as well as (2.26) with Voronoi diameters bounded by $O(2^{2\alpha})$. They further satisfy $(\mathrm{H}^{\mathrm{unif}}_\rho)$, $(\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa})$, or $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$ provided that $\mathcal{P}, \mathcal{I}$ satisfy the corresponding assumption. We focus on the case of $(\mathrm{H}^{\mathrm{unif}}_\rho)$ or $(\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa})$, while the case of $(\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa})$ is analogous. Applying the result (2.23) of Step 1, by definition of $\mathcal{P}_\alpha$, we get

$$\begin{aligned}\mathbb{E}[\zeta] = \mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_\alpha(x_n)}\zeta\Big] \\ + \mathbb{E}\Big[\sum_n \mathbb{1}_{\mathrm{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}}\frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\int_{V_\alpha(x'_n)}\zeta\Big].\end{aligned}$$

As $0 \leq \zeta \leq M$, as Voronoi diameters are bounded by $C2^{2\alpha}$ almost surely, and using stationarity and the independence of $\mathcal{P}, \mathcal{I}$ and $\mathcal{P}', \mathcal{I}'$, the second right-hand side term satisfies

$$\begin{aligned}0 &\leq \mathbb{E}\Big[\sum_n \mathbb{1}_{\mathrm{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}}\frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\int_{V_\alpha(x'_n)}\zeta\Big] \\ &\lesssim M2^{2\alpha d}\,\mathbb{E}\Big[\sum_n \mathbb{1}_{\mathrm{dist}(2^{2\alpha}x'_n,\mathcal{P})\geq 2^{2\alpha+3}}\frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\Big] \\ &= M2^{2\alpha d}\,\mathbb{E}\Big[\sum_n \frac{\mathbb{1}_{0\in B(2^{2\alpha}x'_n)}}{|B|}\Big]\mathbb{P}\big[\mathrm{dist}(0,\mathcal{P}) \geq 2^{2\alpha+3}\big] \\ &= M\lambda(\mathcal{P}')\,\mathbb{P}\big[\mathrm{dist}(0,\mathcal{P}) \geq 2^{2\alpha+3}\big].\end{aligned}$$

Inserting this into the above, we deduce

$$\mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_\alpha(x_n)}\zeta\bigg] \\ \le \mathbb{E}[\zeta] \le \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_\alpha(x_n)}\zeta\bigg] + CM\lambda(\mathcal{P}')\,\mathbb{P}\big[\operatorname{dist}(0,\mathcal{P})\ge 2^{2\alpha+3}\big].$$

and the conclusion (2.23) follows from the monotone convergence theorem. ∎

2.4 Proof of Lemma 2.3

Without loss of generality, we can assume that $\mathcal{E}_{n,D}<\infty$ almost surely as otherwise the claimed estimate (2.13) would be trivial. The variational definition of the effective viscosity (1.24) can be written as

$$\begin{aligned} &E:\bar{\mathbf{B}}E \\ &= |E|^2 + \inf\Big\{\mathbb{E}\big[|\mathrm{D}(\psi)|^2\big] : \psi\in \mathrm{L}^2\big(\Omega; H^1_{\mathrm{loc}}(\mathbb{R}^d)^d\big),\ \nabla\psi \text{ stationary}, \\ &\qquad \operatorname{div}(\psi)=0,\ \big(\mathrm{D}(\psi)+E\big)\big|_{\mathcal{I}}=0,\ \mathbb{E}\big[\mathrm{D}(\psi)\big]=0\Big\}, \end{aligned} \tag{2.28}$$

and the definition (2.5) of $\bar{\mathbf{B}}^1$ as

$$E:\bar{\mathbf{B}}^1E = \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,\infty}\bigg]. \tag{2.29}$$

Note that an energy estimate for (2.11) using Bogovskii's construction yields the uniform bound $\mathcal{E}_{n,\infty}\lesssim|E|^2$. In order to prove (2.13), it remains to compare (2.28) to a superposition of the single-particle problems $\{\mathcal{E}_{n,\infty}\}_n$ and to recognize (2.29). We split the proof into three steps.

Step 1. Upper bound: proof that we have in case of ($\mathrm{H}^{\mathrm{unif}}_\rho$) or ($\mathrm{H}^{\mathrm{mom}}_{\rho,\kappa}$),

$$E:\bar{\mathbf{B}}E \le |E|^2 + \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,D}\bigg], \tag{2.30}$$

and in case of ($\mathrm{H}^{\mathrm{perc}}_{\rho,\kappa}$),

$$E:\bar{\mathbf{B}}E \le |E|^2 + \mathbb{E}\bigg[\sum_{n\in\mathcal{S}} \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,D} + \sum_{q\in\mathcal{S}'}\frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathcal{F}_{q,D}\bigg]. \tag{2.31}$$

We focus on (2.31), the proof of (2.30) being identical. We define almost surely

$$\psi_D := \sum_{n\in\mathcal{S}}\psi_{n,D} + \sum_{q\in\mathcal{S}'}\psi_{q,D} \quad \in H^1_{\mathrm{loc}}(\mathbb{R}^d)^d,$$

where the summands $\psi_{n,D} \in H_0^1(V_n)^d$ and $\psi_{q,D} \in H_0^1(W_q)^d$ are implicitly extended by zero outside V_n and W_q, respectively. Properties of Dirichlet minimizers $\{\psi_{n,D}\}_n$ and $\{\psi_{q,D}\}_q$ ensure that $\nabla\psi_D$ is stationary and that it satisfies $\mathrm{div}(\psi_D) = 0$ and $(\mathrm{D}(\psi_D) + E)|_{\mathcal{I}} = 0$. Assume that $\mathrm{D}(\psi_D) \in \mathrm{L}^2(\Omega)^{d\times d}$ (for otherwise the claim is trivial by (2.24)). Then, appealing to (2.24), we find $\mathbb{E}[\mathrm{D}(\psi_D)] = 0$. We may then use ψ_D as a test function in the variational problem (2.28), to the effect of

$$E : \bar{\mathbf{B}}E \le |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_D)\big|^2\big],$$

and the claim (2.31) now follows from (2.24).

Step 2. Lower bound: proof that

$$E : \bar{\mathbf{B}}E \ge |E|^2 + \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,N}\bigg]. \tag{2.32}$$

By (2.23), we can write

$$\begin{aligned} E : \bar{\mathbf{B}}E &= |E|^2 + \mathbb{E}\big[\big|\mathrm{D}(\psi_E)\big|^2\big] \\ &= |E|^2 + \mathbb{E}\bigg[\sum_n \frac{\mathbb{1}_{0\in I_n}}{|I_n|}\int_{V_n}\big|\mathrm{D}(\psi_E)\big|^2\bigg]. \end{aligned}$$

Using the corrector ψ_E as a test function for the Neumann single-particle problem (2.10), we find $\mathcal{E}_{n,N} \le \int_{V_n}|\mathrm{D}(\psi_E)|^2$ and the claim (2.32) follows.

Step 3. Conclusion. In view of (2.30) and (2.32), it remains to compare $\mathcal{E}_{n,D}$ and $\mathcal{E}_{n,N}$ to $\mathcal{E}_{n,\infty}$. On the one hand, since $\psi_{n,D}$ is an admissible test function for $\mathcal{E}_{n,\infty}$, we have

$$\mathcal{E}_{n,\infty} \le \int_{\mathbb{R}^d}\big|\mathrm{D}(\psi_{n,D})\big|^2 = \int_{V_n}\big|\mathrm{D}(\psi_{n,D})\big|^2 = \mathcal{E}_{n,D}.$$

On the other hand, since the restriction $\psi_{n,\infty}|_{V_n}$ is an admissible test function for $\mathcal{E}_{n,N}$, we have

$$\mathcal{E}_{n,N} \le \int_{V_n}\big|\mathrm{D}(\psi_{n,\infty})\big|^2 \le \mathcal{E}_{n,\infty}.$$

This yields

$$\mathcal{E}_{n,N} \le \mathcal{E}_{n,\infty} \le \mathcal{E}_{n,D},$$

or alternatively,

$$|\mathcal{E}_{n,N} - \mathcal{E}_{n,\infty}| + |\mathcal{E}_{n,D} - \mathcal{E}_{n,\infty}| = \mathcal{E}_{n,D} - \mathcal{E}_{n,N}.$$

Further, note that the minimality of Neumann problems entails

$$\sum_{n:I_n\subset J_{q,\rho}} \mathcal{E}_{n,N} \le \mathcal{F}_{q,D},$$

and thus

$$\mathbb{E}\Big[\sum_{n\notin\mathcal{S}}\frac{\mathbb{1}_{0\in I_n}}{|I_n|}\mathcal{E}_{n,N}\Big]=\mathbb{E}\Big[\sum_{q\in\mathcal{S}'}\frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\sum_{n:I_n\subset J_{q,\rho}}\mathcal{E}_{n,N}\Big]$$
$$\le\mathbb{E}\Big[\sum_{q\in\mathcal{S}'}\frac{\mathbb{1}_{0\in J_{q,\rho}}}{|J_{q,\rho}|}\mathcal{F}_{q,D}\Big].$$

Combining these observations with (2.30) and (2.32), the conclusion (2.13) follows.

2.5 Proof of Lemma 2.4

The bound (2.15) on $\mathcal{F}_{q,D}$ follows from Bogovskii's construction in form of [21, Lemma 4.2]. We turn to the proof of (2.14). By [13, Section 4.1], there exists $w_n \in W_0^{1,\infty}(V_n)^d$ that is an admissible test function for the Dirichlet problem $\mathcal{E}_{n,D}$ such that

$$\mathcal{E}_{n,D}\le\int_{V_n}|\mathrm{D}(w_n)|^2\lesssim\mu(\rho_n),$$

which entails

$$0\le\mathcal{E}_{n,D}-\mathcal{E}_{n,N}\le\mathcal{E}_{n,D}\lesssim\mu(\rho_n).$$

To prove (2.14), it remains to show that in the case $\rho_n\ge 1$ we have

$$\mathcal{E}_{n,D}-\mathcal{E}_{n,N}\lesssim\rho_n^{-d}. \tag{2.33}$$

This amounts to investigating the role of the different boundary conditions on ∂V_n. We assume from now on that $\rho_n\ge 1$ and, without loss of generality, $x_n=0$. We drop the index n to simplify notation and we set $r=\rho_n$ (to avoid confusion with the constant ρ in Assumption (H_ρ) and elsewhere). We split the argument into three steps.

Step 1. Proof that

$$\mathcal{E}_D-\mathcal{E}_N=\int_V|\mathrm{D}(\psi_D-\psi_N)|^2. \tag{2.34}$$

By the Euler–Lagrange equation for ψ_N in form of

$$\int_V\mathrm{D}(\psi_N):\mathrm{D}(\psi_D-\psi_N)=0,$$

we find

$$\int_V|\mathrm{D}(\psi_D)|^2-\int_V|\mathrm{D}(\psi_N)|^2=\int_V\mathrm{D}(\psi_D+\psi_N):\mathrm{D}(\psi_D-\psi_N)$$
$$=\int_V|\mathrm{D}(\psi_D-\psi_N)|^2,$$

that is, (2.34).

Step 2. Proof that

$$\mathcal{E}_D - \mathcal{E}_N \lesssim \Big(\int_{I+\frac{1}{2}B} |\mathrm{D}(\psi_D - \psi_N)|^2 \Big)^{\frac{1}{2}}. \tag{2.35}$$

As in (1.5), the Euler–Lagrange equation for ψ_D takes the following form, in terms of the associated pressure field Σ_D,

$$\begin{cases} -\triangle \psi_D + \nabla \Sigma_D = 0, & \text{in } V \setminus I, \\ \mathrm{div}(\psi_D) = 0, & \text{in } D \setminus I, \\ \psi_D = 0, & \text{on } \partial V, \\ \mathrm{D}(\psi_D + Ex) = 0, & \text{in } I, \\ \int_{\partial I} \sigma(\psi_D + Ex, \Sigma_D)\nu = 0, & \\ \int_{\partial I} \Theta x \cdot \sigma(\psi_D + Ex, \Sigma_D)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{2.36}$$

and similarly the equation for ψ_N is as follows, in terms of the associated pressure Σ_N,

$$\begin{cases} -\triangle \psi_N + \nabla \Sigma_N = 0, & \text{in } V \setminus I, \\ \mathrm{div}(\psi_N) = 0, & \text{in } V \setminus I, \\ \sigma(\psi_N, \Sigma_N)\nu = 0, & \text{on } \partial V, \\ \mathrm{D}(\psi_N + Ex) = 0, & \text{in } I, \\ \int_{\partial I} \sigma(\psi_N + Ex, \Sigma_N)\nu = 0, & \\ \int_{\partial I} \Theta x \cdot \sigma(\psi_N + Ex, \Sigma_N)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{2.37}$$

Testing the above equations for ψ_D and ψ_N, and using boundary conditions, we find the following energy identities,

$$2\int_{V\setminus I} \big|\mathrm{D}(\psi_D)\big|^2 = \int_{V\setminus I} \mathrm{div}\,\big(\sigma(\psi_D, \Sigma_D)\psi_D\big) = -\int_{\partial I} \psi_D \cdot \sigma(\psi_D, \Sigma_D)\nu,$$
$$2\int_{V\setminus I} \big|\mathrm{D}(\psi_N)\big|^2 = \int_{V\setminus I} \mathrm{div}\,\big(\sigma(\psi_N, \Sigma_N)\psi_N\big) = -\int_{\partial I} \psi_N \cdot \sigma(\psi_N, \Sigma_N)\nu,$$

and thus, using boundary conditions for ψ_D, ψ_N on ∂I,

$$2\int_{V\setminus I} \big|\mathrm{D}(\psi_D)\big|^2 = \int_{\partial I} Ex \cdot \sigma(\psi_D, \Sigma_D)\nu,$$
$$2\int_{V\setminus I} \big|\mathrm{D}(\psi_N)\big|^2 = \int_{\partial I} Ex \cdot \sigma(\psi_N, \Sigma_N)\nu.$$

Taking the difference and noticing $|\mathrm{D}(\psi_D)|^2 - |\mathrm{D}(\psi_N)|^2 = 0$ in I, we get

$$\mathcal{E}_D - \mathcal{E}_N = \frac{1}{2}\int_{\partial I} Ex \cdot \sigma(\psi_D - \psi_N, \Sigma_D - \Sigma_N)\nu.$$

Noting that the trace-free condition for E yields $\int_{\partial I} Ex \cdot \nu = 0$, we can add any constant to the pressure field $\Sigma_D - \Sigma_N$ in the right-hand side, and the claim (2.35) then follows from the trace estimate in Lemma 2.5 (ii).

Step 3. Mean-value property with rigid inclusion:

$$\int_{I+\frac{1}{2}B} \big|\mathrm{D}(\psi_D - \psi_N)\big|^2 \lesssim r^{-d} \int_V \big|\mathrm{D}(\psi_D - \psi_N)\big|^2. \tag{2.38}$$

If the difference $\psi := \psi_D - \psi_N$ satisfied the free steady Stokes equation in the whole domain V, then Lemma 2.6 would already yield

$$\int_{I+\frac{1}{2}B} \big|\mathrm{D}(\psi)\big|^2 \lesssim r^{-d} \int_V \big|\mathrm{D}(\psi)\big|^2,$$

that is indeed the claim (2.38). However, ψ is rigid in I and does not satisfy the free steady Stokes equation in the whole domain.[4] To overcome this issue, we shall compare ψ to a suitable proxy: we consider the solution $(\widetilde{\psi}, \widetilde{\Sigma})$ of the following auxiliary Dirichlet problem in V,

$$\begin{cases} -\triangle\widetilde{\psi} + \nabla\widetilde{\Sigma} = 0, & \text{in } V,\\ \mathrm{div}(\widetilde{\psi}) = 0, & \text{in } V,\\ \widetilde{\psi} = \psi, & \text{on } \partial V. \end{cases}$$

Testing this latter equation with $\widetilde{\psi}$ or with ψ, we find

$$2\int_V |\mathrm{D}(\widetilde{\psi})|^2 = \int_{\partial V} \widetilde{\psi} \cdot \widetilde{\sigma}\nu = \int_{\partial V} \psi \cdot \widetilde{\sigma}\nu = 2\int_V \mathrm{D}(\psi) : \mathrm{D}(\widetilde{\psi}),$$

and thus, by the Cauchy–Schwarz inequality,

$$\int_V \big|\mathrm{D}(\widetilde{\psi})\big|^2 \le \int_V \big|\mathrm{D}(\psi)\big|^2. \tag{2.39}$$

Next, for the approximation error, we similarly compute

$$2\int_V \big|\mathrm{D}(\widetilde{\psi} - \psi)\big|^2 = -2\int_V \mathrm{D}(\widetilde{\psi} - \psi) : \mathrm{D}(\psi),$$

and thus, testing the equations (2.36) and (2.37) for $\psi = \psi_D - \psi_N$ with $\widetilde{\psi}$, and using boundary conditions,

$$2\int_V \big|\mathrm{D}(\widetilde{\psi} - \psi)\big|^2 = \int_{\partial I} (\widetilde{\psi} - \psi) \cdot \sigma(\psi, \Sigma)\nu = \int_{\partial I} \widetilde{\psi} \cdot \sigma(\psi, \Sigma)\nu.$$

[4] As shown in Lemma A.2 in Appendix A, the mean-value property can actually be extended in presence of rigid particles. Rather than appealing to this general result here, we provide a self-contained and more elementary approach in the present single-particle setting.

As the boundary conditions for ψ allow to add any rigid motion to $\widetilde{\psi}$ in the right-hand side, and as the incompressibility of $\widetilde{\psi}$ in form of $\int_{\partial I} \widetilde{\psi} \cdot \nu = 0$ allows to add any constant to the pressure field Σ, the trace estimates in Lemma 2.5 lead us to

$$\int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 \lesssim \left(\int_I \left|\mathrm{D}(\widetilde{\psi})\right|^2\right)^{\frac{1}{2}} \left(\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2\right)^{\frac{1}{2}},$$

and thus, decomposing $\psi = \widetilde{\psi} - (\widetilde{\psi} - \psi)$ in the last factor,

$$\int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 \lesssim \int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2.$$

We then deduce

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2 \lesssim \int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2 + \int_V \left|\mathrm{D}(\widetilde{\psi} - \psi)\right|^2 \lesssim \int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2.$$

As $\widetilde{\psi}$ satisfies the free steady Stokes equation in V and as we have $|I + \frac{1}{2}B| \lesssim 1$ and $\mathrm{dist}(I + \frac{1}{2}B, \partial V) \geq \frac{r}{2}$, we may now appeal to Lemma 2.6, to the effect of

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\widetilde{\psi})\right|^2 \lesssim r^{-d} \int_V \left|\mathrm{D}(\widetilde{\psi})\right|^2.$$

The above then becomes

$$\int_{I+\frac{1}{2}B} \left|\mathrm{D}(\psi)\right|^2 \lesssim r^{-d} \int_V \left|\mathrm{D}(\widetilde{\psi})\right|^2,$$

and the claim (2.38) for $\psi = \psi_D - \psi_N$ follows in combination with (2.39).

Step 4. Conclusion. Combining (2.35) and (2.38), we get

$$\mathcal{E}_D - \mathcal{E}_N \lesssim r^{-\frac{d}{2}} \left(\int_V \left|\mathrm{D}(\psi_D - \psi_N)\right|^2\right)^{\frac{1}{2}},$$

and thus, by (2.34),

$$\mathcal{E}_D - \mathcal{E}_N \lesssim r^{-\frac{d}{2}} (\mathcal{E}_D - \mathcal{E}_N)^{\frac{1}{2}},$$

which precisely proves the conclusion (2.33). ∎

2.6 Explicit form of Einstein's formula

This section is devoted to the proof of Proposition 2.2. Under Assumption (Indep), the definition (2.5) of $\bar{\mathbf{B}}^1$ becomes

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P}) \mathbb{E}\left[\int_{\mathbb{R}^d} \left|\mathrm{D}(\psi_E^\circ)\right|^2\right],$$

that is (2.6), in terms of the unique decaying solution ψ_E° of the single-particle problem (2.7). It remains to prove Einstein's formula (2.8) for spherical particles, $I_n = B(x_n, r_n)$, with iid random radii $\{r_n\}_n$. By scaling, the above becomes

$$E : \bar{\mathbf{B}}^1 E = \lambda(\mathcal{P})\mathbb{E}\big[(r_n)^d\big] \int_{\mathbb{R}^d} \big|\mathrm{D}(\widetilde{\psi}_E^\circ)\big|^2,$$

in terms of the unique decaying solution $\widetilde{\psi}_E^\circ$ of the rescaled problem

$$\begin{cases} -\triangle\widetilde{\psi}_E^\circ + \nabla\widetilde{\Sigma}_E^\circ = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \mathrm{div}(\widetilde{\psi}_E^\circ) = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \mathrm{D}(\widetilde{\psi}_E^\circ + Ex) = 0, & \text{in } B, \\ \int_{\partial B} \sigma(\widetilde{\psi}_E^\circ + Ex, \widetilde{\Sigma}_E^\circ)\nu = 0, & \\ \int_{\partial B} \Theta x \cdot \sigma(\widetilde{\psi}_E^\circ + Ex, \widetilde{\Sigma}_E^\circ)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

Alternatively, using the energy identity for this equation,

$$E : \bar{\mathbf{B}}^1 E = \frac{1}{2}\lambda(\mathcal{P})\,\mathbb{E}\big[(r_n)^d\big] \int_{\partial B} Ex \cdot \sigma(\widetilde{\psi}_E^\circ + Ex, \widetilde{\Sigma}_E^\circ)\nu. \tag{2.40}$$

As is well known, e.g. [31, Section 2.1.3], $\widetilde{\psi}_E^\circ$ coincides with the unique solution of

$$\begin{cases} -\triangle\widetilde{\psi}_E^\circ + \nabla\widetilde{\Sigma}_E^\circ = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \mathrm{div}(\widetilde{\psi}_E^\circ) = 0, & \text{in } \mathbb{R}^d \setminus B, \\ \widetilde{\psi}_E^\circ = -Ex, & \text{on } \partial B, \end{cases}$$

and is explicitly given by the following formulas for $|x| \geq 1$,

$$\begin{aligned} \widetilde{\psi}_E^\circ(x) &:= -\frac{d+2}{2}\frac{(x \cdot Ex)x}{|x|^{d+2}} - \frac{1}{2}\bigg(2\frac{Ex}{|x|^{d+2}} - (d+2)\frac{(x \cdot Ex)x}{|x|^{d+4}}\bigg), \\ \widetilde{\Sigma}_E^\circ(x) &:= -(d+2)\frac{x \cdot Ex}{|x|^{d+2}}. \end{aligned}$$

Inserting this into (2.40), a direct computation yields

$$E : \bar{\mathbf{B}}^1 E = \frac{d+2}{2}\,\lambda(\mathcal{P})\,\mathbb{E}\big[(r_n)^d\big]|B||E|^2,$$

that is, Einstein's formula (2.8). ∎