## Chapter 3

# Cluster expansion of the effective viscosity

This chapter is devoted to the higher-order cluster expansion of the effective viscosity $\bar{\mathbf{B}}$: starting from finite-volume approximations, we establish cluster formulas and prove uniform estimates in the large-volume limit. These results are mainly inspired by our previous work [15] on the Clausius–Mossotti conductivity formula, where we introduced the triad consisting of: (1) finite-volume approximation; (2) cluster expansion; (3) uniform $\ell^1 - \ell^2$ energy estimates. We further refine the analysis of [15], in particular improving on error estimates, and properly estimating cluster coefficients in case of uniform particle separation $\ell(\mathcal{P}) \gg 1$. Compared to [15], there are also some new twists due to the rigidity of the inclusions. Henceforth, in the rest of this memoir, we shall assume that particles are uniformly separated in the sense of Assumptions ($\mathrm{H}_\rho$) and ($\mathrm{H}_\rho^{\mathrm{unif}}$).

## 3.1 Finite-volume approximations

In order to make sense of cluster expansions and avoid diverging series, we start by defining finite-volume approximations of the effective viscosity, obtained by a periodization procedure, which will in turn provide an implicit renormalization of cluster coefficients in the large-volume limit. More precisely, we define a restriction $\mathcal{P}_L$ on $Q_L$ of the point process $\mathcal{P}$ via

$$\mathcal{P}_L := \{x_n : n \in N_L\}, \quad N_L := \{n : x_n \in Q_{L,\rho}\}, \quad Q_{L,\rho} := Q_{L-2(\ell(\mathcal{P})\vee(1+\rho))}$$

and we consider the corresponding random set

$$\mathcal{I}_L := \bigcup_{n\in N_L} I_n, \quad I_n = x_n + I_n^\circ. \tag{3.1}$$

For convenience, we choose an enumeration $\mathcal{P}_L = \{x_{n,L}\}_n$ and set $I_{n,L} = x_{n,L} + I_{n,L}^\circ$. Under Assumptions ($\mathrm{H}_\rho$) and ($\mathrm{H}_\rho^{\mathrm{unif}}$), we have the following:

- *Regularity and separation:* For all $L$, the periodized random set $\mathcal{I}_L + L\mathbb{Z}^d$ satisfies the $\rho$-regularity and uniform separation assumptions in ($\mathrm{H}_\rho$) and ($\mathrm{H}_\rho^{\mathrm{unif}}$). Moreover, the periodized point process $\mathcal{P}_L + L\mathbb{Z}^d$ satisfies

$$\ell(\mathcal{P}_L + L\mathbb{Z}^d) \ge \ell(\mathcal{P}) \gtrsim 1.$$

- *Stabilization:* For all $L$, there holds $\mathcal{P}_L|_{Q_{L,\rho}} = \mathcal{P}|_{Q_{L,\rho}}$.

We then define the following finite-volume approximation of the effective viscosity $\bar{\mathbf{B}}$,

$$E : \bar{\mathbf{B}}_L E := \mathbb{E}\Big[ \fint_{Q_L} \big|\mathrm{D}(\psi_{E;L}) + E\big|^2 \Big], \tag{3.2}$$

where $\psi_{E;L} \in \mathrm{L}^2(\Omega; H^1_{\mathrm{per}}(Q_L)^d)$ is almost surely the unique solution in $H^1_{\mathrm{per}}(Q_L)^d$, with vanishing average $\fint_{Q_L} \psi_{E;L} = 0$, of the periodized version of the corrector problem (1.3),

$$\begin{cases} -\triangle \psi_{E;L} + \nabla \Sigma_{E;L} = 0, & \text{in } Q_L \setminus \mathcal{I}_L, \\ \operatorname{div}(\psi_{E;L}) = 0, & \text{in } Q_L \setminus \mathcal{I}_L, \\ \mathrm{D}(\psi_{E;L} + Ex) = 0, & \text{in } \mathcal{I}_L, \\ \int_{\partial I_{n;L}} \sigma_{E;L}\nu = 0, & \forall n, \\ \int_{\partial I_{n;L}} \Theta(x - x_{n;L}) \cdot \sigma_{E;L}\nu = 0, & \forall n,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{3.3}$$

where we use the shorthand notation $\sigma_{E;L} := \sigma(\psi_{E;L} + Ex, \Sigma_{E;L})$ for the Cauchy stress tensor. As a corollary of [18, Theorem 1],[1] in view of the above stabilization property, this finite-volume approximation (3.2) is consistent in the sense of

$$\lim_{L\uparrow\infty} \bar{\mathbf{B}}_L = \bar{\mathbf{B}}. \tag{3.4}$$

As opposed to $\bar{\mathbf{B}}$, we emphasize that the approximation $\bar{\mathbf{B}}_L$ depends only on the finite number of inclusions $\{I_{n,L}\}_n$. Indeed, by ($\mathrm{H}_\rho$), the number of inclusions in $Q_L$ has almost surely a deterministic upper bound $CL^d$. The associated cluster expansion is therefore well defined.

## 3.2 Main results

We start with the cluster expansion of the finite-volume approximation $\bar{\mathbf{B}}_L$, establishing suitable formulas for cluster coefficients and for the remainder. This is analogous to formulas obtained in our previous work on the conductivity problem [15]. While the formula (3.9) for the remainder naturally involves the original corrector with the whole set $\mathcal{P}_L$ of particles, we emphasize that the bound (3.10) only involves correctors associated with finite numbers of inclusions (uniformly in $L$): this is key to the optimal estimates obtained in the sequel and constitutes the first twist w.r.t. [15]. Indeed, this control is based on the rigidity of the inclusions and is therefore not available in the generality considered for the conductivity problem in [15]; it was first observed at second order by Gérard-Varet in [26]. The proof is displayed in Section 3.4.

[1] This requires to replace Dirichlet boundary conditions in [18] by periodic conditions, as is standard in homogenization theory.

**Theorem 3.1** (Finite-volume cluster expansion). *Under Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *finite-volume approximations of the effective viscosity can be expanded for all $L$ and $k \geq 1$,*

$$\bar{\mathbf{B}}_L = \mathrm{Id} + \sum_{j=1}^{k} \frac{1}{j!} \bar{\mathbf{B}}_L^j + R_L^{k+1}, \tag{3.5}$$

*where the coefficients and remainders are defined as follows:*

- *The coefficients $\{\bar{\mathbf{B}}_L^j\}_j$ are given by cluster formulas, cf.* (1.21),

$$E : \bar{\mathbf{B}}_L^j E := j! \sum_{\sharp F = j} \mathbb{E}\Big[ \fint_{Q_L} \delta^F \big( \big| \mathrm{D}(\psi_{E;L}^{\varnothing}) + E \big|^2 \big) \Big], \tag{3.6}$$

*which can be alternatively expressed as*

$$\begin{aligned} E : \bar{\mathbf{B}}_L^j E &= \frac{1}{2} j! \, L^{-d} \sum_{\sharp F = j} \sum_{n \in F} \mathbb{E}\Big[ \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \delta^{F \setminus \{n\}} \sigma_{E;L}^{\{n\}} \nu \Big] \qquad (3.7) \\ &= \frac{1}{2} j! \, L^{-d} \sum_{\sharp F = j} \sum_{n \in F} \mathbb{E}\Big[ \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \big( \psi_{E;L}^{\varnothing} + E(x - x_{n,L}) \big) \cdot \sigma_{E;L}^{F} \nu \Big], \qquad (3.8) \end{aligned}$$

*where, for any $H \subset \mathbb{N}$, $\psi_{E;L}^H$ stands for the solution of the periodized corrector problem* (3.3) *with inclusion set $\mathcal{I}_L$ replaced by $\mathcal{I}_L^H := \bigcup_{n \in H} I_{n,L}$, where we use the shorthand notation $\sigma_{E;L}^H := \sigma(\psi_{E;L}^H + Ex, \Sigma_{E;L}^H)$ for the Cauchy stress tensor, and where we recall the notation of Section* 1.3.1 *for cluster difference operators.*

- *The remainder $R_L^{k+1}$ can be represented as*

$$E : R_L^{k+1} E = \frac{1}{2} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[ \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \big( \psi_{E;L}^{\varnothing} + E(x - x_{n,L}) \big) \cdot \sigma_{E;L} \nu \Big], \tag{3.9}$$

*and is estimated as follows,*

$$\begin{aligned} |E : R_L^{k+1} E| &\lesssim \mathbb{E}\Big[ L^{-d} \sum_n \int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = k \\ n \notin F}} \mathrm{D}(\delta^F \psi_{E;L}^{\varnothing}) \Big|^2 \Big] \\ &\quad + \sum_{j=1}^{k} \mathbb{E}\Big[ L^{-d} \sum_n \Big( \int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = k \\ n \notin F}} \mathrm{D}(\delta^F \psi_{E;L}^{\varnothing}) \Big|^2 \Big)^{\frac{1}{2}} \\ &\quad \times \Big( \int_{I_{n,L} + \rho B} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}\big(\delta^F \big(\psi_{E;L}^{\{n\}} + E(x - x_{n,L})\big)\big) \Big|^2 \Big)^{\frac{1}{2}} \Big]. \end{aligned} \tag{3.10}$$

In view of the short-range setting (1.20), we expect

$$\bar{\mathbf{B}}_L^j = O\big(\lambda_j(\mathcal{P})\big)$$

and we aim to prove uniform-in-$L$ estimates that would allow us to pass to the large-volume limit and to recover a dilute expansion for the original effective viscosity $\bar{\mathbf{B}}$. This is partially achieved in the upcoming theorem, which states fine estimates on cluster coefficients and on the remainder. However, note that we cannot directly obtain uniform-in-$L$ estimates with the desired scalings $O(\lambda_j(\mathcal{P}))$. Instead, the result below is twofold:

— *Uniform estimates:* In (i), we state uniform-in-$L$ estimates, which further display the optimal scaling w.r.t. the minimal distance $\ell = \ell(\mathcal{P})$, but fail to capture the general expected dependence on multi-point intensities $\{\lambda_j(\mathcal{P})\}_j$.

— *Non-uniform estimates:* In (ii), we state non-uniform estimates, which display a logarithmic divergence in the large-volume limit $L \uparrow \infty$, but have the merit of capturing the correct dependence on multi-point intensities.

Uniform estimates in (i) allow to deduce the convergence of cluster coefficients $\{\bar{\mathbf{B}}_L^j\}_j$ in the large-volume limit $L \uparrow \infty$, cf. (3.13) below: this actually *defines* infinite-volume cluster coefficients in a meaningful way, providing an implicit renormalization of diverging series and answering the question raised in Section 1.3.4. As they display the optimal dependence on the minimal distance $\ell = \ell(\mathcal{P})$, these estimates already yield the desired infinite-volume cluster expansion in the large-separation regime $\ell \gg 1$ with $\lambda_j(\mathcal{P})$ replaced by $(\ell^{-d})^j$, which is optimal in some cases (see dilation setting in Theorem 5.4 and Remark 5.5). To treat the general model-free dilute setting, however, uniform estimates need to be further derived with the correct dependence on multi-point intensities: this requires to overcome logarithmic divergences in non-uniform estimates in (ii), which is the subject of Chapter 4. The proof of the present result is split between Sections 3.5, 3.6, 3.7, and 3.8.

**Theorem 3.2** (Cluster estimates and large-volume limit). *Under Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, the coefficients and the remainder of the finite-volume cluster expansion in Theorem* 3.1 *satisfy the following two classes of estimates.*

(i) Uniform estimates: *For all $L$ and $k, j \ge 1$,*

$$|\bar{\mathbf{B}}_L^j| \le j!(C\ell^{-d})^j, \qquad |R_L^{k+1}| \le (C\ell^{-d})^{k+1}. \tag{3.11}$$

(ii) Non-uniform estimates: *For all $L$ and $k, j \ge 1$,*

$$|\bar{\mathbf{B}}_L^j| \lesssim_j \lambda_j(\mathcal{P})\,(\log L)^{j-1}, \qquad |R_L^{k+1}| \lesssim_k \sum_{l=k}^{2k} \lambda_{l+1}(\mathcal{P})(\log L)^l. \tag{3.12}$$

*In particular, as a consequence of (i), for all $k, j \geq 1$, the following large-volume limits are well defined,*

$$\bar{\mathbf{B}}^j := \lim_{L\uparrow\infty} \bar{\mathbf{B}}^j_L, \quad R^{k+1} := \lim_{L\uparrow\infty} R^{k+1}_L, \tag{3.13}$$

*so that the cluster expansion* (3.5) *becomes, for all $k \geq 1$,*

$$\left| \bar{\mathbf{B}} - \left( \mathrm{Id} + \sum_{j=1}^{k} \frac{1}{j!} \bar{\mathbf{B}}^j \right) \right| \leq |R^{k+1}| \leq (C\ell^{-d})^{k+1}.$$

## 3.3 Preliminary lemmas

Henceforth, we fix $E$ with $|E| = 1$ and we skip the associated subscript for notational convenience. Before turning to the proof of Theorems 3.1 and 3.2, we state a series of preliminary lemmas. We start with the following useful reformulation of the corrector equation (1.3), where the rigidity constraint is viewed as generating a source term concentrated at particle boundaries in steady Stokes equations.

**Lemma 3.3** (Reformulation of the corrector equation). *For all $H \subset \mathbb{N}$ we have in $Q_L$,*

$$-\triangle\psi^H_L + \nabla(\Sigma^H_L \mathbb{1}_{Q_L \setminus \mathcal{I}^H_L}) = -\sum_{n\in H} \delta_{\partial I_{n,L}} \sigma^H_L \nu, \tag{3.14}$$

*where $\delta_{\partial I_{n,L}}$ stands for the Hausdorff measure on the boundary of $I_{n,L}$.*[2]

*Proof.* For any test function

$$\phi_L \in C^\infty_{\mathrm{per}}(Q_L)^d,$$

recalling that $\psi^H_L$ is divergence-free in $Q_L$ and that it satisfies $\mathrm{D}(\psi^H_L + Ex) = 0$ in $\mathcal{I}^H_L$, we find

$$\begin{aligned}
&\int_{Q_L} \nabla\phi_L : \nabla\psi^H_L - \int_{Q_L\setminus\mathcal{I}^H_L} \Sigma^H_L \operatorname{div}(\phi_L) \\
&\quad = 2\int_{Q_L} \nabla\phi_L : \mathrm{D}(\psi^H_L) - \int_{Q_L\setminus\mathcal{I}^H_L} \Sigma^H_L \operatorname{div}(\phi_L) \\
&\quad = 2\int_{Q_L} \nabla\phi_L : \mathrm{D}(\psi^H_L + Ex) - \int_{Q_L\setminus\mathcal{I}^H_L} \Sigma^H_L \operatorname{div}(\phi_L) \\
&\quad = \int_{Q_L\setminus\mathcal{I}^H_L} \nabla\phi_L : \sigma(\psi^H_L + Ex, \Sigma^H_L).
\end{aligned}$$

[2] More precisely, we define $\int_{Q_L} \phi_L\, \delta_{\partial I_{n,L}} := \int_{\partial I_{n,L}} \phi_L$ for any test function $\phi_L \in C^\infty_{\mathrm{per}}(Q_L)$.

Since the steady Stokes equation for $\psi_L^H$ reads $\operatorname{div}(\sigma(\psi_L^H+Ex,\Sigma_L^H))=0$ in $Q_L\setminus\mathcal{I}_L^H$, we deduce by integration by parts,

$$\int_{Q_L}\nabla\phi_L:\nabla\psi_L^H-\int_{Q_L\setminus\mathcal{I}_L^H}\Sigma_L^H\operatorname{div}(\phi_L)=-\sum_{n\in H}\int_{\partial I_{n,L}}\phi_L\cdot\sigma(\psi_L^H+Ex,\Sigma_L^H)\nu.$$

By the arbitrariness of $\phi_L$, this proves (3.14) with $\sigma_L^H=\sigma(\psi_L^H+Ex,\Sigma_L^H)$. ∎

Next, the following result provides corresponding Stokes equations for corrector differences, which will be used abundantly in the sequel.

**Lemma 3.4** (Equations for corrector differences). *For all disjoint subsets $F,H\subset\mathbb{N}$ with $F$ finite, we have in $Q_L$,*

$$\begin{aligned}&-\triangle\delta^F\psi_L^H+\nabla\delta^F\big(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big)\\&\qquad=-\sum_{n\in H}\delta_{\partial I_{n,L}}\delta^F\sigma_L^H\nu-\sum_{n\in F}\delta_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\sigma_L^{H\cup\{n\}}\nu.\end{aligned}\tag{3.15}$$

*Proof.* The starting point is equation (3.14) satisfied by $\psi_L^{S\cup H}$,

$$-\triangle\psi_L^{S\cup H}+\nabla\big(\Sigma_L^{S\cup H}\mathbb{1}_{Q_L\setminus\mathcal{I}_L^{S\cup H}}\big)=-\sum_{n\in S\cup H}\delta_{\partial I_{n,L}}\sigma_L^{S\cup H}\nu.$$

Using the definition (1.10) of the difference operator, we deduce

$$-\triangle\delta^F\psi_L^H+\nabla\delta^F\big(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big)=-\sum_{S\subset F}(-1)^{|F\setminus S|}\sum_{n\in S\cup H}\delta_{\partial I_{n,L}}\sigma_L^{S\cup H}\nu,$$

and it remains to reformulate the right-hand side. For that purpose, we decompose

$$\begin{aligned}&-\triangle\delta^F\psi_L^H+\nabla\delta^F\big(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big)\\&=-\sum_{n\in H}\delta_{\partial I_{n,L}}\sum_{S\subset F}(-1)^{|F\setminus S|}\sigma_L^{S\cup H}\nu-\sum_{n\in F}\delta_{\partial I_{n,L}}\sum_{S\subset F}\mathbb{1}_{n\in S}(-1)^{|F\setminus S|}\sigma_L^{S\cup H}\nu.\end{aligned}$$

Changing summation variables and recognizing the definition (1.10) of the difference operator, the conclusion follows. ∎

We now state and prove trace estimates, which constitute an upgraded version of Lemma 2.5. We shall repeatedly appeal to these estimates to control force terms at particle boundaries, which appear in our formulation (3.15) of equations for corrector differences.

**Lemma 3.5** (Trace estimates). *Under Assumptions ($H_\rho$) and ($H_\rho^{\mathrm{unif}}$), for all families $\mathcal{F}$ of finite subsets of $\mathbb{N}$, for all $H\subset\mathbb{N}$ and $n\in\mathbb{N}$ with $n\notin\bigcup_{F\in\mathcal{F}}F$, we have*

$$\inf_{\kappa\in\mathbb{R}^d,\Theta\in\mathbb{M}^{\mathrm{skew}}}\int_{\partial I_{n,L}}\Big|\sum_{F\in\mathcal{F}}\delta^F\psi_L^H-\big(\kappa+\Theta(x-x_{n,L})\big)\Big|^2\lesssim\int_{I_{n,L}}\Big|\sum_{F\in\mathcal{F}}\mathrm{D}(\delta^F\psi_L^H)\Big|^2,$$

*and*

$$\inf_{c\in\mathbb{R}}\int_{\partial I_{n,L}}\Big|\sum_{F\in\mathcal{F}}\delta^F\sigma_L^H-c\,\mathrm{Id}\Big|^2\lesssim\int_{I_{n,L}+\rho B}\Big|\sum_{F\in\mathcal{F}}\mathrm{D}\big(\delta^F(\psi_L^H+Ex)\big)\Big|^2.$$

*Proof.* We split the proof into three steps. We set for abbreviation

$$\psi_L^{\mathcal{F},H}:=\sum_{F\in\mathcal{F}}\delta^F\psi_L^H,\quad\Sigma_L^{\mathcal{F},H}:=\sum_{F\in\mathcal{F}}\delta^F\Sigma_L^H,\quad\sigma_L^{\mathcal{F},H}:=\sum_{F\in\mathcal{F}}\delta^F\sigma_L^H.$$

We also use the shorthand notation $\bar\psi_L^H:=\psi_L^H+Ex$ and $\bar\psi_L^{\mathcal{F},H}:=\sum_{F\in\mathcal{F}}\delta^F\bar\psi_L^H$. This last expression is equal to $\sum_{F\in\mathcal{F}}\delta^F\psi_L^H+Ex$ if $\varnothing\in\mathcal{F}$, and to $\sum_{F\in\mathcal{F}}\delta^F\psi_L^H$ otherwise.

*Step* 1. Proof of the first estimate on $\psi_L^{\mathcal{F},H}$. We appeal to a trace estimate in form of

$$\int_{\partial I_{n,L}}\big|\psi_L^{\mathcal{F},H}-\big(\kappa+\Theta(x-x_{n,L})\big)\big|^2\lesssim\int_{I_{n,L}}\big|\langle\nabla\rangle^{\frac12}\big(\psi_L^{\mathcal{F},H}-\big(\kappa+\Theta(x-x_{n,L})\big)\big)\big|^2,$$

and the conclusion then follows from Poincaré's and Korn's inequalities.

*Step* 2. Proof of the second estimate on $\sigma_L^{\mathcal{F},H}$ in the case $n\notin H$. As $\sigma_L^{\mathcal{F},H}=\sigma(\bar\psi_L^{\mathcal{F},H},\Sigma_L^{\mathcal{F},H})$, a trace estimate yields

$$\int_{\partial I_{n,L}}|\sigma_L^{\mathcal{F},H}-c\,\mathrm{Id}|^2\lesssim\int_{(I_{n,L}+\frac12\rho B)\setminus I_{n,L}}\big|\langle\nabla\rangle^{\frac12}\nabla\bar\psi_L^{\mathcal{F},H}\big|^2+\big|\langle\nabla\rangle^{\frac12}(\Sigma_L^{\mathcal{F},H}-c)\big|^2. \tag{3.16}$$

Given $n\notin H$, as the uniform separation assumption in $(\mathrm{H}_\rho^{\mathrm{unif}})$ ensures that no other particle intersects $I_{n,L}+\rho B$, we note that $(\bar\psi_L^{\mathcal{F},H},\Sigma_L^{\mathcal{F},H})$ satisfies

$$-\triangle\bar\psi_L^{\mathcal{F},H}+\nabla\Sigma_L^{\mathcal{F},H}=0,\quad\text{in } I_{n,L}+\rho B. \tag{3.17}$$

By the local regularity theory for steady Stokes equations, e.g. [25, Theorem IV.4.1], we deduce for all $m\ge0$, for all constants $\kappa\in\mathbb{R}^d$ and $c\in\mathbb{R}$,

$$\begin{aligned}&\|\nabla\bar\psi_L^{\mathcal{F},H}\|_{H^{m+1}(I_{n,L}+\frac12\rho B)}+\|\Sigma_L^{\mathcal{F},H}-c\|_{H^{m+1}(I_{n,L}+\frac12\rho B)}\\&\qquad\lesssim\|\bar\psi_L^{\mathcal{F},H}-\kappa\|_{H^1(I_{n,L}+\rho B)}+\|\Sigma_L^{\mathcal{F},H}-c\|_{\mathrm{L}^2(I_{n,L}+\rho B)}.\end{aligned}$$

Choosing $c:=\fint_{I_{n,L}+\rho B}\Sigma_L^{\mathcal{F},H}$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma_L^{\mathcal{F},H}-c\|_{\mathrm{L}^2(I_{n,L}+\rho B)}\lesssim\|\nabla\bar\psi_L^{\mathcal{F},H}\|_{\mathrm{L}^2(I_{n,L}+\rho B)},$$

so that the above reduces to

$$\begin{aligned}&\|\nabla\bar\psi_L^{\mathcal{F},H}\|_{H^{m+1}(I_{n,L}+\frac12\rho B)}+\|\Sigma_L^{\mathcal{F},H}-c\|_{H^{m+1}(I_{n,L}+\frac12\rho B)}\\&\qquad\lesssim\|\bar\psi_L^{\mathcal{F},H}-\kappa\|_{H^1(I_{n,L}+\rho B)}.\end{aligned}$$

Further choosing $\kappa := \fint_{I_{n,L}+\rho B} \overline{\psi}_L^{\mathcal{F},H}$ and applying Poincaré's inequality, we conclude

$$\begin{aligned}&\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}(I_{n,L}+\frac{1}{2}\rho B)}+\|\Sigma_L^{\mathcal{F},H}-c\|_{H^{m+1}(I_{n,L}+\frac{1}{2}\rho B)}\\&\qquad\lesssim\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{\mathrm{L}^2(I_{n,L}+\rho B)}.\end{aligned}$$

In particular, combining this with (3.16) and noting that the Cauchy stress tensor $\sigma_L^{\mathcal{F},H}$ is unchanged if we add a rigid motion to $\overline{\psi}_L^{\mathcal{F},H}$, the conclusion follows from Korn's inequality.

*Step* 3. Proof of the second estimate on $\sigma_L^{\mathcal{F},H}$ in the case $n \in H$. The starting point is again (3.16). Now, given $n \in H$, we note that $(\overline{\psi}_L^{\mathcal{F},H}, \Sigma_L^{\mathcal{F},H})$ satisfies, instead of (3.17),

$$-\triangle\overline{\psi}_L^{\mathcal{F},H}+\nabla\Sigma_L^{\mathcal{F},H}=0,\quad\text{in }(I_{n,L}+\rho B)\setminus I_{n,L},$$

and $\overline{\psi}_L^{\mathcal{F},H}$ is affine in $I_{n,L}$. By the local regularity theory for the steady Stokes equation near a boundary, e.g. [25, Theorem IV.5.1–5.3], we obtain for all $m \geq 0$, for all constants $\kappa \in \mathbb{R}^d$ and $c \in \mathbb{R}$,

$$\begin{aligned}&\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}+\|\Sigma_L^{\mathcal{F},H}-c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\&\lesssim\|\overline{\psi}_L^{\mathcal{F},H}|_{I_{n,L}}-\kappa\|_{H^{m+\frac{3}{2}}(\partial I_{n,L})}+\|\overline{\psi}_L^{\mathcal{F},H}-\kappa\|_{H^1((I_{n,L}+\rho B)\setminus I_{n,L})}\\&\quad+\|\Sigma_L^{\mathcal{F},H}-c\|_{\mathrm{L}^2((I_{n,L}+\rho B)\setminus I_{n,L})}.\end{aligned}$$

Choosing $c := \fint_{(I_{n,L}+\rho B)\setminus I_{n,L}} \Sigma_L^{\mathcal{F},H}$ and using a local pressure estimate for the steady Stokes equation, e.g. [19, Lemma 3.3], we find

$$\|\Sigma_L^{\mathcal{F},H}-c\|_{\mathrm{L}^2((I_{n,L}+\rho B)\setminus I_{n,L})}\lesssim\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{\mathrm{L}^2((I_{n,L}+\rho B)\setminus I_{n,L})},$$

so that the above reduces to

$$\begin{aligned}&\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}+\|\Sigma_L^{\mathcal{F},H}-c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\&\lesssim\|\overline{\psi}_L^{\mathcal{F},H}|_{I_{n,L}}-\kappa\|_{H^{m+\frac{3}{2}}(\partial I_{n,L})}+\|\overline{\psi}_L^{\mathcal{F},H}-\kappa\|_{H^1((I_{n,L}+\rho B)\setminus I_{n,L})}.\end{aligned}$$

As $\overline{\psi}_L^{\mathcal{F},H}$ is affine in $I_{n,L}$, we have

$$\|\overline{\psi}_L^{\mathcal{F},H}|_{I_{n,L}}-\kappa\|_{H^{m+\frac{3}{2}}(\partial I_{n,L})}\lesssim\|\overline{\psi}_L^{\mathcal{F},H}-\kappa\|_{H^{m+2}(I_{n,L})}=\|\overline{\psi}_L^{\mathcal{F},H}-\kappa\|_{H^1(I_{n,L})},$$

and the above then becomes

$$\begin{aligned}&\|\nabla\overline{\psi}_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}+\|\Sigma_L^{\mathcal{F},H}-c\|_{H^{m+1}((I_{n,L}+\frac{1}{2}\rho B)\setminus I_{n,L})}\\&\lesssim\|\overline{\psi}_L^{\mathcal{F},H}-\kappa\|_{H^1(I_{n,L}+\rho B)}.\end{aligned}$$

Further choosing $\kappa := \fint_{I_{n,L}+\rho B} \bar\psi_L^{\mathcal{F},H}$ and applying Poincaré's inequality, we deduce

$$\begin{aligned}&\|\nabla\bar\psi_L^{\mathcal{F},H}\|_{H^{m+1}((I_{n,L}+\frac12\rho B)\setminus I_{n,L})} + \|\Sigma_L^{\mathcal{F},H} - c\|_{H^{m+1}((I_{n,L}+\frac12\rho B)\setminus I_{n,L})}\\ &\quad\lesssim \|\nabla\bar\psi_L^{\mathcal{F},H}\|_{\mathrm{L}^2(I_{n,L}+\rho B)}.\end{aligned}$$

In particular, combined with (3.16), this yields the conclusion as in Step 2. ∎

## 3.4 Cluster formulas

This section is devoted to the proof of Theorem 3.1. We start by establishing the validity of the expansion (3.5) with coefficients given by formula (3.8) and with the explicit remainder (3.9). The proof is similar to its counterpart for the conductivity problem in our previous work [15].

**Lemma 3.6** (Finite-volume cluster expansion). *Under Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *finite-volume approximations of the effective viscosity can be expanded for all $L$ and $k \geq 1$ as*

$$\bar{\mathbf{B}}_L = \mathrm{Id} + \sum_{j=1}^k \frac{1}{j!}\bar{\mathbf{B}}_L^j + R_L^{k+1}, \qquad (3.18)$$

*where the coefficients $\{\bar{\mathbf{B}}_L^j\}_j$ and the remainder $R_L^{k+1}$ are given by formulas* (3.8) *and* (3.9), *respectively.*

*Proof.* Given $E \in \mathbb{M}_0^{\mathrm{sym}}$ with $|E| = 1$, we recall that we drop the corresponding subscripts in the notation. We split the proof into three steps.

*Step* 1. General strategy. The starting point is formula (3.2) for the finite-volume approximation of the effective viscosity,

$$E : \bar{\mathbf{B}}_L E = 1 + \mathbb{E}\Big[\fint_{Q_L} |\mathrm{D}(\psi_L)|^2\Big].$$

The energy identity for the corrector equation (3.3) takes the form

$$2\int_{Q_L} |\mathrm{D}(\psi_L)|^2 = \sum_n \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L \nu, \qquad (3.19)$$

and thus, further decomposing $\sigma_L = \sigma_L^{\{n\}} + (\sigma_L - \sigma_L^{\{n\}})$, we obtain

$$\begin{aligned}\mathbb{E}\Big[2\int_{Q_L} |\mathrm{D}(\psi_L)|^2\Big] &= \sum_n \mathbb{E}\Big[\int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L^{\{n\}} \nu\Big]\\ &\quad + \sum_n \mathbb{E}\Big[\int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot (\sigma_L - \sigma_L^{\{n\}}) \nu\Big].\end{aligned}$$

In addition, we shall prove below that for all $k \geq 1$,

$$\begin{aligned}\sum_{\sharp F=k}\sum_{n\in F}\mathbb{E}\Big[\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^{\varnothing}+E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\Big]\\ =\sum_{\sharp F=k+1}\sum_{n\in F}\mathbb{E}\Big[\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\psi_L^{\varnothing}\cdot\sigma_L\nu\Big].\end{aligned} \tag{3.20}$$

We note that (3.19) already proves the claim (3.18) for $k=0$. Next, we proceed by induction: if (3.18) holds for some $k \geq 0$, formulas (3.8) and (3.9) for $R_L^{k+1}$, $\bar{\mathbf{B}}_L^{k+1}$ allow us to decompose

$$\begin{aligned}E:R_L^{k+1}E&=\frac{1}{(k+1)!}E:\bar{\mathbf{B}}_L^{k+1}E\\ &+\frac{1}{2}L^{-d}\sum_{\sharp F=k+1}\sum_{n\in F}\mathbb{E}\Big[\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^{\varnothing}+E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\Big].\end{aligned}$$

Inserting identity (3.20), noting that for $\sharp F=k+2$ there holds

$$\delta^{F\setminus\{n\}}\psi_L^{\varnothing}=\delta^{F\setminus\{n\}}\big(\psi_L^{\varnothing}+E(x-x_{n,L})\big),$$

and recognizing formula (3.9) for $R_L^{k+2}$, we deduce

$$R_L^{k+1}=\frac{1}{(k+1)!}\bar{\mathbf{B}}_L^{k+1}+R_L^{k+2},$$

hence the claim (3.18) follows with $k$ replaced by $k+1$. It remains to prove (3.20), which we do in the next two steps.

*Step* 2. Proof that for all $\sharp F=k\geq 1$ and $G\subset F$,

$$\begin{aligned}\sum_{n\in F\setminus G}\int_{\partial I_{n,L}}\big(\psi_L^G+E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\\ =\sum_{n\notin F}\int_{\partial I_{n,L}}(\psi_L^F-\psi_L^G)\cdot\sigma_L\nu.\end{aligned} \tag{3.21}$$

On the one hand, testing equation (3.14) for $\psi_L^G$ with the difference $\psi_L-\psi_L^F$, and using the boundary conditions for $\psi_L$, $\psi_L^F$, $\psi_L^G$ on $\partial I_{n,L}$ with $n\in G\subset F$, we find

$$\int_{Q_L}\nabla(\psi_L-\psi_L^F):\nabla\psi_L^G=-\sum_{n\in G}\int_{\partial I_{n,L}}(\psi_L-\psi_L^F)\cdot\sigma_L^G\nu=0. \tag{3.22}$$

On the other hand, equations (3.14) for $\psi_L$ and $\psi_L^F$ entail

$$\begin{aligned}-\triangle(\psi_L-\psi_L^F)+\nabla\big(\Sigma_L\mathbb{1}_{Q_L\setminus\mathcal{I}_L}-\Sigma_L^F\mathbb{1}_{Q_L\setminus\mathcal{I}_L^F}\big)\\ =-\sum_{n\notin F}\delta_{\partial I_{n,L}}\sigma_L\nu-\sum_{n\in F}\delta_{\partial I_{n,L}}(\sigma_L-\sigma_L^F)\nu,\end{aligned}$$

and thus, testing with $\psi_L^G$ and using the boundary conditions for $\psi_L$, $\psi_L^F$, $\psi_L^G$ on $\partial I_{n,L}$ with $n \in G \subset F$,

$$\begin{aligned}\int_{Q_L} \nabla\psi_L^G : \nabla(\psi_L - \psi_L^F) &= -\sum_{n\notin F}\int_{\partial I_{n,L}} \psi_L^G\cdot\sigma_L\nu - \sum_{n\in F}\int_{\partial I_{n,L}} \psi_L^G\cdot(\sigma_L-\sigma_L^F)\nu \\ &= -\sum_{n\notin F}\int_{\partial I_{n,L}} \psi_L^G\cdot\sigma_L\nu - \sum_{n\in F\setminus G}\int_{\partial I_{n,L}} \psi_L^G\cdot(\sigma_L-\sigma_L^F)\nu \\ &\quad + \sum_{n\in G}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot(\sigma_L-\sigma_L^F)\nu.\end{aligned}$$

Combined with (3.22), this entails

$$\begin{aligned}&\sum_{n\in F\setminus G}\int_{\partial I_{n,L}} \psi_L^G\cdot(\sigma_L-\sigma_L^F)\nu \\ &\quad = \sum_{n\in G}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot(\sigma_L-\sigma_L^F)\nu - \sum_{n\notin F}\int_{\partial I_{n,L}} \psi_L^G\cdot\sigma_L\nu,\end{aligned}$$

or alternatively,

$$\begin{aligned}&\sum_{n\in F\setminus G}\int_{\partial I_{n,L}} \big(\psi_L^G + E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu \\ &\quad = \sum_{n\in F}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot(\sigma_L-\sigma_L^F)\nu - \sum_{n\notin F}\int_{\partial I_{n,L}} \psi_L^G\cdot\sigma_L\nu. \qquad (3.23)\end{aligned}$$

For $G = F$, the left-hand side vanishes, hence

$$\sum_{n\in F}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot(\sigma_L-\sigma_L^F)\nu = \sum_{n\notin F}\int_{\partial I_{n,L}} \psi_L^F\cdot\sigma_L\nu,$$

which allows us to reformulate (3.23) into (3.21).

*Step* 4. Proof of (3.20). Denote by $T_{k,L}$ the left-hand side of (3.20). By the definition (1.10) of the difference operator, we have

$$T_{k,L} = -\sum_{\sharp F=k}\sum_{n\in F}\sum_{G\subset F\setminus\{n\}}(-1)^{|F\setminus G|}\mathbb{E}\Big[\int_{\partial I_{n,L}}\big(\psi_L^G + E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\Big],$$

or alternatively, after changing summation variables,

$$T_{k,L} = -\sum_{\sharp F=k}\sum_{G\subset F}(-1)^{|F\setminus G|}\mathbb{E}\Big[\sum_{n\in F\setminus G}\int_{\partial I_{n,L}}\big(\psi_L^G + E(x-x_{n,L})\big)\cdot(\sigma_L-\sigma_L^F)\nu\Big].$$

We now appeal to (3.21), to the effect of

$$T_{k,L} = -\sum_{\sharp F=k}\sum_{G\subset F}(-1)^{|F\setminus G|}\mathbb{E}\bigg[\sum_{n\notin F}\int_{\partial I_{n,L}}(\psi_L^F-\psi_L^G)\cdot\sigma_L\nu\bigg].$$

Using that $\sum_{G\subset F}(-1)^{|F\setminus G|}=0$ for $F\neq\varnothing$ and recalling the definition (1.10) of the difference operator, this implies

$$\begin{aligned}T_{k,L} &= \sum_{\sharp F=k}\sum_{G\subset F}(-1)^{|F\setminus G|}\mathbb{E}\bigg[\sum_{n\notin F}\int_{\partial I_{n,L}}\psi_L^G\cdot\sigma_L\nu\bigg]\\ &= \sum_{\sharp F=k}\mathbb{E}\bigg[\sum_{n\notin F}\int_{\partial I_{n,L}}\delta^F\psi_L^\varnothing\cdot\sigma_L\nu\bigg],\end{aligned}$$

and the claim (3.20) follows after changing summation variables. ∎

In the above result, we have naturally come up with the definition (3.8) of cluster coefficients $\{\bar{\mathbf{B}}_L^j\}_j$. We now further establish the alternative formulas (3.6) and (3.7). Note that (3.6) coincides with the periodized version of the expected cluster formula (1.21).

**Lemma 3.7** (Equivalent cluster formulas). *Under Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$)*, for all $L$ and $j\geq 1$, the finite-volume cluster coefficient $\bar{\mathbf{B}}_L^j$ defined by formula* (3.8) *is equivalently given by* (3.6) *and* (3.7)*.*

*Proof.* We split the proof into two steps.

*Step* 1. Equivalence of (3.7) and (3.8). It suffices to prove that for all finite $F\subset\mathbb{N}$,

$$\begin{aligned}&\sum_{n\in F}\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^\varnothing+E(x-x_{n,L})\big)\cdot\sigma_L^F\nu\\ &\qquad=\sum_{n\in F}\int_{\partial I_{n,L}}E(x-x_{n,L})\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu.\end{aligned}\tag{3.24}$$

Decomposing $\delta^{F\setminus\{n\}}\psi_L^\varnothing=\delta^{F\setminus\{n\}}\psi_L^{\{n\}}-\delta^F\psi_L^\varnothing$ for $n\in F$ and using the boundary conditions, we find

$$\sum_{n\in F}\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^\varnothing+E(x-x_n)\big)\cdot\sigma_L^F\nu=-\sum_{n\in F}\int_{\partial I_{n,L}}\delta^F\psi_L^\varnothing\cdot\sigma_L^F\nu.$$

Testing equation (3.14) for $\psi_L^F$ with $\delta^F\psi_L^\varnothing$, this becomes

$$\sum_{n\in F}\int_{\partial I_{n,L}}\delta^{F\setminus\{n\}}\big(\psi_L^\varnothing+E(x-x_n)\big)\cdot\sigma_L^F\nu=\int_{Q_L}\nabla\delta^F\psi_L^\varnothing:\nabla\psi_L^F.$$

Now testing equation (3.15) for $\delta^F \psi_L^\varnothing$ with $\psi_L^F$, and using the boundary conditions, we deduce

$$\sum_{n\in F}\int_{\partial I_{n,L}} \delta^{F\setminus\{n\}}\big(\psi_L^\varnothing + E(x-x_n)\big)\cdot\sigma_L^F\nu = -\sum_{n\in F}\int_{\partial I_{n,L}} \psi_L^F\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu$$
$$= \sum_{n\in F}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu,$$

that is, (3.24).

*Step* 2. Equivalence of (3.6) and (3.7). It suffices to prove for all finite $F\subset\mathbb{N}$,

$$\fint_{Q_L}\delta^F\big|\mathrm{D}(\psi_L^\varnothing)\big|^2 = \frac{1}{2}L^{-d}\sum_{n\in F}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\delta^{F\setminus\{n\}}\sigma_L^{\{n\}}\nu. \tag{3.25}$$

Recalling the definition (1.10) of the difference operator, we can write

$$\fint_{Q_L}\delta^F|\mathrm{D}(\psi_L^\varnothing)|^2 = \sum_{G\subset F}(-1)^{|F\setminus G|}\fint_{Q_L}\big|\mathrm{D}(\psi_L^G)\big|^2,$$

which entails, in view of the energy identity for $\psi_L^G$, cf. (3.19),

$$\fint_{Q_L}\delta^F\big|\mathrm{D}(\psi_L^\varnothing)\big|^2 = \frac{1}{2}L^{-d}\sum_{G\subset F}\sum_{n\in G}(-1)^{|F\setminus G|}\int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\sigma_L^G\nu.$$

After changing summation variables and using again the definition (1.10) of the difference operator, this yields the claim (3.25). ■

To conclude the proof of Theorem 3.1, it remains to establish the control (3.10) of the remainder, which is inspired by a recent work of Gérard-Varet [26] and which we prove in the slightly refined form of (3.26) below. This extends the argument of [26] to all $k>2$.

**Lemma 3.8** (Control of the remainder). *Under Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *for all $L$ and $j\ge 1$, the remainder term defined in* (3.9) *can be estimated by*

$$|R_L^{k+1}| \le \mathbb{E}\bigg[L^{-d}\sum_n\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=k\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^\varnothing)\Big|^2\bigg]$$
$$+\sum_{j=1}^k\bigg|\mathbb{E}\bigg[L^{-d}\sum_n\int_{I_{n,L}}\Big(\sum_{\substack{\sharp F=k\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^\varnothing)\Big):\Big(\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\widehat{\psi}_{n,L}^{\{n\}})\Big)\bigg]\bigg|, \tag{3.26}$$

*where in view of* (1.10) *we have defined, with a slight abuse of notation,*

$$\delta^F\widehat{\psi}_{n,L}^{\{n\}} := \sum_{G\subset F}(-1)^{|F\setminus G|}\widehat{\psi}_{n,L}^{G\cup\{n\}},$$

*where for all $H \subset \mathbb{N}$ and $n \in H$ we denote by $\widehat{\psi}_{n,L}^{H}$ the solution of the following Neumann boundary value problem in the inclusion $I_{n,L}$ (unique up to a rigid motion),*

$$\begin{cases} -\triangle \widehat{\psi}_{n,L}^{H} + \nabla \widehat{\Sigma}_{n,L}^{H} = 0, & \text{in } I_{n,L}, \\ \operatorname{div}\big(\widehat{\psi}_{n,L}^{H}\big) = 0, & \text{in } I_{n,L}, \\ \sigma\big(\widehat{\psi}_{n,L}^{H}, \widehat{\Sigma}_{n,L}^{H}\big)\nu = \sigma_{L}^{H}\nu, & \text{on } \partial I_{n,L}. \end{cases} \tag{3.27}$$

*In particular, this yields the bound* (3.10).

*Proof.* We split the proof into two steps, first showing that (3.27) is well posed, and then proving the bound (3.26).

*Step* 1. Proof that the Neumann problem (3.27) is well posed for all $H \subset \mathbb{N}$ and $n \in H$, and that the solution satisfies

$$\int_{I_{n,L}} \big|\mathrm{D}(\widehat{\psi}_{n,L}^{H})\big|^2 \lesssim \int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi_{L}^{H}) + E\big|^2. \tag{3.28}$$

In addition, the proof yields similarly

$$\int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^{F} \widehat{\psi}_{n,L}^{\{n\}})\Big|^2 \lesssim \int_{I_{n,L}+\rho B} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}\big(\delta^{F} (\psi_{L}^{\{n\}} + Ex)\big)\Big|^2.$$

This last estimate entails that the bound (3.10) follows from (3.26).

We turn to the proof of (3.28). The weak formulation of equation (3.27) reads for all $\phi \in H^1(I_{n,L})^d$ with $\operatorname{div}(\phi) = 0$,

$$2 \int_{I_{n,L}} \mathrm{D}(\phi) : \mathrm{D}(\widehat{\psi}_{n,L}^{H}) = \int_{\partial I_{n,L}} \phi \cdot \sigma_{L}^{H}\nu. \tag{3.29}$$

Let us analyze the linear functional defining the right-hand side. Using the incompressibility of $\phi$ in form of $\int_{\partial I_{n,L}} \phi \cdot \nu = 0$, we can add any multiple of the identity matrix to $\sigma_{L}^{H}$. Further noting that the boundary conditions for $\psi_{L}^{H}$ on $\partial I_{n,L}$ with $n \in H$ allow to subtract a rigid motion from the test function $\phi$, we are led to

$$\Big| \int_{\partial I_{n,L}} \phi \cdot \sigma_{L}^{H}\nu \Big| \le \bigg( \inf_{\kappa \in \mathbb{R}^d,\, \Theta \in \mathbb{M}^{\mathrm{skew}}} \int_{\partial I_{n,L}} \big|\phi - \big(\kappa + \Theta(x - x_{n,L})\big)\big|^2 \bigg)^{\frac12} \times \bigg( \inf_{c \in \mathbb{R}} \int_{\partial I_{n,L}} |\sigma_{L}^{H} - c\,\mathrm{Id}\,|^2 \bigg)^{\frac12}.$$

Appealing to the trace estimates of Lemma 3.5, this becomes

$$\Big| \int_{\partial I_{n,L}} \phi \cdot \sigma_{L}^{H}\nu \Big| \lesssim \bigg( \int_{I_{n,L}} \big|\mathrm{D}(\phi)\big|^2 \bigg)^{\frac12} \bigg( \int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi_{L}^{H}) + E\big|^2 \bigg)^{\frac12}. \tag{3.30}$$

This proves that the right-hand side in the weak formulation (3.29) is a continuous linear functional with respect to $\mathrm{D}(\phi) \in \mathrm{L}^2(I_{n,L})^{d\times d}$. The Lax–Milgram theorem then ensures that equation (3.27) is well posed in the sense that it admits a unique solution $\mathrm{D}(\widehat{\psi}^H_{n,L}) \in \mathrm{L}^2(I_{n,L})^{d\times d}$, and the a priori bound (3.28) follows.

*Step* 2. Proof of (3.26). Inserting the energy identity (3.19) and the formula (3.7) for the coefficients, the cluster expansion (3.5) yields the following formula for the remainder,

$$\begin{aligned} E : R_L^{k+1} E &= E : \bar{\mathbf{B}}_L E - 1 - \sum_{j=1}^k \frac{1}{j!} E : \bar{\mathbf{B}}_L^j E \\ &= \frac{1}{2} L^{-d} \mathbb{E}\Big[ \sum_n \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \sigma_L \nu \Big] \\ &\quad - \sum_{j=1}^k \frac{1}{2} L^{-d} \sum_{\sharp F = j} \sum_{n \in F} \mathbb{E}\Big[ \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \delta^{F \setminus \{n\}} \sigma_L^{\{n\}} \nu \Big], \end{aligned}$$

or equivalently, changing summation variables,

$$\begin{aligned} &E : R_L^{k+1} E \\ &= \frac{1}{2} L^{-d} \mathbb{E}\Big[ \sum_n \int_{\partial I_{n,L}} E(x - x_{n,L}) \cdot \Big( \sigma_L - \sum_{j=1}^k \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \sigma_L^{\{n\}} \Big) \nu \Big]. \end{aligned} \tag{3.31}$$

Consider the cluster expansion error

$$\Psi_L^k := \psi_L - \sum_{j=1}^k \sum_{\sharp F = j} \delta^F \psi_L^{\varnothing}, \tag{3.32}$$

$$\Xi_L^k := \Sigma_L \mathbb{1}_{Q_L \setminus \mathcal{I}_L} - \sum_{j=1}^k \sum_{\sharp F = j} \delta^F \big( \Sigma_L^{\varnothing} \mathbb{1}_{Q_L \setminus \mathcal{I}_L^{\varnothing}} \big),$$

and note that in view of (3.15) it satisfies the following equation in $Q_L$,

$$-\triangle \Psi_L^k + \nabla \Xi_L^k = -\sum_n \delta_{\partial I_{n,L}} \Big( \sigma_L - \sum_{j=1}^k \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \sigma_L^{\{n\}} \Big) \nu. \tag{3.33}$$

Testing this equation with $\psi_L$ and using the boundary conditions, the identity (3.31) for the remainder becomes

$$E : R_L^{k+1} E = \mathbb{E}\Big[ \fint_{Q_L} \mathrm{D}(\psi_L) : \mathrm{D}(\Psi_L^k) \Big].$$

Adding and subtracting $\sum_{j=1}^{k}\sum_{\sharp F=j}\delta^F\psi_L^{\varnothing}$ to $\psi_L$, we deduce by (3.32),

$$|E:R_L^{k+1}E|\le\mathbb{E}\bigg[\fint_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\bigg]+\sum_{j=1}^{k}\bigg|\mathbb{E}\bigg[\fint_{Q_L}\mathrm{D}(\Psi_L^k):\sum_{\sharp F=j}\mathrm{D}(\delta^F\psi_L^{\varnothing})\bigg]\bigg|.$$

The conclusion (3.26) then follows from the estimate

$$\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\lesssim\sum_n\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^{\varnothing})\Big|^2,\tag{3.34}$$

and from the identity for all $1\le j\le k$

$$\begin{aligned}\int_{Q_L}&\mathrm{D}(\Psi_L^k):\sum_{\sharp F=j}\mathrm{D}(\delta^F\psi_L^{\varnothing})\\&=\sum_n\int_{I_{n,L}}\Big(\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\psi_L^{\varnothing})\Big):\Big(\sum_{\substack{\sharp F=j-1\\ n\notin F}}\mathrm{D}(\delta^F\widehat{\psi}_{n,L}^{\{n\}})\Big),\end{aligned}\tag{3.35}$$

which we prove in the next two substeps, respectively.

*Substep* 2.1. Proof of (3.34). In view of (3.33), the cluster expansion error $\Psi_L^k$ satisfies

$$-\triangle\Psi_L^k+\nabla\Xi_L^k=0,\quad \mathrm{div}(\Psi_L^k)=0,\quad \text{in } Q_L\setminus\mathcal{I}_L,$$

which entails

$$\begin{aligned}\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2&=\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2+\int_{Q_L\setminus\mathcal{I}_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\\&=\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2-\frac12\sum_n\int_{\partial I_{n,L}}\Psi_L^k\cdot\sigma(\Psi_L^k,\Xi_L^k)\nu.\end{aligned}$$

Hence, using the boundary conditions and the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\lesssim\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2+\sum_n\Big(\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2\Big)^{\frac12}\Big(\int_{I_{n,L}^+}\big|\mathrm{D}(\Psi_L^k)\big|^2\Big)^{\frac12},$$

from which we deduce by Young's inequality,[3]

$$\int_{Q_L}\big|\mathrm{D}(\Psi_L^k)\big|^2\lesssim\sum_n\int_{I_{n,L}}\big|\mathrm{D}(\Psi_L^k)\big|^2.\tag{3.36}$$

[3] As argued in [26], this estimate (3.36) can alternatively be deduced from minimizing properties of Stokes equations for $\Psi_L^k$ in $Q_L\setminus\mathcal{I}_L$ with prescribed symmetric gradient in $\mathcal{I}_L$. We rather give a PDE argument that is more in line with the other arguments of this memoir.

Next, the definition of $\Psi_L^k$ and the rigidity constraint for $\psi_L$ in $I_{n,L}$ yield

$$\mathrm{D}(\Psi_L^k) = -E - \sum_{j=1}^{k} \sum_{\sharp F=j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \quad \text{in } I_{n,L}. \tag{3.37}$$

Distinguishing between the cases $n \in F$ and $n \notin F$, and noting that for $n \in F$ we can decompose

$$\delta^F \psi_L^{\varnothing} = \delta^{F\setminus\{n\}} \psi_L^{\{n\}} - \delta^{F\setminus\{n\}} \psi_L^{\varnothing},$$

we find

$$\sum_{\sharp F=j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) = \sum_{\substack{\sharp F=j \\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) + \sum_{\substack{\sharp F=j-1 \\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\{n\}}) - \sum_{\substack{\sharp F=j-1 \\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}),$$

and thus, in view of the rigidity constraint for $\delta^F \psi_L^{\{n\}}$ in $I_{n,L}$,

$$\sum_{\sharp F=j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) = -E \mathbb{1}_{j=1} + \sum_{\substack{\sharp F=j \\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) - \sum_{\substack{\sharp F=j-1 \\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \quad \text{in } I_{n,L}.$$

Inserting this into (3.37) and recognizing a telescoping sum, we deduce for all $n$,

$$\mathrm{D}(\Psi_L^k) = - \sum_{\substack{\sharp F=j-1 \\ n\notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \quad \text{in } I_{n,L}. \tag{3.38}$$

Combined with (3.36), this yields the claim (3.34).

*Substep* 2.2. Proof of (3.35). Testing equation (3.15) for $\delta^F \psi_L^{\varnothing}$ with $\Psi_L^k$, and changing summation variables, we find

$$\begin{aligned} 2\int_{Q_L} \mathrm{D}(\Psi_L^k) : \sum_{\sharp F=j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) &= -\sum_{\sharp F=j} \sum_{n\in F} \int_{\partial I_{n,L}} \Psi_L^k \cdot \delta^{F\setminus\{n\}} \sigma_L^{\{n\}} \nu \\ &= -\sum_{n} \int_{\partial I_{n,L}} \Psi_L^k \cdot \sum_{\substack{\sharp F=j-1 \\ n\notin F}} \delta^F \sigma_L^{\{n\}} \nu. \end{aligned}$$

In view of equation (3.29) for $\mathrm{D}(\delta^F \hat{\psi}_{n,L}^{\{n\}})$, this can be rewritten as

$$\int_{Q_L} \mathrm{D}(\Psi_L^k) : \sum_{\sharp F=j} \mathrm{D}(\delta^F \psi_L^{\varnothing}) = -\sum_{n} \int_{I_{n,L}} \mathrm{D}(\Psi_L^k) : \sum_{\substack{\sharp F=j-1 \\ n\notin F}} \mathrm{D}(\delta^F \hat{\psi}_{n,L}^{\{n\}}).$$

Combined with (3.38), this yields the claim (3.35). ∎

## 3.5 Uniform $\ell^1-\ell^2$ energy estimates

In order to prove uniform cluster estimates, cf. Theorem 3.2 (i), our main analytical achievement is the following hierarchy of interpolating $\ell^1-\ell^2$ energy estimates for corrector differences, inspired by our previous work [15] on the conductivity problem (which also considers the case of "overlapping particles"; see [20, 30] for refinements in that direction). More precisely, we consider the following quantities, for all $H\subset\mathbb{N}$, all $L$, and $k,j\geq 0$,

$$S_L^H(k,j):=\sum_{\sharp G=k}\fint_{Q_L}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2,$$

$$T_L^H(k,j):=L^{-d}\sum_{\sharp G=k}\sum_{n\notin G\cup H}\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2,$$

and we prove the following result. The novelty with respect to [15] is that we further identify here the optimal dependence on the minimal distance

$$\ell=\ell(\mathcal{P})\gtrsim 1,$$

which appears to be surprisingly challenging and relies on a fine use of elliptic regularity via a duality argument.

**Theorem 3.9** (Uniform $\ell^1-\ell^2$ energy estimates). *Under Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$), *we have for all* $H\subset\mathbb{N}$, *all* $L$, *and* $k,j\geq 0$,

$$S_L^H(k,j)\lesssim\begin{cases}\ell^{-d} & \text{if } k=j=0,\\ (C\ell^{-d})^{2(k+j)-1} & \text{if } k,j\geq 0,\ k+j\geq 1,\end{cases}$$

$$T_L^H(k,j)\lesssim\begin{cases}\ell^{-2d} & \text{if } k=j=0,\\ (C\ell^{-d})^{2(k+j)+1} & \text{if } k,j\geq 0,\ k+j\geq 1.\end{cases}$$

The proof is split into two parts in the following two subsections: to simplify the presentation, we first give a short proof in the spirit of [15] without keeping track of the $\ell$-dependence, and then we establish the estimates in their stated optimal form.

### 3.5.1 Proof of Theorem 3.9 without $\ell$-dependence

This section is devoted to the proof that for all $H\subset\mathbb{N}$, all $L$, and $k,j\geq 0$,

$$S_L^H(k,j)+T_L^H(k,j)\lesssim C^{k+j}. \tag{3.39}$$

For notational convenience, we set $S_L^H(k,j)=T_L^H(k,j)=0$ for $j<0$ or $k<0$. We split the proof into three steps.

*Step* 1. Reduction to $S_L^H$: for all $H \subset \mathbb{N}$ and $L, k, j$,

$$T_L^H(k,j) \lesssim S_L^H(k,j) + S_L^H(k,j-1), \tag{3.40}$$

which entails in particular that it suffices to prove the bound (3.39) for $S_L^H$.

First note that for all maps $f$ and all $n \notin G$ we have

$$\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} f(F\cup G) = \sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}} f(F\cup G) + \sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}} f\big(F\cup G\cup\{n\}\big). \tag{3.41}$$

Using this identity to decompose $T_L^H(j,k)$ and changing summation variables, we find

$$T_L^H(k,j) \lesssim L^{-d} \sum_{\sharp G=k} \sum_{n\notin G\cup H} \int_{I_{n,L}+\rho B} \Big| \sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2$$
$$+ L^{-d} \sum_{\sharp G=k+1} \sum_{n\in G\setminus H} \int_{I_{n,L}+\rho B} \Big| \sum_{\substack{\sharp F=j-1\\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2,$$

and thus, using the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$ and recognizing the definition of $S_L^H$, the claim (3.40) follows.

*Step* 2. Energy estimate for correctors: for all $H \subset \mathbb{N}$,

$$S_L^H(0,0) \lesssim 1. \tag{3.42}$$

As in (3.19), the energy identity for the corrector equation (3.14) for $\psi_L^H$ takes the form

$$2\int_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 = \sum_{n\in H} \int_{\partial I_{n,L}} E(x-x_{n,L})\cdot \sigma_L^H \nu. \tag{3.43}$$

Using the incompressibility constraint $\mathrm{tr}(E) = 0$ to add an arbitrary constant to the pressure in $\sigma_L^H$, as in the proof of (3.30), and then appealing to the trace estimates of Lemma 2.5 (ii), we obtain

$$\int_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \sum_{n\in H} \bigg(\int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi_L^H)+E\big|^2\bigg)^{\frac12}.$$

Since the fattened inclusions $\{I_{n,L}+\rho B\}_n$ are disjoint, the Cauchy–Schwarz inequality then yields, recalling the choice of the periodization (3.1),

$$\int_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \sharp\{n\in H : x_n \in Q_L\}. \tag{3.44}$$

As the right-hand side is bounded by $CL^d$, the claim (3.42) follows. For future reference, we also note that this bound entails, when taking the expectation,

$$\mathbb{E}\Big[\fint_{Q_L} \big|\mathrm{D}(\psi_L)\big|^2\Big] \lesssim \lambda(\mathscr{P}). \tag{3.45}$$

*Step* 3. Key recurrence relation: for all $H \subset \mathbb{N}$ and $k, j \geq 0$,

$$\begin{aligned} S_L^H(k,j) \lesssim \mathbb{1}_{k+j\leq 1} + S_L^H(k+1,j-1) + S_L^H(k,j-1) \\ + S_L^H(k-1,j) + S_L^H(k,j-2) + S_L^H(k-1,j-1), \end{aligned} \tag{3.46}$$

which then leads to the conclusion (3.39) by a direct double induction argument.

Let a finite subset $G \subset \mathbb{N}$ be momentarily fixed. In view of (3.15), the following equation holds in $Q_L$, for any $F \subset \mathbb{N}$ with $F \cap G = \varnothing$,

$$\begin{aligned} -\triangle\delta^{F\cup G}\psi_L^H + \nabla\delta^{F\cup G}\big(\Sigma_L^H \mathbb{1}_{Q_L\setminus \mathcal{I}_L^H}\big) = & -\sum_{n\in H} \delta_{\partial I_{n,L}}\delta^{F\cup G}\sigma_L^H \nu \\ & - \sum_{n\in F\setminus H} \delta_{\partial I_{n,L}}\delta^{(F\setminus\{n\})\cup G}\sigma_L^{H\cup\{n\}}\nu \\ & - \sum_{n\in G\setminus H} \delta_{\partial I_{n,L}}\delta^{F\cup (G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu. \end{aligned}$$

Hence, after summing over $F$ and changing summation variables,

$$\begin{aligned} & -\triangle\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\psi_L^H\Big) + \nabla\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\big(\Sigma_L^H \mathbb{1}_{Q_L\setminus \mathcal{I}_L^H}\big)\Big) \\ & = -\sum_{n\in H}\delta_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\sigma_L^H\nu\Big) - \sum_{n\notin G\cup H}\delta_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j-1\\ F\cap (G\cup\{n\})=\varnothing}} \delta^{F\cup G}\sigma_L^{H\cup\{n\}}\nu\Big) \\ & \quad - \sum_{n\in G\setminus H}\delta_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup (G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu\Big). \end{aligned}$$

Testing this equation with the solution $\sum_{\sharp F=j:F\cap G=\varnothing}\delta^{F\cup G}\psi_L^H$ itself, we obtain the energy identity

$$2\int_{Q_L}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2 = A_L^1(G,j) + A_L^2(G,j) + A_L^3(G,j), \tag{3.47}$$

in terms of

$$A_L^1(G,j) := -\sum_{n\in H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup G}\sigma_L^H\nu\Big), \tag{3.48}$$

$$A_L^2(G,j) := -\sum_{n\notin G\cup H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}\sigma_L^{H\cup\{n\}}\nu\Big),$$

$$A_L^3(G,j) := -\sum_{n\in G\setminus H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup(G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu\Big).$$

We analyze these three contributions separately and we start with the first one. In view of the boundary conditions for $\delta^{F\cup G}\psi_L^H$ on $\partial I_{n,L}$ with $n\in H$, we can rewrite

$$\begin{aligned} A_L^1(G,j) &= \sum_{n\in H}\int_{\partial I_{n,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\big(E(x-x_{n,L})\big)\Big)\cdot\Big(\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\delta^{F\cup G}\sigma_L^H\nu\Big)\\ &= \mathbb{1}_{G=\varnothing,j=0}\sum_{n\in H}\int_{\partial I_{n,L}}E(x-x_{n,L})\cdot\sigma_L^H\nu. \end{aligned}$$

Summing over $G\subset\mathbb{N}$ with $\sharp G=k$, and using the energy identity (3.43), we deduce

$$L^{-d}\sum_{\sharp G=k}A_L^1(G,j) = \mathbb{1}_{k=j=0}\,S_L^H(0,0). \tag{3.49}$$

We turn to the second term $A_L^2$ in (3.47). Using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and then appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned} \big|A_L^2(G,j)\big| \lesssim \sum_{n\notin G\cup H}&\bigg(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\bigg)^{\frac12}\\ &\times\bigg(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}\big(\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex)\big)\Big|^2\bigg)^{\frac12}. \end{aligned} \tag{3.50}$$

Decomposing the second factor via the following identity, for all $n\notin F\cup G\cup H$ and $F\cap G=\varnothing$,

$$\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex) = \mathbb{1}_{G=F=\varnothing}Ex+\delta^{F\cup G}\psi_L^H+\delta^{F\cup G\cup\{n\}}\psi_L^H,$$

summing over $G\subset\mathbb{N}$ with $\sharp G=k$, using the Cauchy–Schwarz inequality, and using the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$, we get

$$\begin{aligned} &L^{-d}\sum_{\sharp G=k}\big|A_L^2(G,j)\big|\\ &\qquad\lesssim \big(S_L^H(k,j)\big)^{\frac12}\big(\mathbb{1}_{k=0,j=1}+T_L^H(k,j-1)+S_L^H(k+1,j-1)\big)^{\frac12}. \end{aligned} \tag{3.51}$$

We turn to the third contribution $A_L^3$ in (3.47). For $n \in G \setminus H$ and $F \cap G = \varnothing$, decomposing

$$\delta^{F\cup G}\psi_L^H = \delta^{F\cup(G\setminus\{n\})}\psi_L^{H\cup\{n\}} - \delta^{F\cup(G\setminus\{n\})}\psi_L^H,$$

and using the boundary conditions, we can rewrite

$$A^3_{L,\ell}(G,j) = \mathbb{1}_{\sharp G=1,j=0} \sum_{n\in G\setminus H} \int_{\partial I_{n,L}} E(x-x_{n,L})\cdot\sigma_L^{H\cup\{n\}}\nu$$
$$+ \sum_{n\in G\setminus H} \int_{\partial I_{n,L}} \Big( \sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup(G\setminus\{n\})}\psi_L^H\Big)\cdot\Big( \sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \delta^{F\cup(G\setminus\{n\})}\sigma_L^{H\cup\{n\}}\nu\Big).$$

Using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and then appealing to the trace estimates of Lemma 3.5, we find

$$\big|A_L^3(G,j)\big| \lesssim \mathbb{1}_{\sharp G=1,j=0} \sum_{n\in G\setminus H}\Big(\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\psi_L^{H\cup\{n\}})+E\big|^2\Big)^{\frac12}$$
$$+ \sum_{n\in G\setminus H}\Big(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup(G\setminus\{n\})}\psi_L^H)\Big|^2\Big)^{\frac12}$$
$$\times\Big(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \mathrm{D}\big(\delta^{F\cup(G\setminus\{n\})}(\psi_L^{H\cup\{n\}}+Ex)\big)\Big|^2\Big)^{\frac12}. \quad (3.52)$$

Decomposing the first right-hand side term and the last factor of the second term via the following identities, for all $n \in G \setminus H$ and $F \cap G = \varnothing$,

$$\psi_L^{H\cup\{n\}} = \psi_L^H + \delta^{\{n\}}\psi_L^H,$$
$$\delta^{F\cup(G\setminus\{n\})}(\psi_L^{H\cup\{n\}}+Ex) = \mathbb{1}_{\sharp G=1,\sharp F=0}Ex + \delta^{F\cup(G\setminus\{n\})}\psi_L^H$$
$$+ \delta^{F\cup(G\setminus\{n\})}\delta^{\{n\}}\psi_L^H, \quad (3.53)$$

summing over $G \subset \mathbb{N}$ with $\sharp G = k$, and using the Cauchy–Schwarz inequality and the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$, this becomes

$$L^{-d}\sum_{\sharp G=k}\big|A_L^3(G,j)\big|$$
$$\lesssim \mathbb{1}_{k=1,j=0}\big(1+S_L^H(0,0)+S_L^H(1,0)\big)^{\frac12}$$
$$+ \big(T_L^H(k-1,j)\big)^{\frac12}\big(\mathbb{1}_{k=1,j=0}+T_L^H(k-1,j)+S_L^H(k,j)\big)^{\frac12}. \quad (3.54)$$

Inserting this into (3.47), together with (3.49) and (3.51), we obtain

$$\begin{aligned} S_L^H(k,j) \lesssim \mathbb{1}_{k=0,j=0}\, S_L^H(0,0) + \mathbb{1}_{k=1,j=0}\big(1 + S_L^H(0,0) + S_L^H(1,0)\big)^{\frac12} \\ + \big(S_L^H(k,j)\big)^{\frac12}\big(\mathbb{1}_{k=0,j=1} + T_L^H(k,j-1) + S_L^H(k+1,j-1)\big)^{\frac12} \\ + \big(T_L^H(k-1,j)\big)^{\frac12}\big(\mathbb{1}_{k=1,j=0} + T_L^H(k-1,j) + S_L^H(k,j)\big)^{\frac12}. \end{aligned}$$

Using Young's inequality to absorb the occurrences of $S_L^H(k,j)$ in the right-hand side into the left-hand side, we are led to

$$\begin{aligned} S_L^H(k,j) \lesssim \mathbb{1}_{k=0,j=0}\, S_L^H(0,0) + \mathbb{1}_{k=0,j=1} + \mathbb{1}_{k=1,j=0}\big(1 + S_L^H(0,0)\big)^{\frac12} \\ + S_L^H(k+1,j-1) + T_L^H(k,j-1) + T_L^H(k-1,j), \end{aligned}$$

and the claim (3.46) now follows in combination with (3.40) and (3.42). ∎

### 3.5.2 Proof of Theorem 3.9 with optimal $\ell$-dependence

It remains to refine the proof of the previous section to capture the optimal dependence on the minimal distance

$$\ell = \ell(\mathcal{P}) \gtrsim 1.$$

The proof involves a new intricate induction argument that combines both $S_L^H$ and $T_L^H$, and the optimal scaling is then captured by a suitable application of elliptic regularity via a duality argument. By the result of the previous section, we may assume $\ell \gg 1$, in which case the uniform separation assumption in ($\mathrm{H}_\rho^{\mathrm{unif}}$) holds in the stronger form of

$$\frac12 \inf_{n\neq m} \operatorname{dist}(I_{n,L}, I_{m,L}) \ge \frac12 \ell - 1 \ge \frac14 \ell \ge \rho, \tag{3.55}$$

and the definition (3.1) of the periodization further ensures

$$\inf_n \operatorname{dist}(I_{n,L}, \partial Q_L) \ge \ell - 1 \ge \frac12 \ell \ge \rho.$$

We split the proof into four steps.

*Step* 1. Energy estimate for correctors: for all $H \subset \mathbb{N}$,

$$S_L^H(0,0) = \fint_{Q_L} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \ell^{-d}, \tag{3.56}$$

$$T_L^H(0,0) = L^{-d} \sum_{n\notin H} \int_{I_{n,L}+\rho B} \big|\mathrm{D}(\psi_L^H)\big|^2 \lesssim \ell^{-2d}. \tag{3.57}$$

By the $\ell$-separation property (3.55), the number of points of the process $\mathscr{P}_L$ in $Q_L$ is bounded by $C(L/\ell)^d$, so that the first estimate (3.56) follows from (3.44). It remains to prove (3.57). For that purpose, first note that for $n \notin H$ the $\ell$-separation property (3.55) entails that the following free steady Stokes equations hold in $I_{n,L} + \frac{1}{4}\ell B \subset Q_L \setminus \mathcal{I}_L^H$,

$$-\triangle\psi_L^H + \nabla\Sigma_L^H = 0, \quad \operatorname{div}(\psi_L^H) = 0, \quad \text{in } I_{n,L} + \frac{1}{4}\ell B.$$

Elliptic regularity in form of Lemma 2.6 then yields

$$\int_{I_{n,L}+\rho B} \left|\mathrm{D}(\psi_L^H)\right|^2 \lesssim \ell^{-d} \int_{I_{n,L}+\frac{1}{4}\ell B} \left|\mathrm{D}(\psi_L^H)\right|^2.$$

Summing this over $n \notin H$ and using the $\ell$-separation property (3.55) in form of the disjointness of the fattened inclusions $\{I_{n,L} + \frac{1}{4}\ell B\}_n$, we deduce

$$\sum_{n\notin H} \int_{I_{n,L}+\rho B} \left|\mathrm{D}(\psi_L^H)\right|^2 \lesssim \ell^{-d} \int_{Q_L} \left|\mathrm{D}(\psi_L^H)\right|^2,$$

and the claim (3.57) now follows from (3.56).

*Step* 2. Recurrence relation for $S_L^H$: for all $H \subset \mathbb{N}$ and $k, j \geq 0$,

$$\begin{aligned} S_L^H(k,j) \lesssim \mathbb{1}_{k+j\leq 1}\ell^{-d} &+ S_L^H(k+1, j-1) \\ &+ T_L^H(k,j) + T_L^H(k, j-1) \\ &+ T_L^H(k-1, j). \end{aligned} \tag{3.58}$$

This provides a refined version of the recurrence relation (3.46), which can indeed be recovered by appealing to (3.40) to bound $T_L^H$ in terms of $S_L^H$. The present refined version will be combined with a recurrence relation for $T_L^H$ in the next step.

Let $G \subset \mathbb{N}$ be momentarily fixed. As in the proof of (3.46), the starting point is identity (3.47), that is,

$$2\int_{Q_L} \Big| \sum_{\substack{\sharp F=j \\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2 = A_L^1(G,j) + A_L^2(G,j) + A_L^3(G,j), \tag{3.59}$$

where we recall that $A_L^1, A_L^2, A_L^3$ are defined in (3.48). We analyze these contributions separately. The first one satisfies (3.49), and thus, combined with the energy estimate (3.56),

$$L^{-d} \sum_{\sharp G=k} A_L^1(G,j) = \mathbb{1}_{k=j=0}\, S_L^H(0,0) \lesssim \mathbb{1}_{k=j=0}\ell^{-d}. \tag{3.60}$$

It remains to prove refined versions of (3.51) and (3.54) for $A_L^2$ and $A_L^3$, and we start with the contribution of $A_L^2$. The starting point is the trace estimate (3.50) used in the proof of (3.51), that is,

$$\big|A_L^2(G,j)\big| \lesssim \sum_{n\notin G\cup H} \bigg(\int_{I_{n,L}} \Big| \sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\bigg)^{\frac12} \times \bigg(\int_{I_{n,L}+\rho B} \Big| \sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}\big(\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex)\big)\Big|^2\bigg)^{\frac12},$$

which we shall now analyze more carefully. Using identity (3.41) to decompose the first factor, and decomposing the second factor via the following identity, for all $n\notin F\cup G\cup H$ and $F\cap G=\varnothing$,

$$\delta^{F\cup G}(\psi_L^{H\cup\{n\}}+Ex) = \mathbb{1}_{G=F=\varnothing}Ex + \delta^{F\cup G}\psi_L^H + \delta^{F\cup G\cup\{n\}}\psi_L^H,$$

we find

$$\begin{aligned}|A_L^2(G,j)| \lesssim \sum_{n\notin G\cup H} \bigg(&\int_{I_{n,L}} \underbrace{\Big| \sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2}_{\clubsuit} \\ &+ \underbrace{\Big| \sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G\cup\{n\}}\psi_L^H)\Big|^2}_{\diamondsuit}\bigg)^{\frac12} \\ &\times \bigg(\mathbb{1}_{\sharp G=0,j=1} + \int_{I_{n,L}+\rho B} \underbrace{\Big| \sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2}_{\spadesuit} \\ &+ \underbrace{\Big| \sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G\cup\{n\}}\psi_L^H)\Big|^2}_{\diamondsuit}\bigg)^{\frac12}.\end{aligned}$$

Summing over $G\subset\mathbb{N}$ with $\sharp G=k$, using Young's inequality, using the separation property in form of the disjointness of the fattened inclusions $\{I_{n,L}+\rho B\}_n$, using that the number of points of the process $\mathcal{P}_L$ in $Q_L$ is bounded by $C(L/\ell)^d$, and reorganizing the terms, we conclude

$$L^{-d}\sum_{\sharp G=k}\big|A_L^2(G,j)\big|\lesssim \ell^{-d}\mathbb{1}_{k=0,j=1}+S_L^H(k+1,j-1)\\ +T_L^H(k,j)+T_L^H(k,j-1), \tag{3.61}$$

where the last three right-hand side terms come from ◇, ♣, ♠, respectively.

We turn to the contribution of $A_L^3$. The starting point is the trace estimate (3.52) used in the proof of (3.54). Further, using the decomposition (3.53), this estimate becomes

$$\begin{aligned}&\big|A_L^3(G,j)\big|\\ &\lesssim \mathbb{1}_{\sharp G=1,j=0}\sum_{n\in G\setminus H}\Big(1+\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\psi_L^H)\big|^2+\big|\mathrm{D}(\delta^{\{n\}}\psi_L^H)\big|^2\Big)^{\frac12}\\ &\quad+\sum_{n\in G\setminus H}\Big(\int_{I_{n,L}}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{n\})}\psi_L^H)\Big|^2\Big)^{\frac12}\\ &\quad\times\Big(\int_{I_{n,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{n\})}\psi_L^H)\Big|^2+\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\Big)^{\frac12}.\end{aligned} \tag{3.62}$$

Summing the first right-hand side term over $G\subset\mathbb{N}$ with $\sharp G=1$, using the Cauchy–Schwarz inequality, recalling that the number of points of the process $\mathcal{P}_L$ in $Q_L$ is bounded by $C(L/\ell)^d$, and appealing to the energy estimate (3.57), we find

$$\begin{aligned}&\sum_{n\notin H}\Big(1+\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\psi_L^H)\big|^2+\big|\mathrm{D}(\delta^{\{n\}}\psi_L^H)\big|^2\Big)^{\frac12}\\ &\quad\lesssim L^{\frac d2}\ell^{-\frac d2}\Big(L^d\ell^{-d}+\sum_{n\notin H}\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\psi_L^H)\big|^2+\sum_{n\notin H}\int_{Q_L}\big|\mathrm{D}(\delta^{\{n\}}\psi_L^H)\big|^2\Big)^{\frac12}\\ &\quad\lesssim L^d\big(\ell^{-2d}+\ell^{-d}S_L^H(1,0)\big)^{\frac12}.\end{aligned}$$

Now summing (3.62) over $G\subset\mathbb{N}$ with $\sharp G=k$, inserting the above estimate for the first right-hand side term, and using the Cauchy–Schwarz inequality, we find

$$L^{-d}\sum_{\sharp G=k}\big|A_L^3(G,j)\big|\lesssim \mathbb{1}_{k=1,j=0}\big(\ell^{-2d}+\ell^{-d}S_L^H(1,0)\big)^{\frac12}\\ +\big(T_L^H(k-1,j)\big)^{\frac12}\big(T_L^H(k-1,j)+S_L^H(k,j)\big)^{\frac12}.$$

Inserting this into (3.59), together with (3.60) and (3.61), we conclude

$$\begin{aligned} S_L^H(k,j) \lesssim\ & \mathbb{1}_{k=0,j\le 1}\ell^{-d} + \mathbb{1}_{k=1,j=0}\big(\ell^{-2d} + \ell^{-d} S_L^H(1,0)\big)^{\frac{1}{2}} \\ & + S_L^H(k+1,j-1) + T_L^H(k,j) + T_L^H(k,j-1) \\ & + \big(T_L^H(k-1,j)\big)^{\frac{1}{2}}\big(T_L^H(k-1,j) + S_L^H(k,j)\big)^{\frac{1}{2}}. \end{aligned}$$

Using Young's inequality to absorb the occurrence of $S_L^H(k,j)$ in the right-hand side into the left-hand side, the claim (3.58) follows.

*Step* 3. Recurrence relation for $T_L^H$: for all $H \subset \mathbb{N}$ and $k, j \ge 0$,

$$\begin{aligned} T_L^H(k,j) \lesssim\ & \mathbb{1}_{k=j=0}\ell^{-2d} + \mathbb{1}_{k+j=1}\ell^{-3d} \\ & + \ell^{-2d}\big(T_L^H(k-1,j) + T_L^H(k,j-1) + T_L^H(k+1,j-2) \\ & + S_L^H(k,j) + S_L^H(k+1,j-1) + S_L^H(k+2,j-2)\big). \end{aligned} \tag{3.63}$$

Let $k, j \ge 0$ be fixed with $k + j \ge 1$ (the case $k = j = 0$ already follows from (3.57)). For $G \subset \mathbb{N}$ and $n \notin G$, the $\ell$-separation property (3.55) implies that the following free steady Stokes equations hold in $I_{n,L} + \frac{1}{4}\ell B$,

$$\begin{aligned} & -\triangle\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}\psi_L^H\Big) + \nabla\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}(\Sigma_L^H \mathbb{1}_{Q_L\setminus \mathcal{I}_L^H})\Big) = 0, \\ & \operatorname{div}\Big(\sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \delta^{F\cup G}\psi_L^H\Big) = 0, \quad \text{in } I_{n,L} + \frac{1}{4}\ell B, \end{aligned}$$

so that elliptic regularity in form of Lemma 2.6 yields

$$T_L^H(k,j) \lesssim L^{-d}\ell^{-d} \sum_{\sharp G=k}\sum_{n\notin G\cup H} \int_{I_{n,L}+\frac{1}{4}\ell B} \Big| \sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2. \tag{3.64}$$

In order to analyze the right-hand side, we shall appeal to elliptic regularity a second time, now via a duality argument. For that purpose, we use the following dual representation

$$\begin{aligned} & \sum_{\sharp G=k}\sum_{n\notin G\cup H} \int_{I_{n,L}+\frac{1}{4}\ell B} \Big| \sum_{\substack{\sharp F=j \\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2 \\ & \quad = \sup_{\alpha,h}\Big\{ I(\alpha,h)^2 : \sum_{\sharp G=k}\sum_{n\notin G\cup H} |\alpha_{n,G}|^2 = 1, \\ & \qquad\qquad \int_{Q_L} |h_{n,G}|^2 = 1,\ \operatorname{supp} h_{n,G} \subset I_{n,L} + \frac{1}{4}\ell B,\ \forall n, G\Big\}, \end{aligned} \tag{3.65}$$

where for any $\alpha=\{\alpha_{n,G}\}_{n,G}\subset\mathbb{R}$ and $h=\{h_{n,G}\}_{n,G}\subset\mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}$ we have set for abbreviation

$$I(\alpha,h):=\sum_{\sharp G=k}\sum_{n\notin G\cup H}\alpha_{n,G}\int_{Q_L}h_{n,G}:\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big).\qquad(3.66)$$

Let $\alpha=\{\alpha_{n,G}\}_{n,G}\subset\mathbb{R}$ and $h=\{h_{n,G}\}_{n,G}\subset\mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}$ be momentarily fixed, satisfying the constraints in (3.65),

$$\sum_{\sharp G=k}\sum_{n\notin G\cup H}|\alpha_{n,G}|^2=1,\qquad\int_{Q_L}|h_{n,G}|^2=1,\qquad\operatorname{supp}h_{n,G}\subset I_{n,L}+\frac14\ell B,\quad\forall n,G.\qquad(3.67)$$

For $n\notin G\cup H$, consider the periodic solution $w_{h,n,G}$ of the following auxiliary steady Stokes problem,

$$\begin{cases}-\triangle w_{h,n,G}+\nabla P_{h,n,G}=\operatorname{div}(h_{n,G}), & \text{in } Q_L\setminus\mathcal{I}_L^H,\\ \operatorname{div}(w_{h,n,G})=0, & \text{in } Q_L\setminus\mathcal{I}_L^H,\\ \mathrm{D}(w_{h,n,G})=0, & \text{in } \mathcal{I}_L^H,\\ \int_{\partial I_{m,L}}\sigma(w_{h,n,G},P_{h,n,G})\nu=0, & \forall m\in H,\\ \int_{\partial I_{m,L}}\Theta(x-x_{m,L})\cdot\sigma(w_{h,n,G},P_{h,n,G})\nu=0, & \forall\Theta\in\mathbb{M}^{\mathrm{skew}},\ \forall m\in H.\end{cases}\qquad(3.68)$$

Note that this problem is well posed since $h_{n,G}$ is supported in

$$I_{n,L}+\frac14\ell B\subset Q_L\setminus\mathcal{I}_L^H.$$

The same argument as for (3.14) shows that $w_{h,n,G}$ satisfies in $Q_L$,

$$-\triangle w_{h,n,G}+\nabla\big(P_{h,n,G}\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H}\big)=\operatorname{div}(h_{n,G})-\sum_{m\in H}\delta_{\partial I_{m,L}}\sigma(w_{h,n,G},P_{h,n,G})\nu,$$

and, appealing to (3.15) and changing summation variables, we also find in $Q_L$,

$$\begin{aligned}-\triangle\Big(&\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}\psi_L^H\Big)+\nabla\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}(\Sigma_L^H\mathbb{1}_{Q_L\setminus\mathcal{I}_L^H})\Big)\\ &=-\sum_{m\in H}\delta_{\partial I_{m,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup G}\sigma_L^H\nu\Big)\\ &\quad-\sum_{m\in G\setminus H}\delta_{\partial I_{m,L}}\Big(\sum_{\substack{\sharp F=j\\ F\cap(G\cup\{n\})=\varnothing}}\delta^{F\cup(G\setminus\{m\})}\sigma_L^{H\cup\{m\}}\nu\Big)\\ &\quad-\sum_{m\notin G\cup H\cup\{n\}}\delta_{\partial I_{m,L}}\Big(\sum_{\substack{\sharp F=j-1\\ F\cap(G\cup\{n,m\})=\varnothing}}\delta^{F\cup G}\sigma_L^{H\cup\{m\}}\nu\Big).\end{aligned}$$

Testing the second of these two equations with the solution of the first one, and vice versa, and using the boundary conditions, we can reformulate $I(\alpha, h)$ in (3.66) as follows, provided $k + j \geq 1$,

$$\begin{aligned} I(\alpha, h) &= -\sum_{\sharp G = k} \sum_{n \notin G \cup H} 2\alpha_{n,G} \int_{Q_L} \mathrm{D}(w_{h,n,G}) : \Big( \sum_{\substack{\sharp F = j \\ F \cap (G \cup \{n\}) = \varnothing}} \mathrm{D}(\delta^{F \cup G} \psi_L^H) \Big) \\ &= I_1(\alpha, h) + I_2(\alpha, h), \end{aligned} \tag{3.69}$$

where we have set

$$\begin{aligned} &I_1(\alpha, h) \\ &:= \sum_{\sharp G = k} \sum_{n \notin G \cup H} \alpha_{n,G} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} w_{h,n,G} \cdot \Big( \sum_{\substack{\sharp F = j \\ F \cap (G \cup \{n\}) = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big), \\ &I_2(\alpha, h) \\ &:= \sum_{\sharp G = k} \sum_{n \notin G \cup H} \alpha_{n,G} \sum_{m \notin G \cup H \cup \{n\}} \int_{\partial I_{m,L}} w_{h,n,G} \cdot \Big( \sum_{\substack{\sharp F = j-1 \\ F \cap (G \cup \{n,m\}) = \varnothing}} \delta^{F \cup G} \sigma_L^{H \cup \{m\}} \nu \Big). \end{aligned}$$

We only treat $I_1(\alpha, h)$ in detail since the argument for $I_2(\alpha, h)$ is similar. Appealing to identity (3.41), we can rewrite

$$\begin{aligned} &I_1(\alpha, h) \\ &= \sum_{\sharp G = k} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} \Big( \sum_{n \notin G \cup H} \alpha_{n,G} w_{h,n,G} \Big) \cdot \Big( \sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big) \\ &- \sum_{\sharp G = k} \sum_{n \notin G \cup H} \alpha_{n,G} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} w_{h,n,G} \cdot \Big( \sum_{\substack{\sharp F = j-1 \\ F \cap (G \cup \{n\}) = \varnothing}} \delta^{F \cup (G \setminus \{m\}) \cup \{n\}} \sigma_L^{H \cup \{m\}} \nu \Big), \end{aligned}$$

or equivalently, after further changing summation variables in the second term,

$$\begin{aligned} I_1(\alpha, h) = &\sum_{\sharp G = k} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} \Big( \sum_{n \notin G \cup H} \alpha_{n,G} w_{h,n,G} \Big) \\ &\qquad \cdot \Big( \sum_{\substack{\sharp F = j \\ F \cap G = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big) \\ &- \sum_{\sharp G = k+1} \sum_{m \in G \setminus H} \int_{\partial I_{m,L}} \Big( \sum_{n \in G \setminus (H \cup \{m\})} \alpha_{n, G \setminus \{n\}} w_{h,n,G \setminus \{n\}} \Big) \\ &\qquad \cdot \Big( \sum_{\substack{\sharp F = j-1 \\ F \cap G = \varnothing}} \delta^{F \cup (G \setminus \{m\})} \sigma_L^{H \cup \{m\}} \nu \Big). \end{aligned}$$

Now using the boundary conditions and the incompressibility constraints to add arbitrary constants to the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 3.5, we are led to

$$\big|I_1(\alpha,h)\big| \lesssim I_{1,1}(\alpha,h) + I_{1,2}(\alpha,h), \tag{3.70}$$

where we have set

$$\begin{aligned} I_{1,1}(\alpha,h) := \sum_{\sharp G=k}\sum_{m\in G\setminus H}\bigg(\int_{I_{m,L}}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2\bigg)^{\frac12} \\ \times\bigg(\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}\big(\delta^{F\cup(G\setminus\{m\})}(\psi_L^{H\cup\{m\}}+Ex)\big)\Big|^2\bigg)^{\frac12}, \\ I_{1,2}(\alpha,h) := \sum_{\sharp G=k+1}\sum_{m\in G\setminus H}\bigg(\int_{I_{m,L}}\Big|\sum_{n\in G\setminus(H\cup\{m\})}\alpha_{n,G\setminus\{n\}}\mathrm{D}(w_{h,n,G\setminus\{n\}})\Big|^2\bigg)^{\frac12} \\ \times\bigg(\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j-1\\ F\cap G=\varnothing}}\mathrm{D}\big(\delta^{F\cup(G\setminus\{m\})}(\psi_L^{H\cup\{m\}}+Ex)\big)\Big|^2\bigg)^{\frac12}. \end{aligned}$$

We start by estimating $I_{1,1}(\alpha,h)$. Decomposing the second factor via the following identity, for all $m\in G\setminus H$ and $F\cap G=\varnothing$,

$$\delta^{F\cup(G\setminus\{m\})}(\psi_L^{H\cup\{m\}}+Ex) = \mathbb{1}_{\sharp G=1,\sharp F=0}Ex + \delta^{F\cup(G\setminus\{m\})}\psi_L^H + \delta^{F\cup G}\psi_L^H,$$

noting that the $\ell$-separation property (3.55) entails that $\sum_{n\notin G\cup H}\alpha_{n,G}w_{h,n,G}$ satisfies the free steady Stokes equations in $I_{m,L}+\frac14\ell B$ for all $m\notin H$, and appealing to elliptic regularity in form of Lemma 2.6, we find

$$\begin{aligned} I_{1,1}(\alpha,h) \lesssim \sum_{\sharp G=k}\sum_{m\in G\setminus H}\bigg(\ell^{-d}\int_{I_{m,L}+\frac14\ell B}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2\bigg)^{\frac12} \\ \times\bigg(\mathbb{1}_{k=1,j=0}+\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup(G\setminus\{m\})}\psi_L^H)\Big|^2 \\ +\int_{I_{m,L}+\rho B}\Big|\sum_{\substack{\sharp F=j\\ F\cap G=\varnothing}}\mathrm{D}(\delta^{F\cup G}\psi_L^H)\Big|^2\bigg)^{\frac12}. \end{aligned} \tag{3.71}$$

Next, the energy estimate for (3.68) yields

$$\sum_{\sharp G=k}\int_{Q_L}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}\mathrm{D}(w_{h,n,G})\Big|^2 \lesssim \sum_{\sharp G=k}\int_{Q_L}\Big|\sum_{n\notin G\cup H}\alpha_{n,G}h_{n,G}\Big|^2,$$

and thus, using the constraints (3.67) on $\alpha, h$, and noting that the $\ell$-separation property (3.55) entails that the $h_{n,G}$'s have disjoint supports for different $n$'s,

$$\sum_{\sharp G=k} \int_{Q_L} \Big| \sum_{n\notin G\cup H} \alpha_{n,G} \mathrm{D}(w_{h,n,G}) \Big|^2 \lesssim \sum_{\sharp G=k} \sum_{n\notin G\cup H} |\alpha_{n,G}|^2 \int_{Q_L} |h_{n,G}|^2 = 1.$$

Inserting this into (3.71), using the Cauchy–Schwarz inequality, the $\ell$-separation property (3.55) in form of the disjointness of the fattened inclusions $\{I_{m,L} + \frac{1}{4}\ell B\}_m$, using that the number of points of the process $\mathcal{P}_L$ in $Q_L$ is bounded by $C(L/\ell)^d$, and changing summation variables, we deduce

$$L^{-d} I_{1,1}(\alpha, h)^2 \lesssim \mathbb{1}_{k=1,j=0} \ell^{-2d} + \ell^{-d} \big( T_L^H(k-1, j) + S_L^H(k, j) \big). \tag{3.72}$$

We turn to a corresponding estimation for $I_{1,2}(\alpha, h)$. For that purpose, we first note that the disjointness of fattened inclusions $\{I_{m,L} + \frac{1}{4}\ell B\}_m$ allows us to decompose

$$\begin{aligned} &\sum_{\sharp G=k+1} \sum_{m\in G\setminus H} \int_{I_{m,L}+\frac{1}{4}\ell B} \Big| \sum_{n\in G\setminus(H\cup\{m\})} \alpha_{n,G\setminus\{n\}} \mathrm{D}(w_{h,n,G\setminus\{n\}}) \Big|^2 \\ &\quad\lesssim \sum_{\sharp G=k+1} \int_{Q_L} \Big| \sum_{n\in G\setminus H} \alpha_{n,G\setminus\{n\}} \mathrm{D}(w_{h,n,G\setminus\{n\}}) \Big|^2 \\ &\qquad + \sum_{\sharp G=k+1} \sum_{m\in G\setminus H} |\alpha_{m,G\setminus\{m\}}|^2 \int_{Q_L} \big| \mathrm{D}(w_{h,m,G\setminus\{m\}}) \big|^2, \end{aligned}$$

and the energy estimate for (3.68) then yields

$$\begin{aligned} &\sum_{\sharp G=k+1} \sum_{m\in G\setminus H} \int_{I_{m,L}+\frac{1}{4}\ell B} \Big| \sum_{n\in G\setminus(H\cup\{m\})} \alpha_{n,G\setminus\{n\}} \mathrm{D}(w_{h,n,G\setminus\{n\}}) \Big|^2 \\ &\quad\lesssim \sum_{\sharp G=k+1} \int_{Q_L} \Big| \sum_{n\in G\setminus H} \alpha_{n,G\setminus\{n\}} h_{n,G\setminus\{n\}} \Big|^2 \\ &\qquad + \sum_{\sharp G=k+1} \sum_{m\in G\setminus H} |\alpha_{m,G\setminus\{m\}}|^2 \int_{Q_L} |h_{m,G\setminus\{m\}}|^2, \end{aligned}$$

from which we deduce, using the constraints (3.67) on $\alpha$, $h$ and recalling that the $h_{n,G}$'s have disjoint supports for different $n$'s,

$$\begin{aligned} &\sum_{\sharp G=k+1} \sum_{m\in G\setminus H} \int_{I_{m,L}+\frac{1}{4}\ell B} \Big| \sum_{n\in G\setminus(H\cup\{m\})} \alpha_{n,G\setminus\{n\}} \mathrm{D}(w_{h,n,G\setminus\{n\}}) \Big|^2 \\ &\quad\lesssim \sum_{\sharp G=k+1} \sum_{n\in G\setminus H} |\alpha_{n,G\setminus\{n\}}|^2 = \sum_{\sharp G=k} \sum_{n\notin G\cup H} |\alpha_{n,G}|^2 = 1. \end{aligned}$$

With this estimate at hand, we may now repeat the same argument as for (3.72) and we obtain

$$L^{-d} I_{1,2}(\alpha,h)^2 \lesssim \mathbb{1}_{k=0,j=1}\ell^{-2d} + \ell^{-d}\big(T_L^H(k,j-1) + S_L^H(k+1,j-1)\big). \tag{3.73}$$

Likewise, the second term $I_2(\alpha,h)$ in (3.69) is easily estimated as follows,

$$L^{-d} I_2(\alpha,h)^2 \lesssim \mathbb{1}_{k=0,j=1}\ell^{-2d} + \ell^{-d}\big(T_L^H(k,j-1) + T_L^H(k+1,j-2) + S_L^H(k+1,j-1) + S_L^H(k+2,j-2)\big). \tag{3.74}$$

Combining these different estimates, that is, (3.70), (3.72), (3.73), and (3.74), inserting them into (3.65), and recalling (3.64), the claim (3.63) follows.

*Step* 4. Conclusion. By a direct double induction argument, starting with (3.57), the recurrence relation (3.63) entails, for all $H \subset \mathbb{N}$ and $k, j \geq 0$,

$$T_L^H(k,j) \lesssim \mathbb{1}_{k=j=0}\ell^{-2d} + \mathbb{1}_{k+j\geq 1}(C\ell^{-d})^{2(k+j)+1} + \sum_{l=0}^{k+j-1}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2(l+1)} S_L^H(k+i-l,j-i). \tag{3.75}$$

Combined with the other recurrence relation (3.58), this yields

$$\begin{aligned} S_L^H(k,j) \lesssim{}& \mathbb{1}_{k=j=0}\ell^{-d} + \mathbb{1}_{k+j\geq 1}(C\ell^{-d})^{2(k+j)-1} + S_L^H(k+1,j-1) \\ &+ \sum_{l=0}^{k+j-1}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+2} S_L^H(k+i-l,j-i) \\ &+ \sum_{l=0}^{k+j-2}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+3} S_L^H(k+i-l-1,j-i). \end{aligned}$$

For $\ell \gg 1$, occurrences of $S_L^H(k,j)$ in the right-hand side can be absorbed into the left-hand side, and we are then left with

$$\begin{aligned} S_L^H(k,j) \lesssim{}& \mathbb{1}_{k=j=0}\ell^{-d} + \mathbb{1}_{k+j\geq 1}(C\ell^{-d})^{2(k+j)-1} + S_L^H(k+1,j-1) \\ &+ S_L^H(k+2,j-2) + \sum_{l=1}^{k+j-1}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+2} S_L^H(k+i-l,j-i) \\ &+ \sum_{l=0}^{k+j-2}(C\ell^{-d})^{2(l+1)}\sum_{i=0}^{2l+3} S_L^H(k+i-l-1,j-i). \end{aligned}$$

By a double induction argument, this relation leads to the conclusion

$$S_L^H(k,j) \lesssim \begin{cases} \ell^{-d} & \text{if } k=j=0, \\ (C\ell^{-d})^{2(k+j)-1} & \text{if } k,j\geq 0,\ k+j\geq 1. \end{cases}$$

Combining this with (3.75) further yields

$$T_L^H(k,j) \lesssim \begin{cases} \ell^{-2d} & \text{if } k=j=0, \\ (C\ell^{-d})^{2(k+j)+1} & \text{if } k,j \geq 0,\ k+j \geq 1. \end{cases}$$

Recalling that the case $\ell \simeq 1$ was already covered in (3.39), this finally concludes the proof of Theorem 3.9. ∎

## 3.6 Uniform cluster estimates

This section is devoted to the proof of Theorem 3.2 (i), based on the interpolating $\ell^1 - \ell^2$ energy estimates of Theorem 3.9. We focus on the bound (3.11) on the remainder $R_L^{k+1}$, while the corresponding bounds on cluster coefficients follow along the same lines. For $k \geq 1$, after changing summation variables, the definition (3.9) of the remainder can be written as

$$E : R_L^{k+1} E = \frac{1}{2} L^{-d} \sum_n \mathbb{E}\Big[ \int_{\partial I_{n,L}} \Big( \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \delta^F \psi_L^{\varnothing} \Big) \cdot \sigma_L \nu \Big].$$

Using the boundary conditions and the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), using the Cauchy–Schwarz inequality, and then appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned} &|E : R_L^{k+1} E| \\ &\quad \lesssim L^{-d} \mathbb{E}\Big[ \sum_n \int_{I_{n,L}} \Big| \sum_{\substack{\sharp F = j-1 \\ n \notin F}} \mathrm{D}(\delta^F \psi_L^{\varnothing}) \Big|^2 \Big]^{\frac{1}{2}} \mathbb{E}\Big[ \sum_n \int_{I_{n,L}+\rho B} \big| \mathrm{D}(\psi_L) + E \big|^2 \Big]^{\frac{1}{2}}. \end{aligned}$$

Recalling the disjointness of the fattened inclusions $\{I_{n,L} + \rho B\}_n$, recognizing the definition of $S_L$ and $T_L^{\varnothing}$, and using that in case $\ell \gg 1$ the $\ell$-separation property (3.55) entails that the number of points of the process $\mathcal{P}_L$ in $Q_L$ is bounded by $C(L/\ell)^d$, we are led to

$$|R_L^{k+1}| \lesssim \mathbb{E}\big[T_L^{\varnothing}(0,k)\big]^{\frac{1}{2}} \big(\ell^{-d} + \mathbb{E}\big[S_L(0,0)\big]\big)^{\frac{1}{2}},$$

and the conclusion (3.11) then follows from Theorem 3.9. ∎

## 3.7 Convergence of finite-volume approximations

This section is devoted to the proof of the convergence result (3.13) in Theorem 3.2. The idea is as follows: if $\{\bar{\mathbf{B}}_L^j\}_j$ could be viewed as derivatives of $\bar{\mathbf{B}}_L$ in some sense, then the convergence of $\bar{\mathbf{B}}_L$ as $L \uparrow \infty$ and the uniform bounds on derivatives $\{\bar{\mathbf{B}}_L^j\}_j$ would ensure the convergence of the latter. We split the proof into two steps, first appealing to a probabilistic argument to view $\{\bar{\mathbf{B}}_L^j\}_j$ as true derivatives, and then concluding by means of standard real analysis.

*Step* 1. Dilution by random deletion. Taking inspiration from [53], given $p \in [0,1]$, we consider a sequence $\{b_n^{(p)}\}_n$ of iid Bernoulli variables, independent of $\mathcal{P}, \mathcal{I}$, with parameter

$$p = \mathbb{P}\big[b_n^{(p)} = 1\big],$$

and we define the corresponding decimated process

$$\mathcal{P}^{(p)} := \{x_n\}_{n \in N^{(p)}}, \quad \mathcal{I}^{(p)} := \bigcup_{n \in N^{(p)}} I_n, \quad N^{(p)} := \{n : b_n^{(p)} = 1\}. \tag{3.76}$$

Similarly, in the periodized setting (3.1), we set

$$\mathcal{P}_L^{(p)} := \{x_{n,L}\}_{n \in N^{(p)}}, \quad \mathcal{I}_L^{(p)} := \bigcup_{n \in N^{(p)}} I_{n,L}.$$

By definition, the decimated processes $\mathcal{P}^{(p)}$, $\mathcal{I}^{(p)}$ satisfy ($\mathrm{H}_\rho$) and ($\mathrm{H}_\rho^{\mathrm{unif}}$) whenever $\mathcal{P}$, $\mathcal{I}$ do, and their periodized versions $\mathcal{P}_L^{(p)}$, $\mathcal{I}_L^{(p)}$ satisfy the same separation and stabilization properties as $\mathcal{P}_L$, $\mathcal{I}_L$ in Section 3.1. We use the notation $\bar{\mathbf{B}}^{(p)}$, $\bar{\mathbf{B}}_L^{(p)}$, $\{\bar{\mathbf{B}}_L^{(p),j}\}_j$, $\{R_L^{(p),k+1}\}_k$ for the effective viscosity, its periodized approximation, cluster coefficients, and cluster remainders associated with decimated processes $\mathcal{I}^{(p)}$, $\mathcal{I}_L^{(p)}$. As a corollary of [18, Theorem 1], as in (3.4), we have for all $p \in [0,1]$,

$$\lim_{L \uparrow \infty} \bar{\mathbf{B}}_L^{(p)} = \bar{\mathbf{B}}^{(p)}. \tag{3.77}$$

In the next two substeps, we shall further prove for all $k, j \geq 1$,

$$\bar{\mathbf{B}}_L^{(p),j} = p^j \bar{\mathbf{B}}_L^j, \tag{3.78}$$

$$|R_L^{(p),k+1}| \leq (Cp\ell^{-d})^{k+1}. \tag{3.79}$$

Combined with the cluster expansion (3.5), this yields for all $L$ and $k \geq 1$,

$$\left| \bar{\mathbf{B}}_L^{(p)} - \left( \mathrm{Id} + \sum_{j=1}^k \frac{p^j}{j!} \bar{\mathbf{B}}_L^j \right) \right| \leq (Cp\ell^{-d})^{k+1}, \tag{3.80}$$

which entails that $\bar{\mathbf{B}}_L^j$ can be seen as the $j$th derivative of the map $p \mapsto \bar{\mathbf{B}}_L^{(p)}$ at $p = 0$. (Note that this estimate further shows that this map is real-analytic; we shall later come back to this observation as part of Theorem 5.4.)

*Substep* 1.1. Proof of (3.78). By definition of decimated processes, the cluster formula (3.6) for $\bar{\mathbf{B}}_L^{(p),j}$ can be written as

$$E : \bar{\mathbf{B}}_L^{(p),j} E = j! \sum_{\sharp F = j} \mathbb{E}\Big[ \mathbb{1}_{F \subset N^{(p)}} \fint_{Q_L} \delta^F \big( |\mathrm{D}(\psi_L^{\varnothing}) + E|^2 \big) \Big].$$

As $N^{(p)}$ is independent of $\mathcal{I}$ and as

$$\mathbb{P}\big[F \subset N^{(p)}\big] = \mathbb{P}\big[b_n^{(p)} = 1,\ \forall n \in F\big] = p^{\sharp F},$$

we get

$$E : \bar{\mathbf{B}}_L^{(p),j} E = j! p^j \sum_{\sharp F = j} \mathbb{E}\Big[ \fint_{Q_L} \delta^F \big( |\mathrm{D}(\psi_L^{\varnothing}) + E|^2 \big) \Big] = p^j E : \bar{\mathbf{B}}_L^j E,$$

that is, (3.78).

*Substep* 1.2. Proof of (3.79). Let $k \geq 1$. By definition of decimated processes, the remainder formula (3.9) for $R_L^{(p),k+1}$ can be written as

$$E : R_L^{(p),k+1} E = \frac{1}{2} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[ \mathbb{1}_{F \subset N^{(p)}} \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{(p)} \nu \Big],$$

or equivalently, using the constraint $F \subset N^{(p)}$ to replace $\sigma_L^{(p)} = \sigma_L^{N^{(p)}}$ by $\sigma_L^{N^{(p)} \cup F}$,

$$\begin{aligned} E &: R_L^{(p),k+1} E \\ &= \frac{1}{2} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[ \mathbb{1}_{F \subset N^{(p)}} \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{N^{(p)} \cup F} \nu \Big]. \end{aligned}$$

In this expression, the integral

$$\int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{N^{(p)} \cup F} \nu$$

does not depend on the value of $\{b_n^{(p)}\}_{n \in F}$ and is thus independent of

$$\mathbb{1}_{F \subset N^{(p)}} = \prod_{n \in F} \mathbb{1}_{b_n^{(p)} = 1},$$

hence we are led to

$$\begin{aligned} E &: R_L^{(p),k+1} E \\ &= \frac{1}{2} p^{k+1} L^{-d} \sum_{\sharp F = k+1} \sum_{n \in F} \mathbb{E}\Big[ \int_{\partial I_{n,L}} \delta^{F \setminus \{n\}} \psi_L^{\varnothing} \cdot \sigma_L^{N^{(p)} \cup F} \nu \Big]. \end{aligned} \tag{3.81}$$

It remains to estimate the right-hand side and deduce (3.79), which is easily done by adapting the proof of Theorem 3.2 (i) in Section 3.6. For that purpose, we first note that, for all $F \subset \mathbb{N}$, using that $\sum_{H' \subset H} (-1)^{|H'|} = 0$ if $H \neq \varnothing$, we have

$$\begin{aligned} \sum_{G \subset F} \delta^G \sigma_L^{(p)} &= \sum_{G \subset F} \sum_{G' \subset G} (-1)^{|G \setminus G'|} \sigma_L^{N^{(p)} \cup G'} \\ &= \sum_{G' \subset F} \Big( \sum_{G'' \subset F \setminus G'} (-1)^{|G''|} \Big) \sigma_L^{N^{(p)} \cup G'} \\ &= \sigma_L^{N^{(p)} \cup F}, \end{aligned}$$

so that formula (3.81) can be decomposed as follows, after changing summation variables,

$$E : R_L^{(p),k+1} E = \frac{1}{2} p^{k+1} L^{-d} \sum_{\sharp F = k} \sum_{n \notin F} \sum_{G \subset F \cup \{n\}} \mathbb{E}\Big[ \int_{\partial I_{n,L}} \delta^F \psi_L^{\varnothing} \cdot \delta^G \sigma_L^{(p)} \nu \Big].$$

Using the following identity, for all maps $f$ and all $n \notin F$,

$$\sum_{G \subset F \cup \{n\}} f(G) = \sum_{G \subset F} f(G) + \sum_{G \subset F} f\big(G \cup \{n\}\big),$$

we deduce

$$\begin{aligned} &E : R_L^{(p),k+1} E \\ &\quad = \frac{1}{2} p^{k+1} L^{-d} \sum_{\sharp F = k} \sum_{G \subset F} \sum_{n \notin F} \mathbb{E}\Big[ \int_{\partial I_{n,L}} \delta^F \psi_L^{\varnothing} \cdot \big( \delta^G \sigma_L^{(p)} + \delta^{G \cup \{n\}} \sigma_L^{(p)} \big) \nu \Big], \end{aligned}$$

or equivalently, further changing summation variables,

$$\begin{aligned} &E : R_L^{(p),k+1} E \\ &\quad = \frac{1}{2} p^{k+1} L^{-d} \sum_{j=0}^{k} \sum_{\sharp G = j} \sum_{n \notin G} \mathbb{E}\Big[ \int_{\partial I_{n,L}} \Big( \sum_{\substack{\sharp F = k-j \\ F \cap (G \cup \{n\}) = \varnothing}} \delta^{F \cup G} \psi_L^{\varnothing} \Big) \\ &\qquad\qquad \cdot \Big( \delta^G \sigma_L^{(p)} + \delta^{G \cup \{n\}} \sigma_L^{(p)} \Big) \nu \Big]. \end{aligned}$$

Using the boundary conditions for

$$\delta^G \sigma_L^{(p)} + \delta^{G \cup \{n\}} \sigma_L^{(p)} = \delta^G \sigma_L^{N^{(p)} \cup \{n\}}$$

and using the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and then appealing to the trace estimates of

Lemma 3.5, we find

$$
\begin{aligned}
&|E : R_L^{(p),k+1} E| \\
&\lesssim p^{k+1} L^{-d} \sum_{j=0}^{k} \sum_{\sharp G=j} \sum_{n\notin G} \mathbb{E}\Big[\int_{I_{n,L}} \Big| \sum_{\substack{\sharp F=k-j\\ F\cap(G\cup\{n\})=\varnothing}} \mathrm{D}(\delta^{F\cup G}\psi_L^{\varnothing})\Big|^2\Big]^{\frac12} \\
&\quad\times \mathbb{E}\Big[\mathbb{1}_{j=0} + \int_{I_{n,L}+\rho B} \big|\mathrm{D}(\delta^{G}\psi_L^{(p)})\big|^2 + \big|\mathrm{D}(\delta^{G\cup\{n\}}\psi_L^{(p)})\big|^2\Big]^{\frac12}.
\end{aligned}
$$

Recalling the disjointness of fattened inclusions $\{I_{n,L}+\rho B\}_n$, recognizing the definition of $S_L^{(p)}$ and $T_L^{\varnothing}$, and using that in case $\ell \gg 1$ the $\ell$-separation property (3.55) entails that the number of points of the process $\mathcal{P}_L$ in $Q_L$ is bounded by $C(L/\ell)^d$, we deduce

$$
\begin{aligned}
&|E : R_L^{(p),k+1} E| \\
&\lesssim p^{k+1} \sum_{j=0}^{k} \mathbb{E}\big[T_L^{\varnothing}(j,k-j)\big]^{\frac12} \big(\mathbb{1}_{j=0}\ell^{-d} + \mathbb{E}\big[S_L^{(p)}(j,0)\big] + \mathbb{E}\big[S_L^{(p)}(j+1,0)\big]\big)^{\frac12}.
\end{aligned}
$$

Now appealing to Theorem 3.9, the claim (3.79) follows.

*Step* 2. Conclusion. While the uniform estimates of Theorem 3.2 (i) ensure that the sequence $\{\overline{\mathbf{B}}_L^j\}_{L\ge1}$ converges as $L\uparrow\infty$ up to extraction of a subsequence, we shall use their interpretation as derivatives of the map $p\mapsto \overline{\mathbf{B}}_L^{(p)}$ at $p=0$, together with some real analysis, to deduce the convergence of the full sequence. We argue by induction: given $k\ge0$, we assume that the limits $\overline{\mathbf{B}}^j=\lim_{L\uparrow\infty}\overline{\mathbf{B}}_L^j$ exist for all $1\le j\le k$, and we shall then prove that the limit

$$
\overline{\mathbf{B}}^{k+1} = \lim_{L\uparrow\infty} \overline{\mathbf{B}}_L^{k+1}
$$

also exists. As $\overline{\mathbf{B}}_L^{k+1}$ is bounded uniformly in $L$ by Theorem 3.2 (i), it has a limit $\overline{\mathbf{C}}^{k+1}$ as $L\uparrow\infty$ up to extraction of a subsequence. Passing to the limit along this subsequence in (3.80), with $k$ replaced by $k+1$, and using (3.77) and the induction assumptions, we get for all $p$,

$$
\bigg|\overline{\mathbf{B}}^{(p)} - \bigg(\mathrm{Id} + \sum_{j=1}^{k} \frac{p^j}{j!}\overline{\mathbf{B}}^j + \frac{p^{k+1}}{(k+1)!}\overline{\mathbf{C}}^{k+1}\bigg)\bigg| \le (Cp)^{k+2},
$$

which proves that $\overline{\mathbf{C}}^{k+1}$ satisfies

$$
\overline{\mathbf{C}}^{k+1} = \lim_{p\downarrow0} \frac{(k+1)!}{p^{k+1}} \bigg(\overline{\mathbf{B}}^{(p)} - \bigg(\mathrm{Id} + \sum_{j=1}^{k}\frac{p^j}{j!}\overline{\mathbf{B}}^j\bigg)\bigg),
$$

where in particular the limit exists. Since the right-hand side does not depend on the choice of the extracted subsequence, we deduce that the limit $\overline{\mathbf{C}}^{k+1}$ is uniquely defined, hence the limit $\overline{\mathbf{B}}^{k+1} := \overline{\mathbf{C}}^{k+1} = \lim_{L\uparrow\infty} \overline{\mathbf{B}}_L^{k+1}$ actually exists. By induction, this concludes the proof of the convergence result (3.13) in Theorem 3.2. ∎

## 3.8 Non-uniform cluster estimates

This section is devoted to the proof of Theorem 3.2 (ii). Taking inspiration from [12, Section 5.A], we proceed by a direct analysis of Green's representation formulas for corrector differences. More precisely, we introduce operators $\{\mathcal{J}_{L;H}^n\}_{n,H}$ that describe the fluid velocity generated by localized force dipoles in the presence of a finite number of rigid inclusions: these are viewed as Stokeslets for the problem with rigid inclusions and lead to a useful decomposition of corrector differences, cf. (3.82) below. The following lemma defines such operators and states their optimal decay properties, which are shown to coincide with the decay for the explicit Stokeslet associated with the problem in free space without rigid particles. This result is a particular case of Lemma A.1, the proof of which is postponed to Appendix A.

**Lemma 3.10** (Decay of Stokeslets with rigid inclusions). *Let Assumptions* ($\mathrm{H}_\rho$) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$) *hold, let $H \subset \mathbb{N}$ be finite and $n \notin H$, and let $(\zeta, P)$ satisfy the following Stokes problem in a neighborhood of $I_{n,L}$,*

$$\begin{cases} -\triangle\zeta + \nabla P = 0, & \text{in } (I_{n,L} + \rho B) \setminus I_{n,L}, \\ \operatorname{div}(\zeta) = 0, & \text{in } (I_{n,L} + \rho B) \setminus I_{n,L}, \\ \mathrm{D}(\zeta) = 0, & \text{in } I_{n,L}, \\ \int_{\partial I_{n,L}} \sigma(\zeta, P)\nu = 0, & \\ \int_{\partial I_{n,L}} \Theta(x - x_{n,L}) \cdot \sigma(\zeta, P)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

*Denote by $\mathcal{J}_{L;H}^n \zeta \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following Stokes problem,*

$$\begin{cases} -\triangle\mathcal{J}_{L;H}^n\zeta + \nabla\mathcal{Q}_{L;H}^n\zeta = -\delta_{\partial I_{n,L}}\sigma(\zeta, P)\nu, & \text{in } Q_L \setminus \mathcal{I}_L^H, \\ \operatorname{div}(\mathcal{J}_{L;H}^n\zeta) = 0, & \text{in } Q_L \setminus \mathcal{I}_L^H, \\ \mathrm{D}(\mathcal{J}_{L;H}^n\zeta) = 0, & \text{in } \mathcal{I}_L^H, \\ \int_{\partial I_{m,L}} \sigma(\mathcal{J}_{L;H}^n\zeta, \mathcal{Q}_{L;H}^n\zeta)\nu = 0, & \forall m \in H, \\ \int_{\partial I_{m,L}} \Theta(x - x_{m,L}) \cdot \sigma(\mathcal{J}_{L;H}^n\zeta, \mathcal{Q}_{L;H}^n\zeta)\nu = 0, & \forall m \in H,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

*Then, we have for all $z \in Q_L$,*

$$\Big(\int_{B(z)} |\mathrm{D}(\mathcal{J}_{L;H}^n\zeta)|^2\Big)^{\frac12} \lesssim_{\sharp H} \langle (z - x_{n,L})_L \rangle^{-d} \Big(\int_{I_{n,L}+\rho B} |\mathrm{D}(\zeta)|^2\Big)^{\frac12}.$$

The above definition of operators $\{\mathcal{J}^n_{L;H}\}_{n,H}$ is motivated by the following observation: for all $F,H\subset\mathbb{N}$ with $F$ finite and nonempty, equations (3.15) for corrector differences entail, in these terms,

$$\delta^F\psi_L^H=\sum_{n\in F\setminus H}\mathcal{J}^n_{L;H}\,\delta^{F\setminus\{n\}}(\psi_L^{H\cup\{n\}}+Ex).\tag{3.82}$$

Iterating this identity allows us to write $\delta^F\psi_L^H$ as a combination of iterations of $\mathcal{J}^n_{L;H}$'s, which are viewed as elementary single-particle contributions. With the above result at hand, we may now conclude with the proof of Theorem 3.2 (ii).

*Proof of Theorem* 3.2 (ii). We focus on the bound (3.12) on the remainder $R_L^{k+1}$, while the corresponding bound on cluster coefficients follows along the same lines. We split the proof into two steps.

*Step* 1. Estimation of corrector differences. For all finite $F,H\subset\mathbb{N}$ with $F$ nonempty, and for all $n\in\mathbb{N}$, recalling the decomposition (3.82) for corrector differences, Lemma 3.10 yields

$$\begin{aligned}&\bigg(\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\delta^F\psi_L^H)\big|^2\bigg)^{\frac12}\\&\qquad\lesssim_{\sharp H}\sum_{m\in F\setminus H}\langle(x_{n,L}-x_{m,L})_L\rangle^{-d}\bigg(\int_{I_{m,L}+\rho B}\big|\mathrm{D}\big(\delta^{F\setminus\{m\}}(\psi_L^{H\cup\{m\}}+Ex)\big)\big|^2\bigg)^{\frac12}.\end{aligned}$$

Iterating this bound, and recalling that the energy estimate (3.44) gives for all finite $G\subset\mathbb{N}$,

$$\int_{Q_L}\big|\mathrm{D}(\psi_L^G)\big|^2\lesssim\sharp G,$$

we deduce for all $n$, setting $k:=\sharp F\geq 1$,

$$\begin{aligned}&\bigg(\int_{I_{n,L}+\rho B}\big|\mathrm{D}(\delta^F\psi_L^\varnothing)\big|^2+\big|\mathrm{D}(\delta^F\psi_L^{\{n\}})\big|^2\bigg)^{\frac12}\\&\lesssim_k\sum_{n_1,\ldots,n_k\in F}^{\neq}\langle(x_{n,L}-x_{n_1,L})_L\rangle^{-d}\langle(x_{n_1,L}-x_{n_2,L})_L\rangle^{-d}\cdots\langle(x_{n_{k-1},L}-x_{n_k,L})_L\rangle^{-d}.\end{aligned}\tag{3.83}$$

*Step* 2. Conclusion. The starting point is the estimate (3.10) in Theorem 3.1 for the cluster remainder,

$$|E:R_L^{k+1}E|\lesssim A_k^\circ+\sum_{j=1}^k A_{j,k},\tag{3.84}$$

in terms of

$$A_k^\circ := \mathbb{E}\Big[L^{-d}\sum_n \int_{I_{n,L}} \Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}} \mathrm{D}(\delta^F \psi_L^\varnothing)\Big|^2\Big],$$

$$A_{j,k} := \mathbb{E}\Big[L^{-d}\sum_n \Big(\int_{I_{n,L}} \Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}} \mathrm{D}(\delta^F \psi_L^\varnothing)\Big|^2\Big)^{\frac12} \times \Big(\int_{I_{n,L}+\rho B} \Big|\sum_{\substack{\sharp F=j-1\\ n\notin F}} \mathrm{D}\big(\delta^F (\psi_L^{\{n\}}+Ex)\big)\Big|^2\Big)^{\frac12}\Big]. \tag{3.85}$$

We shall prove for all $1\le j\le k$,

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} \lambda_{k+l+1}(\mathscr{P})(\log L)^{2l}, \tag{3.86}$$

$$A_{j,k} \lesssim_k \sum_{l=0}^{j-1} \lambda_{k+l+1}(\mathscr{P})(\log L)^{2l+k-j+1}. \tag{3.87}$$

Inserting this into (3.84), the conclusion (3.12) follows. We split the proof into two further substeps, separately proving (3.86) and (3.87).

*Substep* 2.1. Proof of (3.86). Let $k\ge 1$. The deterministic bound (3.83) yields

$$\sum_{\sharp F=k}\Big(\int_{I_{n,L}} \big|\mathrm{D}(\delta^F\psi_L^\varnothing)\big|^2 + \big|\mathrm{D}(\delta^F\psi_L^{\{n\}})\big|^2\Big)^{\frac12} \lesssim_k \sum_{n_1,\ldots,n_k}^{\neq} D_L(x_{n,L},x_{n_1,L},\ldots,x_{n_k,L}), \tag{3.88}$$

where we have set

$$D_L(y_0,y_1,\ldots,y_k) := \prod_{j=0}^{k-1}\big\langle (y_j-y_{j+1})_L\big\rangle^{-d}.$$

Inserting this in the definition (3.85) of $A_k^\circ$, expanding the square, separating the different intersection patterns, and reformulating in terms of multi-points densities, cf. (1.15), we are led to

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} L^{-d}\int_{(Q_L)^{k+l+1}} D_L(x,x_1,\ldots,x_k)D_L(x,x_1,\ldots,x_{k-l},y_1,\ldots,y_l) \times f_{k+l+1}(x,x_1,\ldots,x_k,y_1,\ldots,y_l)\,dx\,dx_1\cdots dx_k\,dy_1\cdots dy_l,$$

hence, in terms of multi-point intensities, appealing to Lemma 1.1 (iii),

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} \lambda_{k+l+1}(\mathcal{P})\, L^{-d} \int_{(Q_L)^{k+l+1}} D_L(x, x_1, \ldots, x_k) \\ \times D_L(x, x_1, \ldots, x_{k-l}, y_1, \ldots, y_l)\, dx\, dx_1 \cdots dx_k\, dy_1 \cdots dy_l.$$

First evaluating integrals over $x_{k-l+1}, \ldots, x_k, y_1, \ldots, y_l$, and noting that

$$\int_{Q_L} \langle (x-y)_L \rangle^{-d}\, dy \lesssim \log L,$$

we find

$$A_k^\circ \lesssim_k \sum_{l=0}^{k} \lambda_{k+l+1}(\mathcal{P}) (\log L)^{2l} L^{-d} \\ \times \int_{(Q_L)^{k-l+1}} D_L(x, x_1, \ldots, x_{k-l})^2\, dx\, dx_1 \cdots dx_{k-l}.$$

Now evaluating the remaining integrals, noting that the square yields an integrable decay, the claim (3.86) follows.

*Substep* 2.2. Proof of (3.87). Let $k \geq j \geq 1$. Inserting (3.88) into the definition (3.85) of $A_{j,k}$, expanding the square, and separating the different intersection patterns, we now find

$$A_{j,k} \lesssim_k \sum_{l=0}^{j-1} L^{-d} \int_{(Q_L)^{k+l+1}} D_L(x, x_1, \ldots, x_k) D_L(x, x_1, \ldots, x_{j-l-1}, y_1, \ldots, y_l) \\ \times f_{k+l+1}(x, x_1, \ldots, x_k, y_1, \ldots, y_l)\, dx\, dx_1 \cdots dx_k\, dy_1 \cdots dy_l,$$

where for notational convenience we define $D_L(x) := 1$. This integral can be evaluated exactly as in the proof of (3.86), and the claim (3.87) follows. ∎