Chapter 4

Renormalization of cluster formulas

This chapter is devoted to the proof of infinite-volume cluster estimates with optimal dependence on multi-point intensities $\{\lambda_j(\mathcal{P})\}_j$. It amounts to improving on the non-uniform cluster estimates (3.12) in Theorem 3.2, which captures the "short-range" dependence on multi-point intensities but displays a logarithmic divergence in the large-volume limit. This requires a better understanding of cluster formulas and of the underlying compensations that make them well defined in the large-volume limit.

4.1 Main results

We explore two different routes for the renormalization of infinite-volume cluster formulas, leading to two complementary results, cf. Theorems 4.1 and 4.3 below. We also discuss the optimality of our cluster estimates, cf. Theorem 4.4.

4.1.1 Implicit renormalization

Our first route relies on a slight algebraic quantification of the convergence of periodic approximations, cf. assumption (QPE) below: it implies a corresponding convergence rate for periodized cluster formulas, cf. (4.2) below, which in turn allows to remove the logarithmic divergence in the non-uniform cluster estimates of Theorem 3.2. This result is particularly general given that the quantitative periodization assumption (QPE) holds under a mere algebraic α-mixing condition for $\mathcal{I}$, cf. Remark 4.2 below. The obtained cluster estimates (4.1) differ from the canonical short-range setting of Lemma 1.2 by some logarithmic factors, which are expected to be optimal in general in link with the long-range nature of hydrodynamic interactions, cf. Theorem 4.4 below. The proof is displayed in Section 4.2.

Theorem 4.1 (Implicit renormalization of cluster formulas). *On top of Assumptions* (H_ρ) *and* (H_ρ^{unif}), *let the following hold:*

(QPE) Quantitative periodization assumption: *There exist $C, \gamma > 0$ such that we have $|\bar{\mathbf{B}}_L^{(p)} - \bar{\mathbf{B}}^{(p)}| \le C L^{-\gamma}$ for all $L \ge 1$ and $p \in [0,1]$, where $\bar{\mathbf{B}}_L^{(p)}$, $\bar{\mathbf{B}}^{(p)}$ refer to the random deletion procedure introduced in Section 3.7, cf.* (3.76).

Then, we have the following estimates for the coefficients and the remainder of the infinite-volume cluster expansion defined by (3.13) *in Theorem 3.2: for all $k, j \ge 1$,*

$$|\bar{\mathbf{B}}^j| \lesssim_j \lambda_j(\mathcal{P}) \big|\log \lambda_j(\mathcal{P})\big|^{j-1}, \tag{4.1}$$

$$|R^{k+1}| \lesssim_k \sum_{l=k+1}^{2k+1} \lambda_l(\mathcal{P}) \big|\log \lambda_{k+1}(\mathcal{P})\big|^{l-1}.$$

In addition, the convergence result (3.13) *for finite-volume approximations can be quantified: for all L and $k, j \geq 1$,*

$$|\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \lesssim_j L^{-2^{-j}\gamma}, \quad |R_L^{k+1} - R^{k+1}| \lesssim_k L^{-2^{-k}\gamma}, \tag{4.2}$$

where γ is the exponent in (QPE).

Remark 4.2 (Quantitative periodization assumption). The validity of (QPE) can be shown to follow from a slight quantitative mixing condition for the inclusion process $\mathcal{I}$, such as the following:

(Mix) *Algebraic α-mixing condition:* There exist $C, \beta > 0$ such that for all Borel subsets $U, V \subset \mathbb{R}^d$ and all events $A \subset \sigma(\mathcal{I}|_U)$ and $B \in \sigma(\mathcal{I}|_V)$ we have

$$\big|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]\big| \leq C \operatorname{dist}(U, V)^{-\beta}. \tag{4.3}$$

More precisely, this condition (Mix) implies (QPE) for some $0 < \gamma \ll \beta$ (depending on β, d) and for all $0 \leq p \leq 1$ (since random deletion preserves (4.3)). This follows from by-now standard quantitative homogenization theory: we refer to Appendix B, where we adapt the techniques developed by Armstrong, Kuusi, Mourrat, and Smart in [3–5] to the present fluid context.

The above result provides optimal cluster estimates and its proof is extremely short, cf. Section 4.2. Yet, it has three main disadvantages, which call for a more detailed analysis.

— *No explicit renormalization:* While infinite-volume cluster formulas take the form of diverging series, cf. Section 1.3.4, cluster coefficients are defined as limits of finite-volume approximations, cf. (3.13). Using straightforward cancellations, we showed that the first-order cluster coefficient $\bar{\mathbf{B}}^1$ can be represented by a summable integral, cf. Proposition 2.2. A similar explicit renormalization was formally performed for the second-order coefficient $\bar{\mathbf{B}}^2$ by Batchelor and Green [7], based on more subtle cancellations. The implicit renormalization approach sheds no light on such questions. We aim to recover the Batchelor–Green renormalized formula for $\bar{\mathbf{B}}^2$ rigorously, as also discussed in [26, 27, 29], and to investigate how explicit renormalizations can be pursued to higher orders.

— *Mixing assumption:* In view of cluster formulas in Theorem 3.1, bounds on the cluster coefficient $\bar{\mathbf{B}}^j$ should only require assumptions on the j-point density. Likewise, in view of (3.10), bounds on the remainder R_L^{k+1} should only require assumptions on the $2k$-point density. Instead, assumptions (QPE) and (Mix) boldly involve the whole law of the inclusion process $\mathcal{I}$, which we aim to refine.

— *Convergence rates:* As the above approach builds on a convergence rate for periodic approximations of the effective viscosity $\bar{\mathbf{B}}$, cf. (QPE), it does not exploit the fact that cluster formulas only involve a finite number of particles at a time and are thus significantly simpler than $\bar{\mathbf{B}}$ itself. In particular, convergence rates for periodic approximations of cluster coefficients are not expected to be worse than for approximations of $\bar{\mathbf{B}}$ (on the contrary!), while the above result (4.2) displays an exponential degradation of the rates for higher-order coefficients.

4.1.2 Explicit renormalization

Our second route to renormalization of cluster formulas aims to remedy the above three issues and we proceed by an explicit analysis of cancellations. As in Proposition 2.2, we assume for convenience that particles have independent shapes, cf. (Indep), which makes cluster formulas somewhat simpler. While for $\bar{\mathbf{B}}^1$ and $\bar{\mathbf{B}}^2$ relatively simple cancellations are enough to turn cluster formulas into summable integrals, higher-order coefficients require a much deeper analysis: we are led to introducing a diagrammatic decomposition of corrector differences that allows to capture relevant cancellations. This fully resolves the higher-order renormalization question that was still open in the physics community. We refer in particular to Section 4.4 for an explicit display of renormalized formulas for $\bar{\mathbf{B}}^2$ and $\bar{\mathbf{B}}^3$, cf. Propositions 4.9 and 4.10: we recover the Batchelor–Green formula for $\bar{\mathbf{B}}^2$ and provide the first renormalized formula for $\bar{\mathbf{B}}^3$. Incidentally, these results only require assumptions on finite-order multi-point densities (instead of mixing assumptions) and Dini-type decay (instead of algebraic), which is beyond the reach of quantitative homogenization methods (and thus of the implicit renormalization approach). Renormalized formulas allow to recover the same cluster estimates (4.1) as obtained above via implicit renormalization and to further prove essentially optimal convergence rates for finite-volume approximations: the convergence rate (4.4) for $\bar{\mathbf{B}}^j$ below only degrades logarithmically when increasing j (as opposed to the exponential degradation in (4.2)), and it is always better (as it should) than the rate for approximations of the effective viscosity $\bar{\mathbf{B}}$ itself (cf. $\gamma \ll \beta$ in Remark 4.2). The proof is displayed in Section 4.4.

Theorem 4.3 (Explicit renormalization of cluster formulas). *On top of Assumptions* (H_ρ) *and* $(\mathrm{H}_\rho^{\mathrm{unif}})$*, let the independence assumption* (Indep) *hold for particle shapes, as well as the following, for some rate function* $\omega \in C_b^\infty(\mathbb{R}^+)$*:*

(Mix_ω) α-Mixing condition with rate ω: *For all Borel subsets* $U, V \subset \mathbb{R}^d$ *and all events* $A \subset \sigma(\mathcal{I}|_U)$ *and* $B \in \sigma(\mathcal{I}|_V)$*, we have*

$$\big|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]\big| \le \omega\big(\mathrm{dist}(U, V)\big).$$

Then, the following hold.

(i) *For all $j \geq 2$, provided ω satisfies the Dini-type condition*

$$\int_1^\infty t^{-1}(\log t)^{j-2}\omega(t)\,dt < \infty,$$

the infinite-volume cluster coefficient $\bar{\mathbf{B}}^j$ can be described by means of summable integrals as detailed in Section 4.4.

(ii) *In case of an algebraic mixing rate $\omega(t) \leq Ct^{-\beta}$ for some $C, \beta > 0$, renormalized formulas lead to the same cluster estimates (4.1) for all $k, j \geq 1$. In addition, the following holds for finite-volume approximations: for all L and $j \geq 1$,*

$$|\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \lesssim_j \frac{(\log L)^{j-1}}{L^{\beta\wedge 1}}. \tag{4.4}$$

In addition, (Mix$_\omega$) can be replaced by a corresponding assumption on the j-point density for results on $\bar{\mathbf{B}}^j$, and on the $(2k+1)$-point density for results on R^{k+1}.

4.1.3 Optimality of cluster estimates

The following result states that logarithmic factors in cluster estimates (4.1) are optimal in general. These factors contrast with the prototypical short-range setting of Lemma 1.2: they are related to the long-range nature of hydrodynamic interactions and appear due to the lack of L^∞-boundedness of Calderón–Zygmund operators. We focus on the second-order coefficient $\bar{\mathbf{B}}^2$ for illustration, but, starting from renormalized formulas, the argument could be extended to higher orders as well. The proof is displayed in Section 4.5.

Theorem 4.4. *About the optimality of estimates on $\bar{\mathbf{B}}_2$, the following statements hold.*

(i) Isotropic setting: *On top of Assumptions (H$_\rho$), (H$_\rho^{\mathrm{unif}}$), and (Indep), assume that the 2-point correlation function $h_2(x,y) := f_2(x,y) - \lambda(\mathcal{P})^2$ satisfies the following decay assumption,*

$$\iint_{B(x)\times B(y)} |h_2| \leq \omega\big(|x-y|\big), \tag{4.5}$$

with some rate ω satisfying the Dini condition $\int_1^\infty t^{-1}\omega(t)\,dt < \infty$. If in addition the point process $\mathcal{P}$ is statistically isotropic, which entails that the correlation function is radial, then the following improved estimate holds,

$$|\bar{\mathbf{B}}^2| \lesssim \lambda_2(\mathcal{P}).$$

(ii) Optimality in the general setting: *There exists an inclusion process $\mathcal{I}$ that satisfies Assumptions (H$_\rho$), (H$_\rho^{\mathrm{unif}}$), (Indep), and (4.5), as well as the local independence condition $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^2 \ll \lambda(\mathcal{P})$, such that we have*

$$|\bar{\mathbf{B}}^2| \simeq \lambda_2(\mathcal{P})\big|\log \lambda_2(\mathcal{P})\big|.$$

4.2 Implicit renormalization of cluster formulas

This section is devoted to the short proof of Theorem 4.1, which we split into two steps. We start with the quantitative convergence result (4.2) for finite-volume approximations of cluster coefficients, which we obtain by quantifying the argument for the corresponding qualitative result (3.13) in Section 3.7. The claimed cluster estimates (4.1) then follow by optimization.

Step 1. Suboptimal convergence result: proof of (4.2). Starting from the cluster expansion (3.5) in Theorem 3.1, the triangle inequality yields for all $k \geq 0$,

$$|R_L^{k+1} - R^{k+1}| \leq |\bar{\mathbf{B}}_L - \bar{\mathbf{B}}| + \sum_{j=1}^{k} |\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j|,$$

so that the convergence rate for the remainder in (4.2) follows from Assumption (QPE) together with the convergence rate for cluster coefficients. It remains to prove the latter, that is, for all $j \geq 1$,

$$|\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \lesssim_j L^{-2^{-j}\gamma}. \tag{4.6}$$

For that purpose, we quantify the induction argument in the proof of the corresponding qualitative convergence result (3.13) in Section 3.7. Let $k \geq 0$ and assume that (4.6) holds for all $1 \leq j \leq k$. Taking the same notation as in Section 3.7 for the random deletion procedure, we recall the cluster expansion (3.80), for all L, p,

$$\left| \bar{\mathbf{B}}_L^{(p)} - \left(\mathrm{Id} + \sum_{j=1}^{k+1} \frac{p^j}{j!} \bar{\mathbf{B}}_L^j \right) \right| \leq (Cp)^{k+2}.$$

Hence, comparing to the corresponding estimate in the large-volume limit, we find

$$\left| (\bar{\mathbf{B}}_L^{(p)} - \bar{\mathbf{B}}^{(p)}) - \sum_{j=1}^{k+1} \frac{p^j}{j!} (\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j) \right| \leq (Cp)^{k+2}.$$

Isolating the difference $\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}$, and using Assumption (QPE) and the induction hypothesis to estimate other contributions, we deduce

$$\begin{aligned} |\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}| &\leq \frac{(k+1)!}{p^{k+1}} \left((Cp)^{k+2} + |\bar{\mathbf{B}}_L^{(p)} - \bar{\mathbf{B}}^{(p)}| + \sum_{j=1}^{k} \frac{p^j}{j!} |\bar{\mathbf{B}}_L^j - \bar{\mathbf{B}}^j| \right) \\ &\lesssim_k p + \sum_{j=0}^{k} p^{j-k-1} L^{-2^{-j}\gamma}. \end{aligned}$$

The choice $p = L^{-2^{-k-1}\gamma}$ then yields $|\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}| \lesssim_k L^{-2^{-k-1}\gamma}$, and the claim (4.6) follows by induction for all $j \geq 1$.

Step 2. Uniform cluster estimates: proof of (4.1). Combining the non-uniform estimates (3.12) of Theorem 3.2 with the suboptimal convergence result (4.2), we find for all $k \geq j \geq 1$,

$$|\bar{\mathbf{B}}^j| \lesssim_j L^{-2^{-j}\gamma} + \lambda_j(\mathscr{P})(\log L)^{j-1},$$
$$|R^{k+1}| \lesssim_j L^{-2^{-k}\gamma} + \sum_{l=k}^{2k} \lambda_{l+1}(\mathscr{P})(\log L)^l,$$

and the conclusion (4.1) follows from the choice $L^{-2^{-j}\gamma} = \lambda_j(\mathscr{P})$ or $L^{-2^{-k}\gamma} = \lambda_{k+1}(\mathscr{P})$, respectively. ∎

4.3 Preliminary to explicit renormalization

Before turning to the explicit renormalization of cluster formulas and to the proof of Theorem 4.3, we start with some preliminary definitions and technical tools: we define multi-point correlation functions, which provide a convenient framework to weaken the α-mixing condition, we revisit the decomposition (3.82) for corrector differences in terms of elementary single-particle contributions, and we state several crucial estimates on the latter.

4.3.1 Multi-point correlation functions

Multi-point correlation functions $\{h_j\}_j$ of the point process $\mathscr{P}$ can be defined inductively from the multi-point densities $\{f_j\}_j$, cf. (1.15), via the following relations:[1] for all $j \geq 1$,

$$f_j(x_1, \ldots, x_j) = \sum_{\pi} \prod_{H \in \pi} h_{\sharp H}(x_H), \tag{4.7}$$

where π runs over all partitions of the index set $\{1, \ldots, j\}$, where H runs over all cells of the partition π, and where for $H = \{i_1, \ldots, i_l\}$ we set

$$x_H := (x_{i_1}, \ldots, x_{i_l}).$$

For the first values of k, these relations read

$$f_1(z) = h_1(z) = \lambda(\mathscr{P}),$$
$$f_2(y,z) = \lambda(\mathscr{P})^2 + h_2(y,z),$$
$$f_3(x,y,z) = \lambda(\mathscr{P})^3 + \lambda(\mathscr{P})\big(h_2(x,y) + h_2(y,z) + h_2(z,x)\big) + h_3(x,y,z),$$

[1] Incidentally, these relations are known as Mayer's *cluster expansions*—although unrelated to the kind of cluster expansions otherwise studied in this work.

from which h_1, h_2, h_3 are easily extracted. More generally, note that the inductive definition (4.7) can be explicitly inverted: for all $j \geq 1$, we find

$$h_j(x_1, \ldots, x_j) := \sum_\pi (\sharp\pi - 1)!\,(-1)^{\sharp\pi-1} \prod_{H\in\pi} f_{\sharp H}(x_H), \tag{4.8}$$

where π runs over all partitions of the index set $\{1, \ldots, j\}$ and where $\sharp\pi$ stands for the number of cells $H \in \pi$. The j-point correlation function h_j is thus a symmetric function on the product $(\mathbb{R}^d)^j$ and is a polynomial combination of multi-point densities $(f_i)_{i\leq j}$. The definition of multi-point intensities (1.16) then entails the following bounds on correlations, for all $j \geq 1$,

$$\sup_{z_1,\ldots,z_j} \fint_{Q_\ell(z_1)\times\cdots\times Q_\ell(z_j)} |h_j| \lesssim_j \bar\lambda_j(\mathcal{P}), \tag{4.9}$$

where we recall the notation (1.17). It is easily checked that the α-mixing assumption (Mix$_\omega$) implies the decay of correlation functions in the following quantitative sense. Since we could not find any precise reference in the literature, we include a short proof below for completeness.

Lemma 4.5. *Assume that the point process $\mathcal{P}$ satisfies the α-mixing condition* (Mix$_\omega$) *with a non-increasing rate function $\omega \in C_b^\infty(\mathbb{R}^+)$. Then, correlation functions satisfy for all $j \geq 2$ and $x_1, \ldots, x_j \in \mathbb{R}^d$,*

$$\int_{B(x_1)\times\cdots\times B(x_j)} |h_j| \leq C^j\, j!\, \min_{i\neq l} \omega\Big(\Big(\frac{1}{j}|x_i - x_l| - 2\Big)_+\Big). \tag{4.10}$$

In this view, it is natural to consider a "truncated" version of the α-mixing condition (Mix$_\omega$) in form of the decay of a finite number of correlation functions only. This is the natural setting for cluster estimates.

(Mix$_\omega^n$) *Mixing assumption with rate ω to order n:* Multi-point correlation functions satisfy for all $2 \leq j \leq n$ and $x_1, \ldots, x_j \in \mathbb{R}^d$,

$$\int_{B(x_1)\times\cdots\times B(x_j)} |h_j| \leq \min_{i\neq l} \omega\big(|x_i - x_l|\big).$$

Proof of Lemma 4.5. We argue by induction: given $j \geq 2$, we assume that the claimed decay estimate (4.10) is already known to hold for $h_2, \ldots, h_{j-1}$, and we prove that it also holds for h_j. Let $x_1, \ldots, x_j \in \mathbb{R}^d$ be fixed. The conclusion (4.10) is trivial when $\max_{i\neq l} \frac{1}{j}|x_i - x_l| \leq 2$, and we may thus assume $\max_{i\neq l} \frac{1}{j}|x_i - x_l| > 2$. Up to relabeling the points, we may further assume that there is $1 \leq j_* < j$ such that

$$\begin{aligned} |x_1 - x_j| &= \max_{i\neq l} |x_i - x_l|, \\ |x_i - x_l| &\geq \frac{1}{j}|x_1 - x_j| > 2 \quad \text{for all } 1 \leq i \leq j_* < l \leq j. \end{aligned} \tag{4.11}$$

(The latter condition is obtained by dividing the space between x_1 and x_j into j stripes of width $\frac{1}{j}|x_1 - x_j|$, by selecting the one that contains none of the points $\{x_i\}_{1<i<j}$, and by distinguishing the points on either side of this stripe.) Let

$$\phi \in C\big((\mathbb{R}^d)^{j_*}\big) \quad \text{and} \quad \phi' \in C\big((\mathbb{R}^d)^{j-j_*}\big)$$

be supported in $B(x_1) \times \cdots \times B(x_{j_*})$ and in $B(x_{j_*+1}) \times \cdots \times B(x_j)$, respectively, with $\|\phi\|_{\mathrm{L}^\infty((\mathbb{R}^d)^{j_*})} = \|\phi'\|_{\mathrm{L}^\infty((\mathbb{R}^d)^{j-j_*})} = 1$. Appealing to a standard covariance inequality, see e.g. [11, Lemma 1.2.3], the α-mixing condition (Mix$_\omega$) then yields

$$\begin{aligned}
&\bigg|\mathrm{Cov}\bigg[\sum_{n_1,\ldots,n_{j_*}}^{\neq} \phi(x_{n_1},\ldots,x_{n_{j_*}});\ \sum_{n_{j_*+1},\ldots,n_j}^{\neq} \phi'(x_{n_{j_*+1}},\ldots,x_{n_j})\bigg]\bigg| \\
&\qquad \le 4\,\omega\bigg(\mathrm{dist}\bigg(\bigcup_{i=1}^{j_*} B(x_i),\ \bigcup_{i=j_*+1}^{j} B(x_i)\bigg)\bigg) \le 4\,\omega\Big(\frac{1}{j}|x_1-x_j|-2\Big). \qquad (4.12)
\end{aligned}$$

Now we expand the covariance in terms of multi-point densities: by (4.11) and the support condition for ϕ, ϕ', we find that the product

$$\phi(x_{n_1},\ldots,x_{n_{j_*}})\phi'(x_{n_{j_*+1}},\ldots,x_{n_j})$$

vanishes whenever $n_i = n_l$ for some $1 \le i \le j_* < l \le j$, hence

$$\begin{aligned}
&\mathrm{Cov}\bigg[\sum_{n_1,\ldots,n_{j_*}}^{\neq} \phi(x_{n_1},\ldots,x_{n_{j_*}});\ \sum_{n_{j_*+1},\ldots,n_j}^{\neq} \phi'(x_{n_{j_*+1}},\ldots,x_{n_j})\bigg] \\
&\qquad = \int_{(\mathbb{R}^d)^j} (\phi\otimes\phi')\,(f_j - f_{j_*}\otimes f_{j-j_*}). \qquad (4.13)
\end{aligned}$$

Recalling the relation (4.7) for density functions in terms of correlations, we get

$$\begin{aligned}
&(f_j - f_{j_*}\otimes f_{j-j_*})(z_1,\ldots,z_j) \\
&\qquad = \sum_\pi \mathbb{1}_{\exists H\in\pi: H\cap\{1,\ldots,j_*\}\neq\varnothing\neq H\cap\{j_*+1,\ldots,j\}} \prod_{H\in\pi} h_{\sharp H}(z_H).
\end{aligned}$$

Combining this with (4.12) and (4.13), and isolating the contribution of the j-point correlation h_j (obtained for $\sharp\pi = 1$), we are led to

$$\begin{aligned}
&\bigg|\int_{(\mathbb{R}^d)^j} (\phi\otimes\phi')h_j\bigg| \\
&\qquad \le \sum_{\pi:\sharp\pi>1} \mathbb{1}_{\exists H\in\pi: H\cap\{1,\ldots,j_*\}\neq\varnothing\neq H\cap\{j_*+1,\ldots,j\}} \prod_{H\in\pi} \int_{B(x_H)} |h_{\sharp H}| \\
&\qquad\quad + 4\,\omega\Big(\frac{1}{j}|x_1-x_j|-2\Big),
\end{aligned}$$

where for $H = \{i_1, \ldots, i_l\}$ we set $B(x_H) := B(x_{i_1}) \times \cdots \times B(x_{i_l})$. In view of (4.11), the induction hypothesis for $\{h_l\}_{l<j}$ entails

$$\begin{aligned}
&\left| \int_{(\mathbb{R}^d)^j} (\phi \otimes \phi')h_j \right| \\
&\quad \le \sum_{\ell=2}^{j} \sum_{i_1+\cdots+i_\ell=j} \binom{j}{i_1, \ldots, i_\ell} \prod_{s=1}^{\ell} \left(C^{i_s} i_s! \, \omega\Big(\frac{1}{j}|x_1 - x_j| - 2\Big)^{i_s} \right) \\
&\qquad + 4\,\omega\Big(\frac{1}{j}|x_1 - x_j| - 2\Big),
\end{aligned}$$

from which we easily infer $|\int_{(\mathbb{R}^d)^j} (\phi \otimes \phi')h_j| \le C^j j!\, \omega(\frac{1}{j}|x_1 - x_j| - 2)$. By the arbitrariness of ϕ, ϕ' and of $x_1, \ldots, x_j$, the conclusion (4.10) follows for h_j. ∎

4.3.2 Estimates on single-particle contributions

For notational simplicity, we henceforth assume that particles are spherical with unit radius, $I_n = B(x_n)$; the adaptation to the general case (Indep) with independent particle shapes is straightforward. As we shall see, the explicit renormalization of $\bar{\mathbf{B}}^j$ is particularly intricate for $j \ge 3$ since cancellations are not as apparent as they are for the first two orders: it will require us to decompose corrector differences into elementary single-particle contributions in the spirit of (3.82). We start by slightly changing the point of view for correctors, focussing on particle positions rather than on particle indices in the notation: given a set $Y \subset Q_L$ of "background" positions such that

$$\operatorname{dist}\big(B(y), B(y')\big) > 2\rho, \quad \operatorname{dist}\big(B(y), \partial Q_L\big) > \rho, \quad \text{for all } y, y' \in Y,\ y \neq y', \tag{4.14}$$

we denote by $\psi_L^Y \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following periodic corrector problem, using the shorthand notation $\sigma_L^Y := \sigma(\psi_L^Y + Ex, \Sigma_L^Y)$,

$$\begin{cases}
-\triangle \psi_L^Y + \nabla \Sigma_L^Y = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\
\operatorname{div}(\psi_L^Y) = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\
\mathrm{D}(\psi_L^Y + Ex) = 0, & \text{in } \bigcup_{y \in Y} B(y), \\
\int_{\partial B(y)} \sigma_L^Y \nu = 0, & \forall y \in Y, \\
\int_{\partial B(y)} \Theta(x - y) \cdot \sigma_L^Y \nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall y \in Y.
\end{cases}$$

Next, similarly to (1.10), for any $z \in Q_L$ and any finite subset $Z \subset Q_L$, provided that the union set $\{z\} \cup Z \cup Y$ satisfies (4.14), we can define corrector differences

$$\delta^{\{z\}} \psi_L^Y := \psi_L^{\{z\} \cup Y} - \psi_L^Y, \quad \delta^Z \psi_L^Y := \sum_{W \subset Z} (-1)^{|Z \setminus W|} \psi_L^{W \cup Y}.$$

Compared with the notation that we use elsewhere in this memoir, this means for all index sets $F, H \subset \mathbb{N}$,

$$\psi_L^H \equiv \psi_L^{\{x_{n,L}\}_{n\in H}}, \quad \delta^F \psi_L^H \equiv \delta^{\{x_{n,L}\}_{n\in F}} \psi_L^{\{x_{n,L}\}_{n\in H}}.$$

For $Y = \{y_1, \ldots, y_m\}$ and $Z = \{z_1, \ldots, z_n\}$, we shall also write for convenience

$$\psi_L^{y_1,\ldots,y_m} := \psi_L^Y, \quad \delta^{z_1,\ldots,z_n} \psi_L^{y_1,\ldots,y_m} := \delta^Z \psi_L^Y. \tag{4.15}$$

Recall that Lemma 3.4 states that the corrector difference $\delta^Z \psi_L^Y$ satisfies

$$-\triangle \delta^Z \psi_L^Y + \nabla \delta^Z \Sigma_L^Y = -\sum_{z\in Z} \delta_{\partial B(z)} \delta^{Z\setminus\{z\}} \sigma_L^{Y\cup\{z\}} \nu \quad \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \tag{4.16}$$

together with the rigidity constraint $\mathrm{D}(\delta^Z \psi_L^Y) = 0$ in $\bigcup_{y\in Y} B(y)$ and with associated boundary conditions. In view of this equation, as in (3.82), we can decompose corrector differences into elementary single-particle contributions that we express in terms of operators $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$ defined as follows: given a "tagged" position $z \in Q_L$, given a pair $(\zeta, P) \in H^1(B_{1+\rho}(z))^d \times \mathrm{L}^2(B_{1+\rho}(z) \setminus B(z))$ satisfying the following Stokes equations in a neighborhood of $B(z)$,

$$\begin{cases} -\triangle \zeta + \nabla P = 0, & \text{in } B_{1+\rho}(z) \setminus B(z), \\ \operatorname{div}(\zeta) = 0, & \text{in } B_{1+\rho}(z) \setminus B(z), \\ \mathrm{D}(\zeta) = 0, & \text{in } B(z), \\ \int_{\partial B(z)} \sigma(\zeta, P)\nu = 0, & \\ \int_{\partial B(z)} \Theta(x - z) \cdot \sigma(\zeta, P)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{4.17}$$

and given a finite subset $Y \subset Q_L$ of "background" positions such that $\{z\} \cup Y$ satisfies (4.14), we denote by $\mathcal{J}_{L;Y}^z \zeta \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following Stokes problem,

$$\begin{cases} -\triangle \mathcal{J}_{L;Y}^z \zeta + \nabla \mathcal{Q}_{L;Y}^z \zeta = -\delta_{\partial B(z)} \sigma(\zeta, P)\nu, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \mathrm{D}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x - y) \cdot \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}},\ \forall y \in Y. \end{cases}$$

These operators $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$ describe the fluid velocity generated by localized force dipoles in the presence of a finite number of rigid inclusions and are thus viewed as Stokeslets for the problem with rigid inclusions. In view of our upcoming analysis (see in particular cancellation properties in Lemma 4.6 below), we further need to extend the definition of $\mathcal{J}^z_{L;Y}$ when the support $B(z)$ of the force dipole intersects rigid inclusions $\bigcup_{y\in Y} B(y)$ or the cell boundary ∂Q_L, which was excluded above by assuming that $\{z\}\cup Y$ satisfies (4.14). A convenient way to proceed is as follows: given $z\in Q_L$ and $Y\subset Q_L$ with only Y satisfying (4.14), we define $\mathcal{J}^z_{L;Y}\zeta\in H^1_{\mathrm{per}}(Q_L)^d$ as the solution of the following Stokes problem,

$$\begin{cases} -\triangle \mathcal{J}^z_{L;Y}\zeta+\nabla \mathcal{Q}^z_{L;Y}\zeta=-\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L\setminus\bigcup_{y\in Y}B(y),\\ \operatorname{div}(\mathcal{J}^z_{L;Y}\zeta)=0, & \text{in } Q_L\setminus\bigcup_{y\in Y}B(y),\\ \mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)=0, & \text{in } \bigcup_{y\in Y\setminus Y_z}B(y),\\ \int_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu=0, & \forall y\in Y\setminus Y_z,\\ \int_{\partial B(y)}\Theta(x-y)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu=0, & \forall\Theta\in\mathbb{M}^{\mathrm{skew}},\ \forall y\in Y\setminus Y_z,\\ \mathcal{J}^z_{L;Y}\zeta=V_z+\Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z}B(y),\\ & \text{for some } V_z\in\mathbb{R}^d,\Theta_z\in\mathbb{M}^{\mathrm{skew}},\\ \sum_{y\in Y_z}\int_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu & \\ =\sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\sigma(\zeta,P)\nu, & \\ \sum_{y\in Y_z}\int_{\partial B(y)}\Theta(x-z)\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu & \\ =\sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\Theta(x-z)\cdot\sigma(\zeta,P)\nu, & \forall\Theta\in\mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{4.18}$$

where $B^L(z):=(B(z)+L\mathbb{Z}^d)\cap Q_L$ stands for the periodization of the ball $B(z)$ in Q_L, where we have set $Y_z:=\{y\in Y: B(y)\cap B^L(z)\neq\varnothing\}$, and where we have implicitly extended (ζ,P) periodically to $B_{1+\rho}(z)+L\mathbb{Z}^d$. We emphasize that these equations are equivalent to the previous simpler ones when $\{z\}\cup Y$ satisfies (4.14) (hence $Y_z=\varnothing$). The solution $\mathcal{J}^z_{L;Y}\zeta$ is only defined up to a rigid motion in Q_L, which we fix by further choosing

$$\int_{Q_L}\mathcal{J}^z_{L;Y}\zeta=0,\qquad \int_{Q_L}\nabla\mathcal{J}^z_{L;Y}\zeta\in\mathbb{M}^{\mathrm{sym}}_0.$$

Note that $\mathcal{J}^z_{L;Y}\zeta$ depends of course on the pair (ζ,P), not only on ζ, but we leave the pressure field implicit in the notation for convenience. We further define

$$\mathcal{J}^z_L\zeta:=\mathcal{J}^z_{L;\varnothing}\zeta,$$

for which the defining Stokes problem (4.18) reduces to

$$-\triangle\mathcal{J}^z_L\zeta+\nabla\mathcal{Q}^z_L\zeta=-\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu,\quad \operatorname{div}(\mathcal{J}^z_L\zeta)=0,\quad \text{in } Q_L, \tag{4.19}$$

and we define $\mathcal{J}^z_Y\zeta$, $\mathcal{J}^z\zeta$ as the corresponding operators on whole space, that is, with $B^L(z)$ and Q_L replaced by $B(z)$ and $\mathbb{R}^d$, respectively, in (4.18) and (4.19).

In these terms, as in (3.82), given $Y, Z \subset Q_L$, provided that $Y \cup Z$ satisfies (4.14), equation (4.16) for corrector differences allows us to decompose

$$\delta^Z \psi_L^Y = \sum_{z \in Z} \mathcal{J}_{L;Y}^z \delta^{Z \setminus \{z\}} (\psi_L^{\{z\} \cup Y} + Ex). \tag{4.20}$$

The above definition (4.18) of $\mathcal{J}_{L;Y}^z$, with the particular choice of the extension to all $z \in Q_L$, is dictated by the following key observation. This constitutes the precise cancellation property that we shall repeatedly use for the explicit renormalization of cluster formulas.

Lemma 4.6 (Cancellation property). *For any $Y \subset Q_L$ satisfying (4.14), and for any function ζ satisfying (4.17) around $z = 0$, we have for $\zeta^z := \zeta(\cdot - z)$,*

$$\int_{Q_L} (\mathcal{J}_{L;Y}^z \zeta^z)\, dz = 0.$$

Proof. Integrating equations (4.18) for $\mathcal{J}_{L;Y}^z \zeta^z$ over z, noting that

$$\int_{Q_L} \Big(\sum_{y \in Y_z} \int_{B(y) \cap \partial B^L(z)} \sigma(\zeta^z, P^z)\nu \Big)\, dz = \sharp Y |B| \int_{\partial B} \sigma(\zeta, P)\nu = 0,$$

and similarly noting that

$$\begin{aligned} &\int_{Q_L} \big(\delta_{\partial B^L(z) \setminus \bigcup_{y \in Y} B(y)} \sigma(\zeta^z, P^z)\nu\big)\, dz \\ &= \int_{Q_L} \int_{\partial B^L(z)} \sigma(\zeta^z, P^z)\nu\, dz - \int_{Q_L} \Big(\sum_{y \in Y_z} \int_{B(y) \cap \partial B^L(z)} \sigma(\zeta^z, P^z)\nu \Big)\, dz \\ &= \Big| Q_L \setminus \bigcup_{y \in Y} B(y) \Big| \int_{\partial B} \sigma(\zeta, P)\nu = 0, \end{aligned}$$

the conclusion easily follows from the uniqueness of the solution to the Stokes problem (4.18). ∎

Next, we establish optimal decay estimates for these operators $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$, which are shown to coincide with the decay for the explicit Stokeslets $\{\mathcal{J}_L^z\}_z$ associated with the problem in free space without rigid inclusions. This result corresponds to Lemma 3.10 and the proof is postponed to Appendix A in form of Lemma A.1.

Lemma 4.7 (Decay of Stokeslets with rigid inclusions). *Let $z \in Q_L$, let (ζ, P) satisfy (4.17) at z, and let $Y \subset Q_L$ satisfy (4.14). Then, we have for all $x \in Q_L$,*

$$\begin{aligned} \Big(\int_{B^L(x)} |\mathrm{D}(\mathcal{J}_{L;Y}^z \zeta)|^2 \Big)^{\frac12} &\lesssim_{\sharp Y} \langle (x-z)_L \rangle^{-d} \Big(\int_{B_{1+\rho}(z)} |\mathrm{D}(\zeta)|^2 \Big)^{\frac12}, \\ \Big(\int_{B(x)} |\mathrm{D}(\mathcal{J}_Y^z \zeta)|^2 \Big)^{\frac12} &\lesssim_{\sharp Y} \langle x-z \rangle^{-d} \Big(\int_{B_{1+\rho}(z)} |\mathrm{D}(\zeta)|^2 \Big)^{\frac12}. \end{aligned}$$

Finally, since we aim at finite-volume approximation error estimates, we need to quantify the difference $\mathcal{J}^z_{L;Y} - \mathcal{J}^z_Y$ between periodized and whole-space Stokeslets. The proof is postponed to Appendix A in form of Lemma A.3. We emphasize that the stated bounds are not optimal, but will be good enough for our purposes.

Lemma 4.8 (Periodization error). *Let $z \in Q_L$, let (ζ, P) satisfy (4.17) at z, and let $Y \subset Q_L$ such that $\{z\} \cup Y$ satisfies (4.14). Then, we have for all $x \in Q_L$,*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2\Big)^{\frac12} \lesssim_{\sharp Y} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2\Big)^{\frac12} \times \big(\mathbb{1}_{|x-z|>\frac{L}{4}}\langle (x-z)_L\rangle^{-d} + \mathbb{1}_{|x-z|\le\frac{L}{4}}\,\mathrm{dist}\big(Y\setminus\{x,z\},\partial Q_L\big)^{-d}\big),$$

where we set for notational convenience $\mathrm{dist}(\varnothing, \partial Q_L) := L$, *and where we denote by $B^L_r(z) = (B_r(z) + L\mathbb{Z}^d) \cap Q_L$ the periodization of the ball $B_r(z)$ in Q_L. In addition,*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2\Big)^{\frac12} \lesssim_{\sharp Y} \big(\langle \mathrm{dist}(x,\partial Q_L)\rangle + \langle \mathrm{dist}\big(Y\setminus\{x\},\partial Q_L\big)\rangle\big)^{-d}.$$

4.4 Explicit renormalization of cluster formulas

This section is devoted to the proof of Theorem 4.3. We first describe the explicit renormalization of the second and third cluster coefficients $\bar{\mathbf{B}}^2$ and $\bar{\mathbf{B}}^3$, cf. Propositions 4.9 and 4.10 below, before turning to the general case, cf. Proposition 4.11. For notational simplicity, we assume that particles are spherical with unit radius, $I_n = B(x_n)$, but we emphasize that the general case follows along the same lines under the independence assumption (Indep). More precisely, it suffices to replace each occurrence of spherical particles below by iid random shapes and to further take the expectation with respect to the latter; we omit the detail.

4.4.1 Explicit renormalization of $\bar{\mathbf{B}}^2$: Batchelor–Green formula

We start with the analysis of $\bar{\mathbf{B}}^2$ and rigorously establish the so-called Batchelor–Green formula [7].

Proposition 4.9 (Batchelor–Green renormalization of $\bar{\mathbf{B}}^2$). *Let (H_ρ) and ($\mathrm{H}^{\mathrm{unif}}_\rho$) hold, and assume for simplicity that particles are spherical with unit radius, $I_n = B(x_n)$. Let also the mixing assumption (Mix^n_ω) hold to order $n = 2$ with some non-increasing* *rate $\omega \in C_b^\infty(\mathbb{R}^+)$ satisfying the Dini condition $\int_1^\infty \frac{1}{t}\,\omega(t)\,dt < \infty$, as well as the doubling condition $\omega(2t) \simeq \omega(t)$ for all $t \geq 0$. Then, the infinite-volume second-order cluster coefficient $\bar{\mathbf{B}}^2$ defined in (3.13) can be expressed as follows,*

$$E : \bar{\mathbf{B}}^2 E = \int_{\mathbb{R}^d} \bigg(\int_{\partial B} \psi^z \cdot \sigma^0 \nu \bigg) h_2(0,z)\,dz + \int_{\mathbb{R}^d} \bigg(\int_{\partial B} \psi^z \cdot \delta^z \sigma^0 \nu \bigg) f_2(0,z)\,dz, \tag{4.21}$$

where both integrals are absolutely converging and where we use the notation (4.15). In addition, the following estimates hold:

(i) Uniform cluster estimate:

$$|\bar{\mathbf{B}}_L^2| \lesssim \lambda_2(\mathscr{P}) + \int_1^\infty \frac{1}{t}\big(\omega(t) \wedge \lambda_2(\mathscr{P})\big)\,dt,$$

hence, in case of an algebraic weight $\omega(t) \leq Ct^{-\beta}$ for some $C, \beta > 0$,

$$|\bar{\mathbf{B}}_L^2| \lesssim \lambda_2(\mathscr{P})\big|\log \lambda(\mathscr{P})\big|.$$

(ii) Periodization error estimate:

$$|\bar{\mathbf{B}}_L^2 - \bar{\mathbf{B}}^2| \lesssim \bigg(\omega(L) + \frac{1}{L}\bigg)\log L + \int_1^\infty \frac{1}{t+L}\,\omega(t)\,dt.$$

(iii) Uniform remainder estimate: *If* (Mix_ω^n) *further holds with $n = 3$, then*

$$|R_L^2| \lesssim \lambda_2(\mathscr{P}) + \int_1^\infty \frac{1}{t}\big(\omega(t) \wedge \lambda_2(\mathscr{P})\big)\,dt + \int_1^\infty \frac{\log t}{t}\big(\omega(t) \wedge \lambda_3(\mathscr{P})\big)\,dt,$$

hence, in case of an algebraic weight $\omega(t) \leq Ct^{-\beta}$ for some $C, \beta > 0$,

$$|R_L^2| \lesssim \lambda_2(\mathscr{P})\big|\log \lambda(\mathscr{P})\big| + \lambda_3(\mathscr{P})\big|\log \lambda(\mathscr{P})\big|^2.$$

Proof. We split the proof into four steps. Given $E \in \mathbb{M}_0^{\mathrm{sym}}$ with $|E| = 1$, for notational convenience, we write $\bar{\mathbf{B}}_L^2$, $\bar{\mathbf{B}}^2$, and R_L^2 for $E : \bar{\mathbf{B}}_L^2 E$, $E : \bar{\mathbf{B}}^2 E$, and $E : R_L^2 E$.

Step 1. Reformulation of $\bar{\mathbf{B}}_L^2$:

$$\begin{aligned}
\bar{\mathbf{B}}_L^2 &= L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) h_2(y,z)\,dydz \\
&\quad + L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu \bigg) f_2(y,z)\,dydz \\
&\quad - \lambda(\mathscr{P})^2 L^{-d} \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) dydz,
\end{aligned} \tag{4.22}$$

where we recall the shorthand notation $Q_{L,\rho} = Q_{L-2(\ell \vee (1+\rho))}$, cf. (3.1).

By definition, cf. (3.8), the finite-volume approximation $\bar{\mathbf{B}}_L^2$ is given by

$$\bar{\mathbf{B}}_L^2 = L^{-d} \sum_{m \neq n} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \psi_L^{\{m\}} \cdot \sigma_L^{\{m,n\}} \nu \bigg].$$

Decomposing

$$\sigma_L^{\{m,n\}} = \sigma_L^{\{n\}} + \delta^{\{m\}} \sigma_L^{\{n\}},$$

this turns into

$$\bar{\mathbf{B}}_L^2 = L^{-d} \sum_{m \neq n} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \psi_L^{\{m\}} \cdot \sigma_L^{\{n\}} \nu \bigg] + L^{-d} \sum_{m \neq n} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \psi_L^{\{m\}} \cdot \delta^{\{m\}} \sigma_L^{\{n\}} \nu \bigg].$$

In terms of multi-point densities, cf. (1.15), recalling the choice of the finite-volume approximation with $\mathcal{P}_L = \{x_n : x_n \in Q_{L,\rho}\}$, cf. (3.1), and using the notation (4.15), we can rewrite

$$\bar{\mathbf{B}}_L^2 = L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) f_2(y,z)\, dy dz + L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu \bigg) f_2(y,z)\, dy dz, \tag{4.23}$$

and it remains to further analyze the first right-hand side term. For that purpose, we note that $\psi_L^z = \psi_L^0(\cdot - z)$ and $\sigma_L^y = \sigma_L^0(\cdot - y)$, so that

$$\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu = \int_{\partial B} \psi_L^0(\cdot + y - z) \cdot \sigma_L^0 \nu.$$

Integrating over z, using the periodicity of ψ_L^0, and recalling that

$$\int_{\partial B} \sigma_L^0 \nu = 0,$$

we deduce

$$\int_{Q_L} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) dz = 0. \tag{4.24}$$

Decomposing

$$f_2(y,z) = \lambda(\mathcal{P})^2 + h_2(y,z)$$

in terms of the 2-point correlation function h_2, and then using this cancellation property (4.24) to reformulate the first right-hand side term in (4.23), the claim (4.22) follows.

Step 2. Uniform estimate: proof of (i). Using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\Big|\int_{\partial B(y)}\psi_L^z\cdot\sigma_L^y\nu\Big|\lesssim\Big(\int_{B(y)}|\mathrm{D}(\psi_L^z)|^2\Big)^{\frac12}\Big(\int_{B_{1+\rho}(y)}|\mathrm{D}(\psi_L^y)|^2\Big)^{\frac12},$$
$$\Big|\int_{\partial B(y)}\psi_L^z\cdot\delta^z\sigma_L^y\nu\Big|\lesssim\Big(\int_{B(y)}|\mathrm{D}(\psi_L^z)|^2\Big)^{\frac12}\Big(\int_{B_{1+\rho}(y)}|\mathrm{D}(\delta^z\psi_L^y)|^2\Big)^{\frac12}.$$

Hence, applying the decay estimates of Lemma 4.7 to $\psi_L^z=\mathcal{J}_L^z(\psi_L^z+Ex)$ and to $\delta^z\psi_L^y=\mathcal{J}_{L;y}^z(\psi_L^{y,z}+Ex)$, combined with the energy estimate (3.44), we get

$$\begin{aligned}\Big|\int_{\partial B(y)}\psi_L^z\cdot\sigma_L^y\nu\Big|&\lesssim\langle(y-z)_L\rangle^{-d},\\ \Big|\int_{\partial B(y)}\psi_L^z\cdot\delta^z\sigma_L^y\nu\Big|&\lesssim\langle(y-z)_L\rangle^{-2d}.\end{aligned}\tag{4.25}$$

Formula (4.22) for $\bar{\mathbf{B}}_L^2$ can then be estimated as follows,

$$\begin{aligned}|\bar{\mathbf{B}}_L^2|\lesssim L^{-d}&\iint_{Q_L\times Q_L}\langle(y-z)_L\rangle^{-d}\big|h_2(y,z)\big|\,dydz\\ &+L^{-d}\iint_{Q_L\times Q_L}\langle(y-z)_L\rangle^{-2d}f_2(y,z)\,dydz\\ &+\lambda(\mathcal{P})^2L^{-d}\iint_{Q_L\times(Q_L\setminus Q_{L,\rho})}\langle(y-z)_L\rangle^{-d}\,dydz.\end{aligned}$$

In terms of the two-point intensity, recalling that $\overline{\lambda}_2(\mathcal{P})=\lambda_2(\mathcal{P})$ by Lemma 1.1 (ii) in view of (Mix$_\omega^n$), we can estimate the 2-point correlation function as follows: appealing both to (4.9) and to the decay assumption (Mix$_\omega^n$), and arguing as in Lemma 1.1 (iii), we find

$$\begin{aligned}&\iint_{Q_L\times Q_L}\langle(y-z)_L\rangle^{-d}\big|h_2(y,z)\big|\,dydz\\ &\qquad\lesssim\iint_{Q_L\times Q_L}\langle(y-z)_L\rangle^{-d}\big(\omega\big(|y-z|\big)\wedge\lambda_2(\mathcal{P})\big)\,dydz.\end{aligned}\tag{4.26}$$

The above then becomes

$$\begin{aligned}|\bar{\mathbf{B}}_L^2|\lesssim L^{-d}&\iint_{Q_L\times Q_L}\langle(y-z)_L\rangle^{-d}\big(\omega\big(|y-z|\big)\wedge\lambda_2(\mathcal{P})\big)\,dydz\\ &+\lambda_2(\mathcal{P})\Big(L^{-d}\iint_{Q_L\times Q_L}\langle(y-z)_L\rangle^{-2d}\,dydz\\ &\qquad+L^{-d}\iint_{Q_L\times(Q_L\setminus Q_{L,\rho})}\langle(y-z)_L\rangle^{-d}\,dydz\Big).\end{aligned}$$

As ω is non-increasing and as $|(y-z)_L| \le |y-z|$, the first right-hand side term is bounded by

$$
\begin{aligned}
& L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \big(\omega(|y-z|) \wedge \lambda_2(\mathcal{P})\big)\, dy dz \\
& \quad \le L^{-d} \iint_{Q_L \times Q_L} \langle (y-z)_L \rangle^{-d} \big(\omega(|(y-z)_L|) \wedge \lambda_2(\mathcal{P})\big)\, dy dz \\
& \quad \lesssim \int_{Q_L} \langle z \rangle^{-d} \big(\omega(|z|) \wedge \lambda_2(\mathcal{P})\big)\, dz \\
& \quad \lesssim \int_1^\infty \frac{1}{t} \big(\omega(t) \wedge \lambda_2(\mathcal{P})\big)\, dt,
\end{aligned}
$$

and the conclusion (i) follows after similarly estimating the other terms.

Step 3. Convergence result: proof of (ii). Comparing identities (4.21) and (4.22), we have

$$
|\overline{\mathbf{B}}_L^2 - \overline{\mathbf{B}}^2| \le A_L^1 + A_L^2 + A_L^3, \tag{4.27}
$$

where we have set for abbreviation

$$
\begin{aligned}
A_L^1 &:= \bigg| L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg) h_2(y,z)\, dy dz \\
& \qquad - \int_{\mathbb{R}^d} \bigg(\int_{\partial B} \psi^z \cdot \sigma^0 \nu \bigg) h_2(0,z)\, dz \bigg|, \\
A_L^2 &:= \bigg| L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi_L^z \cdot \delta^z \sigma_L^y \nu \bigg) f_2(y,z)\, dy dz \\
& \qquad - \int_{\mathbb{R}^d} \bigg(\int_{\partial B} \psi^z \cdot \delta^z \sigma^0 \nu \bigg) f_2(0,z)\, dz \bigg|, \\
A_L^3 &:= \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \bigg| \int_{\partial B(y)} \psi_L^z \cdot \sigma_L^y \nu \bigg|\, dy dz.
\end{aligned}
$$

We estimate these three contributions separately and we start with A_L^1. Noting that stationarity yields $h_2(y,z) = h_2(0, z-y)$, and using that $\psi^z = \psi^{z-y}(\cdot - y)$ and $\sigma^y = \sigma^0(\cdot - y)$, we can write

$$
\begin{aligned}
& L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B(y)} \psi^z \cdot \sigma^y \nu \bigg) h_2(y,z)\, dy dz \\
& \quad = L^{-d} \iint_{Q_{L,\rho} \times Q_{L,\rho}} \bigg(\int_{\partial B} \psi^{z-y} \cdot \sigma^0 \nu \bigg) h_2(0, z-y)\, dy dz \\
& \quad = \int_{\mathbb{R}^d} L^{-d} |Q_{L,\rho} \cap (Q_{L,\rho} + z)| \bigg(\int_{\partial B} \psi^z \cdot \sigma^0 \nu \bigg) h_2(0,z)\, dz,
\end{aligned}
$$

and thus, setting for abbreviation $\gamma_{L,\rho}^2(z) := L^{-d}|Q_{L,\rho} \cap (Q_{L,\rho}+z)|$, we get by the triangle inequality,

$$A_L^1 \lesssim \int_{\mathbb{R}^d} \big(1-\gamma_{L,\rho}^2(z)\big)\bigg|\int_{\partial B} \psi^z \cdot \sigma^0 \nu\bigg|\big|h_2(0,z)\big|\,dz \\ + L^{-d} \iint_{Q_L \times Q_L} \bigg(\bigg|\int_{\partial B(y)} (\psi_L^z - \psi^z)\cdot \sigma_L^y \nu\bigg| + \bigg|\int_{\partial B(y)} \psi^z \cdot (\sigma_L^y - \sigma^y)\nu\bigg|\bigg) \\ \times |h_2(y,z)|\,dy\,dz. \tag{4.28}$$

Appealing to the trace estimates of Lemma 3.5, decomposing

$$\begin{aligned} \psi_L^z - \psi^z &= \mathcal{J}_L^z(\psi_L^z + Ex) - \mathcal{J}^z(\psi^z + Ex) \\ &= \mathcal{J}^z(\psi_L^z - \psi^z) + (\mathcal{J}_L^z - \mathcal{J}^z)(\psi_L^z + Ex), \end{aligned}$$

using the decay estimates of Lemma 4.7, the periodization error estimates of Lemma 4.8, and the energy estimate (3.44), we find

$$\begin{aligned} &\bigg|\int_{\partial B(y)} (\psi_L^z - \psi^z)\cdot \sigma_L^y \nu\bigg| \\ &\lesssim \bigg(\int_{B(y)} \big|\mathrm{D}(\psi_L^z - \psi^z)\big|^2\bigg)^{\frac12}\bigg(\int_{Q_L} \big|\mathrm{D}(\psi_L^y + Ex)\big|^2\bigg)^{\frac12} \\ &\lesssim \bigg(\int_{B(y)} \big|\mathrm{D}\big(\mathcal{J}^z(\psi_L^z - \psi^z)\big)\big|^2 + \big|\mathrm{D}\big((\mathcal{J}_L^z - \mathcal{J}^z)(\psi_L^z + Ex)\big)\big|^2\bigg)^{\frac12} \\ &\lesssim \langle y-z\rangle^{-d}\bigg(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\psi_L^z - \psi^z)\big|^2\bigg)^{\frac12} \\ &\qquad + \Big(\mathbb{1}_{|y-z|>\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d}\Big)\bigg(\int_{Q_L} \big|\mathrm{D}(\psi_L^z + Ex)\big|^2\bigg)^{\frac12} \\ &\lesssim \mathbb{1}_{|y-z|>\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d}, \end{aligned}$$

and similarly,

$$\begin{aligned} \bigg|\int_{\partial B} \psi^z \cdot \sigma^0 \nu\bigg| &\lesssim \langle z\rangle^{-d}, \\ \bigg|\int_{\partial B(y)} \psi^z \cdot (\sigma_L^y - \sigma^y)\nu\bigg| &\lesssim L^{-d}\langle y-z\rangle^{-d}. \end{aligned}$$

Inserting these estimates into (4.28), we get

$$A_L^1 \lesssim \int_{\mathbb{R}^d} \big(1-\gamma_{L,\rho}^2(z)\big)\langle z\rangle^{-d}\big|h_2(0,z)\big|\,dz \\ + L^{-d} \iint_{Q_L \times Q_L} \Big(\mathbb{1}_{|y-z|>\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|y-z|\le\frac{L}{4}} L^{-d}\Big)\big|h_2(y,z)\big|\,dy\,dz.$$

Using the decay assumption (Mix$^n_\omega$) for h_2, noting that

$$1-\gamma_{L,\rho}^2(z) = 1 - L^{-d}\big|Q_{L,\rho}\cap(Q_{L,\rho}+z)\big| \lesssim \frac{|z|}{L}\wedge 1, \tag{4.29}$$

and using that $\int_{Q_L}\langle y\rangle^{-d}\,dy \lesssim \log L$, we conclude after straightforward simplifications,

$$\begin{aligned} A_L^1 &\lesssim \int_{\mathbb{R}^d}\Big(\frac{|z|}{L}\wedge 1\Big)\langle z\rangle^{-d}\omega\big(|z|\big)\,dz + \omega(L)\log L + L^{-d}\int_{Q_{2L}}\omega\big(|z|\big)\,dz \\ &\lesssim \omega(L)\log L + \int_0^\infty \frac{1}{t+L}\,\omega(t)\,dt. \end{aligned} \tag{4.30}$$

We turn to the estimation of the second term A_L^2 in (4.27). By stationarity, as above, we find

$$\begin{aligned} A_L^2 &\lesssim \int_{\mathbb{R}^d}(1-\gamma_{L,\rho}^2(z))\bigg|\int_{\partial B}\psi^z\cdot\delta^z\sigma^0\nu\bigg| f_2(0,z)\,dz \\ &+ L^{-d}\iint_{Q_L\times Q_L}\bigg(\bigg|\int_{\partial B(y)}(\psi_L^z-\psi^z)\cdot\delta^z\sigma_L^y\nu\bigg| + \bigg|\int_{\partial B(y)}\psi^z\cdot(\delta^z\sigma_L^y-\delta^z\sigma^y)\nu\bigg|\bigg) \\ &\qquad\qquad \times f_2(y,z)\,dydz. \end{aligned}$$

Recalling $\psi_L^z-\psi^z = \mathcal{J}^z(\psi_L^z-\psi^z)+(\mathcal{J}_L^z-\mathcal{J}^z)(\psi_L^z+Ex)$, further decomposing

$$\begin{aligned} \delta^z\psi_L^y-\delta^z\psi^y &= \mathcal{J}_{L;y}^z(\psi_L^{y,z}+Ex)-\mathcal{J}_y^z(\psi^{y,z}+Ex) \\ &= \mathcal{J}_y^z(\psi_L^{y,z}-\psi^{y,z})+(\mathcal{J}_{L;y}^z-\mathcal{J}_y^z)(\psi_L^{y,z}+Ex), \end{aligned}$$

and using the trace estimates of Lemma 3.5, the decay estimates of Lemma 4.7, the periodization error estimates of Lemma 4.8, and the energy estimate (3.44), we find

$$\begin{aligned} \bigg|\int_{\partial B}\psi^z\cdot\delta^z\sigma^0\nu\bigg| &\lesssim \langle z\rangle^{-2d}, \\ \bigg|\int_{\partial B(y)}(\psi_L^z-\psi^z)\cdot\delta^z\sigma_L^y\nu\bigg| &\lesssim \big\langle (y-z)_L\big\rangle^{-d}\big(d_L(y)+d_L(z)\big)^{-d} \\ \bigg|\int_{\partial B(y)}\psi^z\cdot(\delta^z\sigma_L^y-\delta^z\sigma^y)\nu\bigg| &\lesssim \langle y-z\rangle^{-2d}\big(d_L(y)+d_L(z)\big)^{-d}+L^{-d}\big\langle (y-z)_L\big\rangle^{-d}. \end{aligned}$$

where we have set for abbreviation $d_L(z) := \langle \mathrm{dist}(z,\partial Q_L)\rangle$. Inserting these estimates into the above, we get

$$\begin{aligned} A_L^2 &\lesssim \int_{\mathbb{R}^d}\big(1-\gamma_{L,\rho}^2(z)\big)\langle z\rangle^{-2d} f_2(0,z)\,dz \\ &\qquad + L^{-d}\iint_{Q_L\times Q_L}\langle (y-z)_L\rangle^{-d} d_L(y)^{-d} f_2(y,z)\,dydz. \end{aligned}$$

In terms of the two-point intensity, appealing to Lemma 1.1 (iii), recalling (4.29), and using that $\int_{Q_L} \langle y\rangle^{-d}dy \lesssim \log L$ and $\int_{Q_L} d_L(y)^{-d}dy \lesssim L^{d-1}$, we deduce

$$\begin{aligned} A_L^2 &\lesssim \lambda_2(\mathscr{P}) \int_{\mathbb{R}^d} \Big(\frac{|z|}{L}\wedge 1\Big)\langle z\rangle^{-2d}dz \\ &\quad + \lambda_2(\mathscr{P})L^{-d} \iint_{Q_L\times Q_L} \big\langle (y-z)_L\big\rangle^{-d} d_L(y)^{-d}\,dydz \\ &\lesssim \lambda_2(\mathscr{P})\frac{\log L}{L}. \end{aligned} \tag{4.31}$$

It remains to estimate the last term A_L^3 in (4.27). Using again the trace estimates of Lemma 3.5, the decay estimates of Lemma 4.7, and the energy estimate (3.44), we find

$$\Big|\int_{\partial B(y)} \psi_L^z\cdot\sigma_L^y\nu\Big| \lesssim \big\langle (y-z)_L\big\rangle^{-d},$$

and thus

$$\begin{aligned} A_L^3 &\lesssim \lambda(\mathscr{P})^2L^{-d} \iint_{Q_{L,\rho}\times(Q_L\setminus Q_{L,\rho})} \big\langle (y-z)_L\big\rangle^{-d}\,dydz \\ &\lesssim \lambda(\mathscr{P})^2\frac{\log L}{L}. \end{aligned}$$

Combining this with (4.27), (4.30), and (4.31), the conclusion (ii) follows.

Step 4. Uniform remainder estimate: proof of (iii). The starting point is the refined estimate (3.26) on remainders, which reads in this case

$$\begin{aligned} |R_L^2| &\le \mathbb{E}\Big[L^{-d}\sum_n \int_{B(x_{n,L})}\Big|\sum_{m:m\ne n} \mathrm{D}(\psi_L^{\{m\}})\Big|^2\Big] \\ &\quad + \Big|\mathbb{E}\Big[L^{-d}\sum_n \int_{B(x_{n,L})}\Big(\sum_{m:m\ne n}\mathrm{D}(\psi_L^{\{m\}})\Big):\mathrm{D}(\widehat\psi_{n,L}^{\{n\}})\Big]\Big|, \end{aligned}$$

where we recall that $\widehat\psi_{n,L}^{\{n\}}$ is defined by (3.27). Expanding the square and separating the different intersection patterns, this can be rewritten as follows, in terms of multi-point densities,

$$\begin{aligned} |R_L^2| &\le L^{-d}\iint_{(Q_{L,\rho})^2}\Big(\int_{B(x)}|\mathrm{D}(\psi_L^y)|^2\Big) f_2(x,y)\,dxdy \\ &\quad + L^{-d}\Big|\iiint_{(Q_{L,\rho})^3}\Big(\int_{B(x)}\mathrm{D}(\psi_L^y):\mathrm{D}(\psi_L^z)\Big) f_3(x,y,z)\,dxdydz\Big| \\ &\quad + L^{-d}\Big|\iint_{(Q_{L,\rho})^2}\Big(\int_{B(x)}\mathrm{D}(\psi_L^y):\mathrm{D}(\widehat\psi_{x,L}^x)\Big) f_2(x,y)\,dxdy\Big|, \end{aligned}$$

where we use the obvious notation for $\widehat{\psi}^x_{x,L}$ such that $\widehat{\psi}^{x_n}_{x_n,L} := \widehat{\psi}^{\{n\}}_{n,L}$. Replacing f_2, f_3 by their expansions (4.7) in terms of correlation functions, and noting that several contributions can be turned into boundary terms using identity $\int_{Q_L} \mathrm{D}(\psi^y_L)\,dy = 0$, we obtain

$$
\begin{aligned}
|R^2_L| \lesssim{}& L^{-d} \iint_{(Q_{L,\rho})^2} \Big(\int_{B(x)} \big|\mathrm{D}(\psi^y_L)\big|^2 \Big) f_2(x,y)\,dxdy \\
&+ L^{-d} \bigg| \iiint_{(Q_{L,\rho})^3} \Big(\int_{B(x)} \mathrm{D}(\psi^y_L) : \mathrm{D}(\psi^z_L) \Big) \big(\lambda(\mathcal{P}) h_2(y,z) + h_3(x,y,z)\big)\,dxdydz \bigg| \\
&+ L^{-d} \bigg| \iint_{(Q_{L,\rho})^2} \Big(\int_{B(x)} \mathrm{D}(\psi^y_L) : \mathrm{D}(\widehat{\psi}^x_{x,L}) \Big) h_2(x,y)\,dxdy \bigg| \\
&+ \lambda(\mathcal{P}) L^{-d} \bigg| \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{B(x)} \mathrm{D}(\psi^y_L) : \mathrm{D}(\psi^z_L) \Big) h_2(x,y)\,dxdydz \bigg| \\
&+ \lambda(\mathcal{P})^3 L^{-d} \bigg| \iiint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})^2} \Big(\int_{B(x)} \mathrm{D}(\psi^y_L) : \mathrm{D}(\psi^z_L) \Big)\,dxdydz \bigg| \\
&+ \lambda(\mathcal{P})^2 L^{-d} \bigg| \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{B(x)} \mathrm{D}(\psi^y_L) : \mathrm{D}(\widehat{\psi}^x_{x,L}) \Big)\,dxdy \bigg|.
\end{aligned}
$$

Using (3.28) to estimate $\widehat{\psi}^x_{x,L}$ in terms of ψ^x_L,

$$
\int_{B(x)} \big|\mathrm{D}(\widehat{\psi}^x_{x,L})\big|^2 \lesssim \int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x_L) + E\big|^2 \lesssim 1,
$$

and appealing to the decay estimates of Lemma 4.7, we deduce

$$
\begin{aligned}
|R^2_L| \lesssim{}& L^{-d} \iint_{(Q_{L,\rho})^2} \langle (x-y)_L \rangle^{-2d} f_2(x,y)\,dxdy \\
&+ L^{-d} \iiint_{(Q_{L,\rho})^3} \langle (x-y)_L \rangle^{-d} \langle (x-z)_L \rangle^{-d} \\
&\qquad \times \big(\lambda(\mathcal{P}) \big|h_2(y,z)\big| + \big|h_3(x,y,z)\big|\big)\,dxdydz \\
&+ L^{-d} \iint_{(Q_{L,\rho})^2} \langle (x-y)_L \rangle^{-d} \big|h_2(x,y)\big|\,dxdy \\
&+ \lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \langle (x-y)_L \rangle^{-d} \langle (x-z)_L \rangle^{-d} \\
&\qquad \times |h_2(x,y)|\,dxdydz \\
&+ \lambda(\mathcal{P})^3 L^{-d} \iiint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})^2} \langle (x-y)_L \rangle^{-d} \langle (x-z)_L \rangle^{-d}\,dxdydz \\
&+ \lambda(\mathcal{P})^2 L^{-d} \iint_{Q_{L,\rho} \times (Q_L \setminus Q_{L,\rho})} \langle (x-y)_L \rangle^{-d}\,dxdy.
\end{aligned}
$$

In terms of multi-point intensities, appealing to Lemma 1.1 (ii)–(iii), and using (4.9) and the decay assumption (Mix^n_ω) to estimate correlation functions as in (4.26), the conclusion (iii) follows after straightforward computations. ∎

4.4.2 Explicit renormalization of $\bar{\mathbf{B}}^3$

The explicit renormalization of $\bar{\mathbf{B}}^2$ above is solely based on the simple and neat cancellation property (4.24). Higher-order cluster formulas require more subtle cancellations, which can only be captured after suitably decomposing corrector differences in terms of elementary single-particle contributions as in (4.20). Before turning to the general case and proving Theorem 4.3, we start with a detailed account of the third-order cluster coefficient $\bar{\mathbf{B}}^3$, which contains all the necessary new ingredients.

Proposition 4.10 (Renormalization of $\bar{\mathbf{B}}^3$). *Let* (H_ρ) *and* ($\text{H}^{\text{unif}}_\rho$) *hold, and assume for simplicity that particles are spherical with unit radius,* $I_n = B(x_n)$*. Let also the mixing assumption* (Mix^n_ω) *hold to order* $n = 3$ *with some non-increasing rate function* $\omega \in C^\infty_b(\mathbb{R}^+)$ *satisfying the Dini-type condition* $\int_1^\infty \frac{\log t}{t}\omega(t)\,dt < \infty$*, as well as the doubling condition* $\omega(2t) \simeq \omega(t)$ *for all* $t \geq 0$*. Then, the infinite-volume third-order cluster coefficient* $\bar{\mathbf{B}}^3$ *defined in* (3.13) *can be expressed as follows,*

$$
\begin{aligned}
&E : \bar{\mathbf{B}}^3 E \\
&= 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} \big(\mathcal{J}^y \mathcal{J}^z_y(\psi^z + Ex)\big)\cdot\sigma^0\nu\Big)\big(\lambda(\mathcal{P})h_2(0,z) + h_3(0,y,z)\big)\,dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} (\mathcal{J}^y \mathcal{J}^z_y \delta^y \psi^z)\cdot\sigma^0\nu\Big)\big(f_3(0,y,z) - \lambda(\mathcal{P}) f_2(y,z)\big)\,dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} \big(\mathcal{J}^y \mathcal{J}^z_y(\psi^z + Ex)\big)\cdot\delta^y\sigma^0\nu\Big) \\
&\qquad\qquad \times \big(f_3(0,y,z) - \lambda(\mathcal{P}) f_2(0,y)\big)\,dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} (\mathcal{J}^z \delta^y \psi^z)\cdot\delta^y\sigma^0\nu\Big) f_3(0,y,z)\,dydz \\
&\quad + 3 \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} (\mathcal{J}^y \mathcal{J}^z_y \delta^y \psi^z)\cdot\delta^y\sigma^0\nu\Big) f_3(0,y,z)\,dydz \\
&\quad + \frac{3}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\partial B} \delta^{y,z}\psi^\varnothing \cdot \delta^{y,z}\sigma^0\nu\Big) f_3(0,y,z)\,dydz,
\end{aligned}
\tag{4.32}
$$

where all the integrals are absolutely convergent and where we use the notation (4.15)*. In addition, the following estimates hold:*

(i) Uniform cluster estimate:

$$|\bar{\mathbf{B}}^3_L| \lesssim \lambda_3(\mathcal{P}) + \int_1^\infty \frac{\log t}{t}\big(\omega(t)\wedge\lambda_3(\mathcal{P})\big)dt,$$

hence, in case of an algebraic weight $\omega(t) \le Ct^{-\beta}$ *for some* $C, \beta > 0$,

$$|\bar{\mathbf{B}}_L^3| \lesssim \lambda_3(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^2.$$

(ii) Periodization error estimate:

$$|\bar{\mathbf{B}}_L^3 - \bar{\mathbf{B}}^3| \lesssim \frac{\log L}{L} + \omega(L)(\log L)^2 + \int_1^\infty \frac{\log t}{t+L}\,\omega(t)\,dt.$$

(iii) Uniform remainder estimate: *If* (Mix$_\omega^n$) *further holds with* $n = 5$, *then*

$$|R_L^3| \lesssim \lambda_3(\mathcal{P}) + \sum_{j=3}^5 \int_1^\infty \frac{(\log t)^{j-2}}{t}\big(\omega(t) \wedge \lambda_j(\mathcal{P})\big)\,dt,$$

hence, in case of an algebraic weight $\omega(t) \le Ct^{-\beta}$ *for some* $C, \beta > 0$,

$$|R_L^3| \lesssim \lambda_3(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^2 + \lambda_4(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^3 + \lambda_5(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^4.$$

Proof. We split the proof into four steps. Given $E \in \mathbb{M}_0^{\mathrm{sym}}$ with $|E| = 1$, for notational convenience, we write $\bar{\mathbf{B}}_L^3$, $\bar{\mathbf{B}}^3$, and R_L^3 for $E : \bar{\mathbf{B}}_L^3 E$, $E : \bar{\mathbf{B}}^3 E$, and $E : R_L^3 E$.

Step 1. Reformulation of $\bar{\mathbf{B}}_L^3$:

$$\begin{aligned}
\bar{\mathbf{B}}_L^3 &= E_L^3 \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \sigma_L^x \nu\bigg) \\
&\qquad\qquad \times \big(\lambda(\mathcal{P}) h_2(x,z) + h_3(x,y,z)\big)\,dx\,dy\,dz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \sigma_L^x \nu\bigg) \\
&\qquad\qquad \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(y,z)\big)\,dx\,dy\,dz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu\bigg) \\
&\qquad\qquad \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(x,y)\big)\,dx\,dy\,dz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu\bigg) f_3(x,y,z)\,dx\,dy\,dz \\
&+ 3L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu\bigg) f_3(x,y,z)\,dx\,dy\,dz \\
&+ \frac{3}{2} L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} \delta^{y,z} \psi_L^\varnothing \cdot \delta^{y,z} \sigma_L^x \nu\bigg) f_3(x,y,z)\,dx\,dy\,dz,
\end{aligned} \tag{4.33}$$

where we henceforth use the shorthand notation

$$\bar\psi_L^Y := \psi_L^Y + Ex,$$

and where E_L^3 stands for boundary terms,

$$\begin{aligned}
&E_L^3 \\
&:= 3\lambda(\mathcal{P})^3 L^{-d} \iiint_{Q_{L,\rho}\times(Q_L\setminus Q_{L,\rho})^2} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z)\cdot\sigma_L^x \nu\bigg)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z)\cdot\sigma_L^x \nu\bigg) h_2(x,y)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z)\cdot\sigma_L^x \nu\bigg) h_2(y,z)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \bar\psi_L^{y,z})\cdot\sigma_L^x \nu\bigg) \\
&\qquad\qquad\qquad\qquad \times f_2(y,z)\, dxdydz \\
&- 3\lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \bar\psi_L^z)\cdot\delta^y\sigma_L^x \nu\bigg) f_2(x,y)\, dxdydz.
\end{aligned}$$

By definition, cf. (3.8), the finite-volume approximation $\bar{\mathbf{B}}_L^3$ is given by

$$\bar{\mathbf{B}}_L^3 = \frac{3}{2} L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}}\psi_L^{\varnothing}\cdot\sigma_L^{\{n,m,p\}}\nu\bigg].$$

Decomposing

$$\sigma_L^{\{n,m,p\}} = \sigma_L^{\{n\}} + \delta^{\{m\}}\sigma_L^{\{n\}} + \delta^{\{p\}}\sigma_L^{\{n\}} + \delta^{\{m,p\}}\sigma_L^{\{n\}},$$

this becomes by symmetry,

$$\begin{aligned}
\bar{\mathbf{B}}_L^3 &= \frac{3}{2} L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}}\psi_L^{\varnothing}\cdot\sigma_L^{\{n\}}\nu\bigg] \\
&\quad + 3L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}}\psi_L^{\varnothing}\cdot\delta^{\{m\}}\sigma_L^{\{n\}}\nu\bigg] \\
&\quad + \frac{3}{2} L^{-d} \sum_{n,m,p}^{\neq} \mathbb{E}\bigg[\int_{\partial B(x_{n,L})} \delta^{\{m,p\}}\psi_L^{\varnothing}\cdot\delta^{\{m,p\}}\sigma_L^{\{n\}}\nu\bigg].
\end{aligned}$$

In terms of multi-point densities, cf. (1.15), recalling the choice of the finite-volume approximation with $\mathcal{P}_L = \{x_n : x_n \in Q_{L,\rho}\}$, cf. (3.1), we can rewrite

$$\begin{aligned}\bar{\mathbf{B}}_L^3 &= \frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz\\&\quad+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^y\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz\\&\quad+\frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^{y,z}\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz. \qquad (4.34)\end{aligned}$$

It remains to further analyze the first two right-hand side terms and we split the proof into two further substeps. To capture cancellations, we shall expand $\delta^{y,z}\psi_L^{\varnothing}$ and $\delta^y\psi_L^x$ in terms of single-particle contributions as in (4.20).

Substep 1.1. Proof that

$$\begin{aligned}&L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz\\&\quad=E_L^{3,1}+2L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)\\&\qquad\qquad\times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\\&\qquad+2L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z})\cdot\sigma_L^x\nu\Big)\\&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dxdydz, \qquad (4.35)\end{aligned}$$

where we recall the shorthand notation $\bar{\psi}_L^Y = \psi_L^Y + Ex$, and where $E_L^{3,1}$ stands for boundary terms,

$$\begin{aligned}E_L^{3,1}&:=2\lambda(\mathcal{P})^3L^{-d}\iiint_{Q_{L,\rho}\times(Q_L\setminus Q_{L,\rho})^2}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)\,dxdydz\\&-2\lambda(\mathcal{P})L^{-d}\iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)h_2(x,y)\,dxdydz\\&-2\lambda(\mathcal{P})L^{-d}\iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^z)\cdot\sigma_L^x\nu\Big)h_2(y,z)\,dxdydz\\&-2\lambda(\mathcal{P})L^{-d}\\&\quad\times\iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_{L,\rho})^2}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z})\cdot\sigma_L^x\nu\Big)f_2(y,z)\,dxdydz.\end{aligned}$$

In view of (4.20), corrector differences can be decomposed as

$$\begin{aligned}\delta^{y,z}\psi_L^{\varnothing}&=\mathcal{J}_L^y\delta^z\psi_L^y+\mathcal{J}_L^z\delta^y\psi_L^z\\&=\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar{\psi}_L^{y,z}+\mathcal{J}_L^z\mathcal{J}_{L;z}^y\bar{\psi}_L^{y,z}, \qquad (4.36)\end{aligned}$$

and thus, further writing

$$\psi_L^{y,z} = \psi_L^z + \delta^y \psi_L^z = \psi_L^z + \mathcal{J}_{L;z}^y \overline{\psi}_L^{y,z}$$
$$= \psi_L^y + \delta^z \psi_L^y = \psi_L^y + \mathcal{J}_{L;y}^z \overline{\psi}_L^{y,z},$$

we deduce

$$\delta^{y,z}\psi_L^{\varnothing} = \mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z + \mathcal{J}_L^z \mathcal{J}_{L;z}^y \overline{\psi}_L^y + \mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \overline{\psi}_L^{y,z} + \mathcal{J}_L^z \mathcal{J}_{L;z}^y \mathcal{J}_{L;y}^z \overline{\psi}_L^{y,z}. \tag{4.37}$$

From this decomposition, we get by symmetry

$$L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} \delta^{y,z}\psi_L^{\varnothing} \cdot \sigma_L^x \nu \bigg) f_3(x,y,z)\, dxdydz$$
$$= 2L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu \bigg) f_3(x,y,z)\, dxdydz$$
$$+ 2L^{-d} \iiint_{(Q_{L,\rho})^3} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \overline{\psi}_L^{y,z}) \cdot \sigma_L^x \nu \bigg) f_3(x,y,z)\, dxdydz. \tag{4.38}$$

We are now in position to exploit cancellations properties. First note that Lemma 4.6 yields, recalling $\psi_L^z = \psi_L^0(\cdot - z)$,

$$\int_{Q_L} (\mathcal{J}_{L;y}^z \overline{\psi}_L^z)\, dz = 0, \tag{4.39}$$

and thus, for all $x, y \in \mathbb{R}^d$,

$$\int_{Q_L} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu \bigg)\, dz = 0.$$

In addition, similarly to what was done in (4.24) for the renormalization of $\overline{\mathbf{B}}_L^2$, writing

$$\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu = \int_{\partial B} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z)(\cdot + x) \cdot \sigma_L^0 \nu,$$

and using the condition $\int_{\partial B} \sigma_L^0 \nu = 0$, we find

$$\int_{Q_L} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu \bigg)\, dx = 0, \tag{4.40}$$

and likewise,

$$\int_{Q_L} \bigg(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \mathcal{J}_{L;z}^y \overline{\psi}_L^{y,z}) \cdot \sigma_L^x \nu \bigg)\, dx = 0.$$

Turning back to (4.38), replacing f_3 by its expansion (4.7) in terms of correlation functions, and using these three cancellation properties, the claim (4.35) follows.

Substep 1.2. Proof that

$$\begin{aligned} L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} \delta^{y,z} \psi_L^{\varnothing} \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz \\ = E_L^{3,2} + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{I}_L^y \mathcal{I}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) \\ \times \big(f_3(x,y,z) - \lambda(\mathcal{P}) f_2(x,y) \big) dx dy dz \\ + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{I}_L^z \mathcal{I}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz \\ + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{I}_L^y \mathcal{I}_{L;y}^z \mathcal{I}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz, \end{aligned} \tag{4.41}$$

where $E_L^{3,2}$ stands for a boundary term,

$$\begin{aligned} E_L^{3,2} := \\ - \lambda(\mathcal{P}) L^{-d} \iiint_{(Q_{L,\rho})^2 \times (Q_L \setminus Q_{L,\rho})} \Big(\int_{\partial B(x)} (\mathcal{I}_L^y \mathcal{I}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) f_2(x,y)\, dx dy dz. \end{aligned}$$

Inserting this into (4.34), together with (4.35), the claim (4.33) follows.

We turn to the proof of (4.41). As this term benefits from some additional decay due to the factor $\delta^y \sigma_L^x$, we only need the following (asymmetric) simpler version of (4.37),

$$\delta^{y,z} \psi_L^{\varnothing} = \mathcal{I}_L^y \mathcal{I}_{L;y}^z \bar{\psi}_L^z + \mathcal{I}_L^z \mathcal{I}_{L;z}^y \bar{\psi}_L^{y,z} + \mathcal{I}_L^y \mathcal{I}_{L;y}^z \mathcal{I}_{L;z}^y \bar{\psi}_L^{y,z},$$

which leads us to

$$\begin{aligned} L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} \delta^{y,z} \psi_L^{\varnothing} \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz \\ = L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{I}_L^y \mathcal{I}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz \\ + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{I}_L^z \mathcal{I}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz \\ + L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{I}_L^y \mathcal{I}_{L;y}^z \mathcal{I}_{L;z}^y \bar{\psi}_L^{y,z}) \cdot \delta^y \sigma_L^x \nu \Big) f_3(x,y,z)\, dx dy dz, \end{aligned}$$

and it remains to analyze the first right-hand side term. For that purpose, we use again the elementary cancellation property (4.39), now in form of

$$\int_{Q_L} \Big(\int_{\partial B(x)} (\mathcal{I}_L^y \mathcal{I}_{L;y}^z \bar{\psi}_L^z) \cdot \delta^y \sigma_L^x \nu \Big) dz = 0,$$

which entails

$$L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\overline{\psi}_L^z)\cdot\delta^y\sigma_L^x\nu\Big)f_3(x,y,z)\,dxdydz$$
$$=L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\overline{\psi}_L^z)\cdot\delta^y\sigma_L^x\nu\Big)$$
$$\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(x,y)\big)\,dxdydz$$
$$-\lambda(\mathcal{P})L^{-d}\iiint_{(Q_{L,\rho})^2\times(Q_L\setminus Q_{L,\rho})}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\overline{\psi}_L^z)\cdot\delta^y\sigma_L^x\nu\Big)$$
$$\times f_2(x,y)\,dxdydz,$$

and the claim (4.41) follows.

Step 2. Uniform estimates: proof of (i). As in the proof of Proposition 4.9 (i), appealing to the trace estimates of Lemma 2.5, the decay estimates of Lemma 4.7, and the energy estimate (3.44), formula (4.33) for $\overline{\mathbf{B}}_L^3$ can be estimated as follows,

$$|\overline{\mathbf{B}}_L^3|\lesssim|E_L^3|$$
$$+L^{-d}\iiint_{(Q_L)^3}\langle(x-y)_L\rangle^{-d}\langle(y-z)_L\rangle^{-d}\big(\lambda(\mathcal{P})|h_2(x,z)|+|h_3(x,y,z)|\big)\,dxdydz$$
$$+L^{-d}\iiint_{(Q_L)^3}\langle(x-y)_L\rangle^{-d}\langle(y-z)_L\rangle^{-2d}\big|f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big|\,dxdydz$$
$$+L^{-d}\iiint_{(Q_L)^3}\langle(x-z)_L\rangle^{-d}\langle(z-y)_L\rangle^{-d}\langle(y-x)_L\rangle^{-d}f_3(x,y,z)\,dxdydz$$
$$+L^{-d}\iiint_{(Q_L)^3}\langle(x-y)_L\rangle^{-2d}\langle(y-z)_L\rangle^{-2d}f_3(x,y,z)\,dxdydz,$$

and, for boundary terms,

$$|E_L^3|\lesssim\lambda(\mathcal{P})^3L^{-d}\iiint_{Q_L\times(Q_L\setminus Q_{L,\rho})^2}\langle(x-y)_L\rangle^{-d}\langle(y-z)_L\rangle^{-d}\,dxdydz$$
$$+\lambda(\mathcal{P})L^{-d}\iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_L)^2}\langle(x-y)_L\rangle^{-d}\langle(y-z)_L\rangle^{-d}|h_2(y,z)|\,dxdydz$$
$$+\lambda(\mathcal{P})L^{-d}\iiint_{(Q_L\setminus Q_{L,\rho})\times(Q_L)^2}\langle(x-y)_L\rangle^{-d}\langle(y-z)_L\rangle^{-2d}f_2(y,z)\,dxdydz.$$

In terms of multi-point intensities, appealing to Lemma 1.1, using both (4.9) and the decay assumption (Mix_ω^n) to estimate correlation functions similarly to (4.26), we deduce after straightforward computations

$$
\begin{aligned}
|\bar{\mathbf{B}}_L^3| &\lesssim |E_L^3| + \lambda_3(\mathcal{P}) + \int_1^\infty \frac{\log t}{t}\big(\omega(t)\wedge\lambda_3(\mathcal{P})\big)dt,\\
|E_L^3| &\lesssim \frac{\log L}{L}\Big(\lambda_3(\mathcal{P}) + \int_1^L \frac{1}{t}\big(\omega(t)\wedge\lambda_3(\mathcal{P})\big)dt\Big),
\end{aligned}
\tag{4.42}
$$

and the conclusion (i) follows.

Step 3. Convergence result: proof of (ii). In terms of $\gamma_{L,\rho}^3(y,z) := L^{-d}|Q_{L,\rho}\cap(y+Q_{L,\rho})\cap(z+Q_{L,\rho})|$, using stationarity and recalling that $\delta^y\psi^z = \mathcal{J}_z^y\bar{\psi}^{y,z}$, the formula (4.32) for the infinite-volume cluster coefficient $\bar{\mathbf{B}}^3$ takes the equivalent form

$$
\begin{aligned}
\bar{\mathbf{B}}^3 = E_L^4
&+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}_y^z\bar{\psi}^z)\cdot\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\\
&+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}_y^z\mathcal{J}_z^y\bar{\psi}^{y,z})\cdot\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dxdydz\\
&+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}_y^z\bar{\psi}^z)\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(x,y)\big)\,dxdydz\\
&+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^z\mathcal{J}_z^y\bar{\psi}^{y,z})\cdot\delta^y\sigma^x\nu\Big)f_3(x,y,z)\,dxdydz\\
&+ 3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y\mathcal{J}_{L;y}^z\mathcal{J}_z^y\bar{\psi}^{y,z})\cdot\delta^y\sigma^x\nu\Big)f_3(x,y,z)\,dxdydz\\
&+ \frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x\nu\Big)f_3(x,y,z)\,dxdydz,
\end{aligned}
$$

where

$$
\begin{aligned}
E_L^4 := 3&\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma_{L,\rho}^3(y,z))\Big(\int_{\partial B}(\mathcal{J}^y\mathcal{J}_y^z\bar{\psi}^z)\cdot\sigma^0\nu\Big)\\
&\qquad\times\big(\lambda(\mathcal{P})h_2(0,z)+h_3(0,y,z)\big)\,dydz\\
+3&\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma_{L,\rho}^3(y,z))\Big(\int_{\partial B}(\mathcal{J}^y\mathcal{J}_y^z\delta^y\psi^z)\cdot\sigma^0\nu\Big)\\
&\qquad\times\big(f_3(0,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dydz\\
+3&\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma_{L,\rho}^3(y,z))\Big(\int_{\partial B}(\mathcal{J}^y\mathcal{J}_y^z\bar{\psi}^z)\cdot\delta^y\sigma^0\nu\Big)\\
&\qquad\times\big(f_3(0,y,z)-\lambda(\mathcal{P})f_2(0,y)\big)\,dydz
\end{aligned}
$$

$$+\,3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\Big(\int_{\partial B}(\mathcal{J}^z\delta^y\psi^z)\cdot\delta^y\sigma^0\nu\Big)f_3(0,y,z)\,dydz$$
$$+\,3\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\Big(\int_{\partial B}(\mathcal{J}^y\mathcal{J}^z_y\delta^y\psi^z)\cdot\delta^y\sigma^0\nu\Big)f_3(0,y,z)\,dydz$$
$$+\,\frac{3}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1-\gamma^3_{L,\rho}(y,z))\Big(\int_{\partial B}\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^0\nu\Big)f_3(0,y,z)\,dydz,$$

so that the identity (4.33) for $\overline{\mathbf{B}}^3_L$ entails

$$\begin{aligned}
&\overline{\mathbf{B}}^3_L-\overline{\mathbf{B}}^3=E^3_L-E^4_L\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\overline{\psi}^z_L)\cdot\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\overline{\psi}^z)\cdot\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\mathcal{J}^y_{L;z}\overline{\psi}^{y,z}_L)\cdot\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\mathcal{J}^y_z\overline{\psi}^{y,z})\cdot\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(y,z)\big)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\overline{\psi}^z_L)\cdot\delta^y\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\overline{\psi}^z)\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times\big(f_3(x,y,z)-\lambda(\mathcal{P})f_2(x,y)\big)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^z_L\mathcal{J}^y_{L;z}\overline{\psi}^{y,z}_L)\cdot\delta^y\sigma^x_L\nu-(\mathcal{J}^z\mathcal{J}^y_z\overline{\psi}^{y,z})\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times f_3(x,y,z)\,dxdydz\\
&+3L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}^y_L\mathcal{J}^z_{L;y}\mathcal{J}^y_{L;z}\overline{\psi}^{y,z}_L)\cdot\delta^y\sigma^x_L\nu-(\mathcal{J}^y\mathcal{J}^z_y\mathcal{J}^y_z\overline{\psi}^{y,z})\cdot\delta^y\sigma^x\nu\Big)\\
&\qquad\qquad\times f_3(x,y,z)\,dxdydz\\
&+\frac{3}{2}L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}\delta^{y,z}\psi^{\varnothing}_L\cdot\delta^{y,z}\sigma^x_L\nu-\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x\nu\Big)\\
&\qquad\qquad\times f_3(x,y,z)\,dxdydz.
\end{aligned}\tag{4.43}$$

The first boundary contribution E^3_L is already estimated in (4.42). Noting that

$$1-\gamma^3_{L,\rho}(y,z)\lesssim\frac{\langle y\rangle}{L}\wedge1+\frac{\langle z\rangle}{L}\wedge1,$$

using the trace estimates of Lemma 3.5, the decay estimates of Lemma 4.7, the energy estimate (3.44), and Lemma 1.1 (iii), and further using (4.9) and the decay assumption (Mix^n_ω) to estimate correlation functions similarly to (4.26), we obtain for the second boundary contribution in (4.43),

$$E^4_L\lesssim\frac{1}{L}\int_1^L(\log t)\,\omega(t)\,dt+\int_L^\infty\frac{\log t}{t}\,\omega(t)\,dt.$$

It remains to estimate the remaining six right-hand side terms in (4.43). We focus on the first term, which is the most involved, and we skip the detail for the last five ones. We split the proof into two further substeps.

Substep 3.1. First periodization error term in (4.43): proof that

$$\Big| L^{-d} \iiint_{(Q_{L,\rho})^3} \Big(\int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu - (\mathcal{J}^y \mathcal{J}_y^z \overline{\psi}^z) \cdot \sigma^x \nu \Big) \times \big(\lambda(\mathcal{P}) h_2(x,z) + h_3(x,y,z)\big)\, dx dy dz \Big| \lesssim \omega(L)(\log L)^2 + \frac{1}{L} \int_1^L (\log t)\omega(t)\, dt. \tag{4.44}$$

Decomposing

$$\begin{aligned}(\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu - (\mathcal{J}^y \mathcal{J}_y^z \overline{\psi}^z) \cdot \sigma^x \nu &= (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot (\sigma_L^x - \sigma^x)\nu + (\mathcal{J}_L^y - \mathcal{J}^y)\mathcal{J}_{L;y}^z \overline{\psi}_L^z \cdot \sigma^x \nu \\ &\quad + \mathcal{J}^y (\mathcal{J}_{L;y}^z - \mathcal{J}_y^z)\overline{\psi}_L^z \cdot \sigma^x \nu + \mathcal{J}^y \mathcal{J}_y^z (\psi_L^z - \psi^z) \cdot \sigma^x \nu,\end{aligned}$$

and appealing to the trace estimates of Lemma 3.5, we find

$$\begin{aligned}&\Big| \int_{\partial B(x)} (\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z) \cdot \sigma_L^x \nu - (\mathcal{J}^y \mathcal{J}_y^z \overline{\psi}^z) \cdot \sigma^x \nu \Big| \\ &\lesssim \Big(\int_{B(x)} \big|\mathrm{D}(\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi_L^x - \psi^x)\big|^2 \Big)^{\frac12} \\ &\quad + \Big(\int_{B(x)} \big|\mathrm{D}\big((\mathcal{J}_L^y - \mathcal{J}^y)\mathcal{J}_{L;y}^z \overline{\psi}_L^z\big)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x)\big|^2 \Big)^{\frac12} \\ &\quad + \Big(\int_{B(x)} \big|\mathrm{D}\big(\mathcal{J}^y (\mathcal{J}_{L;y}^z - \mathcal{J}_y^z)\overline{\psi}_L^z\big)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x)\big|^2 \Big)^{\frac12} \\ &\quad + \Big(\int_{B(x)} \big|\mathrm{D}\big(\mathcal{J}^y \mathcal{J}_y^z (\psi_L^z - \psi^z)\big)\big|^2 \Big)^{\frac12} \Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi^x)\big|^2 \Big)^{\frac12}.\end{aligned}$$

We then appeal to the decay estimates of Lemma 4.7, to the periodization error estimates of Lemma 4.8, and to the energy estimate (3.44), in form of

$$\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\psi_L^x - \psi^x)\big|^2 \Big)^{\frac12} \lesssim L^{-d},$$

$$\Big(\int_{B_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}_L^y \mathcal{J}_{L;y}^z \overline{\psi}_L^z)\big|^2 \Big)^{\frac12} \lesssim \langle (x-y)_L \rangle^{-d} \langle (y-z)_L \rangle^{-d},$$

$$\Big(\int_{B_{1+\rho}(x)} \big|D\big((\mathcal{J}_L^y-\mathcal{J}^y)\mathcal{J}_{L;y}^z\bar\psi_L^z\big)\big|^2\Big)^{\frac12} \lesssim \big\langle (y-z)_L\big\rangle^{-d}\big(\mathbb{1}_{|x-y|>\frac L4}\big\langle (x-y)_L\big\rangle^{-d}+\mathbb{1}_{|x-y|\le\frac L4}L^{-d}\big),$$

$$\Big(\int_{B_{1+\rho}(x)} \big|D\big(\mathcal{J}^y(\mathcal{J}_{L;y}^z-\mathcal{J}_y^z)\bar\psi_L^z\big)\big|^2\Big)^{\frac12} \lesssim \langle x-y\rangle^{-d}\big(\mathbb{1}_{|y-z|>\frac L4}\big\langle (y-z)_L\big\rangle^{-d}+\mathbb{1}_{|y-z|\le\frac L4}L^{-d}\big),$$

$$\Big(\int_{B_{1+\rho}(x)} \big|D\big(\mathcal{J}^y\mathcal{J}_y^z(\psi_L^z-\psi^z)\big)\big|^2\Big)^{\frac12} \lesssim L^{-d}\langle x-y\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d},$$

so that the above becomes

$$\Big|\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar\psi_L^z)\cdot\sigma_L^x\nu-(\mathcal{J}^y\mathcal{J}_y^z\bar\psi^z)\cdot\sigma^x\nu\Big| \lesssim \mathbb{1}_{|x-y|>\frac L4}\big\langle (x-y)_L\big\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d}+\mathbb{1}_{|x-y|\le\frac L4}L^{-d}\big\langle (y-z)_L\big\rangle^{-d} + \mathbb{1}_{|y-z|>\frac L4}\langle x-y\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d}+\mathbb{1}_{|y-z|\le\frac L4}L^{-d}\langle x-y\rangle^{-d}.$$

Further, using (4.9) and the decay assumption (Mix_ω^n) to estimate correlation functions similarly to (4.26), we get by symmetry

$$\begin{aligned}\Big|L^{-d}\iiint_{(Q_{L,\rho})^3}\Big(\int_{\partial B(x)}(\mathcal{J}_L^y\mathcal{J}_{L;y}^z\bar\psi_L^z)\cdot\sigma_L^x\nu-(\mathcal{J}^y\mathcal{J}_y^z\bar\psi^z)\cdot\sigma^x\nu\Big)&\\ \times\big(\lambda(\mathcal{P})h_2(x,z)+h_3(x,y,z)\big)\,dxdydz\Big|&\\ \lesssim L^{-d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|>\frac L4}\big\langle (x-y)_L\big\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d}\omega\big(|x-z|\big)\,dxdydz&\\ +L^{-2d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|\le\frac L4}\big\langle (y-z)_L\big\rangle^{-d}\omega\big(|x-z|\big)\,dxdydz.&\end{aligned}\tag{4.45}$$

We start with the first right-hand side term, which is the most delicate one to estimate. By properties of ω, we may decompose

$$\omega\big(|x-z|\big)\lesssim\mathbb{1}_{|x-z|>\frac L4}\omega(L)+\mathbb{1}_{|x-z|\le\frac L4}\omega\big(|x-z|\big).$$

The first contribution with $|x-z|>\frac L4$ is easily estimated,

$$\omega(L)L^{-d}\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|>\frac L4}\mathbb{1}_{|x-z|>\frac L4}\big\langle (x-y)_L\big\rangle^{-d}\big\langle (y-z)_L\big\rangle^{-d}\,dxdydz \lesssim \omega(L)(\log L)^2,$$

and we turn to the contribution with $|x-z| \le \frac{L}{4}$. For that purpose, we interpolate between two bounds for the integral with respect to y,

$$\begin{aligned}
&\int_{Q_L} \mathbb{1}_{|x-y|>\frac{L}{4}} \mathbb{1}_{|x-z|\le\frac{L}{4}} \big\langle (x-y)_L \big\rangle^{-d} \big\langle (y-z)_L \big\rangle^{-d} \, dy \\
&\quad \lesssim \mathbb{1}_{|x-z|\le\frac{L}{4}} \Big(\big\langle (x-z)_L \big\rangle^{-d} \log\big(2+\big|(x-z)_L\big|\big) \Big) \wedge \big\langle \mathrm{dist}\big(\{x,z\}, \partial Q_L\big) \big\rangle^{-d} \\
&\quad \lesssim \big(\langle x-z \rangle + \mathrm{dist}\big(\{x,z\}, \partial Q_L\big) \big)^{-d} \log\big(2+|x-z|\big),
\end{aligned}$$

where we further used that $(x-z)_L = x-z$ if $|x-z| < \frac{L}{4}$. By symmetry, we then get

$$\begin{aligned}
&L^{-d} \iiint_{(Q_L)^3} \mathbb{1}_{|x-y|>\frac{L}{4}} \mathbb{1}_{|x-z|\le\frac{L}{4}} \big\langle (x-y)_L \big\rangle^{-d} \big\langle (y-z)_L \big\rangle^{-d} \omega\big(|x-z|\big) \, dx dy dz \\
&\lesssim L^{-d} \iint_{(Q_L)^2} \big(\langle x-z \rangle + \mathrm{dist}\big(\{x\}, \partial Q_L\big) \big)^{-d} \log\big(2+|x-z|\big) \omega\big(|x-z|\big) \, dx dz \\
&\lesssim \frac{1}{L} \int_1^L \log(t) \omega(t) \, dt,
\end{aligned} \tag{4.46}$$

where the last bound follows from a straightforward computation, carefully distinguishing between the cases $\langle x-z \rangle \ge \mathrm{dist}(\{x\}, \partial Q_L)$ and $\langle x-z \rangle \le \mathrm{dist}(\{x\}, \partial Q_L)$. Indeed, on the one hand, the part with $\langle x-z \rangle \ge \langle \mathrm{dist}(\{x\}, \partial Q_L) \rangle$ can be estimated by

$$\begin{aligned}
&L^{-d} \iint_{(Q_L)^2} \mathbb{1}_{\mathrm{dist}(\{x\},\partial Q_L)\le\langle x-z\rangle} \langle x-z \rangle^{-d} \log\big(2+|x-z|\big) \omega\big(|x-z|\big) \, dx dz \\
&\quad \lesssim L^{-d} \int_{Q_{2L}} \big(L^{d-1} \langle y \rangle \big) \langle y \rangle^{-d} \log\big(2+|y|\big) \omega\big(|y|\big) \, dy \\
&\quad \lesssim \frac{1}{L} \int_1^L (\log r) \omega(r) \, dr,
\end{aligned}$$

and on the other hand the part with $\langle x-z \rangle \le \mathrm{dist}(\{x\}, \partial Q_L)$ is estimated by

$$\begin{aligned}
&L^{-d} \int_{Q_L} \big\langle \mathrm{dist}\big(\{x\}, \partial Q_L\big) \big\rangle^{-d} \\
&\qquad \times \bigg(\int_{\{z\in Q_L : \langle x-z\rangle \le \mathrm{dist}(\{x\},\partial Q_L)\}} \log\big(2+|x-z|\big) \omega\big(|x-z|\big) \, dz \bigg) \, dx \\
&\quad \lesssim L^{-d} \int_1^L r^{d-1} \langle L-r \rangle^{-d} \bigg(\int_1^{L-r} s^{d-1} (\log s) \omega(s) \, ds \bigg) \, dr \\
&\quad \lesssim \frac{1}{L} \int_{L/2}^L \langle L-r \rangle^{-d} \bigg(\int_1^{L-r} s^{d-1} (\log s) \omega(s) \, ds \bigg) \, dr \\
&\qquad + L^{-d} \int_1^L s^{d-1} (\log s) \omega(s) \, ds
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{L}\int_1^{L/2} s^{d-1}(\log s)\omega(s)\bigg(\int_{L/2}^{L-s}\langle L-r\rangle^{-d}dr\bigg)\,ds\\
&\quad + L^{-d}\int_1^L s^{d-1}(\log s)\omega(s)\,ds\\
&\lesssim \frac{1}{L}\int_1^L (\log s)\omega(s)\,ds,
\end{aligned}
$$

which yields the bound (4.46). It remains to estimate the second right-hand side term in (4.45), for which we directly find

$$
\begin{aligned}
L^{-2d}&\iiint_{(Q_L)^3}\mathbb{1}_{|x-y|\le\frac{L}{4}}\big\langle (y-z)_L\big\rangle^{-d}\omega\big(|x-z|\big)\,dxdydz\\
&\lesssim (\log L)L^{-d}\int_1^L t^{d-1}\omega(t)\,dt \lesssim \frac{1}{L}\int_1^L (\log t)\omega(t)\,dt.
\end{aligned}
$$

Inserting these different estimates into (4.45), the claim (4.44) follows using the doubling property of ω.

Substep 3.2. Conclusion. The next four terms of (4.43) are estimated similarly to the first periodization error term, and we skip most details for brevity. We solely briefly comment on the last term in (4.43), which is slightly different as it involves second-order corrector differences. We claim that

$$
\begin{aligned}
&\bigg|L^{-d}\iiint_{(Q_{L,\rho})^3}\bigg(\int_{\partial B(x)}\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^{y,z}\sigma_L^x\nu-\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x\nu\bigg)f_3(x,y,z)\,dxdydz\bigg|\\
&\quad\lesssim \lambda_3(\mathcal{P})\frac{1}{L}. \qquad (4.47)
\end{aligned}
$$

By (4.36), and arguing similarly to the case of $\delta^{y,z}\psi_L^x$, we can decompose

$$
\begin{aligned}
\delta^{y,z}\psi_L^{\varnothing} &= \mathcal{J}_L^y\mathcal{J}_{L;y}^z\overline{\psi}_L^{y,z}+\mathcal{J}_L^z\mathcal{J}_{L;z}^y\overline{\psi}_L^{y,z},\\
\delta^{y,z}\psi_L^x &= \mathcal{J}_{L;x}^y\mathcal{J}_{L;x,y}^z\overline{\psi}_L^{x,y,z}+\mathcal{J}_{L;x}^z\mathcal{J}_{L;x;z}^y\overline{\psi}_L^{x,y,z},
\end{aligned}
$$

so that, proceeding as in Substep 3.1 above, we can write $\delta^{y,z}\psi_L^{\varnothing}\cdot\delta^{y,z}\sigma_L^x-\delta^{y,z}\psi^{\varnothing}\cdot\delta^{y,z}\sigma^x$ as the difference of four terms, which can each be written as a telescopic sum of six terms involving “elementary” periodization errors. The most delicate of those terms is the following one,

$$
\begin{aligned}
L^{-d}&\iiint_{(Q_L)^3}\bigg(\int_{B(x)}\big|\mathrm{D}(\mathcal{J}^y\mathcal{J}_y^z\overline{\psi}^{y,z})\big|^2\bigg)^{\frac12}\\
&\qquad\times\bigg(\int_{B_{1+\rho}(x)}\big|\mathrm{D}\big(\mathcal{J}_x^z\mathcal{J}_{x,z}^y(\psi_L^{x,y,z}-\psi^{x,y,z})\big)\big|^2\bigg)^{\frac12}f_3(x,y,z)\,dxdydz.
\end{aligned}
$$

By the decay estimates of Lemma 4.7, the periodization error estimates of Lemma 4.8, and the energy estimate (3.44), and further appealing to Lemma 1.1 (iii) to control f_3 in terms of the multi-point intensity, this term is bounded by

$$L^{-d}\lambda_3(\mathscr{P})\iiint_{(Q_L)^3}\langle x-y\rangle^{-d}\langle x-z\rangle^{-d}\langle y-z\rangle^{-2d}\big\langle \operatorname{dist}(y,\partial Q_L)\big\rangle^{-d}\,dxdydz \\ \lesssim \lambda_3(\mathscr{P})\frac{1}{L}.$$

All the other terms can be bounded similarly, and the claim (4.47) follows. This concludes the proof of (ii).

Step 4. Analysis of remainders. Starting from (3.26), expanding the products, and separating the different intersection patterns, we are led to the following, in terms of multi-point intensities,

$$\begin{aligned}|R_L^3| \lesssim L^{-d}\bigg|\int_{(Q_{L,\rho})^5}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^\varnothing):\mathrm{D}(\delta^{y',z'}\psi_L^\varnothing)\bigg)& \\ \times f_5(x,y,z,y',z')\,dxdydzdy'dz'&\bigg| \\ +L^{-d}\bigg|\int_{(Q_{L,\rho})^4}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^\varnothing):\mathrm{D}(\delta^{y,z'}\psi_L^\varnothing)\bigg)f_4(x,y,z,z')\,dxdydzdz'&\bigg| \\ +L^{-d}\bigg|\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^\varnothing):\mathrm{D}(\delta^{z'}\widehat{\psi}_{x,L}^x)\bigg)f_4(x,y,z,z')\,dxdydzdz'&\bigg| \\ +L^{-d}\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}|\mathrm{D}(\delta^{y,z}\psi_L^\varnothing)|^2\bigg)f_3(x,y,z)\,dxdydz& \\ +L^{-d}\bigg|\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^\varnothing):\mathrm{D}(\delta^{y}\widehat{\psi}_{x,L}^x)\bigg)f_3(x,y,z)\,dxdydz&\bigg| \\ +L^{-d}\bigg|\int_{(Q_{L,\rho})^3}\bigg(\int_{B(x)}\mathrm{D}(\delta^{y,z}\psi_L^\varnothing):\mathrm{D}(\widehat{\psi}_{x,L}^x)\bigg)f_3(x,y,z)\,dxdydz&\bigg|.\end{aligned}$$

As in the analysis of $\overline{\mathbf{B}}_L^3$ in Step 1, cancellations are unravelled by decomposing $\delta^{y,z}\psi_L^\varnothing$ in terms of single-particle contributions. We leave the details to the reader. ∎

4.4.3 Higher-order explicit renormalization

Finally, we turn to the general higher-order case. The obtained renormalized formulas are not displayed in the statement as they take the form of intricate diagrammatic expansions and require notation that will be introduced in the proof.

Proposition 4.11 (Higher-order renormalizations). *Let* (H_ρ) *and* ($\mathrm{H}_\rho^{\mathrm{unif}}$) *hold, and assume for simplicity that particles are spherical with unit radius,* $I_n = B(x_n)$. *Let* *also the mixing assumption* (Mix$^n_\omega$) *hold to order* $n = k+1 \ge 2$ *with rate* $\omega \in C_b^\infty(\mathbb{R}^+)$ *satisfying the Dini-type condition* $\int_1^\infty \frac{1}{t}(\log t)^{k-1}\omega(t)\,dt < \infty$*. Then, the infinite-volume* $(k+1)$*th-order cluster coefficient* $\bar{\mathbf{B}}^{k+1}$*, cf.* (3.13)*, can be expressed by means of absolutely convergent integrals. In addition, in case of an algebraic rate* $\omega(t) \le Ct^{-\beta}$ *for some* $C, \beta > 0$*, the following hold.*

(i) Uniform estimate:

$$|\bar{\mathbf{B}}_L^{k+1}| \lesssim \lambda_{k+1}(\mathscr{P})\big|\log\lambda(\mathscr{P})\big|^k.$$

(ii) Convergence result:

$$|\bar{\mathbf{B}}_L^{k+1} - \bar{\mathbf{B}}^{k+1}| \lesssim \frac{(\log L)^k}{L^{\beta\wedge 1}}.$$

(iii) Uniform remainder estimate: *If* (Mix$^n_\omega$) *further holds with* $n = 2k+1$ *with algebraic weight* $\omega(t) \le Ct^{-\beta}$*, then*

$$|R_L^{k+1}| \lesssim \sum_{j=k}^{2k} \lambda_{j+1}(\mathscr{P})\big|\log\lambda(\mathscr{P})\big|^j.$$

Proof. Let $k \ge 1$ be fixed. By definition, cf. (3.8), the periodic approximation $\bar{\mathbf{B}}_L^{k+1}$ is given by

$$\bar{\mathbf{B}}_L^{k+1} = \frac{1}{2}(k+1)!\,L^{-d} \sum_{\sharp F = k+1}\sum_{n\in F}\mathbb{E}\Big[\int_{\partial B(x_{n,L})} \delta^{F\setminus\{n\}}\psi_L^\varnothing \cdot \sigma_L^F\,\nu\Big],$$

and thus, decomposing $\sigma_L^F = \sum_{G\subset F\setminus\{n\}}\delta^G\sigma_L^{\{n\}}$ for $n \in F$, we get by symmetry,

$$\bar{\mathbf{B}}_L^{k+1} = \frac{k+1}{2}\sum_{l=0}^k \binom{k}{l} L^{-d} \sum_{n_0,n_1,\ldots,n_k}^{\neq} \mathbb{E}\Big[\int_{\partial B(x_{n_0,L})} \delta^{\{n_1,\ldots,n_k\}}\psi_L^\varnothing \cdot \delta^{\{n_{l+1},\ldots,n_k\}}\sigma_L^{\{n_0\}}\nu\Big].$$

In terms of multi-point densities, cf. (1.15), recalling the choice (3.1) of the finite-volume approximation, and using the notation (4.15), this becomes

$$\bar{\mathbf{B}}_L^{k+1} = \frac{k+1}{2}\sum_{l=0}^k \binom{k}{l} L^{-d} \int_{(Q_{L,\rho})^{k+1}} \Big(\int_{\partial B(x_0)} \delta^{x_1,\ldots,x_k}\psi_L^\varnothing \cdot \delta^{x_{l+1},\ldots,x_k}\sigma_L^{x_0}\nu\Big) \times f_{k+1}(x_0,\ldots,x_k)\,dx_0\cdots dx_k. \tag{4.48}$$

We now need to capture enough cancellations to make these integrals absolutely summable uniformly in the large-volume limit. For that purpose, similarly to what we did for $\bar{\mathbf{B}}_L^3$ in the proof of Proposition 4.10, we shall proceed to a suitable expansion of $\delta^{x_1,\ldots,x_k}\psi_L^\varnothing$ in terms single-particle contributions. For general order k, it is conveniently expressed in terms of diagrams. We split the proof into six steps.

Step 1. Diagrammatic decomposition of $\delta^{x_1,\ldots,x_k}\psi_L^{\varnothing}$. We start with some terminology and notation:

- We use the standard notation $[k] := \{1,\ldots,k\}$ and for any subset $S \subset [k]$ we define $x_S := (x_i)_{i\in S}$.
- Given a sequence $I = (i_1,\ldots,i_l)$ of indices, the first index i_1 is called the *root* of I, the last index i_l is its *endpoint*, and l is its *length*. The associated index set is denoted by $\langle I\rangle := \{i_1,\ldots,i_l\}$ and we define the *cardinality* of I as $\sharp I := \sharp\langle I\rangle$. An index i is then said to belong to I (for short, $i \in I$) if it belongs to the index set $\langle I\rangle$, and we define $x_I := x_{\langle I\rangle} = (x_i)_{i\in\langle I\rangle}$. Two sequences I and J are said to be *disjoint* if there is no index belonging to both, that is, $\langle I\rangle \cap \langle J\rangle = \varnothing$.
- The *concatenation* of two sequences $I = (i_1,\ldots,i_l)$ and $J = (j_1,\ldots,j_m)$ is denoted by $I \uplus J := (i_1,\ldots,i_l,j_1,\ldots,j_m)$.
- A *string* of indices is defined as any sequence of distinct indices with length ≥ 1.
- Given a string $I = (i_1,\ldots,i_l)$ and an index set S, we define the *elementary contribution of I given S* as the following composition of operators,

$$\mathcal{R}^I_{L;S}(x_{[k]}) := \mathcal{J}^{x_{i_1}}_{L;x_S}\mathcal{J}^{x_{i_2}}_{L;x_{i_1}}\mathcal{J}^{x_{i_3}}_{L;x_{i_1},x_{i_2}}\cdots\mathcal{J}^{x_{i_l}}_{L;x_{i_1},\ldots,x_{i_{l-1}}}, \tag{4.49}$$

  where we recall that the $\mathcal{J}^z_{L,Y}$'s are defined in (4.18).
- A *block* is defined as any sequence B of indices that takes the form

$$B = (b) \uplus I_1 \uplus \cdots \uplus I_r,$$

  where $r \geq 0$ (for $r = 0$ we simply have $B = (b)$) and where $I_1,\ldots,I_r$ are strings of length ≥ 2 with the following property: for all $1 \leq j \leq r$, the endpoint of I_j belongs to $(b) \uplus I_1 \uplus \cdots \uplus I_{j-1}$ but other elements of I_j do not.
- Given a block $B = (b) \uplus I_1 \uplus \cdots \uplus I_r$ and an index set S, we define the *elementary contribution of B given S* as the following composition of operators,

$$\begin{aligned}&\mathcal{C}^B_{L;S}(x_{[k]})\\&:= \mathcal{J}^{x_b}_{L;x_S}\mathcal{R}^{I_1}_{L;\{b\}}(x_{[k]})\mathcal{R}^{I_2}_{L;\langle(b)\uplus I_1\rangle}(x_{[k]})\cdots\mathcal{R}^{I_n}_{L;\langle(b)\uplus I_1\uplus\cdots\uplus I_{n-1}\rangle}(x_{[k]}).\end{aligned} \tag{4.50}$$

In these terms, we claim that $\delta^{x_1,\ldots,x_j}\psi_L^{\varnothing}$ can be decomposed as follows,

$$\begin{aligned}&\delta^{x_1,\ldots,x_k}\psi_L^{\varnothing}\\&= \sum_{r=1}^{k}\sum_{B_1,\ldots,B_r}\mathcal{C}^{B_1}_{L;\varnothing}(x_{[k]})\mathcal{C}^{B_2}_{L;\langle B_1\rangle}(x_{[k]})\cdots\mathcal{C}^{B_r}_{L;\langle B_{r-1}\rangle}(x_{[k]})\,\bar{\psi}_L^{x_{B_r}},\end{aligned} \tag{4.51}$$

where we recall the shorthand notation $\bar{\psi}_L^{x_{B_r}} = \psi_L^{x_{B_r}} + Ex$, and where the sum $\sum_{B_1,\ldots,B_r}$ runs over all r-tuples of disjoint blocks $B_1,\ldots,B_r$ such that

$$\langle B_1 \uplus \cdots \uplus B_r\rangle = [k].$$

Note that this sum (4.51) is obviously finite, uniformly in L. Any sequence of indices of $[k]$ can be viewed as a walk on the vertex set $[k]$, thus inducing a (traversable) graph on $[k]$ where edges are defined by successive elements of the sequence (with possible multiplicities). In this view, each term in (4.51) can be conveniently represented by a corresponding diagram, cf. Figure 4.1; as we shall see, these graphical representations will prove crucial in estimating the different terms. This decomposition of corrector differences can be understood as a variant of the method of reflections [37]: while the latter allows us to decompose multi-particle solutions as an infinite expansion involving iterations of single-particle operators, the present decomposition is always finite, it still involves multi-particle solutions, but it has a simple enough structure to unravel explicit cancellations.

We turn to the proof of (4.51). More precisely, we shall prove the following seemingly simpler statement: for all disjoint index sets $S, T \subset [k]$ with $S \neq \varnothing$, we have

$$\delta^{x_S} \psi_L^{x_T} = \sum_{b \in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S\setminus\{b\}}} \bar{\psi}_L^{x_b} + \sum_{\substack{I \text{ string} \\ \langle I\rangle \subset S}} \sum_{c \in T} \mathcal{R}_{L;T}^{I \uplus (c)} \underbrace{\delta^{x_{S\setminus\langle I\rangle}} \bar{\psi}_L^{x_I, x_T}}_{\clubsuit}. \tag{4.52}$$

We quickly argue that this indeed implies (4.51). First, we iteratively replace the corrector difference $\clubsuit$ in (4.52), using (4.52) itself, to the effect that

$$\begin{aligned}
&\delta^{x_S} \psi_L^{x_T} \\
&= \sum_{l \geq 0} \sum_{\substack{I_1,\ldots,I_l \text{ disjoint strings} \\ \langle I_1\rangle,\ldots,\langle I_l\rangle \subset S}} \sum_{b \in S \setminus \langle I_1 \uplus \cdots \uplus I_l\rangle} \\
&\quad \times \sum_{\substack{c_1,\ldots,c_l \\ \forall j: c_j \in T \cup \langle I_1 \uplus \cdots \uplus I_{j-1}\rangle}} \mathcal{R}_{L;T}^{I_1 \uplus (c_1)}(x_{[k]}) \mathcal{R}_{L;T\cup\langle I_1\rangle}^{I_2 \uplus (c_2)}(x_{[k]}) \\
&\qquad \cdots \mathcal{R}_{L;T\cup\langle I_1 \uplus\cdots\uplus I_{l-1}\rangle}^{I_l \uplus (c_l)}(x_{[k]}) \mathcal{J}_{L;x_{T\cup\langle I_1\uplus\cdots\uplus I_l\rangle}}^{x_b} \delta^{x_{S\setminus\langle I_1\uplus\cdots\uplus I_l \uplus (b)\rangle}} \bar{\psi}_L^{x_b} \\
&\quad + \sum_{l\geq 1} \sum_{\substack{I_1,\ldots,I_l \text{ disjoint strings} \\ \langle I_1\rangle \cup\cdots\cup\langle I_l\rangle = S}} \sum_{\substack{c_1,\ldots,c_l \\ \forall j: c_j \in T\cup\langle I_1\uplus\cdots\uplus I_{j-1}\rangle}} \mathcal{R}_{L;T}^{I_1 \uplus (c_1)}(x_{[k]}) \mathcal{R}_{L;T\cup\langle I_1\rangle}^{I_2 \uplus (c_2)}(x_{[k]}) \\
&\qquad \cdots \mathcal{R}_{L;T\cup\langle I_1\uplus\cdots\uplus I_{l-1}\rangle}^{I_l \uplus (c_l)}(x_{[k]}) \bar{\psi}_L^{x_{S\cup T}}.
\end{aligned}$$

In particular, recalling the notation (4.49) for elementary contributions $\mathcal{R}_{L;S}^{I}$, and recognizing the definition (4.50) of block contributions, we deduce for all disjoint index sets $S, T \subset [k]$ with $S \neq \varnothing$,

$$\begin{aligned}
&\sum_{b\in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S\setminus\{b\}}} \bar{\psi}_L^{x_b} \\
&= \sum_{\substack{B \text{ block} \\ \langle B\rangle \subset S}} \mathcal{C}_{L;T}^{B}(x_{[k]}) \Big(\sum_{b \in S\setminus\langle B\rangle} \mathcal{J}_{L;x_{\langle B\rangle}}^{x_b} \delta^{x_{S\setminus(\langle B\rangle\cup\{b\})}} \bar{\psi}_L^{x_b} \Big) + \sum_{\substack{B \text{ block} \\ \langle B\rangle = S}} \mathcal{C}_{L;T}^{B}(x_{[k]}) \bar{\psi}_L^{x_S}.
\end{aligned}$$

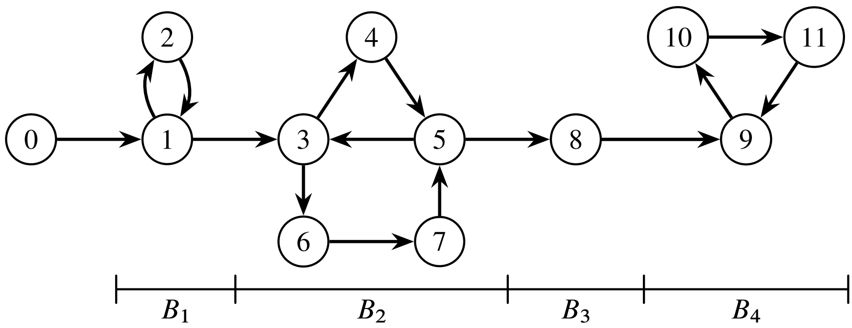


Figure 4.1. Each term in (4.51) can be represented by means of a directed graph on the index set $\{0\} \cup [k]$, where edges are given by pairs of consecutive elements in $B_1 \uplus \cdots \uplus B_r$ with possible multiplicities and where we include the edge $(0, b)$ with b the root of B_1. In this way, an edge to i corresponds to an operator $\mathcal{J}_{L;x_T}^{x_i}$ in (4.51). For instance, the above diagram is associated with blocks $B_1 = (1) \uplus (2,1)$, $B_2 = (3) \uplus (4,5,3) \uplus (6,7,5)$, $B_3 = (8)$, and $B_4 = (9) \uplus (10,11,9)$.

Iterating this identity, starting from (4.20) in form of

$$\delta^{x_S} \psi_L^{\varnothing} = \sum_{b \in S} \mathcal{J}_L^{x_b} \delta^{x_{S \setminus \{b\}}} \bar{\psi}_L^{x_b},$$

the claim (4.51) follows.

We are left with the proof of (4.52). Given disjoint index sets $S, T \subset [k]$ with $S \neq \varnothing$, in view of (4.20), corrector differences can be decomposed as

$$\delta^{x_S} \psi_L^{x_T} = \sum_{b \in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S \setminus \{b\}}} \bar{\psi}_L^{x_b, x_T}.$$

Decomposing $\bar{\psi}_L^{x_b, x_T} = \bar{\psi}_L^{x_b} + (\psi_L^{x_b, x_T} - \psi_L^{x_b})$, this becomes

$$\delta^{x_S} \psi_L^{x_T} = \sum_{b \in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S \setminus \{b\}}} \bar{\psi}_L^{x_b} + \sum_{b \in S} \mathcal{J}_{L;x_T}^{x_b} \delta^{x_{S \setminus \{b\}}} (\psi_L^{x_b, x_T} - \psi_L^{x_b}), \tag{4.53}$$

and it remains to further decompose the last right-hand side term. For that purpose, in view of Lemma 3.4, for all $D \subset S$ with $D \neq \varnothing$, we note that $\delta^{x_{S \setminus D}} (\psi_L^{x_D, x_T} - \psi_L^{x_D})$ satisfies

$$\begin{aligned}
&- \triangle \delta^{x_{S \setminus D}} (\psi_L^{x_D, x_T} - \psi_L^{x_D}) \\
&+ \nabla \delta^{x_{S \setminus D}} \big(\Sigma_L^{x_D, x_T} \mathbb{1}_{Q_L \setminus \bigcup_{i \in D \cup T} B(x_i)} - \Sigma_L^{x_D} \mathbb{1}_{Q_L \setminus \bigcup_{b \in D} B(x_b)} \big) \\
&\quad = - \sum_{b \in D} \delta_{\partial B(x_b)} \delta^{x_{S \setminus D}} (\sigma_L^{x_D, x_T} - \sigma_L^{x_D}) \\
&\qquad - \sum_{i \in S \setminus D} \delta_{\partial B(x_i)} \delta^{x_{S \setminus D \cup \{i\}}} (\sigma_L^{x_D, x_i, x_T} - \sigma_L^{x_D, x_i}) - \sum_{i \in T} \delta_{\partial B(x_i)} \delta^{x_{S \setminus D}} \sigma_L^{x_D, x_T},
\end{aligned}$$

which then allows us to write

$$\begin{aligned}&\delta^{x_{S\setminus D}}\big(\psi_L^{x_D,x_T}-\psi_L^{x_D}\big)\\&\qquad=\sum_{i\in S\setminus D}\mathcal{J}_{L;x_D}^{x_i}\,\delta^{x_{S\setminus D\cup\{i\}}}\big(\psi_L^{x_D,x_i,x_T}-\psi_L^{x_D,x_i}\big)+\sum_{i\in T}\mathcal{J}_{L;x_D}^{x_i}\,\delta^{x_{S\setminus D}}\,\bar\psi_L^{x_D,x_T}.\end{aligned}$$

Using iteratively this identity for D exhausting S, we obtain upon recognizing the definition (4.49) of elementary contributions,

$$\sum_{b\in S}\mathcal{J}_{L;x_T}^{x_b}\,\delta^{x_{S\setminus\{b\}}}\big(\psi_L^{x_b,x_T}-\psi_L^{x_b}\big)=\sum_{\substack{I\text{ string}\\ \langle I\rangle\subset S}}\sum_{c\in T}\mathcal{R}_{L;T}^{I\uplus(c)}(x_{[k]})\,\delta^{x_{S\setminus\langle I\rangle}}\,\psi_L^{x_I,x_T}.$$

Inserting this into (4.53), the claim (4.52) follows.

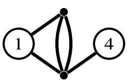
Step 2. Estimation of block contributions and graphical notation. Let B be a block of indices with root b and endpoint f. By definition of elementary block contributions, cf. (4.50), for any index set S that is disjoint from $\langle B\rangle$, Lemma 4.7 yields

$$\Big(\int_{B(z)}\big|\nabla\mathcal{C}_{L;S}^{B}(x_{[k]})\zeta\big|^2\Big)^{\frac12}\lesssim\big\langle(z-x_b)_L\big\rangle^{-d}D_B(x_B)\Big(\int_{B(x_f)}|\nabla\zeta|^2\Big)^{\frac12},\qquad(4.54)$$

where for any sequence $C=(c_1,\ldots,c_m)$ we define

$$D_C(x_C):=\big\langle(x_{c_1}-x_{c_2})_L\big\rangle^{-d}\cdots\big\langle(x_{c_{m-1}}-x_{c_m})_L\big\rangle^{-d}.\qquad(4.55)$$

As such contributions will be combined in intricate ways in the sequel, we introduce a convenient graphical notation. Integration variables are represented by small black circles and frozen variables by small white circles. The index of a frozen variable is occasionally indicated inside the corresponding white circle. A solid line between two vertices i and j represents a factor $\langle(x_i-x_j)_L\rangle^{-d}$. In particular, multiple edges correspond to powers of this factor. For instance, we have

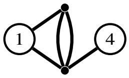


$$=\int_{(Q_L)^2}\big\langle(x_1-x_2)_L\big\rangle^{-d}\big\langle(x_2-x_3)_L\big\rangle^{-2d}\big\langle(x_3-x_1)_L\big\rangle^{-d}\times\big\langle(x_3-x_4)_L\big\rangle^{-d}\,dx_2dx_3.$$

When evaluating integrals with borderline factors $\langle(x_i-x_j)_L\rangle^{-d}$, we naturally obtain logarithmic factors, for which we shall use the shorthand notation

$$\mathcal{L}_L\big((z_i)_{i\in J}\big):=\log\big(2+\max_{i,j\in J}\big|(z_i-z_j)_L\big|\big).$$

This is combined into our graphical notation as follows: a symbolic prefactor $\mathcal{L}$ in front of a diagram indicates that a factor $\mathcal{L}_L(x_S)$ is to be included into the corresponding integral, where S stands for the set of all implemented indices. For instance, for any power $\mu \geq 0$, we have

$\mathcal{L}^{\mu}$ 
$$= \int_{(Q_L)^2} \mathcal{L}_L(x_1, x_2, x_3, x_4)^{\mu} \langle (x_1 - x_2)_L \rangle^{-d} \langle (x_2 - x_3)_L \rangle^{-2d} \times \langle (x_3 - x_1)_L \rangle^{-d} \langle (x_3 - x_4)_L \rangle^{-d} \, dx_2 dx_3.$$

Noting that for any $\gamma > 0$ and $\mu \geq 0$ a direct evaluation of integrals yields

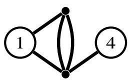
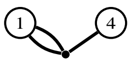
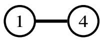
$$\int_{Q_L} \mathcal{L}_L(x, y, z)^{\mu} \langle (x - y)_L \rangle^{-d} \langle (y - z)_L \rangle^{-\gamma} \, dy \lesssim \begin{cases} \langle (x - z)_L \rangle^{-d} \mathcal{L}_L(x, z)^{\mu} & \text{if } \gamma > d, \\ \langle (x - z)_L \rangle^{-d} \mathcal{L}_L(x, z)^{\mu+1} & \text{if } \gamma = d, \\ \langle (x - z)_L \rangle^{-\gamma} \mathcal{L}_L(x, z)^{\mu+1} & \text{if } \gamma < d, \end{cases}$$

we deduce with our graphical notation

$\mathcal{L}^{\mu}$ $\lesssim \mathcal{L}^{\mu+1}$

$\mathcal{L}^{\mu}$  $\lesssim \mathcal{L}^{\mu}$ (4.56)

which allows for instance to estimate graphically

 $\lesssim$  $\lesssim$  $= \langle (x_1 - x_4)_L \rangle^{-d}.$

The counting of logarithmic factors in the sequel will be quite trivial as we shall notice that at most one logarithmic factor appears each time a vertex disappears in the graphical representation. This rough bound can often be improved, but it suffices for our purposes.

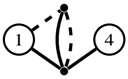
We need to add one more ingredient to our graphical notation. Indeed, in the sequel, we replace the density function f_{k+1} in (4.48) by its expansion (4.7) in terms of correlation functions, and we estimate the latter by appealing both to the decay assumption (Mix$^n_\omega$) and to the uniform bound (4.9). This leads us to combine products of the form $D_C(x_C)$ with products of factors of the form

$$\omega\big((x_i - x_j)_L\big) \wedge \lambda_{p_i}(\mathcal{P})$$

for some $p_i \geq 0$. In our graphical notation, such a factor is represented by a dashed line between vertices i and j. In principle, the value p_i should be included in the notation to precise the value of the edge. For convenience, we rather use a simplified notation: for a diagram with s dashed lines, a symbolic prefactor λ_k° indicates that the dashed lines correspond to factors $(\omega((\cdot)_L) \wedge \lambda_{p_i}(\mathcal{P}))_{1 \leq i \leq s}$ with any $p_1, \ldots, p_s \geq 1$ satisfying

$$p_1 + \cdots + p_s = k,$$

and we take the sum over the different possible choices of such p_i's. For instance,

λ_k° 

$$= \sum_{\substack{p_1,p_2\geq 1\\ p_1+p_2=k}} \int_{(Q_L)^2} \big(\omega\big((x_1-x_2)_L\big)\wedge\lambda_{p_1}(\mathcal{P})\big)\big(\omega\big((x_2-x_3)_L\big)\wedge\lambda_{p_2}(\mathcal{P})\big) \times \big\langle(x_2-x_3)_L\big\rangle^{-d}\big\langle(x_3-x_1)_L\big\rangle^{-d}\big\langle(x_3-x_4)_L\big\rangle^{-d}\,dx_2dx_3.$$

In addition, a symbolic prefactor λ'_k in front of a diagram with s dashed lines indicates that the whole expression is multiplied by a factor $\lambda_{p_0}(\mathcal{P})$ and that the dashed lines correspond to factors $(\omega((\cdot)_L)\wedge\lambda_{p_i}(\mathcal{P}))_{1\leq i\leq s}$ with any $p_0,\ldots,p_s\geq 1$ satisfying $p_0+\cdots+p_s=k$, where we again take the sum over all possible choices. In other words,

$$\big(\lambda'_k\big)=\sum_{p=1}^{k}\lambda_p(\mathcal{P})\times\big(\lambda^\circ_{k-p}\big).$$

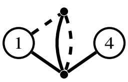
As obviously $\omega((\cdot)_L)\wedge\lambda_{p_i}(\mathcal{P})\leq\lambda_{p_i}(\mathcal{P})$, we get with our notation

λ_k°  $\leq \lambda'_k$ $= \lambda_k(\mathcal{P})$ (4.57)

Next, we combine this with the notation $\mathcal{L}$ for logarithmic factors: in front of a diagram with a prefactor λ°_k, a symbolic prefactor $\mathcal{L}$ indicates that either a factor $\mathcal{L}(x_S)$ is to be included into the corresponding integral, where S stands for the set of implemented indices, or that one of the factors $\omega((x_i-x_j)_L)\wedge\lambda_{p_i}(\mathcal{P})$ is to be replaced by

$$\big(\omega\big((x_i-x_j)_L\big)\mathcal{L}_L(x_i,x_j)\big)\wedge\big(\lambda_{p_i}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|\big),$$

and we take the sum over the two choices. Powers of $\mathcal{L}$ are defined accordingly: for instance, for any $\mu\geq 0$,

$\mathcal{L}^\mu\lambda_k^\circ$ 

$$= \sum_{\substack{\mu_0,\mu_1,\mu_2\geq 0\\ \mu_0+\mu_1+\mu_2=\mu}}\ \sum_{\substack{p_0,p_1,p_2\geq 1\\ p_0+p_1+p_2=k}} \int_{(Q_L)^2}\mathcal{L}_L(x_1,x_2,x_3,x_4)^{\mu_0} \times\Big(\big(\omega\big((x_1-x_2)_L\big)\mathcal{L}_L(x_1,x_2)^{\mu_1}\big)\wedge\big(\lambda_{p_1}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^{\mu_1}\big)\Big) \times\Big(\big(\omega\big((x_2-x_3)_L\big)\mathcal{L}(x_2,x_3)^{\mu_2}\big)\wedge\big(\lambda_{p_2}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^{\mu_2}\big)\Big) \times\big\langle(x_2-x_3)_L\big\rangle^{-d}\big\langle(x_3-x_1)_L\big\rangle^{-d}\big\langle(x_3-x_4)_L\big\rangle^{-d}\,dx_2dx_3.$$

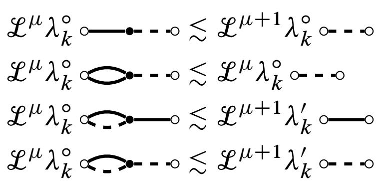
When $\mathcal{L}$ is in front of a diagram with a prefactor λ'_k, we add the possibility of multiplying the whole expression by a factor $|\log\lambda(\mathcal{P})|$: for all $\mu\geq 0$, this means

$$\big(\mathcal{L}^\mu\lambda'_k\big)=\sum_{\nu=0}^{\mu}\sum_{p=1}^{k}\lambda_p(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^\nu\times\big(\mathcal{L}^{\mu-\nu}\lambda^\circ_{k-p}\big).$$

In case of an algebraic rate $\omega(t) = Ct^{-\beta}$ for some $\beta \in (0, d)$, a direct evaluation of integrals yields, for all $\lambda > 0$ and $\mu, \nu \geq 0$,

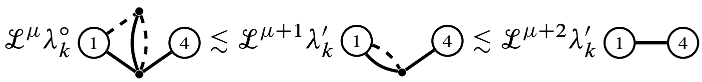
$$\int_{Q_L} \mathcal{L}_L(x,y,z)^{\mu} \langle (x-y)_L \rangle^{-d} \Big(\big(\omega\big((y-z)_L\big) \mathcal{L}_L(y,z)^{\nu}\big) \wedge \big(\lambda \big|\log \lambda\big|^{\nu}\big) \Big)\, dy$$
$$\lesssim \big(\omega\big((x-z)_L\big) \mathcal{L}_L(x-z)^{\mu+\nu+1}\big) \wedge \big(\lambda \big|\log \lambda\big|^{\mu+\nu+1}\big).$$

With our graphical notation, recalling Lemma 1.1 (ii), this estimate and similar computations yield for all $\mu \geq 0$,



(4.58)

which allows for instance to estimate graphically



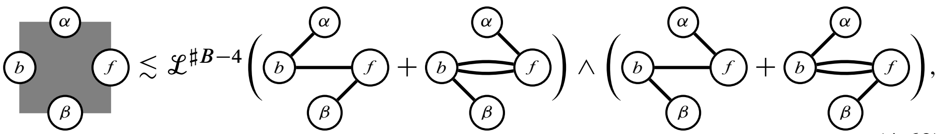
$$\leq \lambda_k(\mathcal{P}) \langle (x_1 - x_4)_L \rangle^{-d} \big(\big|\log \lambda(\mathcal{P})\big| + \mathcal{L}_L(x_1, x_4) \big)^{\mu+2}.$$

This notation will be used abundantly in the sequel.

Step 3. Partial integration on blocks. Let B be a block of indices with root b and endpoint f. We shall establish the following key estimate for partial integrals on B: for all $\alpha, \beta \in \langle B \rangle$,

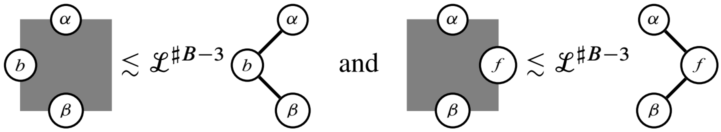
$$\int_{(Q_L)^{\sharp \langle B \rangle \setminus \{b,f,\alpha,\beta\}}} D_B(x_B)\, dx_{\langle B \rangle \setminus \{b,f,\alpha,\beta\}}$$
$$\lesssim_B \mathcal{L}_L(x_b, x_f, x_\alpha, x_\beta)^{\sharp \langle B \rangle \setminus \{b,f,\alpha,\beta\}}$$
$$\times \big(D_{(\alpha,b,f,\beta)}(x_{[k]}) + D_{(\alpha,f,b,\beta)}(x_{[k]}) \langle (x_b - x_f)_L \rangle^{-d} \big)$$
$$\wedge \big(D_{(\alpha,f,b,\beta)}(x_{[k]}) + D_{(\alpha,b,f,\beta)}(x_{[k]}) \langle (x_b - x_f)_L \rangle^{-d} \big). \qquad (4.59)$$

With the above graphical notation, if b, f, α, β are all distinct, this can be written as



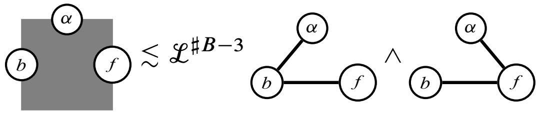
(4.60)

where henceforth gray squares stand for integration on a generic block. As these estimates will be abundantly used in the sequel, we also display the important special case when one further integrates over b or f, which is deduced by applying (4.56),

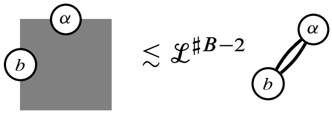


(4.61)

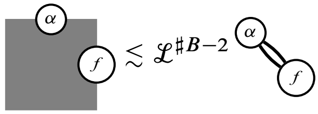
and we further display the special cases when $\alpha = \beta$,



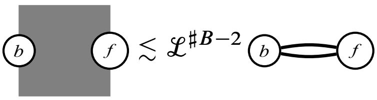
(4.62)



(4.63)



(4.64)



(4.65)

We shall in fact prove a much more precise estimates, see (4.67) below, but the above convenient estimates will be enough for our purposes. Powers of the logarithmic factor in each of these estimates is equal to the difference between the numbers of vertices in the left-hand side and in the right-hand side: indeed, in view of (4.56), each vertex that is integrated yields at most one logarithmic factor. This could in fact be improved in (4.62)–(4.65) based on (4.67), but we shall not need such refinements.

Before turning to the proof, we make a notational comment. A special role is of course played in the above estimates by the root b and by the endpoint f of the block. In the sequel, even when vertices are not labeled explicitly, as e.g. in (4.67), we take the convention that the root and the endpoint are always drawn respectively on the left and on the right sides of the square (or at one of these two locations in case they coincide), while all other distinguished vertices are drawn indistinctly on the upper and lower sides.

We turn to the proof of (4.59). For that purpose, we shall study geometric properties of the graph $\mathfrak{G}$ associated with the block B. Letting

$$B = (b_1, b_2, \ldots, b_l)$$

with $b_1 = b$ and $b_l = f$, we recall that we define vertices of $\mathfrak{G}$ as the elements of the index set $\langle B \rangle = \{b_i\}_{1 \le i \le l}$, and edges of $\mathfrak{G}$ as pairs of consecutive indices (b_i, b_{i+1}) with $1 \le i < l$. Note that $\mathfrak{G}$ is connected and may have multiple edges but no self-loop. We shall repeatedly use the following observation: as edges of $\mathfrak{G}$ are defined from the block B, we note that b and f have odd degree ≥ 3 and that other vertices have even degree ≥ 2 (where the degree of a vertex is the number of unoriented edges containing that vertex; see e.g. Figure 4.2). We split the proof into three further substeps.

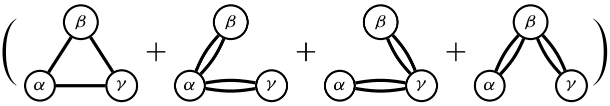
Substep 3.1. Cyclic estimate. In the spirit of (4.55), for a graph $\mathfrak{H}$ on the index set $[k]$, we define

$$D_{\mathfrak{H}}(x_{[k]}) := \prod_{(i,j)\in\mathfrak{H}} \langle (x_i - x_j)_L \rangle^{-d},$$

where the notation $(i,j) \in \mathfrak{H}$ means that (i,j) is an edge of $\mathfrak{H}$. Provided that $\mathfrak{H}$ is Eulerian (that is, provided that $\mathfrak{H}$ is connected and that each vertex has even degree), we claim that for all vertices $\alpha, \beta, \gamma \in \langle \mathfrak{H} \rangle$,

$$\int_{(Q_L)^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{H}}(x_{[k]})\, dx_{\langle\mathfrak{H}\rangle\setminus\{\alpha,\beta,\gamma\}}$$

$$\lesssim_{\mathfrak{H}} \mathcal{L}^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}} \Big(\quad \Big). \qquad (4.66)$$



We will use standard terminology from graph theory, which we recall here for clarity: a walk is a sequence of edges joining a sequence of vertices; a trail is a walk in which all edges are distinct (taking edge multiplicity into account); a path is a trail in which all vertices are also distinct; a circuit is a trail in which the first and last vertices coincide; a cycle is a circuit in which only the first and last vertices coincide.

We turn to the proof of (4.66) and we argue by induction on the size of $\mathfrak{H}$. Assume that α, β, γ are distinct (the cases $\alpha = \beta \neq \gamma$ and $\alpha = \beta = \gamma$ can be treated similarly and are skipped for brevity). The result is straightforward if $\sharp\mathfrak{H} = 3$ as no integral is performed in that case. We turn to the case $\sharp\mathfrak{H} > 3$. As $\mathfrak{H}$ is Eulerian, there is a circuit that covers $\mathfrak{H}$ (that is, a circuit that visits every edge of $\mathfrak{H}$ exactly once). Removing some subcircuits, we deduce that one of the following two possibilities must hold up to a permutation of α, β, γ:

(a) either there is a cycle $\mathfrak{C}$ visiting α, β, γ;

(b) or there is a cycle $\mathfrak{C}_1$ visiting α, β and a cycle $\mathfrak{C}_2$ visiting α, γ such that vertices of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are all distinct except α.

Both cases can be treated similarly and we focus on the first one for brevity. Let $\mathfrak{C}$ be a cycle visiting α, β, γ. Denote by $\mathfrak{H}'$ the (possibly empty) subgraph of $\mathfrak{H}$ induced by the complement of the edge set of the cycle $\mathfrak{C}$. As $\mathfrak{H}$ is Eulerian and as $\mathfrak{C}$ is a cycle, we notice that $\mathfrak{H}'$ is the union of Eulerian subgraphs $\mathfrak{H}'_1, \ldots, \mathfrak{H}'_s$ that are edge-disjoint. We may then decompose

$$D_{\mathfrak{H}}(x_{[k]}) = D_{\mathfrak{C}}(x_{[k]}) D_{\mathfrak{H}'_1}(x_{[k]}) \cdots D_{\mathfrak{H}'_s}(x_{[k]}).$$

For all $1 \le i \le s$, there is a vertex j_i of $\mathfrak{H}'_i$ that also belongs to the cycle $\mathfrak{C}$. Summing separately over repeated variables, we may then estimate

$$\int_{(Q_L)^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{H}}(x_{[k]})\, dx_{\langle\mathfrak{H}\rangle\setminus\{\alpha,\beta,\gamma\}}$$
$$\lesssim \int_{(Q_L)^{\sharp\mathfrak{C}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{C}}(x_{[k]}) \bigg(\prod_{i=1}^{s} \int_{(Q_L)^{\sharp\mathfrak{H}'_i-1}} D_{\mathfrak{H}'_i}(x_{[k]})\, dx_{\langle\mathfrak{H}'_i\rangle\setminus\{j_i\}}\bigg)\, dx_{\langle\mathfrak{C}\rangle\setminus\{\alpha,\beta,\gamma\}}.$$

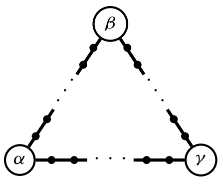
As the $\mathfrak{H}'_i$'s are strict Eulerian subgraphs of $\mathfrak{H}$, an induction argument allows to assume that the claim (4.66) is already known to hold for $\mathfrak{H}$ replaced by any of the $\mathfrak{H}'_i$'s. In particular, upon integration, this entails

$$\int_{(Q_L)^{\sharp\mathfrak{H}'_i-1}} D_{\mathfrak{H}'_i}(x_{[k]})\, dx_{\langle\mathfrak{H}'_i\rangle\setminus\{j_i\}} \lesssim \mathcal{L}^{\sharp\mathfrak{H}'_i-1}.$$

The above then reduces to

$$\int_{(Q_L)^{\sharp\mathfrak{H}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{H}}(x_{[k]})\, dx_{\langle\mathfrak{H}\rangle\setminus\{\alpha,\beta,\gamma\}}$$
$$\lesssim \mathcal{L}^{\sum_i(\sharp\mathfrak{H}'_i-1)} \int_{(Q_L)^{\sharp\mathfrak{C}\setminus\{\alpha,\beta,\gamma\}}} D_{\mathfrak{C}}(x_{[k]})\, dx_{\langle\mathfrak{C}\rangle\setminus\{\alpha,\beta,\gamma\}},$$

where the right-hand side is now simply an integral of the form



Using (4.56) to evaluate the integrals, noting that the number of appearing logarithmic factors is bounded by the length of $\mathfrak{C}$ minus 3 and that the length of $\mathfrak{C}$ is bounded by the total number of vertices minus the number of vertices not in the cycle (that is, $\sum_i(\sharp\mathfrak{H}'_i - 1)$), the claim (4.66) follows. More precisely, we obtain in this way the first right-hand side term in (4.66), while other terms correspond to case (b) above.

Substep 3.2. Path decomposition of the graph $\mathfrak{G}$ associated with a block B. We show that, if $b \neq f$, there exist three edge-disjoint trails $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ that cover $\mathfrak{G}$ (that is, the union of their vertex sets is the vertex set of $\mathfrak{G}$ and the disjoint union of their edge sets is the edge set of $\mathfrak{G}$). We refer to Figure 4.2 for an illustrative example.

As b and f have odd degree ≥ 3 and as all other vertices of $\mathfrak{G}$ have even degree, we can find a trail $\mathfrak{L}^1$ from b to f. Without loss of generality, we can assume that b and f are visited only once by $\mathfrak{L}^1$. Then consider the subgraph $\mathfrak{G}'$ of $\mathfrak{G}$ induced by the complement of the edge set of $\mathfrak{L}^1$. By construction, all vertices of $\mathfrak{G}'$ now have

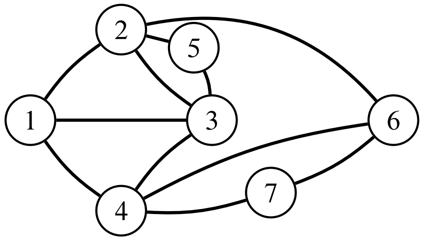


Figure 4.2. This graph represents the block $B = (1) \uplus (2,3,1) \uplus (4,3) \uplus (5,2) \uplus (6,4) \uplus (7,6)$. The path decomposition of Substep 2.3 can be chosen in this case as $\mathfrak{L}^1 = (1,2,6)$, $\mathfrak{L}^2 = (1,4,7,6)$, and $\mathfrak{L}^3 = (1,3,2,5,3,4,6)$.

even degree, and the definition of the block B ensures that $\mathfrak{G}'$ must be connected. This allows to find two other disjoint trails $\mathfrak{L}^2, \mathfrak{L}^3$ from b to f in $\mathfrak{G}'$.

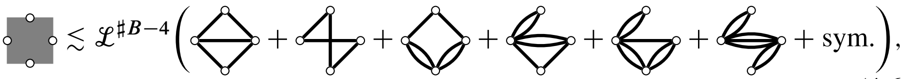
Next, assume that a vertex $\alpha \in \langle \mathfrak{G} \rangle$ is not visited by any of the three constructed trails $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$. Recalling that $\mathfrak{G}$ is connected, a degree argument as above ensures that there exists a circuit $\mathfrak{K}$ from α to itself that is disjoint from the trails $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ and that crosses at least one of them. A detour via $\mathfrak{K}$ is then easily added to those trails in such a way that they remain disjoint and that at least one of them now visits α. Repeating this construction, we are led to edge-disjoint trails $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ that visit all vertices of $\mathfrak{G}$.

Finally, consider the subgraph $\mathfrak{G}''$ of $\mathfrak{G}$ induced by the complement of the union of the edge sets of $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$. By construction, all vertices of $\mathfrak{G}''$ have even degree, which allows us to write $\mathfrak{G}''$ as a union of edge-disjoint circuits. Adding detours via these circuits, we can assume that the trails $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ cover the whole graph $\mathfrak{G}$, and the claim follows.

Substep 3.3. Proof of (4.59). Let $\mathfrak{L}^1, \mathfrak{L}^2, \mathfrak{L}^3$ be three covering edge-disjoint trails from b to f as constructed above. Given a vertex α, distinguishing between the number of paths from b to f to which α belongs, and removing cycles, we get the following four possibilities:

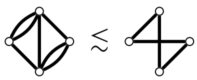
- (a) either there exists a cycle $\mathfrak{C}$ from b or from f that visits α and there exist three paths $\mathfrak{K}^1, \mathfrak{K}^2, \mathfrak{K}^3$ from b to f, such that $\mathfrak{C}, \mathfrak{K}^1, \mathfrak{K}^2, \mathfrak{K}^3$ are edge-disjoint and cross each other only at b or f;
- (b) or there exists a path $\mathfrak{K}^1$ from b to f that visits α and there exist two other paths $\mathfrak{K}^2, \mathfrak{K}^3$ from b to f that do not, such that $\mathfrak{K}^1, \mathfrak{K}^2, \mathfrak{K}^3$ are edge-disjoint and cross each other only at b or f;
- (c) or there exist two paths $\mathfrak{K}^1, \mathfrak{K}^2$ from b to f that visit α and there exists another path $\mathfrak{K}^3$ from b to f that does not, such that $\mathfrak{K}^1, \mathfrak{K}^2, \mathfrak{K}^3$ are edge-disjoint and cross each other only at α, b or f;
- (d) or there exist three paths $\mathfrak{K}^1, \mathfrak{K}^2, \mathfrak{K}^3$ from b to f that visit α and that are edge-disjoint and cross each other only at α, b and f.

Given another vertex β, and distinguishing between corresponding cases, we obtain three distinguished paths $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f that may visit or not α and β, in different possible orders, and we obtain up to two cycles $\mathfrak{C}^1$, $\mathfrak{C}^2$ from b or f visiting α or β. The subgraph of $\mathfrak{G}$ induced by the complement of the union of the edge sets of those three paths and possible cycles is necessarily a disjoint union of Eulerian graphs and can be removed by duplicating variables as in Substep 3.1. It remains to consider the union of those three paths and possible cycles. Considering different patterns and using (4.56) to estimate consecutive edges along each path between frozen vertices b, f, α, β, we are led to

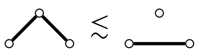


(4.67)

where for brevity "sym." stands for the sum of all other graphs obtained by reflecting the six pictured graphs with respect to the vertical axis, the horizontal axis, or both (which corresponds to permuting α and β, b and f, or both). In fact, the analysis of all possible patterns produces a larger number of terms, but we claim that all others are bounded by the above. For instance, another possible pattern corresponds to the case of three paths $\mathfrak{K}^1$, $\mathfrak{K}^2$, $\mathfrak{K}^3$ from b to f visiting both α and β, where $\mathfrak{K}^1$, $\mathfrak{K}^2$ visit α before β while $\mathfrak{K}^3$ visit them in reverse order: we claim that the corresponding contribution can be bounded as follows,



which is indeed bounded by the right-hand side of (4.67). This bound follows from



which is itself nothing but the triangle inequality

$$\langle (x_1 - x_3)_L \rangle \leq \langle (x_1 - x_2)_L \rangle + \langle (x_2 - x_3)_L \rangle,$$

post-processed into

$$\langle (x_1 - x_3)_L \rangle \leq 2 \langle (x_1 - x_2)_L \rangle \langle (x_2 - x_3)_L \rangle$$

and raised to the power $-d$. A straightforward similar inspection of all other possible patterns shows that the bound (4.67) indeed holds; we skip the detail for brevity.

Finally, removing a few edges in (4.67), we are led in particular to the claim (4.60). The claims (4.61)–(4.65) further follow as straightforward corollaries after integrations using (4.56).

Step 4. Approximate cancellation of translation-invariant averages on given blocks. Let B be a block of indices with root b and endpoint f, and let S, T be disjoint index sets with $(S \cup T) \cap \langle B\rangle = \varnothing$. Let $m := \sharp B$ and $s := \sharp S$. For all x_B, x_S, let $\zeta_{L;x_B}^{x_S} \in H^1_{\mathrm{per}}(Q_L)^d$ satisfy (4.17) at $z = x_f$, and assume that ζ_L is equivariant under translations in the sense that

$$\zeta_{L;x_B+[z]_B}^{x_S+[z]_S}(\cdot + z) = \zeta_{L;x_B}^{x_S}, \qquad \text{for all } z \in \mathbb{R}^d,$$

where $[z]_B$ (resp. $[z]_S$) stands for the element of $(\mathbb{R}^d)^m$ (resp. $(\mathbb{R}^d)^s$) with all coordinates equal to z. Then, for any function h on $(Q_L)^{m+s}$ that is translation-invariant in the sense that $h(x_B + [z]_B, x_S + [z]_S) = h(x_B, x_S)$ for all $z \in \mathbb{R}^d$, we have for any linear functional $F : H^1_{\mathrm{per}}(Q_L)^d \to \mathbb{R}$,

$$\begin{aligned} &\bigg| \int_{(Q_L)^{m+s}} F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big] h(x_B, x_S)\, dx_B dx_S \bigg| \\ &\qquad \le \int_{Q_L} \bigg(\int_{(Q_L+x_b)^{m+s-1}\setminus(Q_L)^{m+s-1}} \big|F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]\big| \\ &\qquad\qquad \times \big(\big|h_L(x_B, x_S)\big| + \big|h(x_B, x_S)\big|\big)\, dx_{\langle B\rangle\setminus\{b\}} dx_S \bigg) dx_b, \end{aligned} \tag{4.68}$$

where we have defined the periodization $h_L(z) := h(z_L)$ where $z_L \in (Q_L)^{m+s}$ stands for the reduction of $z \in (\mathbb{R}^d)^{m+s}$ modulo $(L\mathbb{Z}^d)^{m+s}$. Note that we do not obtain an exact cancellation in general for such a symmetric average on a block, but this bound reduces it to a boundary term.

We turn to the proof of (4.68). Set for abbreviation $C := \langle B\rangle \setminus \{b\}$. By definition of elementary block contributions, cf. (4.50), we can write

$$\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B} = \mathcal{I}^{x_b}_{L;x_T}\xi^{x_C;x_S}_{L;x_b}, \tag{4.69}$$

for some function $\xi^{x_C;x_S}_{L;x_b}$ that satisfies (4.17) at $z = x_b$ and is such that ξ_L is equivariant under translations. The left-hand side of (4.68) then becomes

$$\begin{aligned} &\int_{(Q_L)^{m+s}} F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big] h(x_B, x_S)\, dx_B dx_S \\ &\qquad = \int_{(Q_L)^{m+s}} F\big[\mathcal{I}^{x_b}_{L;x_T}\xi^{x_C;x_S}_{L;x_b}\big] h(x_b, x_C, x_S)\, dx_b dx_C dx_S, \end{aligned}$$

and thus, using the equivariance of ξ under translations,

$$\begin{aligned} &\int_{(Q_L)^{m+s}} F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big] h(x_B, x_S)\, dx_B dx_S \\ &\quad = \int_{(Q_L)^{m+s}} F\big[\mathcal{I}^{x_b}_{L;x_T}\big(\xi^{x_C-[x_b]_C;x_S-[x_b]_S}_{L;0}(\cdot - x_b)\big)\big] h(x_b, x_C, x_S)\, dx_b dx_C dx_S. \end{aligned}$$

Replacing h by its periodization h_L (which we can on $(Q_L)^{m+s}$), changing variables and using periodicity, the above becomes in these terms,

$$\begin{aligned}\int_{(Q_L)^{m+s}} & F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]h(x_B,x_S)\,dx_B dx_S \\ &= \int_{(Q_L)^{m+s}} F\big[\mathcal{J}^{x_b}_{L;x_T}\big(\xi^{x_C;x_S}_{L;0}(\cdot-x_b)\big)\big] \\ &\qquad\qquad \times h_L\big(x_b, x_C+[x_b]_C, x_S+[x_b]_S\big)\,dx_b dx_C dx_S.\end{aligned}$$

If $h_L(x_b,x_C+[x_b]_C,x_S+[x_b]_S)$ were replaced by $h(0,x_C,x_S)$ in the integrand, the cancellation property of Lemma 4.6 would precisely entail that the integral vanishes (this would have been the case if we had considered a periodization in law of $\mathcal{I}$ rather than (3.1)). Adding and subtracting $h(0,x_C,x_S)$, we deduce

$$\begin{aligned}\int_{(Q_L)^{m+s}} & F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]\,h(x_B,x_S)\,dx_B dx_S \\ &= \int_{(Q_L)^{m+s}} F\big[\mathcal{J}^{x_b}_{L;x_T}\big(\xi^{x_C;x_S}_{L;0}(\cdot-x_b)\big)\big] \\ &\quad \times \big(h_L\big(x_b,x_C+[x_b]_C,x_S+[x_b]_S\big)-h(0,x_C,x_S)\big)\,dx_b dx_C dx_S.\end{aligned}$$

If x_b, x_C, x_S are such that

$$(x_b, x_C+[x_b]_C, x_S+[x_b]_S)\in(Q_L)^{m+s},$$

then the definition of the periodization h_L and the translation invariance of h imply that the integrand vanishes. This leads us to the bound

$$\begin{aligned}\bigg|\int_{(Q_L)^{m+s}} & F\big[\mathcal{C}^B_{L;T}(x_{[k]})\zeta^{x_S}_{L;x_B}\big]\,h(x_B,x_S)\,dx_B dx_S\bigg| \\ &\le \int_{Q_L}\bigg(\int_{(Q_L)^{m+s-1}\setminus(Q_L-x_b)^{m+s-1}}\big|F\big[\mathcal{J}^{x_b}_{L;x_T}\big(\xi^{x_C;x_S}_{L;0}(\cdot-x_b)\big)\big]\big| \\ &\quad \times \big(\big|h_L\big(x_b,x_C+[x_b]_C,x_S+[x_b]_S\big)\big|+\big|h(0,x_C,x_S)\big|\big)\,dx_{\langle B\rangle\setminus\{b\}}dx_S\bigg)\,dx_b.\end{aligned}$$

Using again (4.69) and the equivariance of ξ, the claim (4.68) follows.

Step 5. Uniform estimates: proof of (i). The starting point is the decomposition (4.48) of $\bar{\mathbf{B}}^{k+1}_L$. For brevity, we shall focus on the term corresponding to $l=k$ in (4.48), that is,

$$\bar{\mathbf{C}}^{k+1}_L := \frac{k+1}{2}L^{-d}\int_{(Q_{L,\rho})^{k+1}}\bigg(\int_{\partial B(x_0)}\delta^{x_1,\ldots,x_k}\psi^{\varnothing}_L\cdot\sigma^{x_0}_L\nu\bigg)f_{k+1}(x_0,\ldots,x_k)\,dx_0\cdots dx_k,$$

while the other terms are simpler to estimate due to the additional decay given by the factor $\delta^{x_{l+1},\ldots,x_k}\sigma_L^{x_0}$. Inserting the diagrammatic decomposition (4.51), we get

$$\bar{\mathbf{C}}_L^{k+1} = \frac{k+1}{2}\sum_{r=1}^{k}\sum_{B_1,\ldots,B_r}\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r),$$

where we recall that the sum runs over all r-tuples of disjoint blocks $B_1,\ldots,B_r$ such that $\langle B_1\uplus\cdots\uplus B_r\rangle=[k]$, and where we have set for abbreviation

$$\begin{aligned}&\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r) :=\\ &L^{-d}\int_{(Q_{L,\rho})^{k+1}}\bigg(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\bigg)\\ &\qquad\qquad\times f_{k+1}(x_0,x_{[k]})\,dx_0dx_{[k]}.\end{aligned}$$

Let such $B_1,\ldots,B_r$ be fixed. Replacing f_{k+1} by its expansion (4.7) in terms of correlation functions, we find

$$\begin{aligned}&\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r)\\ &=\sum_\pi L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}h_{\sharp H}(x_H)\Big)\\ &\times\bigg(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\bigg)\,dx_0dx_{[k]},\end{aligned}\tag{4.70}$$

where π runs over all partitions of the index set $\{0\}\cup[k]$ and where H runs over all cells of the partition π.

We shall say that a partition π of $\{0\}\cup[k]$ is *covering* for $B_1,\ldots,B_r$ if there is no "separating" index $1\le\alpha\le r$ such that each cell $H\in\pi$ is included either in $\{0\}\cup\bigcup_{i=1}^{\alpha-1}\langle B_i\rangle$ or in $\bigcup_{i=\alpha}^{r}\langle B_i\rangle$. We denote by $\mathcal{K}(B_1,\ldots,B_r)$ the set of such partitions. Using the approximate cancellation property (4.68), and further noting as in (4.40) that

$$\int_{Q_L}\bigg(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\psi_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\bigg)dx_0=0,$$

we note that only covering partitions produce nontrivial terms in (4.70): contributions from non-covering partitions either vanish or are reduced to boundary terms. Therefore, we naturally decompose

$$\begin{aligned}&\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r)\big|\\ &\le\sum_{\pi\in\mathcal{K}(B_1,\ldots,B_r)}\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\big|+\sum_{\pi\notin\mathcal{K}(B_1,\ldots,B_r)}\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\big|,\end{aligned}$$

where we have set for abbreviation

$$
\begin{aligned}
&\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\\
&:= L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}h_{\sharp H}(x_H)\Big)\\
&\times\bigg(\int_{\partial B(x_0)}\big(\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big)\cdot\sigma_L^{x_0}\nu\bigg)\,dx_0dx_{[k]}.
\end{aligned}
$$

We split the proof into two further substeps, separately considering the two types of contributions.

Substep 5.1. Main contributions: in case of an algebraic rate $\omega(t)\le Ct^{-\beta}$ for some $C,\beta>0$, we have for all $\pi\in\mathcal{K}(B_1,\ldots,B_r)$,

$$
\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\big|\lesssim\lambda_{k+1}(\mathcal{P})\big|\log\lambda(\mathcal{P})\big|^k. \tag{4.71}
$$

Without loss of generality, we may assume $\beta\in(0,d)$ (so we can appeal to (4.58)). Using the boundary conditions for $\psi_L^{x_0}$ and the incompressibility constraints to smuggle in arbitrary constants in the different factors, and then appealing to the trace estimates of Lemma 2.5, we find

$$
\begin{aligned}
&\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\big|\\
&\lesssim L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}\big|h_{\sharp H}(x_H)\big|\Big)\\
&\quad\times\bigg(\int_{B(x_0)}\big|\nabla\mathcal{C}_{L;\varnothing}^{B_1}(x_{[k]})\mathcal{C}_{L;\langle B_1\rangle}^{B_2}(x_{[k]})\cdots\mathcal{C}_{L;\langle B_{r-1}\rangle}^{B_r}(x_{[k]})\bar{\psi}_L^{x_{B_r}}\big|^2\bigg)^{\frac12}dx_0dx_{[k]}.
\end{aligned}
$$

For all $1\le l\le r$, denote by b_l the root of B_l and by f_l its endpoint, and set for notational convenience $f_0:=0$. Iterating the bound (4.54), we then get

$$
\begin{aligned}
&\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\big|\\
&\quad\lesssim L^{-d}\int_{(Q_{L,\rho})^{k+1}}\Big(\prod_{H\in\pi}\big|h_{\sharp H}(x_H)\big|\Big)\\
&\qquad\times\bigg(\prod_{l=1}^r\big\langle(x_{f_{l-1}}-x_{b_l})_L\big\rangle^{-d}D_{B_l}(x_{B_l})\bigg)\,dx_0dx_{[k]}.
\end{aligned}\tag{4.72}
$$

Next, we examine the structure of the product of correlation functions. Given a covering partition $\pi\in\mathcal{K}(B_1,\ldots,B_r)$, we can construct a sequence of intertwined pairings $(m_i,m_i')_{1\le i\le s}$ (for some integer $s\ge1$) such that

— $(m_j)_{1\le j\le s}$ and $(m_j')_{1\le j\le s}$ are increasing, $m_1=0$, and $m_s'=r$;

— $m_{i-1}'<m_{i+1}$ for all $1<i<s$, and $m_i\le m_{i-1}'$ for all $1<i\le s$;

— for all i there is a cell $H \in \pi$ such that $H \cap \langle B_{m_i} \rangle \neq \varnothing$ and $H \cap \langle B_{m'_i} \rangle \neq \varnothing$ (with the understanding that $B_0 = \{0\}$).

The construction is as follows: Starting from $m_1 = 0$, we define m'_1 as the maximum index m such that there is $H \in \pi$ with $\{0\} \cap H \neq \varnothing$ and $\langle B_m \rangle \cap H \neq \varnothing$, which is well defined by the covering assumption for π with index $\alpha = 1$. Once m_i and m'_i are defined for some $i \geq 1$, if $m'_i < r$, we define m'_{i+1} as the maximum index m such that there is $H \in \pi$ with $(\{0\} \cup \bigcup_{l \leq m'_i} \langle B_l \rangle) \cap H \neq \varnothing$ and $\langle B_m \rangle \cap H \neq \varnothing$, which is well defined by the covering assumption for π with index $\alpha = m'_i + 1 \leq r$ and satisfies $m'_{i+1} > m'_i$ by construction. Next, we define m_{i+1} as the minimum index m such that there is $H \in \pi$ with $\langle B_m \rangle \cap H \neq \varnothing$ and $\langle B_{m'_{i+1}} \rangle \cap H \neq \varnothing$. We continue the construction until $m'_s = r$ is reached. We claim that by construction we have $m'_{i-1} < m_{i+1} \leq m'_i$ for all i (which, since $(m'_j)_j$ is increasing, implies that $(m_j)_j$ is increasing as well). On the one hand, we indeed have $m_{i+1} \leq m'_i$ by definition of m'_{i+1}. On the other hand, we must have $m_{i+1} > m'_{i-1}$ since the inequality $m_{i+1} \leq m'_{i-1}$ would imply $m'_i = m'_{i+1}$ and contradict the strict monotonicity of the sequence $(m'_j)_j$.

With this construction of intertwined pairings $(m_i, m'_i)_{1 \leq i \leq s}$, we can choose a sequence of distinct blocks $(H_i)_{1 \leq i \leq s}$ of π such that $\langle B_{m_i} \rangle \cap H_i \neq \varnothing$ and $\langle B_{m'_i} \rangle \cap H_i \neq \varnothing$ for all i (recall that $B_0 = \{0\}$). We may then pick indices $j_i, j'_i \in \{0\} \cup [k]$ such that $j_i \in \langle B_{m_i} \rangle \cap H_i$ and $j'_i \in \langle B_{m'_i} \rangle \cap H_i$ for all i. In these terms, appealing both to (4.9) and to the decay assumption (Mix^n_ω) with $n = k + 1$, the product of correlation functions in (4.72) can be bounded for instance as follows (up to integration, as in (4.26)),

$$\prod_{H \in \pi} \left| h_{\sharp H}(x_H) \right| \lesssim \lambda_{p_0}(\mathscr{P}) \prod_{i=1}^{s} \big(\omega(x_{j_i} - x_{j'_i}) \wedge \lambda_{p_i}(\mathscr{P}) \big), \tag{4.73}$$

for some $p_0, \ldots, p_s \geq 1$ with $\sum_{i=0}^{s} p_i = k + 1$. We then define the following concatenations of blocks between paired indices: for $1 \leq i < s$,

$$\begin{aligned} A_i &:= (f_{m'_{i-1}}) \uplus B_{m'_{i-1}+1} \uplus \cdots \uplus B_{m_{i+1}-1} \uplus (b_{m_{i+1}}), \\ A'_i &:= B_{m_{i+1}} \uplus \cdots \uplus B_{m'_i}, \end{aligned}$$

with the convention $m'_0 := 0$, and

$$A_s := (f_{m'_{s-1}}) \uplus B_{m'_{s-1}+1} \uplus \cdots \uplus B_{m_s}, \quad A'_s := \varnothing.$$

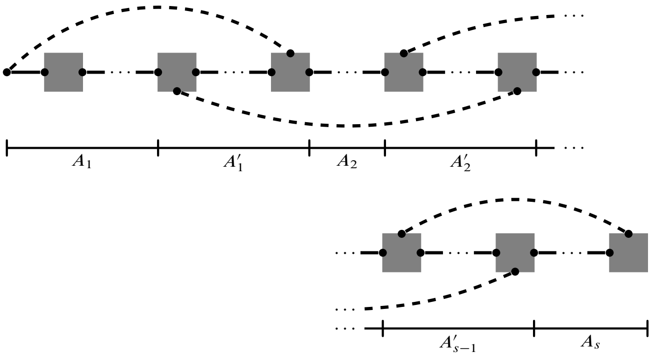
In these terms, inserting (4.73) into (4.72), the integral can be reorganized as

$$\begin{aligned} &\left| \bar{\mathrm{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi) \right| \lesssim \lambda_{p_0}(\mathscr{P}) \\ &\times L^{-d} \int_{(Q_{L,\rho})^{k+1}} \left(\prod_{i=1}^{s} D_{A_i}(x_{A_i}) D_{A'_i}(x_{A'_i}) \big(\omega(x_{j_i} - x_{j'_i}) \wedge \lambda_{p_i}(\mathscr{P}) \big) \right) dx_0 dx_{[k]}. \end{aligned}$$

We emphasize that the coupled indices j_i's and j_i''s belong to A_i''s and can thus intersect A_i's only at their endpoints. In terms of the graphical representation introduced in Step 3, the latter integral can be represented generically in the following way,

$$\left|\bar{\mathbf{C}}_L^{k+1;r}(B_1,\dots,B_r;\pi)\right|$$

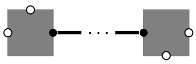
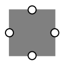
$$\lesssim L^{-d}\lambda'_{k+1}$$



(4.74)

where we further delineate the concatenations of blocks A_i's and A_i''s. In particular, note these take the generic forms

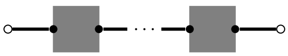
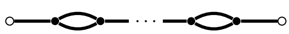
$A_i \equiv$ 

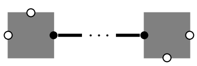
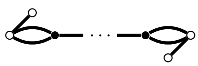
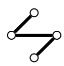
$A_i' \equiv$  or 

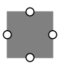
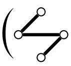
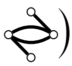
where the second possibility for A_i' corresponds to the case when

$$m_{i+1} = m_i'.$$

In order to estimate (4.74), we first perform integration on A_i's and A_i''s: using (4.60)–(4.65) to estimate the integral on each block, and using (4.56) to estimate consecutive edges, we find

 $\lesssim \mathcal{L}^{[\sharp]}$  $\lesssim \mathcal{L}^{[\sharp]}$

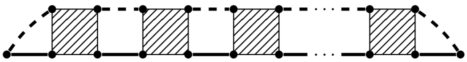
 $\lesssim \mathcal{L}^{[\sharp]}$  $\lesssim \mathcal{L}^{[\sharp]}$ 

 $\lesssim \mathcal{L}^{[\sharp]}\Big($  $+$  $\Big)$

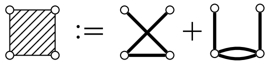
where henceforth we use the shorthand notation $\mathcal{L}^{[\sharp]}$ for a power of the logarithmic factor that can change from an occurrence to another and stands for the difference between the numbers of vertices in the left-hand side and in the right-hand side.

Inserting this into (4.74), we are led to

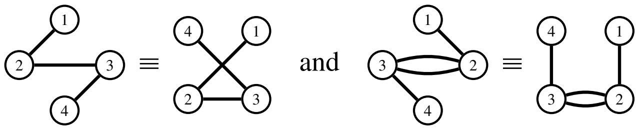
$$\left|\overline{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\right| \lesssim L^{-d}\mathscr{L}^{[\sharp]}\lambda'_{k+1} \tag{4.75}$$



where for abbreviation hatched boxes are given by



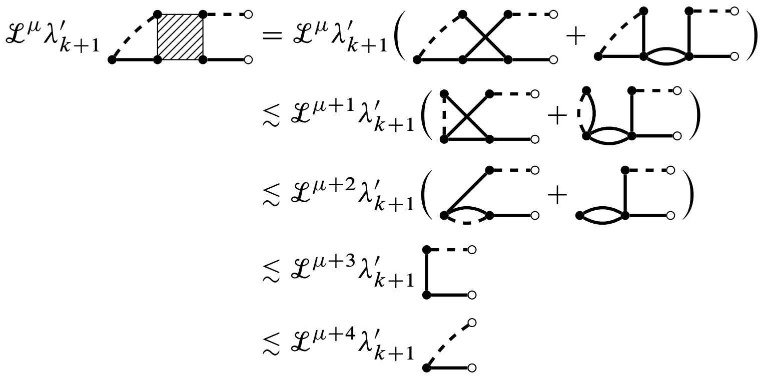
which we obtain by reorganizing the graphs as follows,



It remains to evaluate the right-hand side in (4.75). Using the graphical rules (4.56), (4.57), and (4.58), and noting that direct integrations yield

 $\lesssim \circ$ and $\mathscr{L}^{\mu}\lambda^{\circ}_{k+1}$  $\lesssim \mathscr{L}^{\mu+1}\lambda'_{k+1}\circ$

we can estimate



Iterating this estimate, the right-hand side of (4.75) can now be estimated as follows,

$$\left|\overline{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\right| \lesssim L^{-d}\mathscr{L}^{[\sharp]}\lambda'_{k+1}\bullet$$

As the number of vertices in the left-hand side is equal to $k+1$ while only one vertex remains in the right-hand side, recalling our notation for $\mathscr{L}$ and λ'_k, and noting that free integration yields $\bullet = L^d$, we get

$$\left|\overline{\mathbf{C}}_L^{k+1;r}(B_1,\ldots,B_r;\pi)\right| \lesssim L^{-d}\mathscr{L}^{k}\lambda'_{k+1}\bullet = \lambda_{k+1}(\mathscr{P})\left|\log\lambda(\mathscr{P})\right|^{k},$$

that is, (4.71).

Substep 5.2. Boundary terms: in case of an algebraic rate $\omega(t) \le Ct^{-\beta}$ for $C, \beta > 0$, we have for all $\pi \notin \mathcal{K}(B_1, \ldots, B_r)$,

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi)\big| \\ &\quad \lesssim \begin{cases} \big(\lambda_{k+1}(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^{k-1}\big) \wedge \frac{(\log L)^{k-1}}{L^{\beta\wedge 1}} & \text{if } \beta \neq 1, \\ \big(\lambda_{k+1}(\mathcal{P})\big|\log \lambda(\mathcal{P})\big|^{k}\big) \wedge \frac{(\log L)^{k}}{L} & \text{if } \beta = 1. \end{cases} \end{aligned} \tag{4.76}$$

By definition, given $\pi \notin \mathcal{K}(B_1, \ldots, B_r)$, we can consider the largest separating index $\alpha \le r$ such that each cell $H \in \pi$ is included either in $\{0\} \cup \bigcup_{i=1}^{\alpha-1}\langle B_i\rangle$ or in $\bigcup_{i=\alpha}^{r}\langle B_i\rangle$. Setting $\widehat{B} := B_\alpha \uplus \cdots \uplus B_r$, the choice of α ensures that π can be restricted to a partition $\widehat{\pi}$ of the index subset $\langle \widehat{B}\rangle \subset [k]$ such that $\widehat{\pi}$ is covering for $B_\alpha, \ldots, B_r$. Arguing as in (4.72) and estimating the integrals over the first blocks $\{0\}, B_1, \ldots, B_{\alpha-1}$ brutally as in Section 3.8 without taking any advantage of the decay of correlation functions, we get

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi)\big| \\ &\lesssim (\log L)^\alpha \Big(\prod_{H\in\pi\setminus\widehat{\pi}} \lambda_{\sharp H}(\mathcal{P})\Big) \\ &\times L^{-d}\int_{Q_L}\bigg(\int_{(Q_L+x_b)^{\sharp\widehat{B}-1}\setminus(Q_L)^{\sharp\widehat{B}-1}}\Big(\prod_{H\in\widehat{\pi}}\big|h_{\sharp H}(x_H)\big|\Big) D_{\widehat{B}}(x_{\widehat{B}})\,dx_{\langle\widehat{B}\rangle\setminus\{b\}}\bigg)dx_b \end{aligned}$$

It remains to show that the remaining integral is a boundary term that is algebraically small as $L \uparrow \infty$, so that in particular the logarithmic prefactor $(\log L)^\alpha$ plays no role. For that purpose, we first note that

$$\mathbb{1}_{(Q_L+x_b)^{\sharp\widehat{B}-1}\setminus(Q_L)^{\sharp\widehat{B}-1}}(x_{\langle\widehat{B}\rangle\setminus\{b\}}) \le \sum_{j\in\langle\widehat{B}\rangle\setminus\{b\}} \mathbb{1}_{(Q_L+x_b)\setminus Q_L}(x_j),$$

so the above can be bounded by

$$\begin{aligned} &\big|\bar{\mathbf{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi)\big| \\ &\lesssim (\log L)^\alpha \Big(\prod_{H\in\pi\setminus\widehat{\pi}} \lambda_{\sharp H}\Big) \\ &\quad\times \sum_{j\in\langle\widehat{B}\rangle\setminus\{b\}} L^{-d}\int_{Q_L}\int_{(Q_L+x_b)\setminus Q_L}\bigg(\int_{(Q_L+x_b)^{\sharp\widehat{B}-2}}\Big(\prod_{H\in\widehat{\pi}}\big|h_{\sharp H}(x_H)\big|\Big) \\ &\qquad\qquad\qquad\qquad \times D_{\widehat{B}}(x_{\widehat{B}})dx_{\langle\widehat{B}\rangle\setminus\{b,j\}}\bigg)dx_j\,dx_b. \end{aligned}$$

As $\widehat{\pi}$ is covering for $B_\alpha, \ldots, B_r$, that is, $\widehat{\pi} \in \mathcal{K}(B_\alpha, \ldots, B_r)$, similar arguments based on the graphical representation as in Substep 5.1 allow to estimate the integral over

$\langle \widehat{B}\rangle \setminus \{b, j\}$, to the effect of

$$
\begin{aligned}
&\big|\overline{\mathbf{C}}_L^{k+1;r}(B_1, \ldots, B_r; \pi)\big| \\
&\lesssim (\log L)^{\alpha} \sum_{j \in \langle \widehat{B}\rangle \setminus \{b\}} L^{-d} \int_{Q_L} \int_{(Q_L + x_b)\setminus Q_L} \Big(\mathcal{L}^{\sharp \widehat{B} - 2} \lambda'_{k+1}\, \textcircled{\scriptsize b} \frown \textcircled{\scriptsize j}\Big)\, dx_j\, dx_b.
\end{aligned}
\tag{4.77}
$$

In order to estimate this integral, we note that

$$
\begin{aligned}
&L^{-d} \int_{Q_L} \int_{(Q_L + x_b)\setminus Q_L} \Big(\lambda^{\circ}_{k+1}\, \textcircled{\scriptsize b} \frown \textcircled{\scriptsize j}\Big)\, dx_j\, dx_b \\
&= L^{-d} \int_{Q_L} \int_{(Q_L + x_b)\setminus Q_L} \big\langle (x_b - x_j)_L \big\rangle^{-d} \big(\omega\big((x_b - x_j)_L\big) \wedge \lambda_{k+1}(\mathcal{P})\big)\, dx_j\, dx_b \\
&= L^{-d} \int_{Q_L} \int_{Q_L \setminus (Q_L - x_b)} \langle y \rangle^{-d} \big(\omega(y) \wedge \lambda_{k+1}(\mathcal{P})\big)\, dy\, dx_b \\
&= L^{-d} \int_{Q_L} \langle y \rangle^{-d} \big(\omega(y) \wedge \lambda_{k+1}(\mathcal{P})\big) \big|Q_L \setminus (Q_L - y)\big|\, dy,
\end{aligned}
$$

and thus, using (4.29) in form of $L^{-d}|Q_L \setminus (Q_L - y)| \lesssim \frac{|y|}{L} \wedge 1$, in case of an algebraic rate $\omega(t) \le C t^{-\beta}$ for some $C, \beta > 0$,

$$
\begin{aligned}
&L^{-d} \int_{Q_L} \int_{(Q_L + x_b)\setminus Q_L} \Big(\lambda^{\circ}_{p}\, \textcircled{\scriptsize b} \frown \textcircled{\scriptsize j}\Big)\, dx_j\, dx_b \\
&\lesssim L^{-1} \int_{Q_L} \langle y \rangle^{1-d} \big(\omega(y) \wedge \lambda_p(\mathcal{P})\big)\, dy \\
&\lesssim \begin{cases} \lambda_p(\mathcal{P}) \wedge L^{-\beta \wedge 1} & \text{if } \beta \ne 1, \\ \big(\lambda_p(\mathcal{P}) \big|\log \lambda(\mathcal{P})\big|\big) \wedge \frac{\log L}{L} & \text{if } \beta = 1. \end{cases}
\end{aligned}
$$

Now turning back to the right-hand side in (4.77), repeating the above computation after including logarithmic factors, and noting that $\alpha + \sharp \widehat{B} \le \sharp B = k + 1$, the claim (4.76) follows.

Step 6. Strategy for (ii) and (iii). Both for (ii) and (iii), the arguments are similar to what we already did so far, and require no new insight. We omit lengthy details for brevity.

We start with (ii). In view of the estimation (4.76) for boundary terms, it remains to estimate the convergence of terms corresponding to covering partitions in (4.70) in the large-volume limit. For that purpose, we appeal to the periodization error estimates of Lemma 4.8, as in the proof of Proposition 4.9 (ii).

We turn to (iii). The starting point is the refined estimate (3.26) on R_L^{k+1}. In the spirit of the proof of Proposition 4.9 (iii), a decomposition of the right-hand side in (3.26) can be performed in the same way as what we did above for $\overline{\mathbf{B}}_L^{k+1}$. ∎

4.5 Optimality of cluster estimates

This section is devoted to the proof of Theorem 4.4, which shows that logarithmic factors are optimal in general in our estimation of cluster coefficients, e.g. Proposition 4.11 (i). As will be clear in the proof below, logarithmic factors are related to the lack of continuity of the Helmholtz projection in $\mathrm{L}^\infty(\mathbb{R}^d)$.

Proof of Theorem 4.4. Let Assumptions (H_ρ), ($\mathrm{H}_\rho^{\mathrm{unif}}$), and (Indep) hold, and assume that the correlation function satisfies the Dini condition (4.5). We split the proof into two steps.

Step 1. Proof of (i). Appealing to Proposition 4.9 in form of the explicit formula (4.21) for $\bar{\mathbf{B}}^2$, and estimating the second right-hand side term as in the proof of Proposition 4.9 (i), we find

$$\bigg| E : \bar{\mathbf{B}}^2 E - \int_{\mathbb{R}^d} \mathbb{E}\Big[\int_{\partial I^\circ} \psi^z \cdot \sigma^0 \nu \Big] h_2(0,z)\, dz \bigg| \lesssim \lambda_2(\mathcal{P}), \tag{4.78}$$

where σ^0 and ψ^z are associated respectively with single particles at I° and at $z + I^{\circ\prime}$, where I° and $I^{\circ\prime}$ are iid copies of the same random shape. Replacing ψ^z by its Taylor expansion, using the boundary conditions for σ^0, and using standard decay properties of ψ^z, we find

$$\bigg| \int_{\partial I^\circ} \psi^z \cdot \sigma^0 \nu - \mathrm{D}(\psi^z)(0) : \int_{\partial I^\circ} \sigma^0 \nu \otimes x \bigg| \lesssim \langle z \rangle^{-d-1}.$$

Inserting this into (4.78) together with (4.9), and recalling the shorthand notation

$$\begin{aligned} E : \widehat{\mathbf{B}}^1 E &= \frac{1}{2} \mathbb{E}\Big[\int_{\partial I^\circ} Ex \cdot \sigma^\circ \nu \Big] \\ &= \mathbb{E}\Big[\int_{\mathbb{R}^d} |\mathrm{D}(\psi^\circ)|^2 \Big], \end{aligned}$$

cf. (2.6), we get

$$\bigg| E : \bar{\mathbf{B}}^2 E - (2\widehat{\mathbf{B}}^1 E) : \Big(\int_{\mathbb{R}^d} \mathbb{E}\big[\mathrm{D}(\psi^z)(0)\big] h_2(0,z)\, dz \Big) \bigg| \lesssim \lambda_2(\mathcal{P}). \tag{4.79}$$

Next, we further analyze $\mathrm{D}(\psi^z)(0)$. In view of Lemma 3.3, we note that ψ^z satisfies in $\mathbb{R}^d$,

$$-\triangle \psi^z + \nabla(\Sigma^z \mathbb{1}_{\mathbb{R}^d \setminus (z+I^{\circ\prime})}) = -\delta_{\partial(z+I^{\circ\prime})} \sigma^z \nu.$$

In terms of the Stokeslet G for the free Stokes equation, Green's representation formula then yields

$$\nabla_i \psi^z(0) = -\int_{\partial(z+I^{\circ\prime})} \nabla_i G(-\cdot)\, \sigma^z \nu.$$

Replacing $\nabla_i G$ by its Taylor expansion, using the boundary conditions for σ^z, and using standard decay properties of G, we find

$$\left| \nabla_i \psi^z(0) - \nabla^2_{ij} G(-z) \int_{\partial(z+I^{\circ\prime})} (\cdot - z)_j \, \sigma^z \nu \right| \lesssim \langle z \rangle^{-d-1},$$

and therefore, taking the expectation, noting that $\sigma^z = \sigma^0(\cdot - z)$, and recognizing $\widehat{\mathbf{B}}^1 E$ again,

$$\left| \mathbb{E}\big[\nabla_i \psi^z_k(0)\big] - (2\widehat{\mathbf{B}}^1 E)_{lj} \nabla^2_{ij} G_{kl}(-z) \right| \lesssim \langle z \rangle^{-d-1}.$$

Inserting this into (4.79) together with (4.9) again, we get

$$\left| E : \overline{\mathbf{B}}^2 E - (2\widehat{\mathbf{B}}^1 E)_{lj} (2\widehat{\mathbf{B}}^1 E)_{ki} \left(\text{p. v.} \int_{\mathbb{R}^d} \nabla^2_{ij} G_{kl}(z) h_2(0,z)\, dz \right) \right| \lesssim \lambda_2(\mathcal{P}), \tag{4.80}$$

where the notation p. v. stands for the principal value. It remains to analyze the integral term in the left-hand side. As h_2 satisfies the Dini condition (4.5), this integral is absolutely summable. Further, assuming that the point process $\mathcal{P}$ is statistically isotropic, the correlation function $h_2(0,\cdot)$ is radial. By symmetry, this entails p. v. $\int_{\mathbb{R}^d} \nabla^2_{ij} G_{kl}(z)\, h_2(0,z)\, dz = 0$, and the conclusion (i) follows.

Step 2. Proof of (ii). In view of (4.80), as $2\widehat{\mathbf{B}}^1 E$ does not vanish, it suffices to construct a point process $\mathcal{P}$ that satisfies Assumptions (H_ρ) and (H^{unif}_ρ), has decay of correlations (4.5) with algebraic rate $\omega(t) \le C t^{-\beta}$ for some $C, \beta > 0$, and satisfies the local independence condition $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^2 \ll 1$, such that the integral term in the left-hand side of (4.80) satisfies

$$\left| \text{p. v.} \int_{\mathbb{R}^d} \nabla^2 G(z)\, h_2(0,z)\, dz \right| \gtrsim \lambda(\mathcal{P})^2 \big|\log \lambda(\mathcal{P})\big|, \tag{4.81}$$

with the logarithmic factor. We shall consider spherical particles, $I^\circ = B$, and we start with the construction of the correlation function h_2. For that purpose, first note that we can find a smooth bounded function

$$g : \mathbb{S}^{d-1} \to [0,1]$$

such that

$$\left| \int_{\partial B} \nabla^2 G(e) g(e)\, d\theta(e) \right| \gtrsim 1, \tag{4.82}$$

where $d\theta$ stands for the Lebesgue measure on ∂B, and we then define

$$h(z) := \frac{g\big(\frac{z}{|z|}\big)}{1 + \lambda^2 |z|^{2d+1}}.$$

Using (4.82), a computation in spherical coordinates yields

$$\Big|\int_{|z|>2(1+\rho)} \nabla^2 G(z) h(z)\, dz\Big| = \Big(\int_{2(1+\rho)}^{\infty} r^{-1}(1+\lambda^2 r^{2d+1})^{-1} dr\Big)\Big|\int_{\partial B} \nabla^2 G(e) g(e)\, d\theta(e)\Big| \gtrsim |\log\lambda|,$$

which proves (4.81) if the point process is chosen with intensity $\lambda(\mathcal{P}) = \lambda$ and with two-point correlation function h_2 given by

$$h_2(x,y) + \lambda^2 := \lambda^2 (h(x-y)+1)\mathbb{1}_{|x-y|>2(1+\rho)}.$$

In particular, this choice also yields

$$h_2(0,z)\mathbb{1}_{|z|>2(1+\rho)} \ge 0, \quad \sup_z \int_{Q(z)} \big|h_2(0,\cdot)\big| \lesssim \lambda^2, \quad \big|h_2(0,z)\big| \lesssim \langle z\rangle^{-2d-1}.$$

It remains to prove that this choice of h_2 can be realized as the correlation function of a point process with intensity $\lambda(\mathcal{P}) = \lambda$ and satisfying (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$): this is precisely the subject of Proposition 4.12 below. ■

The construction of a point process with given intensity and given two-point density function is easily done under suitable positivity conditions, e.g. following [43]. In the present setting, more care is needed to further ensure stationarity and ergodicity of the constructed point process. Note that we use here a sufficient positivity condition that is much stronger than the one in [43], but is easier to handle and suffices for our purposes.

Proposition 4.12 (Realizability of point processes). *Let $\lambda, \rho > 0$ and let $h \in \mathrm{L}^\infty(\mathbb{R}^d)$ be nonnegative with $h(x) \to 0$ uniformly as $|x| \uparrow \infty$. Then, there exists a strongly mixing stationary point process $\mathcal{P} = \{x_n\}_n$ on $\mathbb{R}^d$ with intensity λ and two-point density*

$$f_2(x,y) := \lambda^2\big(h(x-y)+1\big)\mathbb{1}_{|x-y|>2(1+\rho)}, \tag{4.83}$$

such that $|x_n - x_m| \ge 2(1+\rho)$ almost surely for all $n \ne m$.

Proof. Let $\mathcal{M}_\rho$ denote the set of locally finite point sets $\{z_n\}_n$ with $|z_n - z_m| \ge 2(1+\rho)$ for all $n \ne m$. It is easily checked that $\mathcal{M}_\rho$ is compact for the topology of convergence of point sets restricted to compact domains (this coincides with the vague topology when viewing point sets $\{z_n\}_n$ as measures $\sum_n \delta_{z_n}$). Consider the space $V := C(\mathcal{M}_\rho)$, and denote by V_0 the dense vector subset of polynomials with continuous coefficients on $\mathcal{M}_\rho$, that is, the subset of functions $P^N : \mathcal{M}_\rho \to \mathbb{R}$ of the form

$$P^N\big(\{z_n\}_n\big) = P_0^N + \sum_{k=1}^{N} \sum_{n_1,\ldots,n_k}^{\ne} P_k^N(z_{n_1},\ldots,z_{n_k}), \tag{4.84}$$

with $P_0^N \in \mathbb{R}$ and $P_k^N \in C_c(S_\rho^k)$ for $1 \le k \le N$, where we use the shorthand notation

$$S_\rho^k := \big\{(z_1, \dots, z_k) \in (\mathbb{R}^d)^k : |z_n - z_m| \ge 2(1+\rho) \text{ for all } n \ne m\big\}.$$

In order to construct a point process with the two-point density f_2 given by (4.83), we shall further prescribe all its multi-point density functions. For convenience, these are chosen in form of Mayer cluster expansions with vanishing higher-order correlations: for all $k \ge 1$,

$$\begin{aligned} & f_k(z_1, \dots, z_k) \\ & := \lambda^k \, \mathbb{1}_{S_\rho^k}(z_1, \dots, z_k) \\ & \times \left(1 + \sum_{j=1}^{k/2} \frac{1}{2^j j!} \sum_{1 \le n_1, \dots, n_{2j} \le k}^{\ne} h(z_{n_1} - z_{n_2}) \cdots h(z_{n_{2\ell-1}} - z_{n_{2j}})\right). \end{aligned} \tag{4.85}$$

Next, we define in these terms a linear map $L : V_0 \to \mathbb{R}$ as follows: for any polynomial P^N of the form (4.84), we set

$$L(P^N) := P_0^N + \sum_{k=1}^N \int_{S_\rho^k} P_k^N f_k. \tag{4.86}$$

We argue that L is a positive linear functional on V_0, hence it is also continuous on V_0 with respect to the topology of V. Indeed, for any polynomial P^N of the form (4.84) with $P^N \ge 0$ pointwise, if we evaluate it at the points of a Poisson point process with intensity λ, and if we compute the expectation, we find

$$P_0^N + \sum_{k=1}^N \lambda^k \int_{S_\rho^k} P_k^N \ge 0,$$

hence, noting that the positivity of h entails $f_k \ge \lambda^k \mathbb{1}_{S_\rho^k}$ for all $k \ge 1$, we get

$$L(P^N) \ge P_0^N + \sum_{k=1}^N \lambda^k \int_{S_\rho^k} P_k^N \ge 0,$$

thus proving the claimed positivity.

As V_0 is dense in V, we can extend L uniquely into a positive linear functional

$$L : V \to \mathbb{R}.$$

Next, by the Riesz–Markov–Kakutani representation theorem, there exists a random element in $\mathcal{M}_\rho$, that is, a random point process $\mathcal{P} = \{x_n\}_n$, such that

$$\mathbb{E}\big[P(\mathcal{P})\big] = L(P) \quad \text{for all } P \in V.$$

Testing this relation with polynomials, and using (4.86), we deduce that for all $k \geq 1$ the k-point density function of the point process $\mathscr{P}$ coincides with f_k. In particular, it has intensity $f_1 = \lambda$ and two-point density

$$f_2(x, y) = \lambda^2\big(h(x-y)+1\big)\mathbb{1}_{|x-y|>2(1+\rho)}$$

as desired. In addition, L is translation-invariant by definition, hence $\mathscr{P}$ is stationary.

It remains to check that $\mathscr{P}$ is strongly mixing. For that purpose, we compute the covariance of $\sigma(\mathscr{P})$-measurable random variables. Choose a polynomial P^N of the form (4.84), and let $R > 0$ be such that P_k^N is supported in $(B_R)^k$ for all $1 \leq k \leq N$. For $|x| > 2R$, as we have $B_R \cap (x + B_R) = \varnothing$, we can compute

$$\begin{aligned} &\operatorname{Cov}\big[P^N(\mathscr{P}+x); P^N(\mathscr{P})\big] \\ &\quad = (P_0^N)^2 + \sum_{k\geq 1}\sum_{j=0}^{k}\int_{S_\rho^k}\big(P_j^N(\cdot+x,\ldots,\cdot+x)\otimes P_{k-j}^N\big)\big(f_k - f_j\otimes f_{k-j}\big). \end{aligned}$$

The definition (4.85) of f_k easily leads to

$$\begin{aligned} &\left|\int_{S_\rho^k}\big(P_j^N(\cdot+x,\ldots,\cdot+x)\otimes P_{k-j}^N\big)\big(f_k - f_j\otimes f_{k-j}\big)\right| \lesssim \\ &\Big(\sup_{z\in B_{2R}} h(x+z)\Big)(2\lambda)^k\big(1+\|h\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\big)^{\frac{k}{2}-1}\|P_j^N\|_{\mathrm{L}^1((\mathbb{R}^d)^j)}\|P_{k-j}^N\|_{\mathrm{L}^1((\mathbb{R}^d)^{k-j})}. \end{aligned}$$

As by assumption $h(x) \to 0$ as $|x| \uparrow \infty$, we get $\operatorname{Cov}[P^N(\mathscr{P}+x); P^N(\mathscr{P})] \to 0$. By a density argument, the same holds if P^N is replaced by any element of V, which proves that $\mathscr{P}$ is strongly mixing. ∎