Chapter 5

Conclusion

In this last chapter, we recall, reformulate, and comment our main findings on the validity of Einstein's formula and of higher-order cluster expansions for the effective viscosity, as obtained in Chapters 2, 3, and 4.

5.1 Cluster expansion of the effective viscosity

We start with the validity of Einstein's formula and the associated error estimates as proved in Chapter 2, cf. Theorem 2.1. The three important features of this result are the generality in terms of probabilistic assumptions (mere qualitative ergodicity under Assumption (H_ρ^{unif})), the sharpness of the error estimate (5.1), and the possibility for particles to touch (under Assumption ($H_{\rho,\kappa}^{\mathrm{mom}}$) or ($H_{\rho,\kappa}^{\mathrm{perc}}$)).

Theorem 5.1 (Einstein's formula). *Under Assumption (H_ρ), provided that Assumption (H_ρ^{unif}), ($H_{\rho,\kappa}^{\mathrm{mom}}$), or ($H_{\rho,\kappa}^{\mathrm{perc}}$) holds for some $\rho > 0$ and $\kappa > 1$, we have*

$$\big|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)\big| \lesssim_\rho \lambda_2(\mathcal{P}) \log\bigg(2 + \frac{\lambda(\mathcal{P})}{\lambda_2(\mathcal{P})\big(\ell(\mathcal{P})+1\big)^d}\bigg) + \begin{cases} 0 & \text{in case of } (H_\rho^{\mathrm{unif}}), \\ \mathcal{K}_\kappa\, \lambda_2(\mathcal{P})^{1-\frac{1}{\kappa}} \lambda(\mathcal{P})^{\frac{1}{\kappa}} & \text{in case of } (H_{\rho,\kappa}^{\mathrm{mom}}) \text{ or } (H_{\rho,\kappa}^{\mathrm{perc}}), \end{cases} \tag{5.1}$$

where $\bar{\mathbf{B}}^1$ satisfies

$$|\bar{\mathbf{B}}^1| \simeq \lambda(\mathcal{P}),$$

and is defined for all $E \in \mathbb{M}_0^{\mathrm{sym}}$ by

$$E : \bar{\mathbf{B}}^1 E := \sum_n \mathbb{E}\bigg[\frac{\mathbb{1}_{0\in I_n}}{|I_n|} \int_{\mathbb{R}^d} \big|\mathrm{D}(\psi_E^{\{n\}})\big|^2\bigg],$$

where $\psi_E^{\{n\}}$ is the unique decaying solution of the single-particle problem (1.5). In particular, the estimate $|\bar{\mathbf{B}} - (\mathrm{Id} + \bar{\mathbf{B}}^1)| = o(\lambda(\mathcal{P}))$ holds provided the point process $\mathcal{P}$ satisfies $\lambda_2(\mathcal{P}) = o(\lambda(\mathcal{P}))$.

In order to address the optimality of this estimate, one needs to identify the next term in the expansion. In Chapters 3 and 4, we have further investigated higher-order expansions of the effective viscosity in form of cluster expansions. The upcoming result, which summarizes Theorems 4.1 and 4.3 in Chapter 4, gives the optimal order of magnitude of the cluster coefficients and of the remainder. The two important features of this result are the generality of the point processes (to be compared with results in Section 5.2 below) and the sharpness of the estimates. The main achievement is the explicit understanding of the needed renormalizations to all orders, solving a problem that was still open in the physics community.

Theorem 5.2 (Cluster expansion in general dilute setting). *On top of* (H_ρ) *and* (H_ρ^{unif}), *assume that the inclusion process is* α*-mixing in the sense of* (Mix) *with algebraic rate* ω. *Then, for all* $k \geq 1$, *the following holds for the effective viscosity,*

$$\Big|\bar{\mathbf{B}} - \mathrm{Id} - \sum_{j=1}^{k} \frac{1}{j!} \bar{\mathbf{B}}^j\Big| \lesssim_k \sum_{l=k+1}^{2k+1} \lambda_l(\mathcal{P}) \big|\log \lambda(\mathcal{P})\big|^{l-1},$$

$$|\bar{\mathbf{B}}^j| \lesssim_j \lambda_j(\mathcal{P}) \big|\log \lambda(\mathcal{P})\big|^{j-1}, \quad \text{for all } 1 \leq j \leq k,$$

where the cluster coefficients $\{\bar{\mathbf{B}}^j\}_j$ *are defined in* (3.13) *by means of finite-volume approximations. If in addition the independence assumption* (Indep) *holds for particle shapes, renormalized formulas can be given for cluster coefficients in form of absolutely convergent multiple integrals, cf. Section* 4.4, *and the following quantitative convergence result holds for finite-volume approximations* $\{\bar{\mathbf{B}}^j_L\}_j$: *in case of an algebraic* α*-mixing rate* $\omega(t) \leq C t^{-\beta}$ *for some* $C, \beta > 0$,

$$|\bar{\mathbf{B}}^j_L - \bar{\mathbf{B}}^j| \lesssim_j \frac{(\log L)^{j-1}}{L^{\beta \wedge 1}}.$$

Note that the bound on $\bar{\mathbf{B}}_2$ in Theorem 5.2 essentially coincides with the estimate on the remainder in Theorem 5.1, which contrasts with the results of Lemma 1.2 in the short-range setting by a logarithmic correction. Optimality of the latter is addressed in Theorem 4.4, which we presently recall.

Theorem 5.3. *About the optimality of estimates on* $\bar{\mathbf{B}}_2$, *the following statements hold.*

(i) Isotropic setting: *On top of Assumptions* (H_ρ), (H_ρ^{unif}), *and* (Indep), *assume that the* 2*-point correlation function* $h_2(x,y) := f_2(x,y) - \lambda(\mathcal{P})^2$ *satisfies the following decay assumption,*

$$\iint_{B(x)\times B(y)} |h_2| \leq \omega\big(|x-y|\big),$$

with some rate ω *satisfying the Dini condition* $\int_1^\infty t^{-1}\omega(t)\,dt < \infty$. *If in addition the point process* $\mathcal{P}$ *is statistically isotropic, which entails that the correlation function is radial, then the following improved estimate holds,*

$$|\bar{\mathbf{B}}^2| \lesssim \lambda_2(\mathcal{P}).$$

(ii) Optimality in the general setting: *There exists an inclusion process* $\mathcal{I}$ *that satisfies Assumptions* (H_ρ), (H_ρ^{unif}), (Indep), *and* (4.5), *as well as the local independence* *condition* $\lambda_2(\mathcal{P}) \simeq \lambda(\mathcal{P})^2 \ll 1$*, such that we have*

$$|\bar{\mathbf{B}}^2| \simeq \lambda_2(\mathcal{P})\big|\log \lambda_2(\mathcal{P})\big|.$$

Based on the explicit renormalization of higher-order cluster coefficients, it appears that Theorem 5.3 (ii) readily extends to higher orders, demonstrating the optimality of cluster estimates in Theorem 5.2.

To conclude this section, let us apply and confront Theorems 5.1, 5.2, and 5.3 to some specific families of inclusion processes displaying multi-point intensities with different scaling laws. We start with the construction.

• *Construction of inclusion processes* $\{\mathcal{I}_{\beta,\lambda}\}_{\beta,\lambda}$*:* We define a family of point processes $\{\mathcal{P}_{\beta,\lambda}\}_{\beta,\lambda}$ with parameters $0 \le \beta \le 1$ and $0 < \lambda \ll 1$ as follows. Consider a hardcore Poisson process $\mathcal{P}' = \{x'_n\}_n$ with radius 6 and with intensity $\lambda(\mathcal{P}') = \lambda$, see e.g. [17, Section 3.4] using Penrose's graphical construction [57]. Next, independently choose a sequence $\{y_n\}_n$ of iid random points that are uniformly distributed in $B_4 \setminus B_3$, and, given $\beta \in [0,1]$, also independently choose a sequence $\{b_{n,\beta}\}_n$ of iid Bernoulli variables with parameter $\lambda^\beta = \mathbb{P}[b_{n,\beta} = 1]$. The desired point processes and spherical inclusion processes are then defined by

$$\mathcal{P}_{\beta,\lambda} := \mathcal{P}' \cup \big\{x'_n + y_n : b_{n,\beta} = 1\big\}, \quad \mathcal{I}_{\beta,\lambda} := \bigcup_{x \in \mathcal{P}_{\beta,\lambda}} B(x).$$

• *Properties of the processes:* $\mathcal{I}_{\beta,\lambda}$ satisfies (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$) (with $\rho = 1$) as well as (Indep). In addition, the point process $\mathcal{P}_{\beta,\lambda}$ is statistically isotropic and α-mixing with exponential rate uniformly with respect to β, λ (e.g. [16, Proposition 1.4 (iii)] and [17, Proposition 3.5]), so that Theorem 5.2 applies. A direct computation shows that the multi-point intensities scale as follows,

$$\lambda(\mathcal{P}_{\beta,\lambda}) \simeq \lambda, \quad \lambda_2(\mathcal{P}_{\beta,\lambda}) \simeq \lambda^{1+\beta}, \quad \lambda_3(\mathcal{P}_{\beta,\lambda}) \simeq \lambda^{2+\beta},$$

and more generally $\lambda_{2k}(\mathcal{P}_{\beta,\lambda}) \simeq_k \lambda^{k(1+\beta)}$ and $\lambda_{2k+1}(\mathcal{P}_{\beta,\lambda}) \simeq_k \lambda^{1+k(1+\beta)}$. In particular the minimal local independence condition $\lambda_3(\mathcal{P}_{\beta,\lambda}) \ll \lambda_2(\mathcal{P}_{\beta,\lambda}) \ll \lambda(\mathcal{P}_{\beta,\lambda})$ holds for $\beta > 0$.

• *Second-order cluster expansion:* We denote by $\bar{\mathbf{B}}_{\beta,\lambda}$ the effective viscosity associated with $\mathcal{I}_{\beta,\lambda}$. Theorem 5.2 implies that

$$\left|\bar{\mathbf{B}}_{\beta,\lambda} - \left(\mathrm{Id} + \bar{\mathbf{B}}^1_{\beta,\lambda} + \frac{1}{2}\bar{\mathbf{B}}^2_{\beta,\lambda}\right)\right| \lesssim \lambda^{2+\beta}|\log \lambda|^2,$$

where $|\bar{\mathbf{B}}^1_{\beta,\lambda}| \simeq \lambda$ and $|\bar{\mathbf{B}}^2_{\beta,\lambda}| \simeq \lambda^{1+\beta}$ (cf. (2.6) and Theorem 5.3 (i)). In particular, discarding $\bar{\mathbf{B}}^2_{\beta,\lambda}$ in the above yields the following (completely new) sharp error estimate for Einstein's formula in this setting: for all $0 \le \beta \le 1$ and $\lambda \ll 1$,

$$\big|\bar{\mathbf{B}}_{\beta,\lambda} - (\mathrm{Id} + \bar{\mathbf{B}}^1_{\beta,\lambda})\big| \simeq \lambda^{1+\beta} \simeq |\bar{\mathbf{B}}^1_{\beta,\lambda}|^{1+\beta}.$$

In this example, Einstein's formula is thus accurate whenever $\beta > 0$, which illustrates the full range of the condition $\lambda_2(\mathcal{P}_{\beta,\lambda}) = o(\lambda(\mathcal{P}_{\beta,\lambda}))$ in Theorem 5.1.

5.2 Summability of the cluster expansion

Finally, we consider the following two specific one-parameter dilution procedures, for which our results can be substantially strengthened using the uniform $\ell^1 - \ell^2$ energy estimates of Theorem 3.9: more precisely, logarithmic corrections in cluster estimates can be removed in that case and the full cluster expansion is absolutely converging.

(Dilat) *Dilution by geometric dilation:* Given a point process $\mathcal{P} = \{x_n\}_n$ and random inclusions $I_n = x_n + I_n^\circ$ satisfying (H_ρ), we consider the dilated process $\mathcal{P}_s = \{sx_n\}_n$ and the corresponding inclusions $I_{n,s} = sx_n + I_n^\circ$. The latter has minimal distance $\ell(\mathcal{P}_s) = s\ell(\mathcal{P}) \simeq s$, still satisfies ($\mathrm{H}_\rho$), and further satisfies ($\mathrm{H}_\rho^{\mathrm{unif}}$) with minimal interparticle distance

$$\inf_{n\neq m} \operatorname{dist}(I_{n,s}, I_{m,s}) \geq \inf_{n\neq m} |sx_n - sx_m| - 2 \geq s\ell(\mathcal{P}) - 2 \gtrsim s,$$

provided $s \gg 1$. Its multi-point intensities take the form

$$\lambda_j(\mathcal{P}_s) = s^{-jd}\lambda_j(\mathcal{P}) \quad \text{for all } j \geq 1.$$

(Delet) *Dilution by random deletion:* Given a point process $\mathcal{P} = \{x_n\}_n$ and random inclusions $I_n = x_n + I_n^\circ$ satisfying (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$), the Bernoulli deletion scheme consists in keeping each inclusion only with given probability $p \in [0, 1]$. More precisely, we attach to the inclusions iid Bernoulli variables $\{b_n^{(p)}\}_n$, independent of $\mathcal{P}, \mathcal{I}$, with parameter

$$p = \mathbb{P}\big[b_n^{(p)} = 1\big],$$

and we define the corresponding decimated process

$$\mathcal{P}^{(p)} := \{x_n\}_{n\in N^{(p)}}, \quad \mathcal{I}^{(p)} := \bigcup_{n\in N^{(p)}} I_n, \quad N^{(p)} := \{n : b_n^{(p)} = 1\}.$$

This decimated process still satisfies (H_ρ) and ($\mathrm{H}_\rho^{\mathrm{unif}}$), and its multi-point intensities are given by

$$\lambda_j(\mathcal{P}^{(p)}) = p^j\lambda_j(\mathcal{P}) \quad \text{for all } j \geq 1.$$

In these one-parameter settings, dilute expansions of the effective viscosity amount to expansions with respect to the dilution parameters s^{-1} or p. Given a random set of particles $\mathcal{I} = \bigcup_n I_n$ centered at the points of $\mathcal{P} = \{x_n\}_n$, we shall consider both dilution procedures at once, defining the dilated decimated process

$$\mathcal{P}_s^{(p)} := \{x_{n,s}\}_{n\in N^{(p)}}, \quad \mathcal{I}_s^{(p)} := \bigcup_{n\in N^{(p)}} I_{n,s}, \quad x_{n,s} := s x_n, \quad I_{n,s} := x_{n,s} + I_n^\circ.$$

As a consequence of Theorem 3.2, together with (3.78) and (3.79) in Section 3.7, we obtain the following summability result and estimates for the cluster expansion of the effective viscosity $\bar{\mathbf{B}}_s^{(p)}$ associated with $\mathcal{I}_s^{(p)}$. In particular, it shows that the scaling of cluster coefficients coincides in this case with that of Lemma 1.2 for the short-range setting: indeed, we have

$$|\bar{\mathbf{B}}_s^{(p),j}| = p^j |\bar{\mathbf{B}}_s^j| \lesssim_j (ps^{-d})^j \simeq \lambda_j(\mathcal{P}_s^{(p)}).$$

We emphasize that no mixing assumption is required here.

Theorem 5.4 (Cluster expansion for one-parameter dilution procedures). *Under* (H$_\rho$) *and* (H$_\rho^{\mathrm{unif}}$)*, for the specific dilution models* (Dilat) *and* (Delet) *above, with dilation parameter s and Bernoulli parameter p, the cluster expansion of the effective viscosity is uniformly summable in the following sense: there exists a constant C (only depending on d, ρ) such that for all $0 \le ps^{-d} < \frac{1}{C}$ the effective viscosity satisfies*

$$\bar{\mathbf{B}}_s^{(p)} = \mathrm{Id} + \sum_{j=1}^{\infty} \frac{p^j}{j!} \bar{\mathbf{B}}_s^j, \quad |\bar{\mathbf{B}}_s^j| \le j!\,(Cs^{-d})^j \quad \textit{for all } j \ge 1, \tag{5.2}$$

where the cluster coefficients $\{\bar{\mathbf{B}}_s^j\}_j$ are defined in (3.13) *by means of finite-volume approximations.*

Remarks 5.5. We comment on the analyticity of the effective viscosity with respect to dilution parameters.

(a) In case of the random deletion model (Delet), the expansion (5.2) yields the local analyticity of $p \mapsto \bar{\mathbf{B}}^{(p)}$ at $p = 0$. Local analyticity can, in fact, be established on the whole interval $0 \le p \le 1$; the reader is referred to [15] for a similar result in the scalar setting.

(b) In case of the dilation model (Dilat), the expansion (5.2) does not yield the analyticity of the map $s^{-d} \mapsto \bar{\mathbf{B}}_s$ since the rescaled coefficients $\{s^{dj}\bar{\mathbf{B}}_s^j\}_j$ also depend on s. By means of multipole expansions, the maps

$$s^{-1} \mapsto s^{dj}\bar{\mathbf{B}}_s^j$$

can be checked to be analytic themselves, as well as $s^{-1} \mapsto \bar{\mathbf{B}}_s$. For a more direct approach to expansions in s^{-1}, we refer to the recent work [59] in the scalar setting; see also [9].

To illustrate Remark 5.5 (b), we display the first term of the monopole expansion for the second-order coefficient $\bar{\mathbf{B}}_s^2$. In particular, as is natural, we note that $\bar{\mathbf{B}}_s^2$ can be expressed to leading order in terms of the single-particle problem only, and it coincides with the formula obtained in [27, Proposition 5.6] in case of spherical inclusions.

Proposition 5.6 (Leading order of monopole expansion). *On top of* (H_ρ) *and* (H_ρ^{unif}), *assume that particles have independent shapes, cf.* (Indep), *and that the two-point correlation function* $h_2 = f_2 - \lambda(\mathscr{P})^2$ *satisfies the decay assumption*

$$\iint_{B(x)\times B(y)} |h_2| \le \omega\big(|x-y|\big),$$

with some rate ω *satisfying the Dini condition* $\int_1^\infty t^{-1}\omega(t)\,dt < \infty$. *Consider the dilated process* $\mathscr{P}_s$, *cf.* (Dilat), *and the associated second-order cluster coefficient* $\bar{\mathbf{B}}_s^2$ *defined in Theorem 5.4. Then, we have*

$$\big|\bar{\mathbf{B}}_s^2 - s^{-2d}\bar{\mathbf{B}}^{2,1}\big| \lesssim s^{-2d-1},$$

and the leading-order contribution $\bar{\mathbf{B}}^{2,1}$ *is given by the following reduced formula,*

$$E : \bar{\mathbf{B}}^{2,1}E = (2\widehat{\mathbf{B}}^1 E) : \bigg(\mathrm{p.\,v.}\int_{\mathbb{R}^d} \mathscr{G}(z)\,h_2(0,z)\,dz\bigg)(2\widehat{\mathbf{B}}^1 E),$$

where $\widehat{\mathbf{B}}^1 E$ *is defined in* (2.6), *where the notation* p. v. *stands for the principal value, and where the* 4*-tensor field* $\mathscr{G}$ *is given by* $M : \mathscr{G}(z)M = M_{jk}M_{lm}\nabla^2_{km}G_{jl}(z)$ *in terms of the standard Stokeslet*

$$G(z) = \frac{|z|^{2-d}}{2(d-2)|\partial B|}\Big(\mathrm{Id} + (d-2)\frac{z\otimes z}{|z|^2}\Big).$$

In case of spherical particles, $I_n = B(x_n)$, *we thus have*

$$E : \bar{\mathbf{B}}^{2,1}E = (d+2)^2|B|\ \mathrm{p.\,v.}\int_{\mathbb{R}^d}\bigg(\frac{d+2}{2}\frac{(z\cdot Ez)^2}{|z|^{d+4}} - \frac{|Ez|^2}{|z|^{d+2}}\bigg)h_2(0,z)\,dz.$$

Proof. Starting from the renormalized formula (4.21) in Proposition 4.9, repeating the proof of (4.80) to decompose the first contribution, and using (4.25) to estimate the second one, we are led to

$$\bigg|E : \bar{\mathbf{B}}_s^2 E - (2\widehat{\mathbf{B}}^1 E)_{lj}(2\widehat{\mathbf{B}}^1 E)_{ki}\bigg(\mathrm{p.\,v.}\int_{\mathbb{R}^d}\nabla^2_{ij}G_{kl}(z)h_2(0,z)\,dz\bigg)\bigg|$$
$$\lesssim \int_{\mathbb{R}^d}\langle z\rangle^{-d-1}\big|h_{2,s}(0,z)\big|\,dz + \int_{\mathbb{R}^d}\langle z\rangle^{-2d} f_{2,s}(0,z)\,dz.$$

Using that the two-point density and the correlation for the dilated process $\mathscr{P}_s$ take the form

$$f_{2,s}(0,z) = s^{-2d} f_2(0,s^{-1}z), \quad h_{2,s}(0,z) = s^{-2d} h_2(0,s^{-1}z),$$

and changing variables, the conclusion follows by scaling. In case of spherical particles, we appeal to the proof of Proposition 2.2 for the explicit computation of $\widehat{\mathbf{B}}^1 E$. ∎

Finally, we revisit a recent result by Gérard-Varet [26] that displays to second order similar estimates as for the random deletion procedure, cf. (5.2), but only assuming some specific structure of the multi-point densities up to order 5, thus contrasting with Theorem 5.4. As a corollary of Proposition 4.11, we establish the following result, which constitutes an extension of [26] to higher orders with new, optimal error bounds. Note indeed that for $k = 2$ the result (5.3) below yields an error bound $O(p^3)$, which improves on the bound $O(p^{\frac{5}{2}})$ obtained in [26].

Corollary 5.7. *Let $\mathcal{P}$ satisfy Assumptions* (H_ρ) *and* $(\mathrm{H}_\rho^{\mathrm{unif}})$*, and let $\mathcal{I}$ satisfy the independence assumption* (Indep)*. Given $k \geq 2$, assume that there exists $0 < p \leq 1$ such that the multi-point density functions of $\mathcal{P}$ can be written as*

$$f_j = p^j f_j^\circ \quad \text{for all } 1 \leq j \leq 2k+1,$$

for some functions $(f_j^\circ)_{1\leq j\leq 2k+1}$. Further, define functions $(h_j^\circ)_{1\leq j\leq 2k+1}$ through the correlation/density relation (4.8) *starting from $(f_j^\circ)_{1\leq j\leq 2k+1}$ and assume that they satisfy* (Mix_ω^n) *to order $n = 2k+1$ with algebraic rate ω. Then, we have*

$$\left|\overline{\mathbf{B}} - \mathrm{Id} - \sum_{j=1}^{k} \frac{1}{j!}\overline{\mathbf{B}}^j\right| \lesssim_k p^{k+1}, \quad |\overline{\mathbf{B}}^j| \lesssim_j p^j \quad \text{for all } 1 \leq j \leq k. \tag{5.3}$$

where the multiplicative constants are independent of p.

Proof. The assumption $f_j = p^j f_j^\circ$ entails $h_j = p^j h_j^\circ$, where h_j° is assumed to satisfy (Mix_ω^n). Further, writing $(f_j^\circ)_j$ in terms of $(h_j^\circ)_j$ by means of (4.7), the assumption (Mix_ω^n) for the latter yields

$$\lambda_j^\circ := \sup_{z_1,\ldots,z_j} \fint_{Q(z_1)\times\cdots\times Q(z_j)} f_j^\circ \lesssim_j 1,$$

where the bound only depends on j, ω, and on the constant function f_1°. In this setting, the bounds of Proposition 4.11 (i)–(iii) now take the form

$$|\overline{\mathbf{B}}^j| \lesssim p^j \overline{\lambda}_j^\circ |\log \lambda^\circ|^{j-1},$$

$$|R^{k+1}| \lesssim \sum_{j=k}^{2k+1} p^{j+1}\overline{\lambda}_{j+1}^\circ |\log \lambda^\circ|^j \lesssim p^{k+1},$$

and the conclusion readily follows. ∎