Appendix A

Stokeslet estimates with rigid inclusions

This appendix is dedicated to the proof of several estimates on the behavior of the fluid velocity generated by a localized force dipole in the presence of a finite number of rigid inclusions. In other words, it concerns the Stokeslet for the Stokes problem with rigid inclusions, and we shall prove in particular Lemmas 3.10, 4.7, and 4.8.

A.1 Main results

For convenience, we start by recalling the notation of Section 4.3.2. Given a set $Y \subset Q_L$ of "background" positions with

$$\operatorname{dist}\big(B(y), B(y')\big) > 2\rho, \ \operatorname{dist}\big(B(y), \partial Q_L\big) > \rho, \ \text{for all } y, y' \in Y, \ y \neq y', \tag{A.1}$$

we denote by $\psi_L^Y \in H^1_{\mathrm{per}}(Q_L)^d$ the solution of the following periodic corrector problem, using the shorthand notation $\sigma_L^Y := \sigma(\psi_L^Y + Ex, \Sigma_L^Y)$,

$$\begin{cases} -\triangle \psi_L^Y + \nabla \Sigma_L^Y = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y),\\ \operatorname{div}(\psi_L^Y) = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y),\\ \mathrm{D}(\psi_L^Y + Ex) = 0, & \text{in } \bigcup_{y\in Y} B(y),\\ \int_{\partial B(y)} \sigma_L^Y \nu = 0, & \forall y \in Y,\\ \int_{\partial B(y)} \Theta(x-y)\cdot \sigma_L^Y \nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \ \forall y \in Y. \end{cases}$$

Next, we turn to elementary single-particle contributions $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$: Given a "tagged" position $z \in Q_L$, given $(\zeta, P) \in H^1(B_{1+\rho}(z))^d \times \mathrm{L}^2(B_{1+\rho}(z) \setminus B(z))$ satisfying the following Stokes equations in a neighborhood of $B(z)$,

$$\begin{cases} -\triangle \zeta + \nabla P = 0, & \text{in } B_{1+\rho}(z) \setminus B(z),\\ \operatorname{div}(\zeta) = 0, & \text{in } B_{1+\rho}(z) \setminus B(z),\\ \mathrm{D}(\zeta) = 0, & \text{in } B(z),\\ \int_{\partial B(z)} \sigma(\zeta, P)\nu = 0, & \\ \int_{\partial B(z)} \Theta(x-z)\cdot \sigma(\zeta, P)\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}, \end{cases} \tag{A.2}$$

and given a finite subset $Y \subset Q_L$ of "background" positions satisfying (A.1), we define $\mathcal{J}_{L;Y}^z \zeta \in H^1_{\mathrm{per}}(Q_L)^d$ as the solution of the following Stokes problem with force dipole localized around z and rigid inclusions around points of Y,

$$\begin{cases} -\triangle \mathcal{J}_{L;Y}^z \zeta + \nabla \mathcal{Q}_{L;Y}^z \zeta = -\delta_{\partial B^L(z)} \sigma(\zeta, P)\nu, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\ \operatorname{div}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\ \mathrm{D}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } \bigcup_{y \in Y \setminus Y^z} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall y \in Y \setminus Y_z, \\ \int_{\partial B(y)} \Theta(x-y) \cdot \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall \Theta \in \mathbb{M}^{\text{skew}},\ \forall y \in Y \setminus Y_z, \\ \mathcal{J}_{L;Y}^z \zeta = V_z + \Theta_z(x-z), & \text{in } \bigcup_{y \in Y_z} B(y), \\ & \text{for some } V_z \in \mathbb{R}^d, \Theta_z \in \mathbb{M}^{\text{skew}}, \\ \sum_{y \in Y_z} \int_{\partial B(y)} \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu & \\ = \sum_{y \in Y_z} \int_{B(y) \cap \partial B^L(z)} \sigma(\zeta, P)\nu, & \\ \sum_{y \in Y_z} \int_{\partial B(y)} \Theta(x-z) \cdot \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu & \\ = \sum_{y \in Y_z} \int_{B(y) \cap \partial B^L(z)} \Theta(x-z) \cdot \sigma(\zeta, P)\nu, & \forall \Theta \in \mathbb{M}^{\text{skew}}, \end{cases} \tag{A.3}$$

where we recall that $B^L(z) = (B(z) + L\mathbb{Z}^d) \cap Q_L$ stands for the periodization of the ball $B(z)$ in Q_L, where we have set $Y_z := \{y \in Y : B(y) \cap B^L(z) \neq \varnothing\}$, and where we have implicitly extended (ζ, P) periodically to $B_{1+\rho}(z) + L\mathbb{Z}^d$. The solution $\mathcal{J}_{L;Y}^z \zeta$ is only defined up to a rigid motion in Q_L, which we fix by further choosing

$$\int_{Q_L} \mathcal{J}_{L;Y}^z \zeta = 0, \quad \int_{Q_L} \nabla \mathcal{J}_{L;Y}^z \zeta \ \in\ \mathbb{M}_0^{\text{sym}}.$$

Note that $\mathcal{J}_{L;Y}^z \zeta$ depends of course on the pair (ζ, P), not only on ζ, but we leave the pressure field implicit in the notation for convenience. We refer to Section 4.3.2 for motivation of the above equations (A.3), and we recall that it reduces to the following simpler equations when $\{z\} \cup Y$ satisfies (A.1) (meaning that z neither gets close to background positions Y nor to the cell boundary ∂Q_L),

$$\begin{cases} -\triangle \mathcal{J}_{L;Y}^z \zeta + \nabla \mathcal{Q}_{L;Y}^z \zeta = -\delta_{\partial B(z)} \sigma(\zeta, P)\nu, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\ \operatorname{div}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y \in Y} B(y), \\ \mathrm{D}(\mathcal{J}_{L;Y}^z \zeta) = 0, & \text{in } \bigcup_{y \in Y} B(y), \\ \int_{\partial B(y)} \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y) \cdot \sigma(\mathcal{J}_{L;Y}^z \zeta, \mathcal{Q}_{L;Y}^z \zeta)\nu = 0, & \forall \Theta \in \mathbb{M}^{\text{skew}},\ \forall y \in Y. \end{cases} \tag{A.4}$$

We further define

$$\mathcal{J}_L^z \zeta := \mathcal{J}_{L;\varnothing}^z \zeta,$$

for which the Stokes problem (A.3) reduces to

$$-\triangle \mathcal{J}_L^z \zeta + \nabla \mathcal{Q}_L^z \zeta = -\delta_{\partial B^L(z)} \sigma(\zeta, P)\nu, \quad \operatorname{div}(\mathcal{J}_L^z \zeta) = 0, \quad \text{in } Q_L, \tag{A.5}$$

and we define $\mathcal{J}_Y^z \zeta$, $\mathcal{J}^z \zeta$ as the corresponding operators on whole space, that is, with $B^L(z)$ and Q_L replaced by $B(z)$ and $\mathbb{R}^d$, respectively, in (A.3) and (A.5).

With the above notation, we start by recalling the statement of Lemma 4.7 regarding the optimal decay properties of the Stokeslets $\{\mathcal{J}_{L;Y}^z\}_{z,Y}$. Note that Lemma 3.10 is a particular case of this result, using notation (4.15), when $\{z\} \cup Y$ satisfies (A.1). The proof is displayed in Section A.2.

Lemma A.1 (Decay of Stokeslets with rigid inclusions). *Let $z \in \mathbb{R}^d$, let (ζ, P) satisfy (A.2) at z, and let $Y \subset Q_L$ satisfy (A.1). Then, we have for all $x \in Q_L$,*

$$\Big(\int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}_{L;Y}^z \zeta)\big|^2 \Big)^{\frac12} \lesssim_{\sharp Y} \langle (x-z)_L \rangle^{-d} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2 \Big)^{\frac12}, \tag{A.6}$$

$$\Big(\int_{B(x)} \big|\mathrm{D}(\mathcal{J}_{Y}^z \zeta)\big|^2 \Big)^{\frac12} \lesssim_{\sharp Y} \langle x-z \rangle^{-d} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2 \Big)^{\frac12}.$$

A similar argument leads us to the following version of the mean-value property for Stokes equations in the presence of a finite number of rigid inclusions. The proof is displayed in Section A.3.

Lemma A.2 (Mean-value property with rigid inclusions). *Let $Y \subset Q_L$ satisfy (A.1) and let $w \in H^1(Q_L)^d$ satisfy the following free steady Stokes equations in Q_L,*

$$\begin{cases} -\triangle w + \nabla P = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y),\\ \operatorname{div}(w) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y),\\ \mathrm{D}(w) = 0, & \text{in } \bigcup_{y\in Y} B(y),\\ \int_{\partial B(y)} \sigma(w,P)\nu = 0, & \forall y \in Y,\\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(w,P)\nu = 0, & \forall y\in Y,\ \forall \Theta \in \mathbb{M}^{\text{skew}}. \end{cases} \tag{A.7}$$

Then, we have for all $B(x) \subset Q_L$,

$$\int_{B^L(x)} \big|\mathrm{D}(w)\big|^2 \lesssim_{\sharp Y} \big\langle \operatorname{dist}(x, \partial Q_L) \big\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(w)\big|^2. \tag{A.8}$$

Finally, we recall the statement of Lemma 4.8 regarding the error $\mathcal{J}_{L;Y}^z - \mathcal{J}_Y^z$ between periodized and whole-space Stokeslets. The proof makes heavy use of the above mean-value property and is displayed in Section A.4. The stated bounds are not optimal: finer estimates are given in the proof, but this simplified statement is good enough for our purposes.

Lemma A.3 (Periodization error). *Let $z \in Q_L$, let (ζ, P) satisfy (A.2) at z, and let $Y \subset Q_L$ be a finite subset such that $\{z\} \cup Y$ satisfies (A.1). Then, we have for all $x \in Q_L$*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2\Big)^{\frac12} \lesssim_{\sharp Y} \Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2\Big)^{\frac12} \times \Big(\mathbb{1}_{|x-z|>\frac{L}{4}}\big\langle (x-z)_L\big\rangle^{-d} + \mathbb{1}_{|x-z|\le\frac{L}{4}}\big\langle \mathrm{dist}(Y\setminus\{x,z\},\partial Q_L)\big\rangle^{-d}\Big), \quad \text{(A.9)}$$

where we recall the notation $\mathrm{dist}(\varnothing,\partial Q_L) = L$ *and* $B^L_r(z) = (B_r(z) + L\mathbb{Z}^d)\cap Q_L$. *In addition,*

$$\Big(\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2\Big)^{\frac12} \lesssim_{\sharp Y} \Big(\big\langle \mathrm{dist}(x,\partial Q_L)\big\rangle + \big\langle \mathrm{dist}\big(Y\setminus\{x\},\partial Q_L\big)\big\rangle\Big)^{-d}. \quad \text{(A.10)}$$

In the above three lemmas, the multiplicative constants in the estimates crucially depend on the finite number of rigid particles: in Lemma A.1, for instance, a quick inspection of the proof shows that the multiplicative constant can be bounded by $C^{\sharp Y}(\sharp Y)!^{3/2}$. Although these deterministic results fail in general for an unbounded number of rigid inclusions, we refer the reader to [19] where corresponding results are proved to hold in a suitable annealed sense in case of a stationary and ergodic random ensemble of rigid inclusions.

A.2 Decay of Stokeslets with rigid inclusions

This section is devoted to the proof of Lemma A.1 (hence of Lemmas 3.10 and 4.7). We argue by comparing $\mathcal{J}^z_{L;Y}\zeta$ to $\mathcal{J}^z_{L;Y_z}\zeta$ (recall $Y_z = \{y \in Y : B(y)\cap B^L(z) \ne \varnothing\}$), which is a variant of the solution $\mathcal{J}^z_L\zeta$ of the corresponding problem without rigid inclusions. Equation (A.3) for $\mathcal{J}^z_{L;Y_z}\zeta$ reads

$$\begin{cases} -\triangle \mathcal{J}^z_{L;Y_z}\zeta + \nabla \mathcal{Q}^z_{L;Y_z}\zeta = -\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L\setminus\bigcup_{y\in Y_z}B(y),\\ \mathrm{div}(\mathcal{J}^z_{L;Y_z}\zeta) = 0, & \text{in } Q_L\setminus\bigcup_{y\in Y_z}B(y),\\ \mathcal{J}^z_{L;Y_z}\zeta = V_z + \Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z}B(y)\\ & \text{for some } V_z\in\mathbb{R}^d, \Theta_z\in\mathbb{M}^{\mathrm{skew}},\\ \sum_{y\in Y_z}\int_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu & \\ = \sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\sigma(\zeta,P)\nu, & \\ \sum_{y\in Y_z}\int_{\partial B(y)}\Theta(x-z)\cdot\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu & \\ = \sum_{y\in Y_z}\int_{B(y)\cap\partial B^L(z)}\Theta(x-z)\cdot\sigma(\zeta,P)\nu, & \forall\Theta\in\mathbb{M}^{\mathrm{skew}}. \end{cases} \quad \text{(A.11)}$$

We split the proof into three steps: we first apply elliptic regularity to unravel the decay properties of $\mathcal{J}^z_{L;Y_z}\zeta$, and then estimate the difference $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta$ in the last two steps. Let $z \in Q_L$, let ζ satisfy (A.2) at z, and let $Y \subset Q_L$ satisfy (A.1).

Step 1. Proof that for all $x \in Q_L$,

$$\Big(\int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\big|^2\Big)^{\frac{1}{2}} \lesssim \langle (x-z)_L\rangle^{-d}\Big(\int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2\Big)^{\frac{1}{2}}. \tag{A.12}$$

The argument is based on elliptic regularity via a duality argument, in a form that is similar to the proof of Theorem 3.9 in Section 3.5.2. By an energy estimate for $\mathcal{J}^z_{L;Y_z}\zeta$, the claim (A.12) is trivial if

$$\big|(x-z)_L\big| \lesssim 1,$$

and we shall focus on the case when

$$r := \frac{1}{2}\big|(x-z)_L\big| > 2(1+\rho). \tag{A.13}$$

By definition (A.11), we then note that $\mathcal{J}^z_{L;Y_z}\zeta$ satisfies the free steady Stokes equation in $B^L_r(x) = (B_r(x) + L\mathbb{Z}^d) \cap Q_L$, which is the periodization of the ball $B_r(x)$ in Q_L. Elliptic regularity in form of Lemma 2.6 then yields

$$\int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\big|^2 \lesssim r^{-d}\int_{B^L_r(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\big|^2. \tag{A.14}$$

Next, by duality, the right-hand side can be written as

$$\int_{B^L_r(x)} \big|\mathrm{D}(\mathcal{J}^n_{L;Y_z}\zeta)\big|^2 = \sup\bigg\{\Big(\int_{Q_L} h : \mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\Big)^2 : h \in \mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}},\ \|h\|_{\mathrm{L}^2(Q_L)} = 1,\ \operatorname{supp} h \subset B^L_r(x)\bigg\}. \tag{A.15}$$

Given a test function $h \in \mathrm{L}^2(Q_L)^{d\times d}_{\mathrm{sym}}$ with $\operatorname{supp} h \subset B^L_r(x)$, let $w_{L;h} \in H^1_{\mathrm{per}}(Q_L)^d$ be the solution of the auxiliary Stokes problem

$$\begin{cases} -\triangle w_{L;h} + \nabla Q_{L;h} = \operatorname{div}(h), & \text{in } Q_L \setminus \bigcup_{y\in Y_z} B(y),\\ \operatorname{div}(w_{L;h}) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y_z} B(y),\\ w_{L;h} = V_z + \Theta_z(x-z), & \text{in } \bigcup_{y\in Y_z} B(y),\\ & \text{for some } V_z \in \mathbb{R}^d, \Theta_z \in \mathbb{M}^{\mathrm{skew}},\\ \sum_{y\in Y_z}\int_{\partial B(y)} \sigma(w_{L;h}, Q_{L;h})\nu = 0, & \\ \sum_{y\in Y_z}\int_{\partial B(y)} \Theta(x-z)\cdot\sigma(w_{L;h}, Q_{L;h})\nu = 0, & \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{A.16}$$

These equations are indeed well posed since by (A.13) the support $B_r^L(x)$ of the force term h does not intersect the rigid inclusions $\bigcup_{y\in Y_z} B(y)$. By Lemma 3.3, $w_{L;h}$ satisfies the following relation in Q_L,

$$-\triangle w_{L;h}+\nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}Q_{L;h}\big)=\operatorname{div}(h)-\sum_{y\in Y_z}\delta_{\partial B(y)}\sigma(w_{L;h},Q_{L;h})\nu. \quad \text{(A.17)}$$

Similarly, the defining equation (A.11) for $\mathcal{J}^z_{L;Y_z}\zeta$ yields in Q_L,

$$\begin{aligned}&-\triangle \mathcal{J}^z_{L;Y_z}\zeta+\nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\mathcal{Q}^z_{L;Y_z}\big)\\&=-\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z} B(y)}\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu-\sum_{y\in Y_z}\delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu. \end{aligned}\quad \text{(A.18)}$$

Testing (A.17) with $\mathcal{J}^z_{L;Y_z}\zeta$ and (A.18) with $w_{L;h}$, we are led to

$$\begin{aligned}\int_{Q_L} h:\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)=&\int_{\partial B^L(z)\setminus\bigcup_{y\in Y_z} B(y)} w_{L;h}\cdot\sigma(\zeta,P)\nu\\&+\sum_{y\in Y_z}\int_{\partial B(y)} w_{L;h}\cdot\sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu\\&+\sum_{y\in Y_z}\int_{\partial B(y)} \mathcal{J}^z_{L;Y_z}\zeta\cdot\sigma(w_{L;h},Q_{L;h})\nu,\end{aligned}$$

and thus, using the boundary conditions in (A.11) and (A.16),

$$\int_{Q_L} h:\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)=\int_{\partial B^L(z)} w_{L;h}\cdot\sigma(\zeta,P)\nu.$$

Recalling that (ζ,P) satisfies (A.2) and is implicitly extended by Q_L-periodicity, using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\Big(\int_{Q_L} h:\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\Big)^2\lesssim\Big(\int_{B^L(z)}|\mathrm{D}(w_{L;h})|^2\Big)\Big(\int_{B_{1+\rho}(z)}|\mathrm{D}(\zeta)|^2\Big). \quad \text{(A.19)}$$

As equation (A.16) entails that $w_{L;h}$ satisfies the free steady Stokes equation in $B_r^L(z)$, elliptic regularity in form of Lemma 2.6 yields

$$\int_{B^L(z)}|\mathrm{D}(w_{L;h})|^2\lesssim r^{-d}\int_{Q_L}|\mathrm{D}(w_{L;h})|^2,$$

and thus, combining this with an energy estimate for (A.16),

$$\int_{B^L(z)}|\mathrm{D}(w_{L;h})|^2\lesssim r^{-d}\int_{Q_L}|h|^2.$$

Combining this with (A.14), (A.15), and (A.19), the claim (A.12) follows.

Step 2. Proof that for all $x \in Q_L$,

$$\int_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)\right|^2 \lesssim \Big(\sum_{y\in\{x\}\cup(Y\setminus Y_z)} \langle (y-z)_L\rangle^{-2d} \Big) \int_{B_{1+\rho}(z)} \left|\mathrm{D}(\zeta)\right|^2. \qquad \text{(A.20)}$$

In view of Lemma 3.3, the defining equation (A.3) for $\mathcal{J}^z_{L;Y}\zeta$ yields in Q_L,

$$-\triangle \mathcal{J}^z_{L;Y}\zeta + \nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y}B(y)}\mathcal{Q}^z_{L;Y}\big) = -\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z}B(y)}\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu - \sum_{y\in Y}\delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu.$$

Subtracting (A.18) entails in Q_L

$$\begin{aligned}
&-\triangle(\mathcal{J}^z_{L;Y}\zeta-\mathcal{J}^z_{L;Y_z}\zeta) + \nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y}B(y)}\mathcal{Q}^z_{L;Y} - \mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z}B(y)}\mathcal{Q}^z_{L;Y_z}\big)\\
&\quad = -\sum_{y\in Y\setminus Y_z}\delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu\\
&\qquad -\sum_{y\in Y_z}\delta_{\partial B(y)}\big(\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu - \sigma(\mathcal{J}^z_{L;Y_z}\zeta,\mathcal{Q}^z_{L;Y_z}\zeta)\nu\big). \qquad \text{(A.21)}
\end{aligned}$$

Testing this equation with $\mathcal{J}^z_{L;Y}\zeta-\mathcal{J}^z_{L;Y_z}\zeta$ itself, and using the boundary conditions in (A.3) and (A.11), we obtain the energy identity

$$2\int_{Q_L}\left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta-\mathcal{J}^z_{L;Y_z}\zeta)\right|^2 = \sum_{y\in Y\setminus Y_z}\int_{\partial B(y)}\mathcal{J}^z_{L;Y_z}\zeta\cdot\sigma(\mathcal{J}^z_{L;Y}\zeta,\mathcal{Q}^z_{L;Y}\zeta)\nu.$$

Further, using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we deduce

$$\int_{Q_L}\left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta-\mathcal{J}^z_{L;Y_z}\zeta)\right|^2 \lesssim \sum_{y\in Y\setminus Y_z}\Big(\int_{B(y)}\left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2\Big)^{\frac12}\Big(\int_{B_{1+\rho}(y)}\left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)\right|^2\Big)^{\frac12}.$$

Decomposing $\mathcal{J}^z_{L;Y}\zeta = (\mathcal{J}^z_{L;Y}\zeta-\mathcal{J}^z_{L;Y_z}\zeta)+\mathcal{J}^z_{L;Y_z}\zeta$ in the last factor, using the triangle inequality and Young's inequality, we are led to

$$\int_{Q_L}\left|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta-\mathcal{J}^z_{L;Y_z}\zeta)\right|^2 \lesssim \sum_{y\in Y\setminus Y_z}\int_{B_{1+\rho}(y)}\left|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\right|^2.$$

The triangle inequality then yields for all $x \in Q_L$,

$$\int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta)\big|^2 \lesssim \int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\big|^2 + \int_{Q_L} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta)\big|^2$$
$$\lesssim \int_{B^L(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\big|^2 + \sum_{y\in Y\setminus Y_z} \int_{B_{1+\rho}(y)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y_z}\zeta)\big|^2,$$

which yields the claim (A.20) in combination with (A.12).

Step 3. Conclusion. We argue by induction on the cardinality of $Y \setminus Y_z$ for (A.6). If $\sharp(Y \setminus Y_z) = 0$, that is, if $Y = Y_z$, the conclusion (A.6) already follows from (A.12). Given $n \geq 1$, we assume that (A.6) holds whenever $\sharp(Y \setminus Y_z) < n$, and we shall show that it also holds when $\sharp(Y \setminus Y_z) = n$. Let $Y \subset Q_L$ be fixed with $\sharp(Y \setminus Y_z) = n$. For any $S \subset Y \setminus Y_z$, the same argument as for (A.21) yields in $Q_L \setminus \bigcup_{y\in S} B(y)$

$$-\triangle(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta) + \nabla\big(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{L;Y} - \mathbb{1}_{Q_L\setminus\bigcup_{y\in Y_z\cup S} B(y)}\mathcal{Q}^z_{L;Y_z\cup S}\big)$$
$$= -\sum_{y\in Y\setminus(Y_z\cup S)} \delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu$$
$$- \sum_{y\in Y_z} \delta_{\partial B(y)}\sigma\big(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta, \mathcal{Q}^z_{L;Y}\zeta - \mathcal{Q}^z_{L;Y_z\cup S}\zeta\big)\nu.$$

As $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta$ is further rigid in $\bigcup_{y\in S} B(y)$, this implies, by definition of $\{\mathcal{J}^y_{L;S}\}_y$,

$$\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta = \sum_{y\in Y\setminus(Y_z\cup S)} \mathcal{J}^y_{L;S}\mathcal{J}^z_{L;Y}\zeta + \sum_{y\in Y_z} \mathcal{J}^y_{L;S}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta),$$

which we may further decompose as

$$\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta = \sum_{y\in Y\setminus(Y_z\cup S)} \mathcal{J}^y_{L;S}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S\cup\{y\}}\zeta)$$
$$+ \sum_{y\in Y\setminus(Y_z\cup S)} \mathcal{J}^y_{L;S}\mathcal{J}^z_{L;Y_z\cup S\cup\{y\}}\zeta + \sum_{y\in Y_z} \mathcal{J}^y_{L;S}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup S}\zeta).$$

Iterating this identity, we find

$$\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z}\zeta$$
$$= \sum_{l=1}^{n} \sum_{y_1,\ldots,y_l\in Y\setminus Y_z}^{\neq} \mathcal{J}^{y_1}_{L}\mathcal{J}^{y_2}_{L;\{y_1\}}\cdots\mathcal{J}^{y_l}_{L;\{y_1,\ldots,y_{l-1}\}}\mathcal{J}^z_{L;Y_z\cup\{y_1,\ldots,y_l\}}\zeta$$
$$+ \sum_{l=1}^{n} \sum_{y_1,\ldots,y_{l-1}\in Y\setminus Y_z}^{\neq} \sum_{y\in Y_z} \mathcal{J}^{y_1}_{L}\mathcal{J}^{y_2}_{L;\{y_1\}}\cdots\mathcal{J}^{y_{l-1}}_{L;\{y_1,\ldots,y_{l-2}\}}\mathcal{J}^{y}_{L;\{y_1,\ldots,y_{l-1}\}}$$
$$\times \big(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{L;Y_z\cup\{y_1,\ldots,y_{l-1}\}}\zeta\big).$$

We now appeal to the induction hypothesis in form of (A.6) for the terms $\mathcal{J}^{y}_{L;\{y_1,\ldots,y_j\}}$ and $\mathcal{J}^{z}_{L;Y_z\cup\{y_1,\ldots,y_j\}}$ for all $1 \le j < n$ and $y \in Y$, to the suboptimal decay estimate (A.20) for $\mathcal{J}^{z}_{L;Y_z\cup\{y_1,\ldots,y_n\}}$ (which only appears in the first right-hand sum when $l = n$). Recalling that

$$\left|(y-z)_L\right| \le 2 \quad \text{for all } y \in Y_z,$$

this yields for all $x \in Q_L$, after straightforward simplifications,

$$\begin{aligned}\left(\fint_{B^L(x)} \left|\mathrm{D}(\mathcal{J}^{z}_{L;Y}\zeta - \mathcal{J}^{z}_{L;Y_z}\zeta)\right|^2\right)^{\frac12} &\lesssim_n \left(\fint_{B_{1+\rho}(z)} \left|\mathrm{D}(\zeta)\right|^2\right)^{\frac12} \\ &\quad\times \sum_{l=0}^{n} \sum_{y_1,\ldots,y_l\in Y\setminus Y_z}^{\neq} \left\langle (x-y_1)_L\right\rangle^{-d}\left\langle (y_1-y_2)_L\right\rangle^{-d}\cdots\left\langle (y_l-z)_L\right\rangle^{-d}.\end{aligned}$$

The conclusion (A.6) now follows from the bound

$$\left\langle (a-b)_L\right\rangle^{-d}\left\langle (b-c)_L\right\rangle^{-d} \lesssim \left\langle (a-c)_L\right\rangle^{-d}$$

for all $a, b, c \in Q_L$. ∎

A.3 Mean-value property with rigid inclusions

This section is devoted to the proof of Lemma A.2. We split the proof into two steps. Let $Y \subset Q_L$ satisfy (A.1) and let $(w, P) \in H^1(Q_L)^d \times \mathrm{L}^2(Q_L)$ satisfy (A.7) in Q_L.

Step 1. Proof that for all $x \in Q_L$,

$$\fint_{B^L(x)} \left|\mathrm{D}(w)\right|^2 \lesssim_{\sharp Y} \Big(\sum_{y\in\{x\}\cup Y} \left\langle \mathrm{dist}(y, \partial Q_L)\right\rangle^{-d}\Big) \fint_{Q_L} \left|\mathrm{D}(w)\right|^2. \tag{A.22}$$

For that purpose, we shall compare w to the solution

$$\tilde{w} \in w + H^1_{\mathrm{per}}(Q_L)^d$$

of the free steady Stokes equations without rigid particles in Q_L,

$$-\triangle\tilde{w} + \nabla\widetilde{P} = 0, \quad \mathrm{div}(\tilde{w}) = 0, \quad \text{in } Q_L. \tag{A.23}$$

In view of Lemma 3.3, the equations (A.7) for w yield the following relation in Q_L,

$$-\triangle w + \nabla(\mathbb{1}_{Q_L\setminus\bigcup_{y\in Y} B(y)}P) = -\sum_{y\in Y} \delta_{\partial B(y)}\sigma(w, P)\nu.$$

Subtracting (A.23), we deduce that the difference $w - \tilde{w} \in H^1_{\mathrm{per}}(Q_L)$ satisfies

$$-\triangle(w - \tilde{w}) + \nabla\big(\mathbb{1}_{Q_L \setminus \bigcup_{y\in Y} B(y)} P - \widetilde{P}\big) = -\sum_{y\in Y} \delta_{\partial B(y)} \sigma(w, P)\nu. \qquad \text{(A.24)}$$

Testing this equation with $w - \tilde{w}$ and using the boundary conditions in (A.7), we obtain the energy identity

$$2\int_{Q_L} \big|\mathrm{D}(w - \tilde{w})\big|^2 = \sum_{y\in Y} \int_{\partial B(y)} \tilde{w} \cdot \sigma(w, \mathcal{Q})\nu.$$

Further, using the boundary conditions and the incompressibility constraints to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we get

$$\int_{Q_L} \big|\mathrm{D}(w - \tilde{w})\big|^2 \lesssim \sum_{y\in Y} \bigg(\int_{B(y)} \big|\mathrm{D}(\tilde{w})\big|^2\bigg)^{\frac{1}{2}} \bigg(\int_{B_{1+\rho}(y)} \big|\mathrm{D}(w)\big|^2\bigg)^{\frac{1}{2}}. \qquad \text{(A.25)}$$

Decomposing

$$w = (w - \tilde{w}) + \tilde{w}$$

in the last factor, using the triangle inequality and Young's inequality, we are led to

$$\int_{Q_L} \big|\mathrm{D}(w - \tilde{w})\big|^2 \lesssim \sum_{y\in Y} \int_{B_{1+\rho}(y)} \big|\mathrm{D}(\tilde{w})\big|^2.$$

and thus, by the triangle inequality, for all $x \in Q_L$,

$$\int_{B^L(x)} \big|\mathrm{D}(w)\big|^2 \lesssim \int_{B^L(x)} \big|\mathrm{D}(\tilde{w})\big|^2 + \sum_{y\in Y} \int_{B_{1+\rho}(y)} \big|\mathrm{D}(\tilde{w})\big|^2. \qquad \text{(A.26)}$$

Rather decomposing $\tilde{w} = w - (w - \tilde{w})$, we note that (A.25) also yields the energy estimate

$$\int_{Q_L} \big|\mathrm{D}(\tilde{w})\big|^2 \lesssim \int_{Q_L} \big|\mathrm{D}(w)\big|^2. \qquad \text{(A.27)}$$

As $\tilde{w}$ satisfies the free steady Stokes equations in Q_L, cf. (A.23), the mean-value property of Lemma 2.6 yields for all $x \in Q_L$,

$$\int_{B^L(x)} \big|\mathrm{D}(\tilde{w})\big|^2 \lesssim \big\langle \mathrm{dist}(x, \partial Q_L)\big\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(\tilde{w})\big|^2,$$

and thus, combined with (A.27),

$$\int_{B^L(x)} \big|\mathrm{D}(\tilde{w})\big|^2 \lesssim \big\langle \mathrm{dist}(x, \partial Q_L)\big\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(w)\big|^2. \qquad \text{(A.28)}$$

Inserting this into (A.26), the claim (A.22) follows.

Step 2. Conclusion. Given $S \subset Y$, we denote by

$$w_S \in w + H^1_{\mathrm{per}}(Q_L)^d$$

the solution of the free steady Stokes problem with rigid inclusions at points of S only,

$$\begin{cases} -\triangle w_S + \nabla P_S = 0, & \text{in } Q_L \setminus \bigcup_{y\in S} B(y), \\ \operatorname{div}(w_S) = 0, & \text{in } Q_L \setminus \bigcup_{y\in S} B(y), \\ \mathrm{D}(w_S) = 0, & \text{in } \bigcup_{y\in S} B(y), \\ \int_{\partial B(y)} \sigma(w_S, P_S)\nu = 0, & \forall y \in S, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(w_S, P_S)\nu = 0, & \forall y \in S,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

In particular, we recover $w_Y = w$ and $w_\varnothing = \tilde{w}$ as defined in (A.23). The result (A.22) of Step 1 yields in this case, for all $x \in Q_L$,

$$\int_{B^L(x)} |\mathrm{D}(w_S)|^2 \lesssim_{\sharp S} \Big(\sum_{y\in\{x\}\cup S} \langle \operatorname{dist}(y, \partial Q_L)\rangle^{-d} \Big) \int_{Q_L} |\mathrm{D}(w_S)|^2.$$

Noting that a similar argument to the case of (A.27) further yields the energy estimate

$$\int_{Q_L} |\mathrm{D}(w_S)|^2 \lesssim \int_{Q_L} |\mathrm{D}(w)|^2,$$

we deduce for all $x \in Q_L$,

$$\int_{B^L(x)} |\mathrm{D}(w_S)|^2 \lesssim_{\sharp S} \Big(\sum_{y\in\{x\}\cup S} \langle \operatorname{dist}(y, \partial Q_L)\rangle^{-d} \Big) \int_{Q_L} |\mathrm{D}(w)|^2. \tag{A.29}$$

We shall now decompose w in terms of this sequence $(w_S)_{S\subset Y}$. Arguing as for (A.24), we note that for any $S \subset Y$ the following relation holds in $Q_L \setminus \bigcup_{y\in S} B(y)$,

$$-\triangle(w - w_S) + \nabla(P - P_S) = -\sum_{y\in Y\setminus S} \delta_{\partial B(y)}\sigma(w, P)\nu.$$

As $w - w_S$ is rigid in $\bigcup_{y\in S} B(y)$, this allows us to decompose

$$w - w_S = \sum_{y\in Y\setminus S} \mathcal{J}^y_{L;S} w,$$

and thus, iterating this identity and starting with $w_\varnothing = \tilde{w}$,

$$w = \tilde{w} + \sum_{l=1}^{\sharp Y} \sum_{y_1,\ldots,y_l\in Y}^{\neq} \mathcal{J}^{y_1}_L \mathcal{J}^{y_2}_{L;\{y_1\}} \cdots \mathcal{J}^{y_l}_{L;\{y_1,\ldots,y_{l-1}\}} w_{\{y_1,\ldots,y_l\}}.$$

Appealing to the decay estimates for $\{\mathcal{J}_{L;S}^{y}\}_{y,S}$ in Lemma A.1, and to (A.28) and (A.29), we get after straightforward simplifications, for all $x \in Q_L$,

$$\int_{B^L(x)} |\mathrm{D}(w)|^2 \lesssim_{\sharp Y} \int_{Q_L} |\mathrm{D}(w)|^2 \\ \times \sum_{l=0}^{\sharp Y} \sum_{y_1,\ldots,y_l \in Y}^{\neq} \langle (x-y_1)_L \rangle^{-2d} \langle (y_1-y_2)_L \rangle^{-2d} \cdots \langle (y_{l-1}-y_l)_L \rangle^{-2d} \langle \mathrm{dist}(y_l, \partial Q_L) \rangle^{-d}.$$

Using that $\langle (a-b)_L \rangle^{-d} \langle (b-c)_L \rangle^{-d} \lesssim \langle (a-c)_L \rangle^{-d}$ for all $a, b, c \in Q_L$, and noting that the infimum over $c \in \partial Q_L$ further yields

$$\langle (a-b)_L \rangle^{-d} \langle \mathrm{dist}(b, \partial Q_L) \rangle^{-d} \lesssim \langle \mathrm{dist}(a, \partial Q_L) \rangle^{-d},$$

the conclusion (A.8) follows. ∎

A.4 Periodization errors

This section is devoted to the proof of Lemma A.3. We split the proof into three steps. Let $z \in Q_L$, let ζ satisfy (A.2) at z, and let $Y \subset Q_L$ be such that $\{z\} \cup Y$ satisfies (A.1).

Step 1. Proof that for all $x \in Q_L$,

$$\int_{B_{1+\rho}^L(x)} |\mathrm{D}(\mathcal{J}_{L;Y}^{z}\zeta - \mathcal{J}_{Y}^{z}\zeta)|^2 \lesssim_{\sharp Y} \int_{B_{1+\rho}(z)} |\mathrm{D}(\zeta)|^2 \\ \times \min \Big\{ \langle (x-z)_L \rangle^{-2d} \wedge \big(\langle \mathrm{dist}(x, \partial Q_L(a)) \rangle^{-d} \langle \mathrm{dist}(z, \partial Q_L(a)) \rangle^{-d} \big) : \\ a \in \mathbb{R}^d,\ x, z \in Q_L(a),\ Y \subset Q_L(a) \Big\}. \qquad \text{(A.30)}$$

It suffices to prove this estimate for $a = 0$, that is,

$$\int_{B_{1+\rho}^L(x)} |\mathrm{D}(\mathcal{J}_{L;Y}^{z}\zeta - \mathcal{J}_{Y}^{z}\zeta)|^2 \lesssim_{\sharp Y} \int_{B_{1+\rho}(z)} |\mathrm{D}(\zeta)|^2 \\ \times \big(\langle (x-z)_L \rangle^{-2d} \wedge \big(\langle \mathrm{dist}(x, \partial Q_L) \rangle^{-d} \langle \mathrm{dist}(z, \partial Q_L) \rangle^{-d} \big) \big),$$

as the claim (A.30) then follows by translating the underlying cell Q_L, which does indeed not change the equations provided that the translated cell still contains the relevant points x, z, Y. Further, noting that Lemma A.1 together with the triangle inequality yields

$$\int_{B_{1+\rho}^L(x)} |\mathrm{D}(\mathcal{J}_{L;Y}^{z}\zeta - \mathcal{J}_{Y}^{z}\zeta)|^2 \lesssim_{\sharp Y} \langle (x-z)_L \rangle^{-2d} \int_{B_{1+\rho}(z)} |\mathrm{D}(\zeta)|^2,$$

it only remains to prove for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \langle \mathrm{dist}(x,\partial Q_L)\rangle^{-d} \langle \mathrm{dist}(z,\partial Q_L)\rangle^{-d} \int_{B_{1+\rho}(z)} \big|\mathrm{D}(\zeta)\big|^2. \tag{A.31}$$

As $\{z\} \cup Y$ satisfies (A.1), we recall that $\mathcal{J}^z_{L;Y}\zeta$ satisfies the simpler Stokes problem (A.4) (and likewise for $\mathcal{J}^z_Y\zeta$). The difference $\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta$ then satisfies the free steady Stokes equations (A.7). Applying the mean-value property of Lemma A.2 to this equation, we get for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \lesssim_{\sharp Y} \langle \mathrm{dist}(x,\partial Q_L)\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2. \tag{A.32}$$

In order to estimate the last integral, taking some inspiration from the proof of (2.33), we note that it is convenient to further compare $\mathcal{J}^z_{L;Y}\zeta$ and $\mathcal{J}^z_Y\zeta$ to the solution of the corresponding Neumann problem in Q_L: we define

$$\mathcal{J}^z_{N;Y}\zeta \in H^1(Q_L)^d$$

as the solution of

$$\begin{cases} -\triangle \mathcal{J}^z_{N;Y}\zeta + \nabla \mathcal{Q}^z_{N;Y}\zeta = -\delta_{\partial B^L(z)}\sigma(\zeta,P)\nu, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y),\\ \mathrm{div}(\mathcal{J}^z_{N;Y}\zeta) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y),\\ \sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu = 0, & \text{on } \partial Q_L,\\ \mathrm{D}(\mathcal{J}^z_{N;Y}\zeta) = 0, & \text{in } \bigcup_{y\in Y} B(y),\\ \int_{\partial B(y)} \sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu = 0, & \forall y \in Y,\\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu = 0, & \forall y \in Y,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases} \tag{A.33}$$

In these terms, we start by estimating

$$\int_{Q_L} \big|\mathrm{D}(\mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_Y\zeta)\big|^2 \le 2\int_{Q_L} \big|\mathrm{D}(H_1)\big|^2 + 2\int_{Q_L} \big|\mathrm{D}(H_2)\big|^2, \tag{A.34}$$

where we have set for abbreviation

$$H_1 := \mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{N;Y}\zeta,$$
$$H_2 := \mathcal{J}^z_Y\zeta - \mathcal{J}^z_{N;Y}\zeta.$$

We denote by P_1, P_2 the corresponding pressure differences. In view of (A.4) and (A.33), (H_1, P_1) satisfies

$$\begin{cases} -\triangle H_1 + \nabla P_1 = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(H_1) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \sigma(H_1, P_1)\nu = \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu, & \text{on } \partial Q_L, \\ \mathrm{D}(H_1) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(H_1, P_1)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(H_1, P_1)\nu = 0, & \forall y \in Y,\ \forall \Theta \in \mathbb{M}^{\text{skew}}, \end{cases} \tag{A.35}$$

for which the energy identity takes the form

$$2\int_{Q_L} \left|\mathrm{D}(H_1)\right|^2 = \int_{\partial Q_L} H_1 \cdot \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu,$$

hence, recalling $H_1 = \mathcal{J}^z_{L;Y}\zeta - \mathcal{J}^z_{N;Y}\zeta$ and the periodicity of $\mathcal{J}^z_{L;Y}\zeta$,

$$2\int_{Q_L} \left|\mathrm{D}(H_1)\right|^2 = -\int_{\partial Q_L} \mathcal{J}^z_{N;Y}\zeta \cdot \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu. \tag{A.36}$$

By Lemma 3.3 and (A.11), $\mathcal{J}^z_{L;Y}\zeta$ satisfies in Q_L

$$\begin{aligned} &-\triangle \mathcal{J}^z_{L;Y}\zeta + \nabla\big(\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{L;Y}\big) \\ &\qquad = -\delta_{\partial B(z)}\,\sigma(\zeta, P)\nu - \sum_{y\in Y} \delta_{\partial B(y)}\sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu, \end{aligned}$$

whereas, by (A.33), $\mathcal{J}^z_{N;Y}\zeta$ satisfies

$$\begin{aligned} &-\triangle \mathcal{J}^z_{N;Y}\zeta + \nabla\big(\mathbb{1}_{\mathbb{R}^d\setminus\bigcup_{y\in Y} B(y)}\mathcal{Q}^z_{N;Y}\big) \\ &\qquad = -\delta_{\partial B(z)}\sigma(\zeta, P)\nu - \sum_{y\in Y} \delta_{\partial B(y)}\sigma(\mathcal{J}^z_{N;Y}\zeta, \mathcal{Q}^z_{N;Y}\zeta)\nu. \end{aligned}$$

Testing the first relation with $\mathcal{J}^z_{N;Y}\zeta$, testing the second one with $\mathcal{J}^z_{L;Y}\zeta$, and using boundary conditions, we find

$$\begin{aligned} &\int_{\partial Q_L} \mathcal{J}^z_{N;Y}\zeta \cdot \sigma(\mathcal{J}^z_{L;Y}\zeta, \mathcal{Q}^z_{L;Y}\zeta)\nu \\ &\quad = 2\int \mathrm{D}(\mathcal{J}^z_{N;Y}\zeta) : \mathrm{D}(\mathcal{J}^z_{L;Y}\zeta) + \int_{\partial B(z)} \mathcal{J}^z_{N;Y}\zeta \cdot \sigma(\zeta, P)\nu \\ &\quad = \int_{\partial B(z)} (\mathcal{J}^z_{N;Y}\zeta - \mathcal{J}^z_{L;Y}\zeta) \cdot \sigma(\zeta, P)\nu, \end{aligned}$$

so that identity (A.36) becomes

$$2\int_{Q_L}\left|\mathrm{D}(H_1)\right|^2=\int_{\partial B(z)}H_1\cdot\sigma(\zeta,P)\nu. \tag{A.37}$$

Using the boundary conditions and the incompressibility constraint to smuggle in arbitrary constants in the different factors, as in the proof of (3.30), and appealing to the trace estimates of Lemma 2.5, we find

$$\int_{Q_L}\left|\mathrm{D}(H_1)\right|^2\lesssim\Big(\int_{B(z)}\left|\mathrm{D}(H_1)\right|^2\Big)^{\frac12}\Big(\int_{B_{1+\rho}(z)}\left|\mathrm{D}(\zeta)\right|^2\Big)^{\frac12}.$$

Applying the mean-value property of Lemma A.2 to equation (A.35) for H_1, and using Young's inequality, we deduce

$$\int_{Q_L}\left|\mathrm{D}(H_1)\right|^2\lesssim\langle\mathrm{dist}(z,\partial Q_L)\rangle^{-d}\int_{B_{1+\rho}(z)}\left|\mathrm{D}(\zeta)\right|^2. \tag{A.38}$$

Likewise, repeating the argument in favor of (A.37), this time for H_2, we obtain

$$2\int_{Q_L}\left|\mathrm{D}(H_2)\right|^2=\int_{\partial B(z)}H_2\cdot\sigma(\zeta,P)\nu+\int_{\partial Q_L}\mathcal{J}_Y^z\zeta\cdot\sigma(\mathcal{J}_Y^z\zeta,\mathcal{Q}_Y^z\zeta)\nu,$$

or equivalently, using the free steady Stokes equations for $\mathcal{J}_Y^z\zeta$ in $\mathbb{R}^d\setminus Q_L$ and integrating by parts to reformulate the second right-hand side term,

$$\begin{aligned}2\int_{Q_L}\left|\mathrm{D}(H_2)\right|^2&=\int_{\partial B(z)}H_2\cdot\sigma(\zeta,P)\nu-2\int_{\mathbb{R}^d\setminus Q_L}\left|\mathrm{D}(\mathcal{J}_Y^z\zeta)\right|^2\\&\le\int_{\partial B(z)}H_2\cdot\sigma(\zeta,P)\nu.\end{aligned}$$

Arguing as for H_1, we may then deduce

$$\int_{Q_L}\left|\mathrm{D}(H_2)\right|^2\lesssim\langle\mathrm{dist}(z,\partial Q_L)\rangle^{-d}\int_{B_{1+\rho}(z)}\left|\mathrm{D}(\zeta)\right|^2.$$

Combined with (A.32), (A.34), and (A.38), this yields the claim (A.31).

Step 2. Proof of (A.9). We claim that the conclusion (A.9) is a simple post-processing of (A.30). As (A.9) trivially follows from (A.30) if $|x-z|>\frac{L}{4}$, it remains to consider the case when $|x-z|\le\frac{L}{4}$. In that case, we can choose $q\in\frac{L}{4}\mathbb{Z}^d$ with $|q|_\infty\le\frac{L}{4}$ such that $x,z\in Q_{\frac12 L}(q)$. We then construct a translation vector a componentwise: First, for all directions $1\le i\le d$ with $q_i=0$, we set $a_i:=0$. Second, for all i with $q_i=\frac{L}{4}$, we set $a_i:=\mathrm{dist}(Y\setminus\{x,z\},P_L^{i,-})$, where $P_L^{i,-}$ is the cubic facet $\{v\in\partial Q_L:v_i=-\frac{L}{2}\}$. Third, for all i with $q_i=-\frac{L}{4}$, we set $a_i:=-\mathrm{dist}(Y\setminus\{x,z\},P_L^{i,+})$, where $P_L^{i,+}$ is the facet $\{v\in\partial Q_L:v_i=\frac{L}{2}\}$. With this construction of a, we find that $Y \setminus \{x,z\}$ is included in the translated cube $Q_L(a)$ (and actually intersects its boundary). Moreover, we find

$$\begin{aligned}\operatorname{dist}\big(x,\partial Q_L(a)\big) &\ge \operatorname{dist}(x,\partial Q_L) + \inf_i |a_i|\\ &\ge \operatorname{dist}(x,\partial Q_L) + \operatorname{dist}\big(Y\setminus\{x,z\},\partial Q_L\big),\end{aligned}$$

and similarly

$$\operatorname{dist}\big(z,\partial Q_L(a)\big) \ge \operatorname{dist}(z,\partial Q_L) + \operatorname{dist}\big(Y\setminus\{x,z\},\partial Q_L\big).$$

In particular, we get

$$\big\langle \operatorname{dist}\big(x,\partial Q_L(a)\big)\big\rangle^{-d}\big\langle \operatorname{dist}\big(z,\partial Q_L(a)\big)\big\rangle^{-d} \le \big\langle \operatorname{dist}\big(Y\setminus\{x,z\},\partial Q_L\big)\big\rangle^{-2d},$$

so that the conclusion (A.9) indeed follows from (A.30).

Step 3. Proof of (A.10). We shall prove the following refined version of (A.10): for all $x \in Q_L$,

$$\begin{aligned}\int_{B^L_{1+\rho}(x)} &\big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2\\ &\lesssim \min\Big\{\big\langle \operatorname{dist}\big(x,\partial Q_L(a)\big)\big\rangle^{-d}\big\langle \operatorname{dist}\big(Y,\partial Q_L(a)\big)\big\rangle^{-d} :\\ &\qquad\qquad a\in\mathbb{R}^d,\ x\in Q_L(a),\ Y\subset Q_L(a)\Big\}.\end{aligned} \tag{A.39}$$

Arguing similarly to Step 2, it is easily seen that the translation a can be suitably chosen so that this estimate yields the conclusion (A.10). In order to prove (A.39), it suffices, in fact, to prove it for $a = 0$, that is,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2 \lesssim \big\langle \operatorname{dist}(x,\partial Q_L)\big\rangle^{-d}\big\langle \operatorname{dist}(Y,\partial Q_L)\big\rangle^{-d}, \tag{A.40}$$

as the claim (A.39) then follows by translating the underlying cell Q_L, which does indeed not change the equations provided that the translated cell still contains x, Y.

We turn to the proof of (A.40). As the difference

$$\psi^Y_L - \psi^Y$$

satisfies a free steady Stokes problem of the form (A.7), we may apply the mean-value property of Lemma A.2 to the effect that for all $x \in Q_L$,

$$\int_{B^L_{1+\rho}(x)} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2 \lesssim \big\langle \operatorname{dist}(x,\partial Q_L)\big\rangle^{-d} \int_{Q_L} \big|\mathrm{D}(\psi^Y_L - \psi^Y)\big|^2. \tag{A.41}$$

In order to estimate the last integral, we argue similarly to Step 1 by further comparing ψ^Y_L, ψ^Y to the solution of the corresponding Neumann problem in Q_L: we define

$\psi_N^Y \in H^1(Q_L)^d$ as the solution of

$$\begin{cases} -\triangle\psi_N^Y + \nabla\Sigma_N^Y = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \operatorname{div}(\psi_N^Y) = 0, & \text{in } Q_L \setminus \bigcup_{y\in Y} B(y), \\ \sigma(\psi_N^Y, \Sigma_N^Y)\nu = 0, & \text{on } \partial Q_L, \\ \mathrm{D}(\psi_N^Y + Ex) = 0, & \text{in } \bigcup_{y\in Y} B(y), \\ \int_{\partial B(y)} \sigma(\psi_N^Y, \Sigma_N^Y)\nu = 0, & \forall y \in Y, \\ \int_{\partial B(y)} \Theta(x-y)\cdot\sigma(\psi_N^Y, \Sigma_N^Y)\nu = 0, & \forall y \in Y,\ \forall \Theta \in \mathbb{M}^{\mathrm{skew}}. \end{cases}$$

In these terms, we start by estimating

$$\int_{Q_L} \left|\mathrm{D}(\psi_L^Y - \psi^Y)\right|^2 \le 2\int_{Q_L} \left|\mathrm{D}(G_1)\right|^2 + 2\int_{Q_L} \left|\mathrm{D}(G_2)\right|^2, \tag{A.42}$$

where we have set for abbreviation

$$G_1 := \psi_L^Y - \psi_N^Y, \quad G_2 := \psi^Y - \psi_N^Y.$$

We denote by R_1, R_2 the corresponding pressure differences. Similarly, as in Step 1, energy identities take the form

$$\begin{aligned} 2\int_{Q_L} \left|\mathrm{D}(G_1)\right|^2 &= \sum_{y\in Y}\int_{\partial B(y)} E(x-y)\cdot\sigma(G_1, R_1)\nu, \\ 2\int_{Q_L} \left|\mathrm{D}(G_2)\right|^2 &= \sum_{y\in Y}\int_{\partial B(y)} E(x-y)\cdot\sigma(G_2, R_2)\nu - 2\int_{\mathbb{R}^d\setminus Q_L} \left|\mathrm{D}(\psi^Y)\right|^2, \end{aligned}$$

and we deduce by means of trace estimates, for both $i = 1, 2$,

$$\int_{Q_L} \left|\mathrm{D}(G_i)\right|^2 \lesssim \sum_{y\in Y}\left(\int_{B_{1+\rho}(y)} \left|\mathrm{D}(G_i)\right|^2\right)^{\frac{1}{2}}.$$

Hence, applying the mean-value property of Lemma A.2 to G_1, G_2, together with Young's inequality,

$$\int_{Q_L} \left|\mathrm{D}(G_i)\right|^2 \lesssim_{\sharp Y} \sum_{y\in Y} \langle \operatorname{dist}(y, \partial Q_L)\rangle^{-d}.$$

Combined with (A.41) and (A.42), this yields the claim (A.40), and concludes the proof. ∎