## Appendix B

# Finite-volume approximation of the effective viscosity

This appendix is devoted to the proof of an algebraic convergence rate for the finite-volume approximation $\bar{\mathbf{B}}_L$ of the effective viscosity $\bar{\mathbf{B}}$ under an algebraic $\alpha$-mixing condition, as announced in Remark 4.2.

**Proposition B.1** (Convergence rate for $\bar{\mathbf{B}}_L$). *On top of* ($H_\rho$), *assume that the algebraic mixing condition* (Mix) *holds. Then there exists* $\gamma \in (0, \beta)$ *(only depending on* $d, \rho$ *and on the mixing exponent* $\beta$*) such that for all* $L$,

$$|\bar{\mathbf{B}}_L - \bar{\mathbf{B}}| \lesssim L^{-\gamma}.$$

The proof displayed below closely follows the monograph [3] by Armstrong, Kuusi, and Mourrat (albeit in the more general version [4] for $\alpha$-mixing coefficients), based on the original argument [5] by Armstrong and Smart. We identify a suitable subadditive quantity $J$ that satisfies all the requirements of [3,4] in the present Stokes context: the definition (B.4) and Lemma B.2 below constitute the only new insight w.r.t. [3], and the conclusion follows from elementary adaptations of the arguments in [3,4]. Although we could have used the same subadditive quantity as in [3], we have chosen to use a subadditive quantity $J$ built on the approximations (2.9) and (2.10) that we used to prove Einstein's formula, that is, in the form of (B.3) below. This choice, which is specific to our problem, makes some of the upcoming arguments technically simpler than in [3], in particular avoiding the use of convex duality.

Let $E \in \mathbb{M}_0$ be fixed with $|E| = 1$. We say that a bounded domain $U \subset \mathbb{R}^d$ is *suitable* if

$$\operatorname{dist}(\mathcal{I} \cap U, \partial U) > \rho.$$

Consider the following weakly closed subsets of $H^1(U)^d$,

$$\mathcal{H}(U) := \big\{u \in H^1(U)^d : \operatorname{div}(\phi) = 0, \text{ and } \mathrm{D}(\phi + Ex) = 0 \text{ on } \mathcal{I} \cap U\big\},$$
$$\mathcal{H}_\circ(U) := H^1_0(U)^d \cap \mathcal{H}(U),$$

and the following minimization problems (note that $\psi_*(U)$ is only defined up to a rigid motion),

$$\psi_*(U) := \arg\min\left\{\int_U \big|\mathrm{D}(\phi)\big|^2 : \phi \in \mathcal{H}(U)\right\}, \tag{B.1}$$

$$\psi_\circ(U) := \arg\min\left\{\int_U \big|\mathrm{D}(\phi)\big|^2 : \phi \in \mathcal{H}_0(U)\right\}. \tag{B.2}$$

Recalling that the fattened inclusions $\{I_n + \rho B\}_n$ are disjoint, we define the modified cubes

$$U_L(x) := \Big( Q_L(x) \setminus \bigcup_{n : x_n \notin Q_L(x)} (I_n + \rho B) \Big) \cup \Big( \bigcup_{n : x_n \in Q_L(x)} (I_n + \rho B) \Big),$$

which satisfy by definition $Q_{L-2(1+\rho)} \subset U_L(x) \subset Q_{L+2(1+\rho)}$ and $\mathcal{I} \cap \partial U_L(x) = \varnothing$. The family $\{U_L(x)\}_{x \in L\mathbb{Z}^d}$ constitutes a partition of $\mathbb{R}^d$. Setting $U_L = U_L(0)$, we then consider the following alternative finite-volume approximations of the effective viscosity $\bar{\mathbf{B}}$,

$$E : \widetilde{\mathbf{B}}_{L,*} E = 1 + \mathbb{E}\Big[ \fint_{U_L} \big| \mathrm{D}\big(\psi_*(U_L)\big) \big|^2 \Big],$$
$$E : \widetilde{\mathbf{B}}_{L,\circ} E = 1 + \mathbb{E}\Big[ \fint_{U_L} \big| \mathrm{D}\big(\psi_\circ(U_L)\big) \big|^2 \Big]. \tag{B.3}$$

Since $\mathcal{H}_\circ(U_L) \subset \mathcal{H}(U_L)$, we have $E : \widetilde{\mathbf{B}}_{L,*} E \le E : \widetilde{\mathbf{B}}_{L,\circ} E$. We then define a random set function $J$ for suitable sets $U$ via

$$J(U) := \fint_U \big| \mathrm{D}\big(\psi_\circ(U)\big) \big|^2 - \big| \mathrm{D}\big(\psi_*(U)\big) \big|^2. \tag{B.4}$$

The following lemma collects elementary properties of $J$. In particular, item (iii) states that $U \mapsto |U| J(U)$ is subadditive.

**Lemma B.2** (Properties of $J$). (i) *Recalling the definition* (B.3) *of finite-volume approximations* $\widetilde{\mathbf{B}}_{L,*}, \widetilde{\mathbf{B}}_{L,\circ}$ *of the effective viscosity, there exists* $C > 0$ *such that*

$$E : \widetilde{\mathbf{B}}_{L,*} E - C L^{-1} \le E : \bar{\mathbf{B}} E \le E : \widetilde{\mathbf{B}}_{L,\circ} E + C L^{-1}, \tag{B.5}$$
$$E : \widetilde{\mathbf{B}}_{L,*} E - C L^{-1} \le E : \bar{\mathbf{B}}_{L+2(1+\rho)} E \le E : \widetilde{\mathbf{B}}_{L,\circ} E + C L^{-1}. \tag{B.6}$$

(ii) *For all suitable* $U$,

$$J(U) = \fint_U \big| \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big) \big|^2. \tag{B.7}$$

(iii) *For all disjoint suitable sets* $U^1, \ldots, U^k$, *setting* $U = \operatorname{int}(\bigcup_j \overline{U^j})$,

$$|U| J(U) \le \sum_{j=1}^k |U^j| J(U^j). \tag{B.8}$$

*In addition, setting* $\delta\psi(U) := \psi_\circ(U) - \psi_*(U)$,

$$\sum_{j=1}^k \big\| \mathrm{D}\big(\delta\psi(U) - \delta\psi(U^j)\big) \big\|^2_{\mathrm{L}^2(U^j)} = \sum_{j=1}^k |U^j| \big( J(U^j) - J(U) \big). \tag{B.9}$$

*Proof.* We split the proof into three steps.

*Step* 1. Proof of (i). We start with the proof of (B.6), that is, the comparison of $\widetilde{\mathbf{B}}_{L,*}$, $\widetilde{\mathbf{B}}_{L,\circ}$ with the periodic approximation $\overline{\mathbf{B}}_{L+2(1+\rho)}$. First, we extend $\psi_\circ(U_L)$ by zero on $Q_{L+2(1+\rho)} \setminus U_L^-$, which makes it a $Q_{L+2(1+\rho)}$-periodic function, and thus, testing variational problems,

$$\fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \le \fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2 = \frac{|U_L|}{\big(L+2(1+\rho)\big)^d} \fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2,$$

which yields, in view of $|L^{-d}|U_L|-1| \lesssim L^{-1}$,

$$\fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \le (1+CL^{-1}) \fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2.$$

Second, as the restriction $\psi_{L+2(1+\rho)}|_{U_L}$ belongs to $\mathcal{H}(U_L)$ and is thus an admissible test function in (B.1), we similarly obtain

$$\begin{aligned}\fint_{U_L} \big|\mathrm{D}\big(\psi_*(U_L)\big)\big|^2 &\le \fint_{U_L} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \\ &\le \frac{\big(L+2(1+\rho)\big)^d}{|U_L|} \fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2 \\ &\le (1+CL^{-1}) \fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2.\end{aligned}$$

The claim (B.6) follows from the combination of these two estimates with the following energy bounds, cf. (3.45),

$$\mathbb{E}\bigg[\fint_{Q_{L+2(1+\rho)}} \big|\mathrm{D}(\psi_{L+2(1+\rho)})\big|^2\bigg] + \mathbb{E}\bigg[\fint_{U_L} \big|\mathrm{D}(\psi_\circ(U_L))\big|^2\bigg] + \mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big] \lesssim \lambda(\mathcal{P}). \tag{B.10}$$

We turn to the proof of (B.5). Since the restriction $\psi|_{U_L}$ belongs to $\mathcal{H}(U_L)$ and is thus an admissible test function in (B.1), we find by stationarity of $\mathrm{D}(\psi)$,

$$\begin{aligned}\mathbb{E}\bigg[\fint_{U_L} \big|\mathrm{D}\big(\psi_*(U_L)\big)\big|^2\bigg] \le \mathbb{E}\bigg[\fint_{U_L} \big|\mathrm{D}(\psi)\big|^2\bigg] \le \mathbb{E}\bigg[\frac{L^d}{|U_L|} \fint_{Q_L} \big|\mathrm{D}(\psi)\big|^2\bigg] \\ \le (1+CL^{-1})\mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big].\end{aligned}$$

For the converse inequality, we appeal to a cut-and-paste argument. The starting point is the following convergence, cf. [18],

$$\mathbb{E}\big[\big|\mathrm{D}(\psi)\big|^2\big] = \lim_{k\uparrow\infty} \mathbb{E}\bigg[\fint_{U_{kL}} \big|\mathrm{D}\big(\psi_\circ(U_{kL})\big)\big|^2\bigg].$$

Since $\widetilde{\psi}_\circ(U_{kL}) := \sum_j \psi_\circ(U_L(z_j))\mathbb{1}_{U_L(z_j)}$ belongs to $\mathcal{H}_0(U_{kL})$, where $\{U_L(z_j)\}_j$ is a partition of $U_{kL}$, we obtain for all $k$, by stationarity of $z \mapsto U_L(z)$,

$$\mathbb{E}\Big[\fint_{U_{kL}} \big|\mathrm{D}\big(\psi_\circ(U_{kL})\big)\big|^2\Big] \le \sum_j \mathbb{E}\Big[\frac{|U_L(z_j)|}{|U_{kL}|}\fint_{U_L(z_j)} \big|\mathrm{D}\big(\psi_\circ(U_L(z_j))\big)\big|^2\Big]$$
$$\le (1+CL^{-1})\mathbb{E}\Big[\fint_{U_L} \big|\mathrm{D}\big(\psi_\circ(U_L)\big)\big|^2\Big].$$

The claim (B.5) follows from the combination of these three properties with the above energy bounds (B.10).

*Step* 2. Proof of (ii). By definition,

$$J(U) = \fint_U \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big) : \mathrm{D}\big(\psi_\circ(U) + \psi_*(U)\big).$$

Since $\psi_\circ(U), \psi_*(U) \in \mathcal{H}(U)$, the difference $\psi_\circ(U) - \psi_*(U)$ is a suitable test function for the Euler–Lagrange equation of the minimization problem (B.1) defining $\psi_*(U)$, which yields

$$\int_U \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big) : \mathrm{D}\big(\psi_*(U)\big) = 0,$$

and the claim (B.7) follows.

*Step* 3. Proof of (iii). We start with the proof of (B.8). Since the minimization problem (B.2) defines a subadditive set function due to the gluing property of $\mathcal{H}_0(U)$, and since the minimization problem (B.1) defines a superadditive function due to the restriction property of $\mathcal{H}(U)$, the function $J$ is subadditive as the difference of a subadditive and of a superadditive function.

We turn to the proof of (B.9). The starting point is (B.7) for $U^j$, which yields

$$\begin{aligned} &|U^j|J(U^j) - \int_{U^j} \big|\mathrm{D}\big(\delta\psi(U)\big)\big|^2 \\ &\quad = \int_{U^j} \mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big) : \mathrm{D}\big(\delta\psi(U^j) + \delta\psi(U)\big) \\ &\quad = \int_{U^j} \big|\mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big)\big|^2 \\ &\qquad + 2\int_{U^j} \mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big) : \mathrm{D}\big(\delta\psi(U)\big). \end{aligned} \tag{B.11}$$

We decompose the second right-hand side term into $2\sum_{k=1}^4 I_{k,j}$, in terms of

$$I_{1,j} = \int_{U^j} \mathrm{D}\big(\psi_\circ(U^j) - \psi_\circ(U)\big) : \mathrm{D}\big(\psi_\circ(U)\big),$$

$$I_{2,j} = -\int_{U^j} \mathrm{D}\big(\psi_\circ(U^j) - \psi_\circ(U)\big) : \mathrm{D}\big(\psi_*(U)\big),$$
$$I_{3,j} = \int_{U^j} \mathrm{D}\big(\psi_*(U)\big) : \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big),$$
$$I_{4,j} = -\int_{U^j} \mathrm{D}\big(\psi_*(U^j)\big) : \mathrm{D}\big(\psi_\circ(U) - \psi_*(U)\big).$$

Since $\psi_\circ(U)|_{U^j}, \psi_*(U)|_{U^j} \in \mathcal{H}(U^j)$, the difference $(\psi_\circ(U) - \psi_*(U))|_{U^j}$ is a suitable test function for the Euler–Lagrange equation for $\psi_*(U^j)$, which yields $I_{4,j} = 0$. Likewise, since $\psi_\circ(U), \sum_j \psi_\circ(U^j)\mathbb{1}_{U^j} \in \mathcal{H}_0(U) \subset \mathcal{H}(U)$, we find both $\sum_j I_{1,j} = 0$ and $\sum_j I_{2,j} = 0$. In addition, since $\psi_\circ(U), \psi_*(U) \in \mathcal{H}(U)$, we find $\sum_j I_{3,j} = 0$. This entails

$$\sum_j \int_{U^j} \mathrm{D}\big(\delta\psi(U^j) - \delta\psi(U)\big) : \mathrm{D}\big(\delta\psi(U)\big) = 0.$$

Summing (B.11) over $j$, inserting the above, and recalling the identity (B.7), the claim (B.9) follows. ∎

For all $n \geq 0$, we set $U^n := U_{3^n}$ and define the discrepancy

$$\tau_n := \mathbb{E}\big[J(U^n)\big] - \mathbb{E}\big[J(U^{n+1})\big].$$

In contrast with [3], the set $U^n$ is now random, so that subadditivity does not directly imply $\tau_n \geq 0$. This is however true up to an error $O(3^{-n})$, as we briefly argue. Choose a partition $\{U_j^n := U_{3^n}(z_j)\}_j$ of the set $U^{n+1}$. Taking the expectation of (B.9) applied to this decomposition of $U^{n+1}$, we find

$$\begin{aligned} 0 &\leq \mathbb{E}\Big[\sum_j \big\|\mathrm{D}\big(\delta\psi(U^{n+1}) - \delta\psi(U_j^n)\big)\big\|^2_{\mathrm{L}^2(U_j^n)}\Big] \\ &= \sum_j \mathbb{E}\big[|U_j^n|\big(J(U_j^n) - J(U^{n+1})\big)\big], \end{aligned} \tag{B.12}$$

whereas by the deterministic bounds $|3^d|U^n| - |U^{n+1}|| \lesssim 3^{n(d-1)}$ and $J(U_j^n) \lesssim 1$ we have for some constant $C \simeq 1$,

$$\begin{aligned} &\sum_j \mathbb{E}\big[|U_j^n|\big(J(U_j^n) - J(U^{n+1})\big)\big] \\ &\qquad \lesssim 3^{nd}\big(\mathbb{E}\big[J(U^n)\big] - \mathbb{E}\big[J(U^{n+1})\big] + C3^{-n}\big). \end{aligned} \tag{B.13}$$

The combination of (B.12) and (B.13) yields the claim in form of

$$\overline{\tau}_n := \tau_n + C3^{-n} \geq 0. \tag{B.14}$$

The crux of the approach is the following control of the variance of averages of $\mathrm{D}(\delta\psi(U))$ in terms of $\tau_n$. In view of Lemma B.2, the proof is identical to that of [3, Lemma 2.13] (albeit in the $\alpha$-mixing version of [4], further arguing as in (B.14) and absorbing the additional error term).

**Lemma B.3.** *There exist $C, \varepsilon > 0$ (only depending on $d, \rho, \beta$) such that for all $n$,*

$$\mathrm{Var}\Big[\fint_{U^n} \mathrm{D}\big(\delta\psi(U^n)\big)\Big] \le C 3^{-\varepsilon n} + C \sum_{m=0}^{n} 3^{-\varepsilon(n-m)} \overline{\tau}_m.$$

Recall the following version of Korn's inequality: for any bounded domain $D \subset \mathbb{R}^d$, for all divergence-free fields $v \in \mathrm{L}^2(D)$, we have

$$\inf_{\substack{\kappa\in\mathbb{R}^d\\ \Theta\in\mathbb{M}^{\mathrm{skew}}}} \int_D \big|v(x) - \kappa - \Theta x\big|^2\, dx \lesssim_D \big\|\mathrm{D}(v)\big\|^2_{H^{-1}(D)},$$

where the multiplicative constant only depends on the regularity of $D$. In contrast with Poincaré's inequality, the infimum over $\Theta \in \mathbb{M}^{\mathrm{skew}}$ allows to have the symmetrized gradient in the right-hand side instead of the full gradient. By the so-called multiscale Poincaré inequality in [3, Proposition 1.12], using the above Korn inequality instead of [3, Lemma 1.13], Lemma B.3 yields the following estimate as in [3, Lemma 2.15]. This is simpler than the statement in [3] since there is no convex duality involved.

**Lemma B.4.** *There exist $C, \varepsilon > 0$ (only depending on $d, \rho, \beta$) such that for all $n$,*

$$\mathbb{E}\Big[\inf_{\substack{\kappa\in\mathbb{R}^d\\ \Theta\in\mathbb{M}^{\mathrm{skew}}}} \fint_{U^{n+1}} \big|\delta\psi(U^{n+1})(x) - \kappa - \Theta x\big|^2 dx\Big] \le C 3^{2n}\Big(3^{-\varepsilon n} + \sum_{m=0}^{n} 3^{-\varepsilon(n-m)} \overline{\tau}_m\Big).$$

Next, we deduce the following estimate on $J$ as in [3, Lemma 2.16] by means of the Caccioppoli inequality. As the latter inequality in the present Stokes context involves the pressure, the proof slightly differs from [3] and is included below.

**Lemma B.5.** *There exist $C, \varepsilon > 0$ (only depending on $d$, $\rho$, $\beta$) such that for all $n$,*

$$\mathbb{E}\big[J(U^n)\big] \le C 3^{-\varepsilon n} + C \sum_{m=0}^{n} 3^{-\varepsilon(n-m)} \overline{\tau}_m.$$

*Proof.* Caccioppoli's inequality in form of e.g. [19, Section 4.4, Step 1] yields for all $K \ge 1$, for any constants $c \in \mathbb{R}$, $\kappa \in \mathbb{R}^d$, and $\Theta \in \mathbb{M}^{\mathrm{skew}}$,

$$\begin{aligned}\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2 &\lesssim K^2 3^{-2n} \fint_{U^{n+1}} \big|\delta\psi(U^{n+1})(x) - \kappa - \Theta x\big|^2 dx \\ &\quad + K^{-2} \fint_{U^{n+1}} \big|\delta\Sigma(U^{n+1}) - c\big|^2 \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}}, \end{aligned} \tag{B.15}$$

where $\delta\Sigma(U^{n+1})$ is the difference of the pressures associated with $\psi_\circ(U^{n+1})$ and $\psi_*(U^{n+1})$. Appealing to a local pressure estimate in form of e.g. [19, Lemma 3.3], and recalling Lemma B.2 (ii), we find

$$\inf_{c\in\mathbb{R}} \fint_{U^{n+1}} \big|\delta\Sigma(U^{n+1})-c\big|^2 \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{I}} \lesssim \fint_{U^{n+1}} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2 = J(U^{n+1}). \quad \text{(B.16)}$$

Taking the infimum over $c$, $\kappa$, $\Theta$ in (B.15), taking the expectation, inserting (B.16), and using Lemma B.4, we obtain

$$\mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] \lesssim K^2\bigg(3^{-\varepsilon n} + \sum_{m=0}^{n} 3^{-\varepsilon(n-m)}\overline{\tau}_m\bigg) + K^{-2}\mathbb{E}\big[J(U^{n+1})\big],$$

and thus, in view of (B.14),

$$\begin{aligned}\mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] \\ \lesssim K^2\bigg(3^{-\varepsilon n} + \sum_{m=0}^{n} 3^{-\varepsilon(n-m)}\overline{\tau}_m\bigg) + K^{-2}\Big(\mathbb{E}\big[J(U^n)\big] + 3^{-n}\Big).\end{aligned} \quad \text{(B.17)}$$

Next, we argue that

$$\mathbb{E}\big[J(U^n)\big] \lesssim \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] + \overline{\tau}_n. \quad \text{(B.18)}$$

For that purpose, we first note that the definition of $J$ yields

$$\begin{aligned}&\mathbb{E}\big[J(U^n)\big] - \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] \\ &\quad = \mathbb{E}\bigg[\fint_{U^n} \mathrm{D}\big(\delta\psi(U^n) - \delta\psi(U^{n+1})\big) : \mathrm{D}\big(\delta\psi(U^n) + \delta\psi(U^{n+1})\big)\bigg] \\ &\quad \lesssim \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^n) - \delta\psi(U^{n+1})\big)\big|^2\bigg]^{\frac12} \Big(\mathbb{E}\big[J(U^n)\big] + \mathbb{E}\big[J(U^{n+1})\big]\Big)^{\frac12}.\end{aligned}$$

In order to control the first factor, we appeal to (B.12) and (B.13) in form of

$$\mathbb{E}\bigg[\sum_j \big\|\mathrm{D}\big(\delta\psi(U_j^n) - \delta\psi(U^{n+1})\big)\big\|^2_{\mathrm{L}^2(U_j^n)}\bigg] \lesssim 3^{nd}\overline{\tau}_n.$$

Further, using the definition (B.14) of $\overline{\tau}_n$ to reformulate the second factor, we deduce

$$\mathbb{E}\big[J(U^n)\big] - \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] \lesssim (\overline{\tau}_n)^{\frac12}\Big(\mathbb{E}\big[J(U^n)\big] + \overline{\tau}_n\Big)^{\frac12},$$

and the claim (B.18) follows.

Choosing $K \simeq 1$ large enough, (B.17) and (B.18) combine to

$$\mathbb{E}\big[J(U^n)\big] \lesssim \mathbb{E}\bigg[\fint_{U^n} \big|\mathrm{D}\big(\delta\psi(U^{n+1})\big)\big|^2\bigg] + \overline{\tau}_n \lesssim 3^{-(\varepsilon\wedge 1)n} + \sum_{m=0}^{n} 3^{-\beta(n-m)}\overline{\tau}_m,$$

and the conclusion follows. ∎

We may now proceed to the proof of Proposition B.1, which follows from Lemma B.5 by iteration.

*Proof of Proposition* B.1. Set $F_n := 3^{-\frac12\varepsilon n}\sum_{m=0}^{n} 3^{\frac12\varepsilon m}\mathbb{E}\big[J(U^m)\big]$. In terms of $\tau_n$, recognizing a telescoping sum, we find

$$\begin{aligned} F_n - F_{n+1} &= 3^{-\frac12\varepsilon n}\sum_{m=0}^{n} 3^{\frac12\varepsilon m}\mathbb{E}\big[J(U^m)\big] - 3^{-\frac12\varepsilon(n+1)}\sum_{m=0}^{n+1} 3^{\frac12\varepsilon m}\mathbb{E}\big[J(U^m)\big] \\ &= 3^{-\frac12\varepsilon n}\sum_{m=0}^{n} 3^{\frac12\varepsilon m}\tau_m - 3^{-\frac12\varepsilon(n+1)}\mathbb{E}\big[J(U^0)\big], \end{aligned}$$

and thus, using (B.14) and $\mathbb{E}\big[J(U^0)\big] \lesssim 1$,

$$F_n - F_{n+1} \geq 3^{-\frac12\varepsilon n}\sum_{m=0}^{n} 3^{\frac12\varepsilon m}\overline{\tau}_m - C3^{-\frac12\varepsilon n}. \tag{B.19}$$

Similarly, we find

$$\begin{aligned} F_{n+1} &\leq 3^{-\frac12\varepsilon(n+1)}\sum_{m=1}^{n+1} 3^{\frac12\varepsilon m}\mathbb{E}\big[J(U^m)\big] + C3^{-\frac12\varepsilon(n+1)} \\ &\leq 3^{-\frac12\varepsilon(n+1)}\sum_{m=1}^{n+1} 3^{\frac12\varepsilon m}\big(\mathbb{E}\big[J(U^{m-1})\big] + C3^{-(m-1)}\big) + C3^{-\frac12\varepsilon(n+1)} \\ &\leq F_n + C3^{-\frac12\varepsilon n}, \end{aligned}$$

which, by Lemma B.5, turns into

$$\begin{aligned} F_{n+1} &\leq C3^{-\frac12\varepsilon n}\sum_{m=0}^{n} 3^{\frac12\varepsilon m}\bigg(3^{-\varepsilon m} + \sum_{k=0}^{m} 3^{-\varepsilon(m-k)}\overline{\tau}_k\bigg) + C3^{-\frac12\varepsilon n} \\ &\leq C3^{-\frac12\varepsilon n} + C3^{-\frac12\varepsilon n}\sum_{m=0}^{n} C3^{\frac12\varepsilon m}\overline{\tau}_m. \end{aligned}$$

Combining this with (B.19), we obtain

$$F_{n+1} \leq C(F_n - F_{n+1}) + C3^{-\frac12\varepsilon n},$$

and thus

$$F_{n+1} \le \frac{C}{C+1}(F_n + 3^{-\frac{1}{2}\varepsilon n}).$$

By iteration, this yields $F_n \le C3^{-\gamma n}$ for some $\gamma > 0$, and thus $\mathbb{E}[J(U^n)] \le C3^{-\gamma n}$ and $\tau_n \le C3^{-\gamma n}$. Since $\mathbb{E}[J(U^n)] = E : (\widetilde{\mathbf{B}}_{3^n,\circ} - \widetilde{\mathbf{B}}_{3^n,*})E$, this implies

$$0 \le E : (\widetilde{\mathbf{B}}_{L,\circ} - \widetilde{\mathbf{B}}_{L,*})E \le L^{-\gamma}.$$

Combined with Lemma B.2 (i), this yields the conclusion. ■