Foreword

This book on "The Structure of Pro-Lie Groups" by Karl H. Hofmann and Sidney A. Morris presents a well-rounded theory of "groups that can be approximated by Lie groups." To put this theory into perspective, one may start 150 years ago, in 1872, when Felix Klein presented his Erlangen Program, roughly asserting that geometric structures are determined by their automorphism groups. Klein's perspective immediately led to the question of how these symmetry groups look and if one could understand their structure. For several decades, the subsequent development split into two strands: discrete groups, whose geometric properties were pursued by Felix Klein, and continuous groups, nowadays called Lie groups, whose basic properties were developed by Sophus Lie.

The classification of Lie groups started with the work of Wilhelm Killing and Élie Cartan in the end of the 19th century. Their work resulted in the classification of the simple real Lie groups. Supplemented by results like the Levi-splitting, this laid the ground for a pretty good understanding of the structure of connected Lie groups in terms of their Lie algebras.

Using finiteness instead of finite dimension, a powerful structure theory can also be developed for finite groups. It culminated in the classification of finite simple groups in the late 20th century. Following the general philosophy that finiteness may often be replaced by compactness, leads to the class of compact groups, a topic of another voluminous monograph of the same authors. The work of Fritz Peter and Hermann Weyl in the 1920s implied that compact groups are projective limits of compact Lie groups, which in turn are extensions of finite groups by compact connected Lie groups. Yamabe's theorem later showed that any locally compact group G, whose component group is compact, is also a projective limit of Lie groups.

The present monograph describes the maximal context in which a powerful Lie theoretic structure theory of groups can be developed: the class of pro-Lie groups, i.e., projective limits of finite-dimensional Lie groups. It leads the reader far beyond finite-dimensional groups, but pro-Lie groups still have a Lie algebra. It enjoys the key structural features well-known in finite dimensions, such as a Levi splitting into a "solvable" part and a direct sum of finite-dimensional simple ones. What happens modulo the identity component cannot be controlled by the Lie algebra, so that it mostly is assumed that the component group is compact. Pro-Lie groups are rarely infinite-dimensional manifolds, hence Lie groups in the infinite-dimensional context. This happens only if they are locally contractible, and for this class of infinite-dimensional Lie groups the present book also provides a fully developed structure theory.

It is truly amazing to see that the minimal assumption of being pro-Lie is enough for such a fine structure theory, but this is not for free. It requires several hundred pages of hard work to put everything into place and to eventually reach the simply formulated beautiful cornerstones of the structure of pro-Lie groups.

The mathematical community has to be grateful to Karl H. Hofmann and Sidney A. Morris who took on the task of developing a systematic theory of pro-Lie groups and, eventually, to present their well-rounded theory in this useful book that now appears in its second expanded edition. It compiles numerous results that have been obtained over the past decades and this comprehensive presentation makes this knowledge easily accessible for present and future generations of mathematicians.

Erlangen, November 2022 Karl-Hermann Neeb