Preface to the Second Edition

The first edition of this book was named "The Lie Theory of Connected Pro-Lie Groups." This second edition is named "The Structure of Pro-Lie Groups." There are three reasons for the name change:

(i) it makes it clearer that this book is a sequel to our book [128] "The Structure of Compact Groups;"

(ii) our aim is to describe the topological and algebraic structure of pro-Lie groups (which is achieved using Lie theory);

(iii) the results described in the first edition were restricted to the case of connected pro-Lie groups whereas the results in this edition cover *almost* connected pro-Lie groups.

The extension from connected to almost connected topological groups, that is, to topological groups G with G/G_0 is compact, where G_0 is the connected component of the identity, is important since the class of almost connected pro-Lie groups includes all compact groups and all connected pro-Lie groups. Moreover, each locally compact group has an open almost connected subgroup.

The material that is new in this edition also includes an enlarged discussion of *weakly complete* topological vector spaces, collected in Appendix A2. This extension is motivated by the fact, that the underlying topological vector spaces of all *pro-Lie algebras* are indeed weakly complete. All Lie algebras of pro-Lie groups are pro-Lie algebras. And so, pro-Lie algebra theory and the representation theory of pro-Lie algebras are based squarely on the class of weakly complete vector spaces. To some extent the same emphasis had already been seen in the development of the *structure of compact groups* in [128].

The recent development of pro-Lie theory has been spurred by a rush of publications following the appearance of the first edition of this book. This is exemplified by [11, 47, 48, 70, 74, 76, 77, 87, 99, 112, 113, 125–127].

The first three chapters of the book have been reorganized, sometimes even shortened by comparison with the first edition.

The substantial extension of our structure theory of connected pro-Lie groups to a structure theory of almost connected pro-Lie groups is spread over the whole book and culminates in Chapters 11, 12, and 13.

What in the first edition had been Chapter 1 dealt in great detail with limits, notably with projective ones. While these are absolutely central to the concept of pro-Lie theory, they are a tool and not a part of that theory. We therefore relocated this chapter as Appendix A1.

But core topics of the entire book remain, firstly, the possible connection between a topological group and its Lie algebra via an exponential function, leading, notably, to the definition of a classical Lie group, and secondly, the precise definition of a pro-Lie group. These matters now are located in the first two chapters. Indeed, for a pro-Lie group we have, at the end of these two chapters, three reasonable and useful equivalent definitions for the concept of a pro-Lie group. Their equivalence, however, is by no means obvious. Nor is the important fact that a closed subgroup of a pro-Lie group is a pro-Lie group (the closed subgroup theorem).

A third core subject for the category of pro-Lie groups is the issue of quotient morphisms. It begins in Chapter 3, where it is shown, in particular, that each quotient morphism between pro-Lie groups produces a quotient morphism between their respective pro-Lie algebras.

Another crucial result is deferred to a later point in Chapter 8, namely, that every surjective morphism between two *almost connected* pro-Lie groups is a quotient morphism. This is Theorem 8.60, the open mapping theorem for almost connected pro-Lie groups.

This new edition presents in Chapters 12 and 13 an important structural result concerning any *almost connected* pro-Lie group G.

Firstly, G has a maximal compact subgroup M, and every maximal compact subgroup is conjugate to M.

Secondly, $G = G_0M$ and $M_0 = G_0 \cap M$ is a maximal compact subgroup of G_0. The quotients $G/G_0 \cong M/M_0$ are profinite.

Thirdly, G contains a closed subspace V whose explicitly given description shows it to be homeomorphic to a weakly complete vector space $\mathbb{R}^J$ for some set J in such a fashion that

$$(v, m) \mapsto vm\colon V \times M \to G \text{ is a homeomorphism.}$$

Since very precise and explicit descriptions of the structure of compact groups like M are available from [128], this is indeed a very explicit description of the topological structure of an almost connected pro-Lie group. In particular, it is seen immediately, that

Every almost connected pro-Lie group is homotopy equivalent to a compact group.

Note that a pro-Lie group P does not necessarily have an open subgroup G which is almost connected, not even in the case that P is abelian. We indicate this at an early point in the book by exhibiting the formidable example of a countable nondiscrete prodiscrete group structure on the free abelian group $\mathbb{Z}^{(\mathbb{N})}$ in which every compact subset is finite (see Proposition 4.2).

For the following observations, we keep in mind that every locally compact almost connected group is a pro-Lie group by Yamabe's theorem (cf. [170, 239, 240]).

One almost immediate consequence of the results we have seen is that an almost connected pro-Lie group is *finite-dimensional* (for almost any definition of topological dimension one likes) if and only if the topological vector space factor V is finite-dimensional (happening precisely when V is locally compact) and M is finite-dimensional – this situation being extensively discussed in [128, 9.52–9.56 and 10.38–10.40]. So

Every almost connected finite-dimensional pro-Lie group is locally compact.

Thus imposing the hypothesis of finite dimensionality on almost connected pro-Lie groups leads at once into territory well-known in the community familiar with locally compact groups.

However, the category of totally disconnected pro-Lie groups, alternatively called *prodiscrete* groups, is largely unknown and unexplored. Every prodiscrete group is characterized by the fact that its Lie algebra is zero. Historically, the theory of *locally compact* prodiscrete groups, including all profinite groups and all totally disconnected locally compact abelian groups, has received considerable attention and investigation due to their emergence out of Galois theory. It contributed to what appears to be a distinct separation of the two research fields of almost connected locally compact groups and that of prodiscrete locally compact groups – a separation that is reflected likewise in their research communities. The emergence of a research community dealing with prodiscrete groups in general is not discernible at this time.

There remains one task for the authors of this second edition of a book on a branch of conceptually and technically intricate mathematics: Namely, to explain to the curious reader the role of special portion of the book which we called *panoramic overview*. As authors we were always deeply conscious of the difficulties and possible impediments inherent in the mathematics of pro-Lie theory. Accordingly, in working on the first edition we felt that it would be wise to offer the readers an overview of the results of the book unfettered by the complications of their proofs and technical details. So we attempted to open the book with a hopefully easily accessible but thorough summary of the book, which might provide a comprehensive impression of the various aspects of a structure theory of pro-Lie groups and give an impression of the essential results that might reward the users provided they penetrated deeply into the subject.

At first, we seriously considered omitting the "Panoramic Overview" in the second edition. Since we expanded the material in the appendices where we explicitly present auxiliary material that we use in the development of our structure theory, as an alternative we asked ourselves whether the overview might be placed somewhere in the back of the book in the spirit of an appendix. This we have done and called it "Postscript."

To make it possible for readers to dive straight into this book irrespective of whether they have studied our earlier book, "The Structure of Compact Groups" [128], we have added at the beginning a selection of results on the structure theory of compact groups and of pro-Lie groups. Our emphasis in this selection is to make it brief and easy to read.

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Karl H. Hofmann
Sidney A. Morris