Preface to the First Edition

Sophus Marius Lie (1842–1899) laid the foundation of the theory named *Lie theory* in honor of its creator. Several mathematicians, likewise prominent in the history of modern mathematics, contributed to its inception in the decades following 1873, which was the year in which Lie started to occupy himself intensively in the study of what he called continuous groups, notably: Friedrich Engel, Wilhelm Killing, Élie Cartan, Henri Poincaré, and Hermann Weyl. From the beginning, however, the advance of Lie theory bifurcated into two separate major highways, which is the reason why the words *Lie theory* mean different things to different people. Lie himself aimed at accomplishing for the solution of differential equations (in the widest sense) what Évariste Galois and Lie's countryman Niels Henrik Abel achieved for the solution of algebraic equations: A profound understanding and, to the best extent possible, a classification in terms of groups. Even though Lie considered himself a "geometer," he created a territory of analysis that is called *Lie theory* by those working in it, and that is represented by the well-known text by Peter J. Olver entitled "Applications of Lie Groups to Differential Equations" [Springer, Berlin, New York, 1986]. We should say in the beginning that the project of Lie theory which we shall discuss in this book, in philosophy and thrust, does not belong to this line.

A second highway was taken by Wilhelm Killing and Élie Cartan. It led to a study of what soon became known as Lie algebras, of the group and structure theory of Lie groups, and to the geometry of homogeneous spaces. The latter notably yielded the classification of symmetric spaces by Élie Cartan. At long last it merged into the encyclopedic attempt by Nicolas Bourbaki of the nineteen hundred sixties and seventies, to summarize what had been achieved, and to the emergence of an immense collection of textbooks at all levels. In 1973, Jean Dieudonné quipped *"Les groupes de Lie sont devenus le centre des Mathématiques; on ne peut rien faire de sérieux sans eux.* (Lie groups have moved to the center of mathematics. One cannot seriously undertake anything without them. Gazette des Mathématiciens, Société Mathématique de France, Octobre 1974, p. 77.) By and large, in this line of "Lie theory" the words meant the structure theory of Lie algebras and Lie groups, and in particular how the latter is based on the former.

The term *Lie group* originally meant *finite-dimensional Lie group* and most people understand the words in this sense today. However, even Sophus Lie spoke of "unendliche Gruppen" by which he meant something like infinite-dimensional Lie groups. But reasonable concepts of dimension were not yet available in the 19th century before topology was on its way. And indeed Lie's attempts in this direction did not appear to have gotten off the ground.

The significance of Lie's discoveries was emphasized by David Hilbert by raising the question in 1900 whether (in later terminology) a locally euclidean topological group is in fact an analytic group in the sense of Lie. This was the fifth of his famous 23 problems which foreshadowed so much of the mathematical creativity of the 20th century. It required half a century of effort on the part of several generations of eminent mathematicians until it was settled in the affirmative. Partial solutions came along as the structure of topological groups was understood better and better: Hermann Weyl and his student Fritz Peter in 1923 laid the foundations of the representation and structure theory of compact groups, and a positive answer to Hilbert's fifth problem for compact groups was a consequence, drawn by John von Neumann in 1932. Lev Semyonovich Pontryagin and Egbert Rudolf van Kampen developed in 1932, respectively, 1936, the duality theory of locally compact abelian groups laying the foundations for an abstract harmonic analysis flourishing throughout the second half of the 20th century and providing the central method for attacking the structure theory of compact abelian groups via duality. Again a positive response to Hilbert's question for locally euclidean abelian groups followed in the wash.

One of the most significant and seminal papers in topological group theory was published in 1949 by Kenkichi Iwasawa, some three years before Hilbert's problem was finally settled by the concerted contribution of Andrew Mattei Gleason, Dean Montgomery, Leon Zippin, and Hidehiko Yamabe. It was Iwasawa who clearly recognized for the first time that the structure theory of locally compact groups reduced to that of compact groups and finite-dimensional Lie groups *provided* one knew that they happen to be approximated by finite-dimensional Lie groups in the sense of projective limits, in other words, if they were pro-Lie groups in our parlance. And this is what Yamabe established in 1953 for all locally compact groups which have a compact factor group modulo their identity component – almost connected locally compact groups as we shall say. The most influential monograph collecting these results was the book by Montgomery and Zippin of 1955 with the title "Topological Transformation Groups." The theories of compact groups and of abelian locally compact groups had introduced in the first half of the century classes of groups with an explicit structure theory without the restriction of finite-dimensionality, and in the middle of the century these results opened up an explicit development for numerous results on the structure theory of locally compact groups.

What are the coordinates of our book in this historical thread?

It was recognized in 1957 by Richard Kenneth Lashof that any locally compact group G has a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is appropriately defined, then the exponential function $\exp\colon \mathfrak{g} \to G$ is supplied along with the definition. Yet the fact that these observations are the nucleus of a complete and rich, although infinite-dimensional Lie theory was never exploited. The present book is devoted to the foundations, and the exploitation of such a Lie theory. At a point in the overall historical development where infinite-dimensional Lie theories gain increasing acceptance and attract much interest, this appears to be timely. The Lie theory we unfold is based on projective limits, both on the group level and on the Lie algebra level. We shall find it very helpful that category theory, as a tool for the "working mathematician" as Saunders Mac Lane formulated it, is so well developed that we see immediately what we need, and we shall exploit it. In our case, we need the theory of limits in a complete category, that is, in a category in which all limits exist, and we need the theory of pairs of adjoint functors, which is closely linked with limits.

The machinery of projective limits is familiar to mathematicians dealing with profinite groups in their work on Galois theory and arithmetics, quite generally. But the apparatus of projective limits is also familiar to mathematicians dealing with compact groups, their representation theory and abstract harmonic analysis. Indeed, all group theoreticians working on the structure theory of locally compact groups encounter projective limits sooner or later. In this book we shall call projective limits of projective systems (or, as some authors say, inverse systems) of finite-dimensional Lie groups *pro-Lie groups*. That is, pro-Lie groups relate to finite-dimensional Lie groups exactly as profinite groups relate to finite groups.

However, in the theory of locally compact groups, one encounters a special kind of projective limit, namely, limit situations where limit maps and bonding maps are *proper*, that is, are closed continuous homomorphisms between locally compact groups having compact kernels. Some authors call such maps perfect. This type of projective limit has a significant element of compactness already built into its definition, and it is this type of limit that has shaped the intuitions of group theoreticians for fifty years or more.

From the vantage point of category theory, however, such a restriction is entirely unnatural, as is indeed the entire focus on locally compact pro-Lie groups: The class of locally compact groups is not even closed under the formation of products – as the example of the groups $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{Z}^{\mathbb{N}}$ shows immediately. Mathematicians will be naturally attracted to the problem of eliminating the focus on locally compact groups. As one proceeds in the direction of pro-Lie groups in general, however, one comes to realize that the restriction to locally compact groups is unnatural also for reasons that are entirely interior to the mathematics of topological groups and Lie groups. For several years we have been engaged in the laying of the foundations of a general theory of the category of pro-Lie groups. The results are presented in this book. On the first 60 pages (in the first edition), the reader will find a panoramic overview of what is contained in its 14 chapters (in the first edition), and the user of the book should get a more compact overview by perusing its table of contents.

The Lie theory of finite-dimensional Lie groups works because for a *connected* Lie group G, its Lie algebra $\mathfrak{g}$ and its exponential function $\exp\colon \mathfrak{g} \to G$ largely determine the structure of G. We hasten to add that, except for the case that G is simply connected, they do not do so completely. As the title of our book indicates, we focus on a Lie theory for connected pro-Lie groups. As a consequence, our structure theory is one that is mainly concerned with connected pro-Lie groups, sometimes going a bit further, but rarely much beyond almost connected groups. In view of Yamabe's theorem, the structure theory of connected or almost connected pro-Lie groups applies at once to connected or even almost connected locally compact groups.

There are several key elements to the structure theory of pro-Lie groups.

Firstly, a thorough understanding of the working of projective limits is needed without the crutch of thinking in terms of proper maps all the time. Appendix A1 (in the second edition) deals with many facets of this issue. But only after Chapter 2 (in the second edition) will we have understood all aspects of what this means for the very definition of pro-Lie groups itself.

Secondly, the entire theory depends on our accepting that pro-Lie groups, even though not being Lie groups, nevertheless have a working Lie theory, complete with the appropriate Lie algebras which we shall call *pro-Lie algebras* and working exponential functions that mediate between pro-Lie groups and their Lie algebras. Indeed, we must become aware at an early stage that there is a good Lie algebra functor from the category of pro-Lie groups to the category of pro-Lie algebras. One of the very positive side effects of facing wider categories than the conventional ones in developing a Lie theory is that this enlargement of scope forces us to realize in great clarity that the Lie algebra functor is opposed by a Lie group functor that encapsulates lucidly the contents of Lie's third fundamental theorem. This applies to the classical situation as well, but it is not recognized there because the theory of universal covering Lie groups, while providing topologically satisfying results in general, tends to obscure the precise functorial set-up. Since for pro-Lie groups a classical covering theory is impossible as one knows from the theory of compact connected abelian groups, it is mandatory that one understands the functorial background of a more general universal covering theory. We shall discuss this in Chapters 1, 3, 5 and 7 (in the second edition).

Thirdly, the success of the structure theory of pro-Lie groups depends in a large measure on our success in dealing with the structure theory of pro-Lie algebras. This pervades the whole book, but most of this is done in our rather long Chapter 6 (in the second edition). The point is that the topological vector spaces underlying pro-Lie algebras are what we call weakly complete topological vector spaces, because they are exactly the duals of real vector spaces given the weak $*$-topology, that is, the topology of pointwise convergence of linear functionals. Since the vector space duality is crucial for this class of topological vector spaces and hence for the structure theory of pro-Lie algebras, we present the essential features of the linear algebra of weakly complete topological vector spaces in an appendix, namely, Appendix A2 (in the second edition). The relevance of weakly complete topological vector spaces in the structure theory of pro-Lie groups themselves is evidenced in that chapter in which we discuss the structure of commutative pro-Lie groups, and that is Chapter 4 (in the second edition).

With all of these foundations done, the Lie and structure theory of pro-Lie groups can proceed, as it does in Chapters 8–14 (in the second edition). This preface is not the place to go into the details, but we shall present to our readers in the beginning of the book (in the first edition), in our panoramic overview, the results which we obtain.

One of the lead motives of our structure theory is to reduce the structure of connected pro-Lie groups in the optimal extent possible to the structure theory of compact connected groups, weakly complete topological vector spaces, and finite-dimensional Lie groups. We will prove some major structure theorems which expose that we, in essence, achieve this goal.

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