ON DISCRETE FOURIER UNIQUENESS SETS IN EUCLIDEAN SPACE

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ABSTRACT

This paper presents a new construction of a discrete Fourier uniqueness set in Euclidean space.

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1. INTRODUCTION

This paper gives a new construction of a closed discrete Fourier uniqueness set in $\mathbb{R}^d$. Let us start with a definition of Fourier uniqueness. For a Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$, its Fourier transform is defined as

$$\hat{f}(y) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x y}\, dx, \quad y \in \mathbb{R}^d.$$

Definition 1.1. A set $X \subset \mathbb{R}^d$ is a *Fourier uniqueness set* if for any Schwartz function f the conditions

$$f|_X \equiv 0 \quad \text{and} \quad \hat{f}|_X \equiv 0$$

imply $f \equiv 0$.

In [4] we have shown that the set $X = \{\operatorname{sign}(n)\sqrt{|n|}\}_{n\in\mathbb{Z}}$ is essentially a uniqueness set in $\mathbb{R}$. More precisely, we have proven that the conditions $f|_X \equiv 0$, $\hat{f}|_X \equiv 0$, together with one more linear constraint $f'(0) = 0$, imply the vanishing of f on the whole real line. M. Stoller [5] has extended this result to $\mathbb{R}^d$ in the following way. For a positive real number r, let $S(r)$ denote the sphere in $\mathbb{R}^d$ with center at the origin and radius r. Stoller has proven that the set $X := \bigcup_{n=1}^{\infty} S(\sqrt{n})$ is a Fourier uniqueness set in $\mathbb{R}^d$ for $d \geq 5$. The following theorem is proven in [5].

Theorem 1.2. *Let $d \geq 5$ be an integer. Suppose that $f : \mathbb{R}^d \to \mathbb{C}$ is a Schwartz function such that $f|_{S(\sqrt{n})} \equiv 0$ and $\hat{f}|_{S(\sqrt{n})} \equiv 0$ for all $n \in \mathbb{Z}_{\geq 1}$. Then, f is identically zero.*

Moreover, Stoller and J. P. G. Ramos have recently shown the existence of a closed discrete Fourier uniqueness set in $\mathbb{R}^d$ [6, THEOREM 2, REMARK 1.1].

A natural question is: How "big" is this discrete Fourier uniqueness set? More precisely, for a closed discrete subset $X \subset \mathbb{R}^d$ we would like to analyze the function $M_X(r)$, $r \in \mathbb{R}_{>0}$, that counts the number of elements of X inside of the ball of radius r about the origin. For the Fourier uniqueness set X constructed in [6, THEOREM 2, REMARK 1.1], the function $M_X(r)$ grows superexponentially in r.

This paper aims to construct a closed discrete Fourier uniqueness set X such that the function $M_X(r)$ grows at most polynomially in r.

1.1. Construction of a discrete Fourier uniqueness set

In this paper we will show that for a family of sufficiently uniformly distributed finite subsets $X_n \subset S(1)$, $n \in \mathbb{Z}_{\geq 1}$, the union

$$X := \bigcup_{n\geq 1} \sqrt{n}\, X_n \tag{1.1}$$

is a Fourier uniqueness set. Let us give one possible quantitative description of the term "uniformly distributed."

Definition 1.3. A finite subset $X \subset S(1)$ is a *spherical design of strength* s if, for all polynomials p in d variables and total degree at most s, the following holds:

$$\int_{S(1)} p(\zeta)\, d\zeta = \frac{1}{|X|} \sum_{x \in X} p(x).$$

Here $d\zeta$ denotes the Lebesgue measure on $S(1)$ normalized so that $\int_{S(1)} 1\, d\zeta = 1$.

The main result of this paper is

Theorem 1.4. *For each dimension d, there exist positive constants $\tilde{A} = \tilde{A}(d)$ and $\tilde{B} = \tilde{B}(d)$ with the following property. If $(X_n)_{n=1}^{\infty}$ is a collection of finite subsets of $S(1)$ such that each set X_n is a spherical design of strength $\tilde{B} n^{\tilde{A}}$ then the set*

$$X := \bigcup_{n \geq 1} \sqrt{n}\, X_n$$

is a Fourier uniqueness set.

It is known [1] that for a dimension d, there exists a constant c_d such that for all nonnegative integers s, there exists a spherical design of strength s with at most $c_d\, s^d$ points. Therefore, the above theorem implies the existence of a closed discrete Fourier uniqueness set X with a polynomially bounded function $M_X(r)$.

2. AUXILIARY RESULTS FROM FOURIER ANALYSIS

Our proof of Theorem 1.4 relies on several facts from Fourier analysis and the theory of modular forms. First, we will use the following statements about the decomposition of a Schwartz function in $\mathbb{R}^d$. Let $\mathcal{H}_m = \mathcal{H}_m(\mathbb{R}^d)$ be the space of homogenous harmonic polynomials of total degree m on $\mathbb{R}^d$. Let $\mathcal{B}_m$ be an orthonormal basis of $\mathcal{H}_m$ with respect to the standard L_2 product on the unit sphere $S(1)$. Set $\mathcal{B} := \bigcup_{m \geq 0} \mathcal{B}_m$. Each Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$ has the unique decomposition

$$f(x) = \sum_{p \in \mathcal{B}} p(x)\, g_p\big(\|x\|\big),$$

where g_p are radial Schwartz functions. For $p \in \mathcal{B}$, we denote

$$f_p(x) := p(x)\, g_p\big(\|x\|\big). \tag{2.1}$$

Theorem 2.1. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz function. For $p \in \mathcal{B}$ and $n \in \mathbb{Z}_{\geq 1}$, we set*

$$\phi_{p,n} = \phi_{p,n}(f) := \sup_{x \in S(\sqrt{n})} \big|f_p(x)\big|.$$

For all $\alpha, \beta > 0$ we have

$$\sup_{p \in \mathcal{B}, n \in \mathbb{Z}_{\geq 1}} \big(\deg(p)^{\alpha}\, n^{\beta}\, \phi_{p,n}\big) < \infty.$$

Proof. We have

$$\phi_{p,n} = \sup_{x \in S(\sqrt{n})} \big|f_p(x)\big| = n^{\frac{\deg(p)}{2}} \big|g_p(\sqrt{n})\big| \sup_{\zeta \in S(1)} \big|p(\zeta)\big|.$$

The number $g_p(\sqrt{n})$ can be computed as follows:

$$\begin{aligned}\int_{S(1)} & f(\sqrt{n}\zeta)\,\overline{p(\zeta)}\,d\zeta \\ &= \int_{S(1)} g_p(\sqrt{n})\,p(\sqrt{n}\zeta)\,\overline{p(\zeta)}\,d\zeta \\ &= n^{\frac{\deg(p)}{2}}\,g_p(\sqrt{n}).\end{aligned} \tag{2.2}$$

Therefore

$$\begin{aligned}\phi_{p,n} &= \left|\int_{S(1)} f(\sqrt{n}\zeta)\|p(\zeta)\|\,d\zeta\right| \cdot \sup_{\zeta\in S(1)}\left|p(\zeta)\right| \\ &\le \sup_{x\in S(\sqrt{n})}\left|f(x)\right| \sup_{\zeta\in S(1)}\left|p(\zeta)\right|^2.\end{aligned} \tag{2.3}$$

Note that there exist positive constants C_1 and C_2 depending only on dimension d such that $\sup_{\zeta\in S(1)}|p(\zeta)| \le C_1 \deg(p)^{C_2 d}$ for all $p \in \mathcal{B}$. This gives us the estimate

$$\phi_{p,n} \le C_1 \deg(p)^{2C_2} \sup_{x\in S(\sqrt{n})}\left|f(x)\right|. \tag{2.4}$$

Let β be a fixed positive number. Since f is a Schwartz function, we have

$$\sup_{x\in\mathbb{R}^d} \|x\|^{\beta}\left|f(x)\right| < \infty. \tag{2.5}$$

Estimates (2.4) and (2.5) imply

$$\sup_{p\in\mathcal{B},n\in\mathbb{Z}_{\ge 1}} \left(\deg(p)^{-2C_2}\,n^{\beta}\,\phi_{p,n}\right) < \infty. \tag{2.6}$$

Our next goal is to replace $-2C_2$ with an arbitrary positive constant α. Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_1^2}$ be the Laplace operator on $\mathbb{R}^d$. For a point $x \in \mathbb{R}^d \setminus \{0\}$, we define its polar coordinates $r = \|x\|$ and $\zeta = \frac{x}{\|x\|}$. Consider the following differential operator:

$$\Delta_{S^{d-1}} f := r^2\,\Delta f - (d-1)\,r\,\frac{\partial}{\partial r}\,f - r^2\,\frac{\partial^2}{\partial r^2}\,f.$$

An important property of this operator is that it maps Schwartz functions to Schwartz functions. Indeed, we compute in polar coordinates $x = r\,\zeta$ that

$$\begin{aligned}r\frac{\partial}{\partial r} f(r\zeta_1,\dots,r\zeta_d) &= r\,\zeta_1\,\frac{\partial}{\partial x_1} f + \cdots + r\,\zeta_d\,\frac{\partial}{\partial x_d} f \\ &= x_1\,\frac{\partial}{\partial x_1} f + \cdots + x_d\,\frac{\partial}{\partial x_d} f\end{aligned}$$

and, analogously,

$$\begin{aligned}r^2\frac{\partial^2}{\partial r^2} f(r\zeta_1,\dots,r\zeta_d) &= r^2\,\zeta_1^2\,\frac{\partial^2}{\partial x_1^2} f + \cdots + r^2\,\zeta_d^2\,\frac{\partial^2}{\partial x_d^2} f \\ &= x_1^2\,\frac{\partial^2}{\partial x_1^2} f + \cdots + x_d^2\,\frac{\partial^2}{\partial x_d^2} f.\end{aligned}$$

Thus, if f is a Schwartz function, so is $\Delta_{S^{d-1}} f$. Suppose that g is a radial Schwartz function and p is a homogenous harmonic polynomial on $\mathbb{R}^d$ of total degree $\deg(p)$. Then a straightforward computation shows that

$$\Delta_{S_d}\big(g(r)\,p(x)\big) = -\deg(p)\,\big(\deg(p)+d-2\big)\,g(r)\,p(x). \tag{2.7}$$

We define $\lambda_m := -m(m+d-2)$. Clearly, $|\lambda_m| \sim m^2$ as m goes to infinity.

Now let α be a positive integer. Given a Schwartz function f, we define a new Schwartz function $\tilde{f} := \Delta^{\alpha}_{S^{d-1}} f$. Suppose that f has decomposition $f = \sum_{p\in\mathcal{B}} f_p$, then by equation (2.7) the new function $\tilde{f}$ has decomposition $\tilde{f} = \sum_{p\in\mathcal{B}} \tilde{f}_p$ where $\tilde{f}_p = \lambda^{\alpha}_{\deg(p)} f_p$. Also the numbers $\widetilde{\phi}_{p,n} := \max_{x\in S(\sqrt{n})} |\tilde{f}_p(x)|$ satisfy

$$\widetilde{\phi}_{p,n} = |\lambda_{\deg(p)}|^{\alpha}\,\phi_{p,n}.$$

Finally, we apply estimate (2.6) to the function $\tilde{f}$ and derive

$$\sup_{p\in\mathcal{B},n\in\mathbb{Z}_{\geq 1}} \big(\deg(p)^{2\alpha-2C_2}\,n^{\beta}\,\phi_{p,n}\big) < \infty$$

for arbitrary positive α and β. This finishes the proof of the theorem. ■

3. AUXILIARY RESULTS FROM THE THEORY OF MODULAR FORMS

Let k be a half-integer. We denote by $S_k(\Gamma(2),\chi_k)$ the space of holomorphic cusp forms h satisfying the transformation rule

$$\begin{cases} h(\tau+2) = h(\tau), \\ \tilde{h}(\tau) := (-i\tau)^k\,h(\tau), \\ \tilde{h}(\tau+2) = \tilde{h}(\tau). \end{cases}$$

The following statement is known as the Voronoi summation formula.

Theorem 3.1. *Let h be a cusp form in $S_{d/2}(\Gamma(2),\chi_d)$ and let $\tilde{h}(\tau) := (-i\tau)^{-d/2}\,h(\frac{-1}{\tau})$. Then, for a radial Schwartz function $f:\mathbb{R}^d\to\mathbb{C}$, the following summation formula holds:*

$$\sum_{n=1}^{\infty} f(\sqrt{n})\,c_h(n) = \sum_{n=1}^{\infty} \hat{f}(\sqrt{n})\,c_{\tilde{h}}(n).$$

For a half-integer k and a positive number ϵ, we define

$$N(k,\epsilon) := \left(\frac{\epsilon\,\Gamma(k-1/2)}{(2\pi)^{k-1}\,\zeta(k-2)\,4\pi}\right)^{1/k}.$$

A straightforward consequence of the Stirling formula is that

$$N(k,\epsilon) \sim \frac{k}{2\pi e} \quad \text{as} \quad k\to\infty.$$

The main technical tool in our proof of Theorem 1.4 is the following statement about the space of modular forms $S_k(\Gamma(2),\chi_k)$.

Theorem 3.2. *Fix a number $\epsilon \in (0, 1/2)$ and for a half-integer k set $N(k) := \lfloor N(k,\epsilon) \rfloor$. For each half-integral weight $k \geq 5/2$, there exist elements $(h_m)_{m=1}^{N(k)-1}$ in the space $S_k(\Gamma(2), \chi_d)$ such that:*

(1) *the function h_m has the Fourier expansion*

$$h_m(\tau) = e^{\pi i m \tau} + \sum_{\substack{n \in \mathbb{Z} \\ n \geq N(k)}} c_{h_m}(n)\, e^{\pi i n \tau};$$

(2) *the function $\tilde{h}_m := (-i\tau)^{-k}\, h_m(\frac{-1}{\tau})$ has the Fourier expansion*

$$\tilde{h}_m(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ n \geq N(k)}} c_{\tilde{h}_m}(n)\, e^{\pi i n \tau};$$

(3) *the Fourier coefficients $c_{h_m}(n)$ and $c_{\tilde{h}_m}(n)$ satisfy the following estimates:*

$$\left|c_{h_m}(n)\right| \leq C\, m^{-k/2+\alpha}\, n^{k/2+\alpha},$$
$$\left|c_{\tilde{h}_m}(n)\right| \leq C\, m^{-k/2+\alpha}\, n^{k/2+\alpha}.$$

Here C and α are positive constants independent of k, m, and n, and depending on ϵ.

4. PROOF OF THEOREM 3.2

Let $P_{k,\chi,m}$ be the Poincaré series for the group $\Gamma(2)$ and multiplier system χ_k (see [3, P. 47, EQUATION (3.2)]). The Fourier coefficients of the Poincaré series can be explicitly computed by the Petersson formula,

$$c_{P_{k,\chi,m}}(n) = \delta_{m,n} + \sum_{c>0} S(m,n,c)\, \mathcal{J}_c(m,n). \tag{4.1}$$

Here $\mathcal{J}_c(m,n)$ is the following sum:

$$\mathcal{J}_c(m,n) = \frac{2\pi}{i^k c} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

the function J_ν is the Bessel J-function given by the power series

$$J_\nu(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\,\Gamma(\ell+1+\nu)} \left(\frac{x}{2}\right)^{\nu+2\ell}.$$

And $S(m,n,c)$ is the Kloosterman sum defined in [3, P. 51, EQUATION (3.13)]. The following estimate can be found in [5].

Lemma 4.1. *For a half-integer weight $k \geq 5/2$ and positive integers m, n, the Fourier coefficients of Poincaré series satisfy:*

$$\left|c_{P_{k,n}}(m) - \delta_{m,n}\right| \leq \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \varepsilon^{-2}\, n^{1+\varepsilon}\, m^{1+\varepsilon}\, C,$$
$$\left|c_{\tilde{P}_{k,n}}(m)\right| \leq \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \varepsilon^{-2}\, n^{1+\varepsilon}\, m^{1+\varepsilon}\, C.$$

Here C is an absolute constant and ε is any number in the interval $(0, \frac{1}{8}]$.

Lemma 4.2. *For a half-integer weight $k \geq 5/2$ and positive integers m, n lying in the interval $[1, N(k,\epsilon)]$, the Fourier coefficients of Poincaré series satisfy:*

(1) $|c_{P_{k,n}}(m) - \delta_{m,n}|\,(\frac{n}{m})^{\frac{k-1}{2}} \leq \frac{\epsilon}{N(k,\epsilon)}$;

(2) $|c_{\tilde{P}_{k,n}}(m)|\,(\frac{n}{m})^{\frac{k-1}{2}} \leq \frac{\epsilon}{N(k,\epsilon)}$.

Proof. Part (1) of the lemma is an immediate consequence of Stirling's formula.

The Mehler–Sonine formula [2] gives the following integral representation of the Bessel J-function:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{izs}\,(1-s^2)^{\nu-\frac{1}{2}}\,ds, \quad \nu > \frac{-1}{2},\ z \in \mathbb{C}.$$

This integral representation implies an estimate

$$\left|J_\nu(z)\right| \leq \frac{(z/2)^\nu\, 2}{\Gamma(\nu+1/2)\sqrt{\pi}}.$$

Also, we use the trivial estimate for the Klostermann sums (see [3, EQUATION (3.13)])

$$\left|S(m,n,c)\right| < c^2.$$

We combine these two estimates with the Petersson formula (4.1) for the Fourier coefficients of the Poincaré series and obtain

$$\begin{aligned}\left|c_{P_{k,n}}(m) - \delta_{m,n}\right| \left(\frac{n}{m}\right)^{\frac{k-1}{2}} &\leq 2\pi \sum_{c>0} c \left| J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right| \\ &\leq 4\pi \sum_{c>0} c \left|\frac{(2\pi/c)^{k-1}}{\Gamma(\nu+1/2)}\right| (mn)^{\frac{k-1}{2}} \\ &\leq 4\pi\, \frac{\zeta(k-2)\,(2\pi)^{k-1}}{\Gamma(k-1/2)}\,(mn)^{\frac{k-1}{2}}. \qquad (4.2)\end{aligned}$$

Note that $\sqrt{mn} \leq N(k,\epsilon)$, therefore inequality (4.2) and our choice of the function $N(k,\epsilon)$ imply part (2) of the lemma. Proof of part (3) is analogous. ∎

Proof of Theorem 3.2. Fix a half-integral weight k and $\epsilon \in (0, 1/2)$ and set $N := \lfloor N(k,\epsilon) \rfloor$. Consider a matrix $A = (a_{m,n})_{m,n=1}^{2N}$ with entries defined by the coefficients of the Poincaré series $\mathcal{P}_m := \mathcal{P}_{k,m}$ as

$$a_{m,n} = \begin{cases} c_{\mathcal{P}_m}(n)\,(\frac{m}{n})^{\frac{k-1}{2}} & \text{if } m,n \in [1,N], \\ c_{\tilde{\mathcal{P}}_m}(n-N)(\frac{m}{n-N})^{\frac{k-1}{2}} & \text{if } m \in [1,N],\ n \in [N+1,2N], \\ c_{\tilde{\mathcal{P}}_{m-N}}(n)(\frac{m-N}{n})^{\frac{k-1}{2}} & \text{if } m \in [N+1,2N],\ n \in [1,N], \\ c_{\tilde{\mathcal{P}}_{m-N}}(n-N)(\frac{m-N}{n-N})^{\frac{k-1}{2}} & \text{if } m,n \in [N+1,2N]. \end{cases}$$

From Lemma 4.2, we know that A is diagonally dominated and therefore invertible. Moreover, the inverse matrix $B = (b_{m,n})_{m,n=1}^{2N} := A^{-1}$ satisfies

$$|b_{m,n} - \delta_{m,n}| < \sum_{k=1}^{\infty} (2\epsilon)^k = \frac{2\epsilon}{1-2\epsilon}. \qquad (4.3)$$

Consider modular forms

$$h_\ell := \ell^{\frac{1-k}{2}} \sum_{n=1}^{N} (b_{\ell,n}\, P_n + b_{\ell,n+N}\, \tilde{P}_n)\, n^{\frac{k-1}{2}}, \quad \ell = 1, \ldots, N. \tag{4.4}$$

From the definition of coefficients $b_{\ell,n}$, we see

$$c_{h_\ell}(m) = \delta_{\ell,m} \quad \text{for } \ \ell, m = 1, \ldots N.$$

For the functions $\tilde{h}_\ell(\tau) := (-i\tau)^{-k}\, h_\ell(-1/\tau)$, we find

$$\tilde{h}_\ell := \ell^{\frac{1-k}{2}} \sum_{n=1}^{N} (b_{\ell,n}\, \tilde{P}_n + b_{\ell,n+N}\, P_n)\, n^{\frac{k-1}{2}}, \quad \ell = 1, \ldots, N. \tag{4.5}$$

The matrix A has symmetries $a_{m,n} = a_{m+N,n+N}$ and $a_{m+N,n} = a_{m,n+N}$ for $m, n = 1, \ldots, N$. Same symmetries are inherited by B, namely $b_{m,n} = b_{m+N,n+N}$, $b_{m+N,n} = b_{m,n+N}$ under same assumptions on indices m and n. Hence, we can rewrite (4.5) as

$$\tilde{h}_\ell := \ell^{\frac{1-k}{2}} \sum_{n=1}^{N} (b_{\ell+N,n}\, P_n + b_{\ell+N,n+N}\, \tilde{P}_n)\, n^{\frac{k-1}{2}}, \quad \ell = 1, \ldots, N.$$

Thus, we see that

$$c_{\tilde{h}_\ell}(m) = \delta_{\ell,m+N} = 0 \quad \text{for } \ \ell, m = 1, \ldots, N.$$

Finally, we prove part (3) of the theorem. Let ℓ and m be integers such that $\ell \in [1, N]$ and $m \in (N, \infty)$. We apply definition (4.4) and estimate the mth Fourier coefficient of h_ℓ as

$$\left|c_{h_\ell}(m)\right| \ell^{\frac{k-1}{2}}\, m^{\frac{1-k}{2}} \le \sum_{n=1}^{N} \Big(|b_{\ell,n}| \left|c_{P_n}(m)\right| n^{\frac{k-1}{2}}\, m^{\frac{1-k}{2}} + |b_{\ell,n+N}| \left|c_{\tilde{P}_n}(m)\right| n^{\frac{k-1}{2}}\, m^{\frac{1-k}{2}} \Big).$$

Now we apply Lemma 4.1 and estimate (4.3) in order to obtain

$$\left|c_{h_\ell}(m)\right| \ell^{\frac{k-1}{2}}\, m^{\frac{1-k}{2}} \le \frac{2}{1-2\epsilon} \sum_{n=1}^{N} \varepsilon^{-2}\, n^{1+\varepsilon}\, m^{1+\varepsilon}\, C \le \frac{2C}{(1-2\epsilon)\varepsilon^2}\, N^{2+\varepsilon}\, m^{1+\varepsilon}.$$

Analogously, we show that

$$\left|c_{\tilde{h}_\ell}(m)\right| \ell^{\frac{k-1}{2}}\, m^{\frac{1-k}{2}} \le \frac{2C}{(1-2\epsilon)\varepsilon^2}\, N^{2+\varepsilon}\, m^{1+\varepsilon}.$$

This finishes the proof of Theorem 3.2. ■

5. PROOF OF THEOREM 1.4

Lemma 5.1. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of subsets of $S(1)$ such that X_n is a spherical design of strength $D(n)$ and let $X := \bigcup_{n=1}^{\infty} \sqrt{n} X_n$. Suppose that f is a Schwartz function such that $f|_X = 0$. There exist an absolute positive constant C independent of f and X and a positive number β, which depends linearly on dimension d, such that for all $p \in \mathcal{B}$ and $n \in \mathbb{Z}_{\ge 1}$,*

$$\phi_{p,n} \le C \deg(p)^{\beta} \sum_{\substack{q \in \mathcal{B} \\ \deg(q) > D(n) - \deg(p)}} \phi_{q,n}.$$

Proof. By (2.3), we have

$$\phi_{p,n} = \left|\int_{S(1)} f(\sqrt{n}\zeta)\, p(\zeta)\, d\zeta\right| \cdot \sup_{\zeta\in S(1)} \left|p(\zeta)\right|.$$

For $M \in \mathbb{Z}_{\geq 0}$, we define the "head" of f as

$$h_M := \sum_{\substack{p\in\mathcal{B}\\ \deg(p)\leq M}} f_p$$

and the "tail" as

$$t_M := \sum_{\substack{p\in\mathcal{B}\\ \deg(p)> M}} f_p.$$

The integral in (2.3) can be written as

$$\int_{S(1)} f(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta = \int_{S(1)} \big(h_M(\sqrt{n}\zeta) + t_M(\sqrt{n}\zeta)\big)\, \|p(\zeta)\|\, d\zeta.$$

For a finite set $Y \subset S(1)$ and a function $g : S(1) \to \mathbb{C}$, we will use the notation

$$\int_Y g(\zeta)\, d\zeta := \frac{1}{|Y|}\sum_{y\in Y} g(y).$$

Suppose the integer M is chosen so that $M + \deg(p) \leq D(n)$. Then, our assumption that the set X_n is a spherical design of strength $D(n)$ implies that

$$\int_{S(1)} h_M(\sqrt{n}\zeta))\, \|p(\zeta)\|\, d\zeta = \int_{X_n} h_M(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta.$$

Thus, we can write the integral (2.2) as

$$\begin{aligned}
&\int_{X_n} h_M(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta + \int_{S(1)} t_M(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta\\
&\quad= \int_{X_n} (f - t_M)(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta + \int_{S(1)} t_M(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta\\
&\quad= \int_{X_n} f(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta + \left(\int_{S(1)} - \int_{X_n}\right) t_M(\sqrt{n}\zeta)\, \|p(\zeta)\|\, d\zeta. \qquad (5.1)
\end{aligned}$$

The first summand in the above line vanishes by the assumption that $f|_{X_n} = 0$. Therefore, we can estimate the integral (2.2) in the following way:

$$\left|\int_{S(1)} f(\sqrt{n}\zeta)\, p(\zeta)\, d\zeta\right| \leq 2 \sup_{\zeta\in S(1)} \left|p(\zeta)\right| \sup_{x\in S(\sqrt{n})} \left|t_M(x)\right|. \qquad (5.2)$$

We observe that

$$\sup_{x\in S(\sqrt{n})} \left|t_M(x)\right| \leq \sum_{\substack{q\in\mathcal{B}\\ \deg(q)> M}} \phi_{q,n}.$$

This finishes the proof of Lemma 5.1. ■

Theorems 3.1 and 3.2 give us other inequalities for the numbers $(\phi_{p,n})_{p\in\mathcal{B},n\in\mathbb{Z}_{\geq 1}}$.

Lemma 5.2. *Fix $\epsilon \in (0,1/2)$ and set $N(k) := \lfloor N(k,\epsilon) \rfloor$. Suppose that a Schwartz function f is an eigenfunction of the Fourier transform. There exists an absolute positive constant C big enough such that for all $p \in \mathcal{B}$ and all positive integers $m \le N(\deg(p)+d/2)$, we have*

$$\phi_{p,m} \le C\, m^{\alpha-\frac{d}{4}} \sum_{\substack{n\in\mathbb{Z}\\ n>N(\deg(p)+d/2)}} n^{\alpha+\frac{d}{4}}\, \phi_{p,n}.$$

Proof. Let f be a Schwartz function in $\mathbb{R}^d$. As described in Section 2, this function has a decomposition

$$f(x) = \sum_{p\in\mathcal{B}} f_p(x), \qquad f_p(x) = p(x)\, g_p\big(|x|\big).$$

Here for each homogenous harmonic polynomial $p \in \mathcal{B}$, the function $g_p : \mathbb{R}_{\ge 0} \to \mathbb{C}$ is such that the function $x \mapsto g_p(|x|)$ on $\mathbb{R}^d$ is a radial Schwartz function. A known result in analysis implies that $x \mapsto g_p(|x|)$ is a Schwartz function on any Euclidean space $\mathbb{R}^s$. We denote by $\mathcal{F}_s$ the s-dimensional Fourier transform and have

$$\mathcal{F}_d(f_p)(x) = \mathcal{F}_d\big(p(x)\, g_p\big(|x|\big)\big) = (-i)^{\deg(p)}\, p(y)\, \mathcal{F}_{d+2\deg(p)}(g_p)\big(|y|\big).$$

Let $\{h_m\}_{m=1}^{N(d/2+\deg(p))} \subset S_{d/2+\deg(p)}(\Gamma(2),\chi)$ be the modular forms constructed in Theorem 3.2. By Theorem 3.1, for each integer m on the interval $[1,\dots,N(d/2+\deg(p))]$, we have the following linear relation between values of g_p:

$$\sum_{n=1}^{\infty} g_p(\sqrt{n})\, c_{h_m}(n) = \sum_{n=1}^{\infty} \mathcal{F}_{d+2\deg(p)}(g_p)(\sqrt{n})\, c_{\widetilde{h}_m}(n).$$

Therefore for each point ζ on the sphere $S(1)$, we have

$$\sum_{n=1}^{\infty} g_p(\sqrt{n})\, p(\sqrt{n}\,\zeta)\, n^{\frac{-\deg(p)}{2}}\, c_{h_m}(n)$$
$$= (-i)^{\deg(p)} \sum_{n=1}^{\infty} \mathcal{F}_{d+2\deg(p)}(g_p)(\sqrt{n})\, p(\sqrt{n}\,\zeta)\, n^{\frac{-\deg(p)}{2}}\, c_{\widetilde{h}_m}(n).$$

This is equivalent to

$$\sum_{n=1}^{\infty} f_p(\sqrt{n}\zeta)\, n^{\frac{-\deg(p)}{2}}\, c_{h_m}(n) = (-i)^{\deg(p)} \sum_{n=1}^{\infty} \widehat{f_p}(\sqrt{n}\zeta)\, n^{\frac{-\deg(p)}{2}}\, c_{\widetilde{h}_m}(n).$$

Conditions (1) and (2) of Theorem 3.2 imply that for an integer m in the interval $[1, N(d/2+\deg(p))]$ and a point ζ on the sphere $S(1)$,

$$f_p(\sqrt{m}\,\zeta)\, m^{\frac{-\deg(p)}{2}} = \sum_{n=1}^{\infty} \big(f_p(\sqrt{n}\,\zeta)\, c_{h_m}(n) + (-i)^{\deg(p)}\, \widehat{f_p}(\sqrt{n}\,\zeta)\, c_{\widetilde{h}_m}(n)\big)\, n^{\frac{-\deg(p)}{2}}.$$

Now condition (3) of Theorem 3.2 and the assumption that f is an eigenfunction of the Fourier transform imply that

$$\Big|f_p(\sqrt{m}\,\zeta)\, m^{\frac{-\deg(p)}{2}}\Big| \le C \sum_{n=N(d/2+\deg(p))+1}^{\infty} \big|f_p(\sqrt{n}\,\zeta)\big|\, n^{\frac{-\deg(p)}{2}}\, n^{\frac{d}{4}+\frac{\deg(p)}{2}+\alpha}\, m^{-\frac{d}{4}-\frac{\deg(p)}{2}+\alpha}.$$

■

We set $\tilde{\alpha} := \alpha + d/4$. For all $p \in \mathcal{B}$ and all positive integers $m \le N(\deg(p) + d/2)$, we have

$$\phi_{p,m} \le C\, m^{\tilde{\alpha}} \sum_{\substack{n \in \mathbb{Z} \\ n > N(\deg(p)+d/2)}} n^{\tilde{\alpha}}\, \phi_{p,n}.$$

Now, we are ready for the final step in the proof of Theorem 1.4. In particular, we will define the positive constants $\tilde{A}(d)$ and $\tilde{B}(d)$. We will show that for a suitable choice of $\tilde{A}(d)$ and $\tilde{B}(d)$ the growth condition of Theorem 2.1, combined with the inequalities of Lemmas 5.1 and 5.2, implies the vanishing of the numbers $(\phi_{p,n})_{p \in \mathcal{B}, n \in \mathbb{Z}_{\ge 0}}$.

For each $\epsilon \in (0, 1/2)$, there exists a sufficiently small positive number b such that

$$N(k, \epsilon) \ge bk, \quad k \in \frac{1}{2}\mathbb{Z}_{\ge 1}.$$

For a polynomial $p \in \mathcal{B}$, we set

$$\mathcal{N}(p) := b \deg(p).$$

Note that

$$\mathcal{N}(p) \le N\big(\deg(p) + d/2\big).$$

Let C' and γ be positive numbers (depending on dimension d) such that $\dim \mathcal{H}_m \le C' m^{\gamma}$. Note that $\gamma = d - 2$ is admissible. We will need the following technical statement.

Lemma 5.3. *For each dimension d, we consider $D(n) := \tilde{B}\, n^{\tilde{A}}$, where*

$$\tilde{B} > 2 \max\left(b + \frac{1}{b}, \frac{C\,C'}{b^{\beta+\gamma+1}}\right), \qquad \tilde{A} = 2\tilde{\alpha} + \beta + \gamma + 3.$$

Then

(1) *for $p, q \in \mathcal{B}$ and $n \in \mathbb{Z}_{\ge 1}$, the conditions $n \ge \mathcal{N}(p)$ and $\deg(q) \ge D(n) - \deg(p)$ imply $n \le \mathcal{N}(q)$.*

(2) *for all positive integers m and all $q \in \mathcal{B}$ with $m \ge \mathcal{N}(q)$, we have*

$$\sum_{\substack{n \in \mathbb{Z}_{\ge 1}, p \in \mathcal{B}: \\ n \ge \mathcal{N}(p) \\ D(n) - \deg(p) \le \deg(q)}} C \cdot \deg(p)^{\beta} \cdot n^{2\tilde{\alpha}+1} < m.$$

Proof. Part (1) of the lemma follows immediately from our choice of $\tilde{A}$ and $\tilde{B}$. Indeed, we observe that $\tilde{A} > 1$ and $\tilde{B} > \frac{1}{b}$. Therefore we have

$$\mathcal{N}(q) = b \deg(q) \ge b\big(2\tilde{B}n^{\tilde{A}} - \deg(p)\big) > b\left(\frac{2n}{b} - \frac{n}{b}\right) = n.$$

We rewrite the sum in part (2) in the following way:

$$\sum_{\substack{n \in \mathbb{Z}_{\ge 1},\ p \in \mathcal{B}: \\ n \ge \mathcal{N}(p) \\ D(n) - \deg(p) \le \deg(q)}} C \cdot \deg(p)^{\beta} \cdot n^{2\tilde{\alpha}+1} = \sum_{\substack{n \in \mathbb{Z}_{\ge 1} \\ D(n) - \frac{n}{b} \le \deg(q)}} \sum_{\substack{p \in \mathcal{B}: \\ \deg(p) \le \frac{n}{b} \\ \deg(p) \ge D(n) - \deg(q)}} C \cdot \deg(p)^{\beta} \cdot n^{2\tilde{\alpha}+1}.$$

Now we use that $D(n) - \frac{n}{b} \geq \frac{1}{2}\tilde{B}\, n^{\tilde{A}}$ and estimate the above expression by

$$\leq \sum_{\substack{n \in \mathbb{Z}_{\geq 1} \\ \frac{1}{2}\tilde{B}\, n^{\tilde{A}} \leq \deg(q)}} \sum_{\substack{p \in \mathcal{B}: \\ \deg(p) \leq \frac{n}{b} \\ \deg(p) \geq D(n) - \deg(q)}} C \cdot \deg(p)^{\beta} \cdot n^{2\tilde{\alpha}+1}.$$

Next we use the fact that the dimension of $\mathcal{H}_{\deg(p)}$ is bounded by $C' \deg(p)^{\gamma}$ and bound the sum in part (2) by

$$\leq \sum_{\substack{n \in \mathbb{Z}_{\geq 1} \\ \frac{1}{2}\tilde{B}\, n^{\tilde{A}} \leq \frac{m}{b}}} \sum_{\substack{s \in \mathbb{Z}_{\geq 1}: \\ D(n) - \frac{m}{b} \leq s \leq \frac{n}{b}}} C\, C'\, s^{\beta+\gamma} \cdot n^{2\tilde{\alpha}+1}.$$

This sum does not exceed

$$\sum_{\substack{n \in \mathbb{Z}_{\geq 1} \\ n \leq (\frac{2m}{b\,\tilde{B}})^{1/\tilde{A}}}} C\, C' \left(\frac{n}{b}\right) \left(\frac{n}{b}\right)^{\beta+\gamma} n^{2\tilde{\alpha}+1}.$$

Finally, we crudely estimate each term of this sum by substituting $n \mapsto (\frac{2m}{b\,\tilde{B}})^{1/\tilde{A}}$ and bounding the number of terms by $(\frac{2m}{b\,\tilde{B}})^{1/\tilde{A}}$. This gives us an upper bound

$$\frac{C\, C'}{b^{\beta+\gamma}} \left(\frac{2m}{b\,\tilde{B}}\right)^{\frac{2\tilde{\alpha}+\beta+\gamma+3}{\tilde{A}}}.$$

Now, our choice of $\tilde{A}$ and $\tilde{B}$ guarantees that the sum in part (2) of the lemma is less than m. ∎

Proof. We are ready to complete the proof of Theorem 1.4. Let $(X_n)_{n=1}^{\infty}$ be a collection of spherical designs on the sphere $S(1)$. We suppose that for each n the design X_n has strength $D(n) = \tilde{B}\, n^{\tilde{A}}$, where $\tilde{A}$ and $\tilde{B}$ are defined in the Lemma 5.3. We will show that $X = \bigcup_n \sqrt{n} X_n$ is a Fourier uniqueness set. Suppose that $f : \mathbb{R}^d \to \mathbb{C}$ is a Schwartz function that satisfies

$$f|_X \equiv 0 \quad \text{and} \quad \hat{f}|_X \equiv 0. \tag{5.3}$$

Then for each $n \in \mathbb{Z}_{\geq 1}$, we have

$$f|_{\sqrt{n}X_n} = \hat{f}|_{\sqrt{n}X_n} = 0.$$

Without loss of generality, we assume that f is an eigenfunction of the Fourier transform.

Consider the sum

$$\sum_{\substack{p \in \mathcal{B},\, n \in \mathbb{Z}: \\ n \geq \mathcal{N}(p)}} \phi_{p,n}\, n^{\tilde{\alpha}+1}. \tag{5.4}$$

By Theorem 2.1, this sum of nonnegative numbers converges to a finite limit.

By Lemma 5.1, we can estimate the sum (5.4) as

$$\sum_{\substack{p \in \mathcal{B}, n \in \mathbb{Z}: \\ n \geq \mathcal{N}(p)}} \phi_{p,n}\, n^{\tilde{\alpha}+1} \leq \sum_{\substack{p \in \mathcal{B}, n \in \mathbb{Z}: \\ n \geq \mathcal{N}(p)}} n^{\tilde{\alpha}+1}\, C \deg(p)^{\beta} \cdot \sum_{\substack{q \in \mathcal{B}: \\ \deg(q) > D(n) - \deg(p)}} \phi_{q,n}.$$

We have chosen the numbers $\tilde{A}$ and $\tilde{B}$ so that the conditions $n \geq \mathcal{N}(p)$ and $\deg(q) \geq D(n) - \deg(p)$ imply $n \leq \mathcal{N}(q)$. We apply Lemma 5.2 and estimate

$$\sum_{\substack{p\in\mathcal{B},n\in\mathbb{Z}:\\ n\geq\mathcal{N}(p)}} \phi_{p,n}\, n^{\tilde{\alpha}+1} \leq \sum_{\substack{p\in\mathcal{B},n\in\mathbb{Z}:\\ n\geq\mathcal{N}(p)}} n^{\tilde{\alpha}+1}\, C\deg(p)^{\beta} \cdot \sum_{\substack{q\in\mathcal{B}:\\ \deg(q)>D(n)-\deg(p)}} \sum_{\substack{m\in\mathbb{Z}:\\ m\geq\mathcal{N}(q))}} m^{\tilde{\alpha}}\, n^{\tilde{\alpha}}\, \phi_{q,m}.$$

Here, C is a new constant equal to the product of the constant C from Lemma 5.1 and the constant C from Lemma 5.2. We change the order of summation and arrive at

$$\sum_{\substack{p\in\mathcal{B},n\in\mathbb{Z}:\\ n\geq\mathcal{N}(p)}} \phi_{p,n}\, n^{\tilde{\alpha}+1} \leq \sum_{\substack{m\in\mathbb{Z},q\in\mathcal{B}:\\ m\geq\mathcal{N}(q))}} m^{\tilde{\alpha}}\, \phi_{q,m} \sum_{\substack{p\in\mathcal{B},n\in\mathbb{Z}:\\ n\geq\mathcal{N}(p)\\ D(n)-\deg(p)\leq\deg(q)}} C\, n^{2\tilde{\alpha}+1}\deg(p)^{\beta}.$$

By Lemma 5.3, the inner sum on the right-hand side of this inequality satisfies

$$\sum_{\substack{p\in\mathcal{B},n\in\mathbb{Z}:\\ n\geq\mathcal{N}(p)\\ D(n)-\deg(p)\leq\deg(q)}} C\, n^{2\tilde{\alpha}+1}\deg(p)^{\beta} < m.$$

This inequality is guaranteed by our choice of function D. Suppose that the nonnegative numbers $(\phi_{q,m})_{\substack{m\in\mathbb{Z},q\in\mathcal{B}\\ m\geq\mathcal{N}(q)}}$ are not all zero. Then

$$\sum_{\substack{p\in\mathcal{B},n\in\mathbb{Z}:\\ n\geq\mathcal{N}(p)}} \phi_{p,n}\, n^{\tilde{\alpha}+1} < \sum_{\substack{q\in\mathcal{B},m\in\mathbb{Z}:\\ m\geq\mathcal{N}(q)}} \phi_{q,m}\, m^{\tilde{\alpha}+1}.$$

This is a contradiction. Therefore, our assumptions on the Schwartz function f imply that $\phi_{q,m} = 0$ whenever $m \geq \mathcal{N}(q)$. Moreover, Lemma 5.2 implies that $\phi_{q,n} = 0$ for all $q \in \mathcal{B}$ and $n \in \mathbb{Z} \geq 0$. Finally, we deduce from Theorem 1.2 that for all harmonic polynomials p in the basis $\mathcal{B}$ the functions f_p in the decomposition (2.1) of the Schwartz function f vanish. Therefore, f is also identically zero. This finishes the proof of Theorem 1.4. ■

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