REPRESENTATIONS OF p-ADIC GROUPS OVER COMMUTATIVE RINGS

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ABSTRACT

Motivated by the Langlands program in representation theory, number theory, and geometry, the theory of representations of a reductive p-adic group, originally in complex vector spaces, has been widely developed in modules over a commutative ring during the last two decades. This article surveys basic results obtained during this period, assuming some familiarity with the representation theory connected to the Langlands program. Addressed to a broader audience, the 2022 ICM Noether Lecture should be accessible without prerequisites and convey intuition on the most striking results.

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1. INTRODUCTION

The theory of representations of a p-adic group G, for instance, $\mathrm{GL}(n, \mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the p-adic completion of $\mathbb{Q}$ is an essential part of the Langlands program. At the beginning, it was a question of studying representations in a complex vector space. But the development of its links with number theory and geometry has required studying continuous representations in vector spaces defined over other fields than $\mathbb{C}$. There are many possibilities for such a generalization. It is easy to replace $\mathbb{C}$ by an algebraic closure $\mathbb{Q}_\ell^{\mathrm{ac}}$ of a local field $\mathbb{Q}_\ell$, where ℓ is a prime different from p. The choice of a field isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{\mathrm{ac}}$ identifies continuous complex representations of G and continuous ℓ-adic representations. A more difficult case is that of $\ell = p$ because the topology of a p-adic group and of $\mathbb{Q}_p$ is the same. One even considers representations with values not in a vector space, but in a module over some commutative ring like $\mathbb{Z}[1/p]$ or $\mathbb{Z}/p^i\mathbb{Z}, i \geq 1$. The representations over these different categories of coefficient rings are now essential in the theory of automorphic forms. Their theory has been widely developed since the beginning of the 21st century, and different versions of the local Langlands correspondence have emerged.

We review the main basic results for representations over coefficient rings[1] different from $\mathbb{C}$. In an attempt to make this paper accessible to readers with a wide range of backgrounds, we give fairly complete definitions of all terminology. Proofs are omitted, yet we give a short indication of the key points, we cite sources and provide examples. For the theory before 2002, the reader may consult our book[2] and our article in the proceedings of the Bejing ICM. The subject has remained confined in research articles since these last two decades, and we hope that this survey provides a navigable route to the literature.

2. NOTATION

We work with a triple (F, G, R) where F is the basic field, G the reductive p-adic group, and R the coefficient ring. We assume that F is a local non-archimedean field of ring of integers O_F, uniformizer p_F, and residue field k_F of characteristic p with q elements, G is the group $\underline{G}(F)$ of F-points of a connected reductive F-group $\underline{G}$, endowed with the topology generated by the open pro-p-subgroups[3] and R is a commutative ring.[4]

An R-representation V of G will always be *smooth* (continuous for the discrete topology on R). It is *admissible* if for all open compact subgroups K of G, the R-module V^K of vectors fixed by K is finitely generated.

The absolute Galois group Gal_E of a field E is the group of automorphisms of an algebraic closure E^{ac} fixing E. For a prime number r, $\mathbb{F}_r$ denotes a field with r elements,

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1. That we are aware of, without geometry or derived functors.
2. Représentations ℓ-modulaires d'un groupe réductif p-adique avec ℓ différent de p, Birkhaüser, 1996.
3. Called a connected reductive p-adic group, but beware that some authors use this terminology only when F contains $\mathbb{Q}_p$.
4. A ring is supposed to have a unit.

$\mathbb{Z}_r$ denotes the ring of integers in the field $\mathbb{Q}_r$ of r-adic numbers, and $\mathbb{Z}_r^{\mathrm{ac}}$ denotes the ring of integers of $\mathbb{Q}_r^{\mathrm{ac}}$. We always denote by ℓ a prime number different from p.

The parabolic and parahoric subgroups play an essential role in the theory of representations of a reductive p-adic group.

The parabolic subgroups appear for the first time in the section on parabolic induction. We fix a maximal split[5] torus T of G of G-centralizer Z and a minimal parabolic subgroup $B = ZU$ of unipotent radical U and opposite $B^{\mathrm{op}} = ZU^{\mathrm{op}}$. A *standard parabolic subgroup* of G is a parabolic subgroup containing B, that is, $P = MN = MB$, with unipotent radical N contained in U and Levi subgroup M containing Z. The opposite parabolic subgroup $P^{\mathrm{op}} = MN^{\mathrm{op}} = MB^{\mathrm{op}}$ is not standard.

The Weyl group W_G is equal to the quotient of the G-normalizer of T by Z. We denote by $Z^+ \subset Z$ the submonoid of elements contracting U by conjugation, Z^- those contracting U^{op}, $T^+ = T \cap Z^+$, $T^- = T \cap Z^-$. The group G is split if $T = Z$ and quasisplit if Z is a torus.[6]

The parahoric subgroups appear for the first time in the section on Hecke algebras. We fix a *special parahoric subgroup* K of G and a *pro-p-Iwahori subgroup* $\tilde{J}$ of G, as follows. We choose a special point x_0 of the apartment $\mathfrak{A}$ of T in the adjoint Bruhat–Tits building of G. The parahoric subgroup of G fixing the alcove in $\mathfrak{A}$ of vertex x_0 associated to B is an Iwahori subgroup J of G. Then K is the parahoric subgroup fixing x_0 and $\tilde{J}$ is the maximal open normal pro-p subgroup of J. For a standard parabolic subgroup $P = MN$ of G, $M^0 = M \cap K$ is a special parahoric subgroup of M and $\tilde{J}_M = \tilde{J} \cap M$ is a pro-p Iwahori subgroup of M. We denote $N^0 = K \cap N$.

The pro-p-Iwahori subgroups of G are all G-conjugate, but in general there are only finitely many G-conjugacy classes of special parahoric subgroups of G.

Examples. There are two conjugacy classes of special parahoric subgroups of $\mathrm{SL}(2, F)$.

The special parahoric subgroups of $\mathrm{GL}(n, F)$ are conjugate to $\mathrm{GL}(n, O_F)$.

The inverse image by the quotient map $\mathrm{GL}(n, O_F) \to \mathrm{GL}(n, k_F)$ of the (strictly) upper triangular group of $\mathrm{GL}(n, k_F)$ is a (pro-p) Iwahori subgroup of $\mathrm{GL}(n, F)$.

The split torus T has a unique parahoric subgroup, equal to the maximal compact subgroup $T^0 = T \cap K = T \cap J$, and the quotient T/T^0 is isomorphic to the group $X_*(T)$ of cocharacters of T via p_F. The compact mod center connected reductive group Z has a unique parahoric subgroup $Z^0 = Z \cap K = Z \cap J$, and the quotient Z/Z^0 is a commutative finitely generated group (Thomas Haines and Sean Rostami [83]).

5 This means, by a common abuse of notation, that $T = \underline{T}$ where $\underline{T}$ is a maximal F-split torus of $\underline{G}$.

6 When G is not quasisplit, Z is not commutative.

3. CHANGE OF BASIC FIELD

The basic field F is a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$. It is called a *p-adic field* in characteristic 0 and a *local function field* in characteristic p. Many geometric methods demand F to be a local function field. For example, the proof by Bao Chau Ngo[7] of the fundamental lemma, essential in the Langlands theory, which asserts an equality between certain linear combinations of integral orbitals over the Lie algebras of G and of endoscopic groups. On the other hand, when F is a local function field, the harmonic analysis is full of traps, there are inseparable semisimple elements, there is no exponential map to pass to the Lie algebra and G has no cocompact discrete subgroup (except for type A), G is not a p-adic Lie group.

But the basic field F appears only through the residual field in many constructions (endoscopy, buildings, Iwahori Hecke algebras). This is a key to transfer properties between basic fields of different characteristics. For instance, Jean-Loup Waldspurger **[201]** proved that the fundamental lemma for F of characteristic p implies the fundamental lemma for F of characteristic 0. There is another proof using the general transfer principle of Cluckers and Loeser in model theory and motivic integration **[31, 32]**. In the other direction, the fundamental lemma for the automorphic induction for $\mathrm{GL}(n, F)$ proved by Guy Henniart and Rebecca Herb for F of characteristic 0 was transferred to F of characteristic p by Henniart and Bertrand Lemaire **[102]** using *close local fields*. For a positive integer m, two non-archimedean local fields are m-closed, if their rings of integers modulo the mth power of their respective maximal ideals are isomorphic. The Deligne–Kazhdan philosophy can be loosely stated as follows: the representation theory of Galois groups (or of reductive groups) over m-close local fields is the same "up to level m". For instance, Radhika Ganapathy **[74]** proved that for two m-close local fields F, F' and $\underline{G}$ split, the category of complex representations of $\underline{G}(F)$ generated by their invariants by the m-filtration subgroup of an Iwahori subgroup is equivalent to the same category for representations of $\underline{G}(F')$. For $\underline{G}$ not split, she made sense of a natural connected reductive group $\underline{G}'$ over F' associated to $\underline{G}$, first when $\underline{G}$ is quasisplit (an F-form of a split group) and then when $\underline{G}$ is general (an inner form of a quasisplit group) **[75, 3.A AND 5.A]**.

The local field $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ and $\mathbb{Q}$ is a globalization of $\mathbb{Q}_p$. The local case is simpler than the global. The ring $\mathbb{Z}_p$ has only one prime ideal, namely $p\mathbb{Z}_p$, but the ring $\mathbb{Z}$ has infinitely many prime ideals. The absolute Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is simple compared to $\mathrm{Gal}_{\mathbb{Q}}$. In the same vein, the local field F is the completion of a (non-unique) global field[8] E and E is a globalization of F, the local group G is a localization of the group H of rational points of a connected reductive group over a global field, and H is a globalization of G.[9] An automorphic irreducible $\mathbb{C}$-representation V_A of the adelic group

7 Fields medal in 2020.

8 A global field is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(T)$.

9 For F of characteristic p, Wee-Teck Gan, Luis Lomeli **[72]**, for F of characteristic 0, Shahidi (A proof of Langland's Conjecture on Plancherel measures; Complementary Series of $\mathfrak{p}$-adic groups, The Annals of Math., Series 2, Vol. 132, 2 (1990), 273–330) when G is quasisplit, implying the general case as in **[72]**.

$H(A)$ gives by localization an irreducible $\mathbb{C}$-representation V of G and V_A is a globalization of V. The study of automorphic representations uses the theory of representations of local reductive groups. In the other direction, some theorems of representations of local groups are proved by embedding the local case into a global one.

The classical local Langlands correspondence introduced by Langlands in 1967–1970 is a generalization of local class field theory from abelian Galois groups to non-abelian Galois groups. The absolute Galois group Gal_{k_F} of the finite field k_F is topologically generated but the Frobenius $\mathrm{Frob}(x) = x^q$. The subgroup of elements in Gal_F with image an integral power of Frob in the natural quotient map $\mathrm{Gal}_F \to \mathrm{Gal}_{k_F}$ is the *Weil group* W_F of F.[10] The reciprocity map of local class field theory $F^* \tilde{\to} W_F^{ab}$ identifies the irreducible R-representations of $\mathrm{GL}(1, F)$ with the one-dimensional R-representations of W_F when R is an algebraically closed field. Langlands proposed a parametrization of the irreducible $\mathbb{C}$-representations of G in terms of $\mathbb{C}$-representations of W_F.

When $G = \mathrm{GL}(n, F)$, the complex local Langlands correspondence is a theorem which has been generalized to representations over $R = \mathbb{F}_\ell^{\mathrm{ac}}$, $\ell \neq p$.[11] The first proofs of local class field theory were global. Today the proofs of the local Langlands correspondence for $\mathrm{GL}(n, F)$ needs global arguments, except for $n = 2$ and $R = \mathbb{C}$, where there is a local proof (Colin Bushnell and Henniart **[21]**). When F has characteristic 0, Peter Scholze **[174]** gave a new local characterization of the complex local Langlands correspondence; a local Langlands correspondence over $R = \mathbb{F}_p^{\mathrm{ac}}$ is to-day a very active research area.[12]

The geometrization of a (semisimple) local Langlands correspondence for all F, G and $R = \mathbb{Z}_\ell$ for almost all $\ell \neq p$, obtained by Laurent Fargues and Scholze in 2021, is entirely local.

4. CHANGE OF COEFFICIENT RING

Many features of complex representations of G use harmonic analysis only apparently and can be generalized to representations over other coefficient rings. For instance,

(a) The theory of discrete series and tempered complex representations has an algebraic and combinatorial flavor.[13] It was extended by Dat **[38]** to an algebraically closed field R of characteristic different from p with a nontrivial valuation.

(b) The proof of the classification of the irreducible complex representations of an inner form of $\mathrm{GL}(n, F)$ by Tadic for $F \supset \mathbb{Q}_p$ uses harmonic analysis (the simple trace formula). Alberto Minguez and Vincent Sécherre **[139]** gave an

10 The kernel I_F of the quotient map is an extension of $\prod_{\ell \neq p} \mathbb{Z}_\ell$ by a pro-p group P_F.

11 Proved when $R = \mathbb{C}$ by Gérard Laumon, Michael Rapoport, and Ulrich Stuhler in 1993 if $F \supset \mathbb{F}_p((t))$, and if $F \supset \mathbb{Q}_p$ by Michael Harris and Richard Taylor in 2001 (Guy Henniart gave another proof), and extended by Vignéras in 2001 to $R = \mathbb{F}_\ell^{\mathrm{ac}}$, $\ell \neq p$.

12 There is nothing for F and R of characteristic p, to the best of my knowledge.

13 The asymptotic behavior of coefficients may be derived from the central exponents of the Jacquet modules.

algebraic proof for all F and all algebraic closed fields R of characteristic different from p.

A prime $\ell \neq p$ not dividing the order of a torsion element of G is called *banal*[14] for G. A field R is of *banal characteristic* for G if its characteristic is 0 or ℓ banal for G. A general principle is that the properties of complex representations of G described in purely algebraic terms transfer to representations of G over fields R of banal characteristic.

Example. The banal primes for $\mathrm{GL}(m, F)$ are those coprime with $q^i - 1$ for $1 \leq i \leq m$.

The R-representations of G form a locally small abelian Grothendieck category $\mathrm{Mod}_R(G)$ (Vignéras [199]). For a commutative ring S which is an R-algebra, the R-representations of G are related to the S-representations of G by the scalar extension[15]

$$S \otimes_R - : \mathrm{Mod}_R(G) \to \mathrm{Mod}_S(G)$$

and, by the restriction its right adjoint, an S-representation is considered as an R-representation. One says that an S-representation of G in the image of the scalar extension *descends* to R, or *is defined* on R.

When R is a field, many properties on admissible irreducible R-representations of G still assume R to be algebraically closed although this is not necessary. A good tool to show this is the bijection (Henniart–Vignéras [106], [107, SECTION 2])

$$V \mapsto BC(V)$$

- from the isomorphism classes of irreducible admissible R-representations V of G,
- onto the Galois orbits[16] $BC(V)$ of the isomorphism classes of the irreducible admissible R^{ac}-representations of G defined on a finite extension of R.

Here $BC(V)$ is the set of isomorphism classes of the irreducible subquotients V^{ac} of

$$R^{\mathrm{ac}} \otimes_R V \simeq \oplus^d \oplus_{V^{\mathrm{ac}} \in BC(V)} W(V^{\mathrm{ac}}),$$

where d is the reduced dimension of the division R-algebra $\mathrm{End}_{RG} V$ over its center E_V, the length of the R^{ac}-representation $R^{\mathrm{ac}} \otimes_R V$ of G is $d[E_V : R]$, the number of elements of $BC(V)$ is $[E_V^s : R]$ where E_V^s is the maximal separable subextension of E_V/R, and $W(V^{\mathrm{ac}})$ is an indecomposable R^{ac}-representation of G of irreducible subquotients isomorphic to V^{ac} and of length $[E_V : E_V^s]$. Any $V^{\mathrm{ac}} \in BC(V)$ is V-isotypic as an R-representation of G, and is defined on a maximal subfield of $\mathrm{End}_{RG} V$ (Justin Trias [188]).

Any irreducible admissible R^{ac}-representation of G is absolutely irreducible and has a central character by the Schur's lemma. If the characteristic of R is different from p,

14 See [51], Lemma 5.22 and Corollary 5.23 for other characterizations.
15 Also called base change or induction.
16 An orbit under the group $\mathrm{Aut}_R(R^{\mathrm{ac}})$ of R-automorphisms of R^{ac}.

any irreducible R^{ac}-representation of G is admissible and defined on a finite extension of R **[107]**.

As G is locally a pro-p group, there is no Haar measure on G with values in a commutative ring R where p is not invertible and the R-representations of G present new phenomena. To understand them is a crucial question.

For a field R of characteristic p, any irreducible R-representation V of G with $\dim_R V^K < \infty$ for some open pro-p subgroup K of G, is admissible (Vytautas Paskunas **[155]**, a simple proof is given in Henniart **[101]**). For any open pro-p subgroup K of G, any nonzero representation of G has a nonzero vector invariant by K (like for finite groups).

Irreducible implies admissible when $G = \mathrm{GL}(2, \mathbb{Q}_p)$. Indeed, one reduces to $R = \mathbb{F}_p^{\mathrm{ac}}$; in this case irreducible implies that the center acts by a character (Laurent Berger **[14]**) hence is admissible by Barthel-Livne and Breuil **[16]**.

But, there exists an irreducible non-admissible $\mathbb{F}_p^{\mathrm{ac}}$-representation of $\mathrm{GL}(2, F)$ for an unramified extension F of $\mathbb{Q}_p$ (Daniel Le **[136]**). One does not know if any infinite-dimensional irreducible non-admissible $\mathbb{F}_p^{\mathrm{ac}}$-representation of G has a central character, because its dimension is equal to the cardinal of $\mathbb{F}_p^{\mathrm{ac}}$ and the classical proof with the Schur's lemma does not apply.

It happens that a property of admissible irreducible representations of G over a field R transfers to representations of G over any coefficient field of the same characteristic. This is the case in the following examples:

(i) In characteristic different from p, for the classification of cuspidal irreducible R-representations of G by compact induction (Henniart–Vignéras **[107]**).

(ii) In characteristic p, for the classification of non-cuspidal[17] admissible irreducible R-representations of G, for the classification of non-supersingular simple modules of the pro-p-Iwahori Hecke R-algebra of G (Noriyuki Abe, Henniart, Florian Herzig, and Vignéras **[8]**, Henniart–Vignéras **[106]**), for the existence of a supersingular admissible irreducible R-representation of G when $F \supset \mathbb{Q}_p$ (Herzig, Karol Koziol, and Vignéras **[110]**).

For a prime r,[18] an *r-adic representation* of G is a representation of G on a $\mathbb{Q}_r^{\mathrm{ac}}$-vector space which is continuous for the r-adic topology on the vector space. A p-adic representation of G may be not smooth, but an ℓ-adic representation of G is smooth if $\ell \neq p$. In this article, an R-representation of G is supposed always to be smooth. A $\mathbb{Q}_r^{\mathrm{ac}}$-representation of G is a smooth r-adic representation of G. The choice of an isomorphism

$$\mathbb{C} \simeq \mathbb{Q}_r^{\mathrm{ac}}$$

identifies the complex representations of G and the $\mathbb{Q}_r^{\mathrm{ac}}$-representations of G.

A *mod r representation* of G is a $\mathbb{F}_r^{\mathrm{ac}}$-representation of G.

17 Cuspidal and supersingular will be defined later.

18 Letter ℓ is reserved for the primes different from p, think $r = \ell$ or p.

An admissible $\mathbb{Q}_r^{\mathrm{ac}}$-representation V of G is called *integral* if V is defined on a finite extension $E/\mathbb{Q}_r$ and V contains an G-stable $\mathbb{Z}_r^{\mathrm{ac}}$-lattice L,[19] admissible as a $\mathbb{Z}_r^{\mathrm{ac}}$-representation of G and descending to O_E.[20] The mod r representation $\mathrm{red}_r(L) = L \otimes_{\mathbb{Z}_r^{\mathrm{ac}}} \mathbb{F}_r^{\mathrm{ac}}$ of G is called the reduction of L.

By the strong Brauer–Nesbitt theorem (Vignéras **[189]**), when $r = \ell \neq p$, the $\mathbb{Z}_\ell^{\mathrm{ac}}[G]$-module L is finitely generated, of reduction $\mathrm{red}_\ell(L)$ of finite length, and the image of $\mathrm{red}_\ell(L)$ in the Grothendieck group of finite length $\mathbb{F}_\ell^{\mathrm{ac}}$-representations of G does not depend on the choice of L; it is called the *reduction* of V. Two finite-length integral ℓ-adic representations of G are said to be *congruent modulo* ℓ when their reductions are isomorphic.

This does not hold true for $\mathbb{Q}_p^{\mathrm{ac}}$-representations of G. For example, an irreducible $\mathbb{Q}_p^{\mathrm{ac}}$-representation $V = \mathrm{ind}_K^G W$ of $G = \mathrm{PGL}(n, F)$ compactly induced from a representation W of $K = \mathrm{PGL}(n, O_F)$ contains an admissible G-stable $\mathbb{Z}_p^{\mathrm{ac}}$-lattice L defined on some O_E as above, of infinite length reduction, and another one L' of finite length reduction. Take $L = \mathrm{ind}_K^G W_{\mathbb{Z}_p^{\mathrm{ac}}}$ for a K-stable $\mathbb{Z}_p^{\mathrm{ac}}$-lattice $W_{\mathbb{Z}_p^{\mathrm{ac}}}$ of W and $L' = V \cap \mathrm{ind}_\Gamma^G 1_{\mathbb{Z}_p^{\mathrm{ac}}}$ for a small enough discrete cocompact subgroup Γ of G.

5. PARABOLIC INDUCTION

For any triple (F, G, R) (as in the Notation section) and any parabolic subgroup P of G of Levi quotient M, the *parabolic induction*[21]

$$\mathrm{ind}_P^G : \mathrm{Mod}_R(M) \to \mathrm{Mod}_R(G)$$

allows constructing representations of G from representations of the smaller connected reductive p-adic group M. The parabolic induction has excellent properties, it commutes with small direct sums[22] (Vignéras **[199]**); for p nilpotent in R, it is fully faithful (Vignéras **[199]**); for a field R, the parabolic induction respects finite length representations with admissible subquotients (this depends on the classification of admissible irreducible representations if the characteristic of R is p).

The parabolic induction is exact and has a *left adjoint* L_P^G called the Jacquet functor, equal to the coinvariant functor $(-)_N$ with respect to the unipotent radical N of P, and a *right adjoint*[23] R_P^G (Vignéras **[199]**). By adjointness, L_P^G is right exact and R_P^G is left exact. The scalar extension commutes with the three parabolic functors (Henniart–Vignéras **[106]**).

For p invertible in R, the *second adjunction*

$$R_P^G = \delta_P L_{P^{\mathrm{op}}}^G,$$

19 A free $\mathbb{Z}_r^{\mathrm{ac}}$-submodule of scalar extension V to $\mathbb{Q}_r^{\mathrm{ac}}$.

20 The ring O_E is principal but not $\mathbb{Z}_r^{\mathrm{ac}}$. The definition bypasses this difficulty.

21 $\mathrm{ind}_P^G(W)$ is the R-module of locally constant functions $f : G \to W$ such that $f(mng) = mf(g)$ for $m \in M$, $n \in N$, $g \in G$, where G acts by right translation.

22 When R is a field of characteristic p, ind_P^G commutes with direct products **[169]**.

23 By **[116, 8.3.27]**, as $\mathrm{Mod}_R(G)$ is a locally small abelian Grothendieck category and ind_P^G is right exact and commutes with small direct sums.

where δ_P is the modulus of P,[24] is a deep property proved this year by Dat, David Helm, Robert Kurinczuk, and Gilbert Moss [52, COROLLARY 1.3], originally proved by Bernstein when $R = \mathbb{C}$. When R is noetherian, the parabolic induction ind_P^G respects admissibility, the second adjunction implies that $\mathrm{Mod}_R(G)$ is noetherian, that the parabolic induction respects projective (resp. finitely generated) R-representations [52, COROLLARIES 1.4, 1.5], and that L_P^G respects admissibility. The functor L_P^G is exact, preserves infinite direct sums [40], and when R is a field, L_P^G respects finite length because L_P^G respects the property of being finitely generated, and an admissible finitely generated R-representation of G has finite length (the proof uses the Moy–Prasad unrefined types when R is algebraically closed but being algebraically closed is not necessary).

For a field R of characteristic p, the adjoint functors L_P^G and R_P^G send an admissible irreducible R-representation of G to 0 or to an admissible irreducible R-representation of M. Irreducibility is necessary, an example of an admissible R-representation V of G with $L_P^G(V)$ not admissible is given in (Abe–Henniart–Vignéras [10]). But contrary to the complex case, the functors L_P^G and R_P^G fail to be exact (for R_P^G, see Emerton [61] and Koziol [122]), ind_P^G does not preserve finitely generated representations, R_P^G does not preserve infinite direct sums (Abe–Henniart–Vignéras [10, SECTION 4.5]).

When p is nilpotent in the commutative ring R, the right adjoint R_P^G respects admissibility (Abe–Henniart–Vignéras [10]); it is equal to the Emerton's functor $\mathrm{Ord}_{P^{\mathrm{op}}}^G$ of ordinary parts on admissible R-representations.[25] If, moreover, R is artinian, Matthew Emerton [61] extended the functor of ordinary parts to a δ-functor, expected to coincide with the derived functors when the characteristic of F is 0.

Example. When $G = \mathrm{SL}(2, \mathbb{Q}_p)$, Koziol [122] showed that the derived functors of R_B^G and Ord_B^G are equal on any absolutely irreducible $\mathbb{F}_p^{\mathrm{ac}}$-representations of G.

When the characteristic of F is p, surprisingly, R_P^G is exact on admissible $\mathbb{F}_p^{\mathrm{ac}}$-representations of G (Julien Hauseux [88]).

A representation of G over a field R is called *unramified* when it is trivial on the subgroup G^0 of G generated by its compact subgroups.[26] The group $\Psi_R(G)$ of unramified R-characters $\psi : G \to R^*$ of G is a torus. One says that (F, G, R) satisfies *generic irreducibility* property if for any parabolic subgroup P of G of Levi M and any irreducible R-representation W of M, the set of $\psi \in \Psi_R(M)$ such that $\mathrm{ind}_P^G(W \otimes \psi)$ is irreducible is Zariski-dense in $\Psi_R(M)$.

Generic irreducibility property is probably true for any F, G and any field R. It is known for R of characteristic p (Abe–Henniart–Vignéras [10]) or when $F \subset \mathbb{Q}_p$ and R algebraically closed of characteristic different from p (Dat [38]).

24 $\delta_P(m) = |\det \mathrm{Ad}_{\mathrm{Lie}\,N}(m)| \in q^{\mathbb{Z}}$.

25 There is no description of R_P^G on non-admissible representations.

26 This coincides with the classical definition (Henniart–Lemaire [103, 2.12 REMARQUE 1]).

Dat [38, THEOREM 3.11] extended the complex *Langlands quotient theorem* to any algebraically closed field R of characteristic different from p with a nontrivial valuation ν (for example, $\mathbb{Q}_\ell^{ac}$).

An admissible R-representation V of G is *ν-tempered* (Dat [38, DEFINITION 3.2]) if for any standard parabolic subgroup $P = MN$ such that $L_P^G(V) \neq 0$, any exponent χ in $L_P^G(V)$ satisfies $-\nu(\delta_P^{-1/2}\chi) \in \overline{{}^+\mathcal{A}_P^*}$.[27] It is called *discrete* if $-\nu(\delta_P^{-1/2}\chi) \in {}^+\mathcal{A}_P^*$. The *exponents* of $L_P^G(V)$ are the R-characters of the split component A_M of the center of M appearing in $L_P^G(V)$ seen as an R-representation of A_M.

Theorem 5.1 (Dat–Langlands quotient theorem).

(i) *When $P = MN$ is a standard parabolic subgroup of G, W is a ν-tempered irreducible R-representation of M, and $\psi \in \Psi_R(M)$ satisfies $-\nu(\psi) \in (\mathcal{A}_P^*)^+$, then the R-representation* $\mathrm{ind}_P^G(W \otimes \psi)$ *has a unique irreducible quotient $J(M, W, \psi)$.*

(ii) *Any irreducible R-representation V of G is isomorphic to $J(M, W, \psi)$ for a unique triple (P, W, ψ).*

From the Dat's theory of ν-tempered representations, David Hansen, Tasho Kaletha, and Jared Weinstein deduced (see [84, C.2.2)]):

The Grothendieck group of finite length ℓ-adic representations of G is generated by representations of the form $\mathrm{ind}_P^G(W \otimes \psi)$, for a standard parabolic subgroup $P = MN$ of G, an integral irreducible ℓ-adic representation W of M and an unramified ℓ-adic character ψ of M.

6. ADMISSIBLE REPRESENTATIONS AND DUALITY

The classification of irreducible admissible R-representations of G is an objective of the local Langlands program. There are few finite-dimensional representations when G is not compact modulo the center, and admissibility is a crucial finiteness property.

When R is a noetherian commutative ring, a subrepresentation of an admissible R-representation of G is admissible. A quotient of an admissible R-representation of G is admissible [195] and the category $\mathrm{Mod}_R(G)^a$ of admissible R-representations of G is abelian if p is invertible in R, or if R is a finite field of characteristic p and $F \supset \mathbb{Q}_p$.[28]

Example. When $F \supset \mathbb{F}_p((T))$ and p is not invertible in R, there exists an admissible representation with a non-admissible quotient (Abe–Henniart–Vignéras [10]).

27 Let $\Delta(M)$ denote the set of simple roots of T in M, $\Delta(P)$ the set of simple roots in P of T_M, $\mathcal{A}^* = X \otimes_{\mathbb{Z}} \mathbb{R}$ where X is the lattice of rational characters of T, and $(\cdot,\cdot)$ a W_G-invariant scalar product on $\mathcal{A}^*$. Then $\overline{{}^+\mathcal{A}_P^*} = \sum_{\alpha\in\Delta(P)} \mathbb{R}_{\geq 0}\,\alpha$ and $(\mathcal{A}_P^*)^+$ is the cone $\{x \in \mathcal{A}^*, (x,\alpha) = 0 \text{ for } \alpha \in \Delta(M), (x,\alpha) > 0 \text{ for } \alpha \in \Delta(P)\}$.

28 The completed group algebra of $R[K]$ is noetherian when $F \supset \mathbb{Q}_p$ but not when $F \supset \mathbb{F}_p((T))$.

Let R be a field. The *smooth dual* $V^\vee$ of an R-representation V of G is the smooth part of the contragredient action of G on the linear dual $V^* = \mathrm{Hom}_R(V, R)$.[29]

For R of characteristic different from p, the smooth dual is an autoduality on $\mathrm{Mod}_R(G)^a$. In particular, $V^\vee$ is irreducible if and only if V is irreducible. The smooth dual and the parabolic induction and its left adjoint satisfy[30]:

$$(\mathrm{ind}_P^G W)^\vee \simeq \mathrm{ind}_P^G(W^\vee \delta_P), \quad L_P^G(V^\vee) \simeq \big(L_{P^{\mathrm{op}}}^G(V)\big)^\vee,$$

for any R-representation W of M and any admissible R-representation V of G.

For R of characteristic p, the smooth dual of any infinite dimensional admissible irreducible R-representation of G is zero! For F of characteristic 0, Jan Kohlhaase **[117]** developed a higher smooth duality theory on $\mathrm{Mod}_R(G)^a$. He studied the ith smooth duality functors $S^i : \mathrm{Mod}_R(G)^a \to \mathrm{Mod}_R(G)^a$ for $0 \le i \le d = \dim_{\mathbb{Q}_p} G$ under tensor product, inflation and induction and proved that for $V \in \mathrm{Mod}_R(G)^a$, the integer

$$d(V) = \max\{i \mid S^i(V) \neq 0)\}$$

satisfies

(i) $d(V) = 0$ if and only if V is finite dimensional,

(ii) $d(\mathrm{ind}_P^G W) = d(W) + \dim_{\mathbb{Q}_p} N$, for a parabolic subgroup $P = MN$ and $W \in \mathrm{Mod}_R(M)^a$,

(iii) $d(V) = 1$ and $S^1(V)$ coincides with the Colmez's contragredient introduced for the p-adic Langlands correspondence for $G = \mathrm{GL}(2, Q_p)$, $R = \mathbb{F}_p^{\mathrm{ac}}$, and V irreducible of infinite dimension; for the Steinberg representation St_G which is irreducible, $S^1(\mathrm{St}_G)$ is indecomposable of length 2!

For G unramified,[31] K a hyperspecial subgroup of G, $W \in \mathrm{Mod}_{\mathbb{F}_p^{\mathrm{ac}}}(K)$ and $i > \dim_{\mathbb{Q}_p} U$, we have $S^i(\mathrm{ind}_K^G W) = 0$ (Claus Sorensen **[185]**).

7. SUPERCUSPIDAL SUPPORT

An R-representation V of G is called *cuspidal* if it is killed by the left and right adjoints of the parabolic induction

$$L_P^G(V) = R_P^G(V) = 0,$$

for all parabolic subgroups $P \neq G$.

When p is invertible in R, the second adjunction implies that V is cuspidal if and only if $L_P^G(V) = 0$ for any proper parabolic subgroup P of G. Any irreducible R-

29 The smooth dual is the set of linear forms on V fixed by some open subgroup of G.

30 The normalized induction $\mathrm{ind}_P^G(W \otimes \delta_P^{1/2})$ commutes with the smooth dual, the second isomorphism is equivalent to the second adjunction.

31 G is quasisplit and splits over some unramified extension of F.

representation V of G is a subrepresentation of $\mathrm{ind}_P^G W$ for some cuspidal irreducible R-representation W. Assuming that R is an algebraically closed field,[32] the pair (M, W) is unique modulo G-conjugation; the G-conjugation class of (M, W) is called the *cuspidal support* of V. Twisting the cuspidal support by unramified characters, we get the *inertial cuspidal support* Ω of V. So, Ω is the set of (M', W') which are G-conjugate to $(M, W \otimes \Psi)$ for some $\psi \in \Psi_R(M)$. The subgroup of $w \in W_G$ fixing M acts on the R-representations of M. Let H be the group of $w \in W_G$ such that $W^w \simeq W \otimes \psi$ for some $\psi \in \Psi_R(M)$ and S the (finite) group of $\psi \in \Psi_R(M)$ such that $W \otimes \psi \simeq W$. Then, Ω is an algebraic variety with regular functions $\mathcal{O}(\Omega) = (R[M/M^0]^S)^H$.

When p is not invertible in R, one needs both L_P^G and R_P^G to define cuspidality. For a field R of characteristic p, the trivial representation 1_G of G and the Steinberg representation $\mathrm{St}_G = \mathrm{ind}_B^G(1_Z)/\sum_{P \supsetneq B} \mathrm{ind}_P^G(1_M)$ satisfy, for any parabolic subgroup P of Levi M,

$$L_P^G(1_G) = 1_M, \quad R_P^G(1_G) = 0, \quad L_P^G(\mathrm{St}_G) = 0, \quad R_P^G(\mathrm{St}_G) = \mathrm{St}_M .$$

The Steinberg representation is not a subrepresentation of $\mathrm{ind}_P^G W$ for any cuspidal admissible irreducible R-representation W. Any irreducible R-representation V of G is a subquotient of $\mathrm{ind}_B^G W$ for some R-representation W of Z (for R algebraically closed, see Abe–Henniart–Herzig–Vignéras **[8, IV.1]**). This is very different from the complex case!

An admissible irreducible R-representation of G which is not isomorphic to a subquotient of a proper parabolically induced representation $\mathrm{ind}_P^G W$ for all $P \neq G$, W an admissible irreducible R-representation of M, is called *supercuspidal*.[33]

A cuspidal irreducible admissible R-representation is always supercuspidal if R is a field of characteristic 0 or p, but not if the characteristic of R is $\ell \neq p$!

Example. When $G = \mathrm{GL}(2, \mathbb{Q}_p)$, $R = \mathbb{F}_\ell^{\mathrm{ac}}$, ℓ divides $p+1$, the unique infinite dimensional irreducible subquotient of the representation $\mathrm{ind}_B^G 1_Z$ indecomposable of length 3 is cuspidal and non-supercuspidal.

Any admissible irreducible R-representation V of G is a subquotient of $\mathrm{ind}_P^G W$ for some supercuspidal admissible irreducible R-representation W.

For a field R of characteristic p, (P, W) is unique modulo G-conjugation. This follows from the classification.

For a field R of characteristic different from p, the G-conjugation class of (M, W) is called a *supercuspidal support* of V. Contrary to the cuspidal support, the supercuspidal support is not always unique if the characteristic of R is $\ell \neq p$.

Examples. The supercuspidal support is not unique when $R = \mathbb{F}_\ell^{\mathrm{ac}}$, ℓ divides q^2+1 and G is the finite group $\mathrm{Sp}_8(\mathbb{F}_q)$ (Olivier Dudas **[58]**) or $\mathrm{Sp}_8(F)$ (Dat **[49]**).

32 Being algebraically closed is probably not necessary.

33 One does not need to suppose that W is irreducible when R is an algebraically closed field of characteristic different from p (Dat **[49]**).

The supercuspidal support is unique if R has characteristic 0, or G is an inner form of $\mathrm{GL}(n, F)$ (Minguez–Sécherre [141]), or G is the unramified unitary group $U(2,1)$, $p \neq 2$ (Kurinczuk [126]), when R is algebraically closed (this is probably not necessary).

When R is algebraically closed, the twist by unramified characters of a supercuspidal support of V is called an *inertial supercuspidal support* of V; if all the irreducible R-representations of G have a unique supercuspidal support, the *Bernstein variety* $\mathcal{B}_R(G)$ is the disjoint union of the inertial supercuspidal supports of the irreducible R-representations of G.

An irreducible $\mathbb{Q}_\ell^{ac}$-representation of G is integral if and only if its supercuspidal support is integral (Dat–Helm–Kurinczuk–Moss [52, COROLLARY 1.6]). Is any irreducible mod ℓ representation of G a subquotient of the reduction of an integral irreducible ℓ-adic representation?

For a field R of banal characteristic for G, any cuspidal irreducible R-representation of G is supercuspidal and projective in the category of R-representations of G with a given central character. The reduction of an integral cuspidal irreducible ℓ-adic representation of G is irreducible and cuspidal, and any cuspidal irreducible mod ℓ-representation of G lifts[34] to an integral cuspidal irreducible ℓ-adic representation of G (Dat–Helm–Kurinczuk–Moss, to appear). The reduction of an integral irreducible ℓ-adic representation of G may be reducible. Does any irreducible mod ℓ representation of G lift to an integral irreducible ℓ-adic representation of G?

8. HECKE ALGEBRAS

Hecke $\mathbb{Z}$-algebras appear everywhere in the theory of representations of G, giving algebraic proofs of properties proved earlier with harmonic analysis. An open subgroup K of G which is compact, or compact modulo the center of G, defines a *Hecke ring*

$$\mathcal{H}(G, K) = \mathrm{End}_{\mathbb{Z}[G]} \mathbb{Z}[K \backslash G],$$

naturally isomorphic to the opposite of $\mathbb{Z}[K\backslash G/K]$. For any commutative ring R, the Hecke R-algebra $\mathcal{H}_R(G, K) = \mathrm{End}_{R[G]} R[K\backslash G]$ is the scalar extension to R of $\mathcal{H}(G, K)$.

Finiteness property of $\mathcal{H}_R(G, K)$. The center $\mathcal{Z}_R(G, K)$ of $\mathcal{H}_R(G, K)$ is a finitely generated R-algebra and $\mathcal{H}_R(G, K)$ is a finitely generated $\mathcal{Z}_R(G, K)$-module, if R is a noetherian $\mathbb{Z}_\ell$-algebra.

This theorem of Dat–Helm–Kurinczuk–Moss [52] is the key of the proof of the second adjunction. It was proved by Deligne and Bernstein for complex Hecke algebras. It is equivalent to another statement, involving the endomorphism ring $\mathcal{Z}_R(G)$ of the identity functor of $\mathrm{Mod}_R(G)$, called the *Bernstein center*:

For R as above, any finitely generated R-representation V of G is $\mathcal{Z}_R(G)$-admissible and the natural image of $\mathcal{Z}_R(G) \to \mathrm{End}_{R[G]} V$ is a finitely generated R-algebra.

34 Is the reduction modulo ℓ of an integral cuspidal irreducible ℓ-adic representation of G.

The highly nontrivial proof uses the Fargues–Scholze local version of the Vincent Lafforgue's theory of excursion operators [65].

The finiteness theorem is true for the Iwahori and the pro-p Iwahori Hecke rings ($R = \mathbb{Z}$ and $K = J$ or $\tilde{J}$) (Vignéras [196]). Is it true for any Hecke ring?

The *K-invariant functor*

$$V \mapsto V^K \simeq \mathrm{Hom}_{R[G]}\big(R[K\backslash G], V\big) : \mathrm{Mod}_R(G) \to \mathrm{Mod}\,\mathcal{H}_R(G,K)$$

and its left adjoint $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{H}_R(G,K)} R[K\backslash G]$ relate the R-representations of G and the right $\mathcal{H}_R(G,K)$-modules. From now on, an $H_R(G,K)$-module will be a right module.

When R is a field and the order of any finite quotient of K is invertible in R, the K-invariant functor induces a bijection between the (isomorphism classes of) irreducible R-representations V of G with $V^K \neq 0$ and the (isomorphism classes of) simple $\mathcal{H}_R(G,K)$-modules.

If R is a field of characteristic different from p, an irreducible R-representation of G is admissible (Henniart–Vignéras [107, THEOREM 3.2]), any simple $\mathcal{H}_R(G,K)$-module has finite dimension. For any field R, a simple module of the Iwahori or pro-p Iwahori Hecke algebra has finite dimension.

Let $\mathrm{Mod}_R(G)(K)$ denote the category of R-representations of G generated by their K-invariant vectors. When any subrepresentation of any representation in $\mathrm{Mod}_R(G)(K)$ belongs to $\mathrm{Mod}_R(G)(K)$, the category $\mathrm{Mod}_R(G)(K)$ is abelian and equivalent by the K-invariant functor to

$$\mathrm{Mod}_R(G)(K) \tilde{\to} \mathrm{Mod}\,\mathcal{H}_R(G,K).$$

This is the case if $R = \mathbb{C}$ and K is an Iwahori subgroup J by a classical result of Borel, or a pro-p Iwahori subgroup $\tilde{J}$ (Vignéras [196]). The category $\mathrm{Mod}_{\mathbb{C}}(G)(J)$ is an indecomposable factor of $\mathrm{Mod}_{\mathbb{C}}(G)(\tilde{J})$, and $\mathrm{Mod}_{\mathbb{C}}(G)(\tilde{J})$ is a factor of $\mathrm{Mod}_{\mathbb{C}}\,G)$.

For R of characteristic p, the category $\mathrm{Mod}_{\mathbb{C}}(G)(\tilde{J})$ is not abelian in general. However, it is abelian if $R = \mathbb{F}_p^{\mathrm{ac}}$ and $G = \mathrm{GL}(2,\mathbb{Q}_p)$ or $\mathrm{SL}(2,\mathbb{Q}_p)$, $p \neq 2$ (Ollivier [146],[35] Koziol [119], Ollivier–Schneider [151]).

For a prime r, a $\mathbb{Q}_r$-representation V of G is called *locally integral* if for some finite extension $E/\mathbb{Q}_r$, V^K admits a $\mathcal{H}(G,K)$-stable O_E-lattice for all open compact subgroups K of G.

An integral irreducible $\mathbb{Q}_r^{\mathrm{ac}}$-representation is clearly locally integral. The converse is true if $r = \ell \neq p$ [38]. The equivalence between integral and locally integral for irreducible $\mathbb{Q}_p^{\mathrm{ac}}$-representations of G is an open question. It is the analogue of the Breuil–Schneider conjecture [20] restricted to smooth representations (Hu [113], Sorensen [183, 184]).

A finite length $\mathbb{Q}_p^{\mathrm{ac}}$-representation V of G is locally integral if and only if (Dat [41])

$$\nu(\delta_P^{-1/2}\chi) \in \rho_P - \overline{{}^{+}\mathcal{A}_P^{*}}$$

for any standard parabolic subgroup $P = MN$ of G with $L_P^G(V) \neq 0$, and any exponent χ of $L_P^G(V)$.[36]

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35 Supposing that a uniformizer of F acts trivially.

36 ρ_P is half the sum of the roots of A_M in Lie P. The formula can be simplified!

9. REPRESENTATIONS OVER A FIELD OF CHARACTERISTIC DIFFERENT FROM p

For any commutative ring R, an R-representation W of an open subgroup K of G defines an R-representation $\operatorname{ind}_K^G W$ of G by *compact induction*.[37]

Example. $\operatorname{ind}_K^G 1_K = R[K\backslash G]$ for the trivial R-representation 1_K of K.

Assume that R is a field of characteristic different from p, until the end of this section.

All cuspidal irreducible R-representations of G are conjectured to be compactly induced from open subgroups of G compact modulo the center of G.

For R algebraically closed, the conjecture has been proved for the level 0[38] cuspidal representations of any G or when

- G has rank 1 (Martin Weissman [203]),
- G is an inner form of $\mathrm{GL}(n, F)$ (Minguez–Sécherre [141]), or of $\mathrm{SL}(n, F)$ (Peyi Cui [36, 37]),
- G is a classical group (Stevens [187], Stevens–Kurinczuk–Skodlerak [131]) or a quaternionic form of G (Skodlerak [181]), if $p \neq 2$.
- G splits on a moderately ramified extension of F and p does not divide the order of the absolute Weyl group (Fintzen [66]).

Being algebraically closed is not necessary and there is an explicit list $\mathcal{X}$ of pairs (K, W) of G where K is an open subgroup of G compact modulo the center and W an R-representation of K such that $\operatorname{ind}_K^G W$ is irreducible cuspidal satisfying (Henniart–Vignéras [107]):

(a) any cuspidal irreducible R-representation of G is isomorphic to $\operatorname{ind}_K^G W$ for some $(K, W) \in \mathcal{X}$ unique modulo G-conjugation,

(b) $\operatorname{ind}_K^G W$ and W have the same intertwining algebra

$$\operatorname{End}_{R[K]} W \simeq \operatorname{End}_{R[G]} \operatorname{ind}_K^G W,$$

(c) $\operatorname{ind}_K^G W$ is supercuspidal if and only if W is supercuspidal, for the "natural notion of supercuspidality" of W,[39]

(d) $\mathcal{X}$ is stable by automorphisms of R.

Until the end of this section, assume R algebraically closed and $G = \mathrm{GL}(m, D)$ where D is a central division algebra of dimension d^2 over F, $n = md$.

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37 The R-module of functions $f : G \to W$ supported on finitely many cosets Kg, satisfying $f(kg) = \rho(k) f(g)$ for $k \in K$, $g \in G$ where G acts by right translation.

38 Definition in the section on Bernstein blocks.

39 Fintzen gave another proof when G is moderately ramified and p does not divide the order of the absolute Weyl group.

Minguez and Sécherre [140] classified the irreducible R-representations of G with a given supercuspidal support by "supercuspidal multisegments," and those with a given cuspidal support by "aperiodic cuspidal multisegments." This generalizes the Bernstein–Zelevinski classification of complex irreducible representations of $\mathrm{GL}(n, F)$. For R of characteristic ℓ, the proof uses the theory of ℓ-modular types (Minguez–Sécherre [141]) and deep results on affine Hecke algebras of type A at roots of unity.

Any irreducible ℓ-modular representation of G is a subquotient of the reduction of an integral irreducible ℓ-adic representation [140]. In the other direction, any irreducible ℓ-modular representation V of G lifts to an ℓ-adic representation when it is supercuspidal or "banal" or unramified[40] (Dat [38], Minguez–Sécherre [139,140,142]) or when it is cuspidal and $G = \mathrm{GL}(n, F)$. Contrary to the case $G = \mathrm{GL}(n, F)$, some irreducible cuspidal ℓ-modular representation of G may not lift and the reduction of a integral cuspidal irreducible ℓ-adic-representation of G may be reducible; but its irreducible components are cuspidal and in the same inertial class.

Example. When $q = 8$, $\ell = 3$, $d = 2$, any integral irreducible ℓ-adic representation of D^* containing an homomorphism $\chi : O_D^* \to (\mathbb{Q}_\ell^{\mathrm{ac}})^*$ trivial on $1 + P_D$ such that $\chi \neq \chi^q$ has dimension 2 and its reduction is reducible. When $q = 4$, $\ell = 17$, $d = 2$, there exists an irreducible cuspidal ℓ-modular representation of $\mathrm{GL}(2, D)$ not lifting to $\mathbb{Q}_\ell^{\mathrm{ac}}$ (Minguez–Sécherre [143]).

Let $\mathcal{D}_\mathbb{C}(G)$ denote the set of isomorphism classes of the essentially square integrable irreducible (or discrete series) complex representations of G. The complex *local Jacquet–Langlands correspondence*

$$\mathrm{JL}_\mathbb{C} : \mathcal{D}_\mathbb{C}\big(\mathrm{GL}(m, D)\big) \xrightarrow{\sim} \mathcal{D}_\mathbb{C}\big(\mathrm{GL}(n, F)\big)$$

is a bijection characterized by a character relation on matching elliptic regular conjugacy classes. Fixing an isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{\mathrm{ac}}$, the complex local Jacquet–Langlands correspondence gives an ℓ-adic local Jacquet–Langlands correspondence

$$\mathrm{JL}_{\mathbb{Q}_\ell^{\mathrm{ac}}} : \mathcal{D}_{\mathbb{Q}_\ell^{\mathrm{ac}}}\big(\mathrm{GL}(m, D)\big) \xrightarrow{\sim} \mathcal{D}_{\mathbb{Q}_\ell^{\mathrm{ac}}}\big(\mathrm{GL}(n, F)\big)$$

independent of the isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{\mathrm{ac}}$, and respecting integrality. Minguez and Sécherre [143] proved that two integral representations of $\mathcal{D}_{\mathbb{Q}_\ell^{\mathrm{ac}}}(\mathrm{GL}(m, D))$ are congruent modulo ℓ if and only if their transfers to $\mathrm{GL}(n, F)$ are congruent modulo ℓ. But there is no ℓ-modular local Jacquet–Langlands correspondence compatible with the ℓ-adic local Jacquet–Langlands correspondence by reduction, as, for example, when $d = 2$ and $q + 1 \equiv 0$ modulo ℓ, the trivial representation $1_{\mathbb{Q}_\ell^{\mathrm{ac}}}$ of D^* corresponds to the Steinberg $\mathrm{St}_{\mathbb{Q}_\ell^{\mathrm{ac}}}$ of $\mathrm{GL}(2, F)$ of reduction modulo ℓ of length 2 (Dat [43]). However, the Badulescu morphism [13]

$$\mathrm{LJ}_{\mathbb{Q}_\ell^{\mathrm{ac}}} : \mathcal{G}r_{\mathbb{Q}_\ell^{\mathrm{ac}}}\big(\mathrm{GL}(n, F)\big) \to \mathcal{G}r_{\mathbb{Q}_\ell^{\mathrm{ac}}}\big(\mathrm{GL}(m, D)\big),$$

40 $V^{\mathrm{GL}(m, O_D)} \neq 0$, equivalent to V irreducibly parabolically induced from an unramified character of a Levi subgroup [142].

where $\mathcal{G}r_R(G)$ is the Grothendieck group of finite length R-representations of G, gives by reduction an ℓ-modular Badulescu morphism

$$\mathrm{LJ}_{\mathbb{F}_\ell^{\mathrm{ac}}} : \mathcal{G}r_{\mathbb{F}_\ell^{\mathrm{ac}}}\big(\mathrm{GL}(n,F)\big) \to \mathcal{G}r_{\mathbb{F}_\ell^{\mathrm{ac}}}\big(\mathrm{GL}(m,D)\big).$$

Sécherre and Stevens [180] introduced the interesting notions of mod ℓ inertial supercuspidal support and linkage for irreducible complex representations π, π' of G.

(a) Picking an isomorphism $\mathbb{C} \simeq \mathbb{Q}_\ell^{\mathrm{ac}}$ one supposes that π is an ℓ-adic representation of G. The inertial cuspidal support of π contains an integral cuspidal representation τ. The *mod ℓ inertial supercuspidal support* of π is the inertial supercuspidal support of any irreducible component of $r_\ell(\tau)$; it depends only on the isomorphism class of π.

(b) π, π' are *linked* if there are prime numbers $\ell_1, \dots, \ell_r$ different from p, and irreducible complex representations $\pi = \pi_0, \pi_1, \dots, \pi_r = \pi'$ such that, for each $i \in \{1, \dots, r\}$, the representations π_{i-1}, π_i have the same mod ℓ_i inertial supercuspidal support.

When π, π' are essentially square integrable, they are linked if and only if their images by the local Jacquet–Langlands correspondence $\mathrm{JL}_{\mathbb{C}}$ are linked if and only if (Dotto [55]) they have the same semisimple endoclass (a type invariant). When $G = \mathrm{GL}(n,F)$ and π, π' are cuspidal, they have the same *endoclass* if and only if the associated irreducible representations of Weil group W_F by the local Langlands correspondence share an irreducible component when restricted to the wild inertia group.

10. BERNSTEIN BLOCKS

For a commutative ring R, a nontrivial idempotent e in the Bernstein center $\mathcal{Z}_R(G)$ decomposes the abelian category

$$\mathrm{Mod}_R(G) = e\big(\mathrm{Mod}_R(G)\big) \times (1-e)\big(\mathrm{Mod}_R(G)\big)$$

into a direct product of two abelian full subcategories. When the idempotent $e \in \mathcal{Z}_R(G)$ is primitive, the subcategory $e(\mathrm{Mod}_R(G))$, where e acts by the identity, is indecomposable (no nontrivial factors) and called a *block*.

Bernstein and Deligne factorized $\mathrm{Mod}_{\mathbb{C}}(G)$ into blocks. Their arguments are valid for any algebraically closed field R of characteristic 0. The decomposition is based on the uniqueness of the supercuspidal support. We have

$$\mathrm{Mod}_R(G) = \prod_{\Omega \in \mathcal{B}_R(G)} \mathrm{Mod}_R(G)_\Omega$$

over the connected components Ω of the Bernstein variety $\mathcal{B}_R(G)$. The *Bernstein block* $\mathrm{Mod}_R(G)_\Omega$ consists of the R-representations of G all of whose irreducible subquotients have inertial supercuspidal support Ω. The center of the block $\mathrm{Mod}_R(G)_\Omega$ is the ring of regular functions on the variety Ω.

When G is an inner form of $\mathrm{GL}(n,F)$, two complex discrete series of G in the same block are inertially equivalent. The complex Jacquet–Langlands correspondence commutes with twisting by characters, and yields a bijection between the blocks containing discrete series. Andrea Dotto [55] parametrized these blocks by two algebraic invariants (one is the endo-class) and obtained a complete algebraic description of the Jacquet–Langlands correspondence at the level of inertial classes.

For an algebraically closed field R of characteristic different from p, the Deligne–Bernstein decomposition remains true (Sécherre and Stevens [179]). Bastien Drevon and Vincent Sécherre [57] described the block decomposition of the abelian category of finite length R-representations of G. Unlike the case of all R-representations of G, several non-isomorphic supercuspidal supports may correspond to the same block. A supercuspidal block is equivalent to the principal block of the multiplicative group of a suitable division algebra.

When R is an algebraically closed field of characteristic ℓ banal for G, it is expected that the Deligne–Bernstein decomposition remains true and that the reduction modulo ℓ gives a bijection between the blocks of ℓ-adic representations of G and the blocks of mod ℓ representations of G.

When $R = W(\mathbb{F}_\ell^{ac})$ is the Witt ring of $\mathbb{F}_\ell^{ac}$ and $G = \mathrm{GL}(n,F)$, Helm [96–98] showed that the block decomposition of $\mathrm{Mod}_{\mathbb{F}_\ell^{ac}}(G)$ lifts to a block decomposition of $\mathrm{Mod}_{W(\mathbb{F}_\ell^{ac})}(G)$,

$$\mathrm{Mod}_{W(\mathbb{F}_\ell^{ac})}(G) = \prod_{\Omega \in \mathcal{B}_{\mathbb{F}_\ell^{ac}(G)}} \mathrm{Mod}_{\mathbb{Z}_\ell^{ac}}(G)_\Omega.$$

The block $\mathrm{Mod}_{W(\mathbb{F}_\ell^{ac})}(G)_\Omega$ consists of the $W(\mathbb{F}_\ell^{ac})$-representations of G such that any irreducible subquotient V

- has a supercuspidal support in Ω modulo isomorphism, if $\ell V = 0$,
- is such that the reduction modulo ℓ of an integral element in the inertial class of the supercuspidal support of V is in Ω modulo isomorphism, if $\ell V = V$.

The center of $\mathrm{Mod}_{W(\mathbb{F}_\ell^{ac})}(G)$ is naturally isomorphic to the ring of endomorphisms of the Gelfand–Graev representation of G,[41] and the center of $\mathrm{Mod}_{W(\mathbb{F}_\ell^{ac})}(G)_\Omega$ is a finitely generated, reduced, ℓ-torsion free $W(\mathbb{F}_\ell^{ac})$-algebra.

The *principal block* of $\mathrm{Mod}_R(G)$ contains the trivial R-representation of G. When $R = \mathbb{C}$, the principal block is equivalent to the category of modules over the Iwahori Hecke $\mathbb{C}$-algebra. The blocks have been computed in a large number of examples with the theory of types. Many blocks are equivalent to the principal block of another group G'.

Example. For an algebraically closed field R of characteristic different from p and G an inner form of $\mathrm{GL}(n,F)$, each block of $\mathrm{Mod}_R(G)$ is equivalent to the principal block of a product of general linear groups [179].

41 $\mathrm{ind}_U^{\mathrm{GL}(n,F)} \psi$, where ψ is a generic $W(\mathbb{F}_\ell^{ac})$-character of the unipotent radical U of a Borel subgroup of $\mathrm{GL}(n,F)$.

When $R = \mathbb{Q}_\ell^{ac}, \mathbb{Z}_\ell^{ac}$ or $\mathbb{F}_\ell^{ac}$, Dat explained the known coincidences between the blocks of $\mathrm{Mod}_R(G)$ and predicted many more by a functoriality principle involving dual groups [47, 48].

For a commutative ring R where p is invertible, there is a decomposition of $\mathrm{Mod}_R(G)$ by the Moy–Prasad *depth* [40, APPENDIX A].

An R-representation V of G has *depth* 0 if $V = \sum_x V^{\tilde{G}_x}$ is the sum of its invariants $V^{\tilde{G}_x}$ by the pro-p radicals $\tilde{G}_x$ of the subgroups of G fixing the vertices of the adjoint Bruhat–Tits building of G. The possible depths form a sequence of non-negative rational numbers $r_0 = 0 < r_1 < \cdots$. The category $\mathrm{Mod}_R(G)^{(r)}$ of R-representations of G of depth r is abelian with an explicit finitely generated projective generator but is generally not a block. We have

$$\mathrm{Mod}_R(G) = \prod_{n \in \mathbb{N}} \mathrm{Mod}_R(G)^{(r_n)}.$$

When $p = 0$ in R, the Bernstein center $\mathcal{Z}_R(G)$ of G is as small as possible, equal to the Bernstein center of the center $Z(G)$ of G (see Ardakov–Schneider [12] when R is a field, but their proofs are valid for a commutative ring, see also Dotto [55])

$$\mathcal{Z}_R\big(Z(G)\big) = \varprojlim_K R\big[Z(G)/K\big], \quad K \subset Z(G) \text{ open compact subgroup.}$$

When $E/\mathbb{Q}_p$ is a finite extension of ring of integers O_E, the category of locally finite representations (equal to the union of their subrepresentations of finite length) of $\mathrm{GL}(2, \mathbb{Q}_p)$ on O_E-torsion modules with a central character decomposes as a product of blocks with a noetherian center (Paskunas and Shen-Nin Tung [159]).

11. SATAKE ISOMORPHISM

The structure of the Hecke ring of any special parahoric subgroup K of G is understood via the *Satake transform*

$$\mathrm{Sat} : \mathcal{H}(G, K) \to \mathcal{H}(Z, Z^0), \quad \mathrm{Sat}(f)(z) = \sum_{u \in U^0 \backslash U} f(uz) \text{ for } z \in Z.$$

It is an injective ring homomorphism, and as $\mathcal{H}(Z, Z^0) \simeq \mathbb{Z}[Z/Z^0]$ is commutative, it shows that the Hecke ring $\mathcal{H}(G, K)$ is commutative. A basis of the image of Sat is

$$S_\lambda = \sum_{\lambda' \in W(\lambda)} \delta^{1/2}(\lambda/\lambda') e_{\lambda'} \quad \text{for } \lambda \in Z^+/Z^0,$$

where $e_\lambda \in \mathcal{H}(Z, Z^0)$ corresponds to λ (Henniart–Vignéras [105], [104, PROPOSITION 2.3]). This shows that modulo isomorphism, the commutative Hecke ring $\mathcal{H}(G, K)$ does not depend on the choice of K.

By scalar extension to a commutative ring R, the Satake transform extends to a map $\mathrm{Sat} : \mathcal{H}_R(G, K) \to \mathcal{H}_R(Z, Z^0)$. For $R = \mathbb{C}$, it is well known that $\delta_B^{1/2}$ Sat induces an isomorphism

$$\mathcal{H}_\mathbb{C}(G, K) \simeq \mathbb{C}\big[Z/Z^0\big]^{W_G}.$$

An all-important special case was singled out by Langlands, that is, where G is unramified and where K is a hyperspecial maximal compact subgroup of G. Langlands interpreted the Satake isomorphism as giving a parametrization of the isomorphism classes of complex irreducible representations of G with a nonzero K-fixed vector, by certain semisimple conjugacy classes in a complex group $\hat{G}$ "dual" to G.

For a field R of characteristic p, Sat induces an isomorphism (Henniart–Vignéras [105])[42]

$$\mathcal{H}_R(G,K) \simeq R[Z^+/Z^0].$$

Instead of focusing on the trivial R-representation 1_K of K, one can consider two finitely generated R-representations W, W' of K and the Hecke R-bimodule

$$\mathcal{H}_R(G,K,W,W') \simeq \mathrm{Hom}_{R[G]}(\mathrm{ind}_K^G W, \mathrm{ind}_K^G W').$$

It is realized as a set of compactly supported functions $f : G \to \mathrm{Hom}_R(W,W')$ with a certain K-biinvariance. In the case $W = W'$, it is an algebra called an *Hecke algebra with weight* W that we rather write $\mathcal{H}_R(G,K,W)$; the Hecke algebra with trivial weight is the Hecke R-algebra $\mathcal{H}_R(G,K)$. For any standard parabolic subgroup $P = MN$, the Satake transform generalizes to an injective map

$$\mathrm{Sat}_M : \mathcal{H}_R(G,K,W,W') \to \mathcal{H}_R(M,M^0,W_{N^0},W'_{N^0}),$$
$$\mathrm{Sat}_M(f)(m)(\overline{v}) = \sum_{n\in N^0\backslash N} \overline{f(nm)(v)}$$

for $m \in M$, $v \in W$, where $v \to \overline{v}$ is the quotient map $W \to W_{N^0}$ (similarly for $W' \to W'_{N^0}$). The functional approach of Sat_M (Henniart–Vignéras [104, SECTION 2]) is a motivation to prefer it to another generalization considered in (see Herzig [109] when G is split, Henniart–Vignéras [105])

$$\mathrm{Sat}'_M : \mathcal{H}_R(G,K,W,W') \to \mathcal{H}_R(M,M^0,W^{N^0},W'^{N^0}),$$
$$\mathrm{Sat}'_M(f)(z)(v) = \sum_{u\in U^0\backslash U} f(uz)(v)$$

for $v \in W^{N^0}$. The maps Sat'_M and Sat_M are related by taking duals [104].

When R is an algebraically closed field of characteristic p and W, W' are irreducible, the generalized Satake transforms play a role in the modulo p and p-adic Langlands correspondence. In this situation W_{U^0}, W'_{U^0} have dimension 1, the Hecke bimodule $\mathcal{H}_R(G,K,W,W')$ is nonzero if and only if the R-characters of Z^0 on W_{U^0}, W'_{U^0} are Z-conjugate. For $M = Z$, there are explicit bases $(S_\lambda^{W,W'})$ of the image of Sat_Z, and $(T_\lambda^{W,W'})$ of $\mathcal{H}_R(G,K,W,W')$ such that

$$\mathrm{Sat}_Z(T_\lambda^{W,W'}) = S_\lambda^{W,W'}$$

for $\lambda \in Z^+(W,W')/Z^0$ where $Z^+(W,W')$ is a certain union of cosets of Z^0 in Z^+ (Abe–Herzig–Vignéras [11]). The proof relies on the theory the pro-p-Iwahori Hecke R-algebra.

42 With Z^-/Z^0 instead of Z^+/Z^0, but these monoids are isomorphic.

A simple consequence is the "change of weight"[43] which is an important step in the proof of the classification of admissible irreducible R-representations of G. There is also a change of weight the pro-p-Iwahori Hecke algebra giving another proof for the change of weight for G (Abe [4]). For an Hecke algebra $\mathcal{H}_R(G, K, W)$ with irreducible weight W, one gets an explicit inverse of the Satake isomorphism (Henniart–Vignéras [104])[44]:

$$\mathrm{Sat}_Z : \mathcal{H}_R(G, K, W) \xrightarrow{\sim} \mathcal{H}_R(Z^+, Z^0, W_{U^0}).$$

For G quasisplit, $\mathcal{H}_R(Z^+, Z^0, W_{U^0}) \simeq R[Z^+/Z^0]$, hence $\mathcal{H}_R(G, K, W)$ is commutative and does not depend on the choice of (K, W) modulo isomorphism. For G general, the center of $\mathcal{H}_R(G, K, W)$ contains a finitely generated subalgebra $\mathcal{Z}_T$ isomorphic to $R[T^+/T^0]$, and $\mathcal{H}_R(G, K, W)$ is a finitely generated $\mathcal{Z}_T$-module.

One chooses an element s in the center of M which strictly contracts N by conjugation. There is a unique element $T_s \in \mathcal{H}_R(M, M^0, W_{N^0})$ with support $M^0 s$ such that $T_s(s)$ is the identity on W_{N^0}. The generalized Satake transform

$$\mathrm{Sat}_M : \mathcal{H}_R(G, K, W) \hookrightarrow \mathcal{H}_R(M, M^0, W_{N^0})$$

is a localization at T_s.[45] The natural intertwiner

$$I_V : \mathrm{ind}_K^G W \to \mathrm{ind}_P^G(\mathrm{ind}_{M^0}^M W_{N^0})$$

is injective and its localization at T_s is bijective when W satisfies a regularity assumption[46] (Herzig [108], Abe [3], Henniart–Vignéras [104]).

For a field R of characteristic p, the *supersingularity* of an admissible irreducible R-representation V of G is defined with the Satake homomorphism (Abe–Henniart–Herzig–Vignéras [8]). First, assuming R algebraically closed, an homomorphism $\mathcal{Z}_R(G, K, W) \to R$ from the center $\mathcal{Z}_R(G, K, W)$ of an Hecke algebra $\mathcal{H}_R(G, K, W)$ with irreducible weight is said to be supersingular if it does not extend to the center of $\mathcal{H}_R(M, M^0, W_{N^0})$ via the Satake homomorphism for any $P \neq G$. As V is admissible, there exists some irreducible representation W of K such that $\mathrm{Hom}_{R[G]}(\mathrm{ind}_K^G W, V) \neq 0$. If $\mathrm{Hom}_{R[G]}(\mathrm{ind}_K^G W, V)$ as a module over the center of $\mathcal{H}_R(G, K, W)$ contains an eigenvector with a supersingular eigenvalue, V is called supersingular. This does not depend on the choice of (K, W). For R not algebraically closed, V is called supersingular if some admissible irreducible R^{ac}-representation V^{ac} of G which is V-isotypic as an R-representation, is supersingular. This does not depend on the choice of V^{ac} (Henniart–Vignéras [106]).

For G unramified and K hyperspecial, using the geometric Satake equivalence, Xinwein Zhu [209] identified the Hecke ring $\mathcal{H}(G, K)$ with a ring associated to the Vinberg

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43 The change of weight theorem is an isomorphism between two compactly induced representations.

44 This isomorphism for Sat' is proved when G is split in Herzig [109], and in general in Henniart–Vignéras [105].

45 This means that the image of Sat_M contains T_s and that its localization at T_s is $\mathcal{H}_R(M, M^0, W_{N^0})$.

46 Meaning that the map $\mathcal{H}_R(M, M^0, V_{N^0}) \otimes_{\mathcal{H}_R(G,K,V)} \mathrm{ind}_K^G V \to \mathrm{ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0})$ is bijective, if the kernel of $V \to V_{N^0}$ contains $kV^{(N^{\mathrm{op}})^0}$ for all $k \in K \setminus P^0 (P^{\mathrm{op}})^0$.

monoid of $\hat{G}$ and formulated a canonical Satake isomorphism. He proved that the commutative $\mathbb{Z}$-algebra $\mathcal{H}(G, K)$ is finitely generated. He extended his formulation to an Hecke algebra $\mathcal{H}_{O_E}(G, K, W)$ with weight a finite free O_E-module W arising from an irreducible algebraic representation $E \otimes_{O_E} W$ of G, where E/F is a finite extension.

For F of characteristic 0 and R a field of characteristic p, Claudius Heyer [112, THEOREM 4.3.2] defined a derived generalized Satake homomorphism.

For F of characteristic 0, G split, K hyperspecial and $R = \mathbb{Z}/p^a\mathbb{Z}$, $a \geq 1$, Niccolo Ronchetti [167] established a Satake homomorphism for the derived Hecke $\mathbb{Z}/p^a\mathbb{Z}$-algebra of (G, K) (a graded associative $\mathbb{Z}/p^a\mathbb{Z}$-algebra whose degree 0 subalgebra is $\mathcal{H}_{\mathbb{Z}/p^a\mathbb{Z}}(G, K)$). The relation with the Heyer derived Satake homomorphism is unclear.

12. PRO-p IWAHORI HECKE RING

The Iwahori Hecke ring $\mathcal{H}(G, J)$ and the pro-p Iwahori Hecke ring $\mathcal{H}(G, \tilde{J})$ modulo isomorphism depend only on G, because the Iwahori subgroups of G are conjugate, as well as the pro-p Iwahori subgroups.

They are both natural generalizations of affine Hecke $\mathbb{Z}$-algebras. We will focus on the pro-p Iwahori Hecke ring which is more involved, that we will denote by $\mathcal{H}(G)$, but all the results apply to Iwahori Hecke rings with some simplifications.

Our motivation to study the pro-p Iwahori Hecke ring instead of the Iwahori Hecke ring comes from the theory of mod p representations.[47] Any nonzero mod p representation of G has a nonzero $\tilde{J}$-fixed vector, and the pro-p radical of any parahoric subgroup of G is contained in some G-conjugate of $\tilde{J}$.

For any commutative ring R, the pro-p Iwahori Hecke R-algebra $\mathcal{H}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{H}(G)$ is a specialization of the *generic pro-p Iwahori Hecke $R[\mathbf{q}_*]$-algebra $\mathcal{H}(G)(\mathbf{q}_*, c_*)$* of G, introduced by Nicolas Schmidt [170, 171] when G is split (Vignéras [198] in general). The $\mathbf{q}_*$ are finitely many indeterminates and the finitely many $c_* \in R[\mathbf{q}_*]$ satisfy simple conditions. The general principle is that one proves properties of the generic pro-p Iwahori Hecke $R[\mathbf{q}_*]$-algebra by specializing all $\mathbf{q}_*$ to 1, and then one transfers them to $\mathcal{H}_R(G)$ by specialization.

Example. The affine Yokonuma–Hecke algebra defined by Maria Chlouveraki and Loic Poulain d'Andecy is a generic pro-p Iwahori Hecke algebra (Chlouveraki and Sécherre [30]).

The main features[48] of affine Hecke R-algebras generalize to the generic pro-p Iwahori Hecke R-algebra, and by specialization to $\mathcal{H}_R(G)$. The $R[\mathbf{q}_*]$-module $\mathcal{H}(G)(\mathbf{q}_*, c_*)$ is free with an Iwahori–Matsumoto basis of elements satisfying braid relations and quadratic relations, with "alcove walk bases" associated to the Weyl chambers. There are product for-

47 Flicker [69] studied the pro-p Iwahori Hecke complex algebra when G is unramified.

48 The Iwahori Matsumoto presentation, the Bernstein basis, the Bernstein–Lusztig relations, the description of the center, and the geometric proofs of Görtz [78].

mulas involving different alcove walk bases, and Bernstein–Lusztig relations from which one deduces an explicit canonical $R[\mathbf{q}_*]$-basis of the center [196].

Finiteness properties of the pro-p Iwahori ring $\mathcal{H}(G)$.

(i) The center $\mathcal{Z}(G)$ of $\mathcal{H}(G)$ is a finitely generated $\mathbb{Z}$-algebra and $\mathcal{H}(G)$ is a finitely generated $\mathcal{Z}(G)$-module.

(ii) $\mathcal{Z}(G)$ contains a canonical subring $\mathcal{Z}_T$ isomorphic to the affine semigroup $\mathbb{Z}$-algebra $\mathbb{Z}[T^+/T^0]$, and the $\mathcal{Z}_T$-modules $\mathcal{Z}$ and $\mathcal{H}$ are finitely generated.

(iii) The elements of the Iwahori–Matsumoto basis[49] of $\mathcal{H}(G)$ are invertible in $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \mathcal{H}(G)$.

(iv) For any commutative ring R, the center of $\mathcal{H}_R(G)$ is $\mathcal{Z}_R(G) = R \otimes_{\mathbb{Z}} \mathcal{Z}(G)$.

For any field R, any simple $\mathcal{H}_R(G)$-module is finite dimensional by (i) and (iv) [101].

Xuhua He and Radhika Ganapathy [93] gave an Iwahori–Matsumoto presentation of the Hecke ring $\mathcal{H}(G, J_n)$ of the nth congruence subgroup J_n of J for any $n \in \mathbb{N}_{>0}$.

For a standard parabolic subgroup $P = MN$, although $\mathcal{H}_R(M)$ is not contained in $\mathcal{H}_R(G)$, there is a *parabolic induction*

$$\operatorname{ind}_{\mathcal{H}(M)}^{\mathcal{H}(G)} = - \otimes_{\mathcal{H}_R(M)} X_{G,P} : \operatorname{Mod} \mathcal{H}_R(M) \to \operatorname{Mod} \mathcal{H}_R(G), \quad X_{G,P} = \operatorname{ind}_P^G\big(R[\tilde{J}_M \backslash M]\big)$$

of right adjoint $\operatorname{Hom}_{\mathcal{H}_R(G)}(X_{G,P}, -)$ and of left adjoint a certain localization (hence the left adjoint is exact, a surprise when p is not invertible in R as the functor $(-)_N$ for representations is not exact). The parabolic induction and its right adjoint for the group and for the pro-p Iwahori Hecke algebra correspond to each other via the pro-p Iwahori invariant functors. The same holds true for the left adjoint functor if R is a field of characteristic different from p, but Abe gave a counterexample for $G = \mathrm{GL}(2, \mathbb{Q}_p)$ and R of characteristic p (Ollivier–Vignéras [154]). The parabolic induction is isomorphic to

$$\operatorname{ind}_{\mathcal{H}(P)}^{\mathcal{H}(G)} = - \otimes_{\mathcal{H}(P)} \mathcal{H}(G) : \operatorname{Mod} \mathcal{H}_R(M) \to \operatorname{Mod} \mathcal{H}_R(G),$$

where $\mathcal{H}(P) = \mathbb{Z}[(\tilde{J} \cap P)\backslash G/(\tilde{J} \cap P)]$ is the parabolic pro-p Iwahori Hecke ring of P for two ring homomorphisms $\mathcal{H}(M) \leftarrow \mathcal{H}(P) \to \mathcal{H}(G)$ (Heyer [111]).

For an algebraically closed field R of characteristic p and an irreducible R-representation W of a special parahoric subgroup K containing $\tilde{J}$, an inverse Satake-type isomorphism

$$f : \mathcal{H}_R(Z^-, Z^0, W^{U^0}) \tilde{\to} \mathcal{H}_R(G, K, W)$$

is obtained by composition of two natural algebra isomorphisms (Ollivier [149] when G is split, Vignéras [200] in general). The first isomorphism is associated to a "good" alcove walk basis

$$\mathcal{H}_R(Z^-, Z^0, W^{U^0}) \tilde{\to} \operatorname{End}_{\mathcal{H}_R(G)}\big(W^{\tilde{J}} \otimes_{\mathcal{H}_R(K,\tilde{J})} \mathcal{H}_R(G)\big).$$

49 The Iwahori–Matsumoto basis of $\mathcal{H}(G)$ is given by the characteristic functions of the double cosets of G modulo $\tilde{J}$.

The dimension of $W^{\tilde{J}}$ is 1. The second isomorphism

$$\mathrm{End}_{\mathcal{H}_R(G)}\big(W^{\tilde{J}} \otimes_{\mathcal{H}_R(K,\tilde{J})} \mathcal{H}_R(G)\big) \tilde{\to} \mathcal{H}_R(G,K,W)$$

is associated to a natural $H_R(G)$-module isomorphism $W^{\tilde{J}} \otimes_{\mathcal{H}_R(K,\tilde{J})} \mathcal{H}_R(G) \tilde{\to} (\mathrm{ind}_K^G W)^{\tilde{J}}$. When G is split, f is the inverse of the generalized Satake isomorphism Sat'_Z (Ollivier [149]).

13. MODULES OF PRO-p IWAHORI HECKE ALGEBRAS OVER A FIELD IN CHARACTERISTIC p

There is a numerical mod p local Langlands correspondence for the pro-p Iwahori Hecke algebra of $\mathrm{GL}(n,F)$ (Vignéras [191]). The following two sets have the same (finite) cardinality[50]:

(a) the isomorphism classes of the n-dimensional irreducible $\mathbb{F}_p^{\mathrm{ac}}$-representations of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$ with a fixed value of the determinant of the action of a Frobenius;

(b) the isomorphism classes of the supersingular simple modules $\mathcal{H}_{\mathbb{F}_p^{\mathrm{ac}}}(\mathrm{GL}(n,F))$ with a fixed action of p_F embedded diagonally.

When $F \supset \mathbb{Q}_p$, this was significantly improved by Grosse-Kloenne [80,81]. He constructed an exact and fully faithful functor from the category of finite length supersingular $\mathcal{H}_{\mathbb{F}_{p^d}}(\mathrm{GL}(n,F))$-modules to the category of $\mathbb{F}_q^d$-representations of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$, if $p^d \geq q$.[51]

We recall that the pro-p Iwahori Hecke ring $\mathcal{H}(G)$ of G is a finitely generated module over a central subring $\mathcal{Z}_T \simeq \mathbb{Z}[T^+/T^0]$. A nonzero (right) $\mathcal{H}_R(G)$-module $\mathcal{V}$ is called

ordinary if the action on $\mathcal{V}$ of any $z \in \mathcal{Z}_T$ corresponding to a non-invertible element of the semigroup T^+/T^0 is invertible;

supersingular if for any $v \in \mathcal{V}$ and any $z \in \mathcal{Z}_T$ corresponding to a non-invertible element of T^+/T^0, there exists $n \in \mathbb{N}$ such that $z^n v = 0$.

Let R be an algebraically closed field of characteristic p.

Classification of simple $\mathcal{H}_R(G)$-modules. The supersingular simple $\mathcal{H}_R(G)$-modules are classified (Vignéras [200]). The simple $\mathcal{H}_R(G)$-modules are classified in terms of the simple supersingular $\mathcal{H}_R(M)$-modules for the Levi subgroups M of the parabolic subgroups of G (Abe [6]; being algebraically closed is not necessary, see Henniart–Vignéras [106]):

For a standard parabolic subgroup $P = MN$ of G and a simple supersingular $\mathcal{H}_R(M)$-module $\mathcal{W}$, there is a notion of extension $e_{P'}(\mathcal{W})$ of $\mathcal{W}$ to $\mathcal{H}_R(M')$ for a parabolic

50 Equal to the number of irreducible unitary polynomials of degree n in $k_F[X]$.

51 $F^{\mathrm{sep}} = F^{\mathrm{ac}}$ as the characteristic of F is 0.

subgroup $P' = M'N'$ of G containing P. There is a maximal P' with this property, denoted by $P(\mathcal{W})$. For a parabolic subgroup Q with $P \subset Q \subset P(\mathcal{W})$, there is a generalized Steinberg $\mathcal{H}_R(M(\mathcal{W}))$-module

$$\mathrm{st}_Q^{P(\mathcal{W})}(\mathcal{W}) = \mathrm{ind}_{\mathcal{H}(Q)}^{\mathcal{H}(G)}\big(e_Q(\mathcal{W})\big) / \sum_{Q \subsetneq Q' \subset Q(\mathcal{W})} \mathrm{ind}_{\mathcal{H}(Q')}^{\mathcal{H}(G)}\big(e_{Q'}(\mathcal{W})\big).$$

The triple $(P, \mathcal{W}, Q)$ is called standard. The $\mathcal{H}_R(G)$-module

$$I_{\mathcal{H}(G)}(P, \mathcal{W}, Q) = \mathrm{ind}_{\mathcal{H}(P(\mathcal{W}))}^{\mathcal{H}(G)}\big(\mathrm{st}_Q^{P(\mathcal{W})}(\mathcal{W})\big)$$

is simple, and any simple $\mathcal{H}_R(G)$-module is isomorphic to $I_{\mathcal{H}(G)}(P, \mathcal{W}, Q)$ for some standard triple $(P, \mathcal{W}, Q)$ unique modulo G-conjugation. It is ordinary if and only if $P = B$.

Extensions. The extensions between simple $\mathcal{H}_R(G)$-modules

$$\mathrm{Ext}^i_{\mathcal{H}(G)}\big(I_{\mathcal{H}_R(G)}(P_1, \mathcal{W}_1, Q), I_{\mathcal{H}(G)}(P_2, \mathcal{W}_2, Q_2)\big), \quad i \geq 0,$$

are either 0, or extensions between supersingular simple modules of a specialization of a generic pro-p Iwahori Hecke algebra which is not of a pro-p Iwahori Hecke R-algebra (Abe [2]). In more details, considering the central characters, the extensions are 0 if $P_1 \neq P_2$. When $P = P_1 = P_2$, following the construction of the simple modules, we have

$$\mathrm{Ext}^i_{\mathcal{H}_R(G)}\big(I_{\mathcal{H}(G)}(P, \mathcal{W}_1, Q), I_{\mathcal{H}(G)}(P, \mathcal{W}_2, Q_2)\big) \simeq \mathrm{Ext}^i_{\mathcal{H}_R(M')}\big(\mathrm{st}_{Q_1'}^{P'}(\mathcal{W}_1), \mathrm{st}_{Q_2'}^{P'}(\mathcal{W}_2)\big)$$

for some P', Q_1', Q_2',

$$\mathrm{Ext}^i_{\mathcal{H}_R(G)}\big(\mathrm{st}_{Q_1}^G(\mathcal{W}_1), \mathrm{st}_{Q_2}^G(\mathcal{W}_2)\big) \simeq \mathrm{Ext}^{i-r}_{\mathcal{H}_R(G)}\big(e_G(\mathcal{W}_1), e_G(\mathcal{W}_2)\big)$$

for some explicit $r \in \mathbb{N}_{\geq 0}$, and using results of Ollivier–Schneider [150],

$$\mathrm{Ext}^i_{\mathcal{H}_R(G)}\big(e_G(\mathcal{W}_1), e_G(\mathcal{W}_2)\big) \simeq \mathrm{Ext}^i_{\mathcal{H}_R(M)/I}(\mathcal{W}_1, \mathcal{W}_2)$$

for some ideal I of $\mathcal{H}_R(M)$ acting on $\mathcal{W}_1$, $\mathcal{W}_2$ by 0. Abe computed explicitly Ext^1 for two supersingular simple $\mathcal{H}_R(M)/I$-modules.

- When $G = \mathrm{GL}(2, F)$, Cédric Pépin and Tobias Schmidt proved:

  (i) The 2-dimensional supersingular simple $\mathcal{H}_{\mathbb{F}_p^{\mathrm{ac}}}(G)$-modules can be realized through the equivariant cohomology of the flag variety of the dual group $\hat{G}$ over $\mathbb{F}_p^{\mathrm{ac}}$ [160].

  (ii) There is a version in families of the Breuil's semisimple mod p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ [161].

  (iii) There is a Kazhdan–Lusztig theory for the generic pro-p Iwahori Hecke $\mathbb{Z}[\mathbf{q}]$-algebra of G, where the role of $\hat{G}$ is taken by the Vinberg monoid $V_{\hat{G}}$ and its flag variety; the monoid comes with a fibration $V_{\hat{G}} \to \mathbb{A}^1$ and the dual parametrization of $H_{\mathbb{F}_p^{\mathrm{ac}}}(G)$-modules is achieved by working over the 0-fiber. They introduce a generic pro-p antispherical module and a generic pro-p Satake homomorphism for a generic spherical Hecke $\mathbb{Z}[\mathbf{q}]$-algebra [162].

14. REPRESENTATIONS OVER A FIELD OF CHARACTERISTIC p

In this section, R is a field of characteristic p. The admissible irreducible R-representations of G are classified in terms of the supersingular admissible irreducible R-representations of the Levi subgroups of G (Abe–Henniart–Herzig–Vignéras [8] for R algebraically closed, Henniart–Vignéras [106] for R not algebraically closed).

Classification. The representation $\mathrm{ind}_P^G W$ parabolically induced from an irreducible admissible supersingular R-representation W of a Levi subgroup M of a parabolic subgroup P of G, has multiplicity 1 and irreducible subquotients

$$I_G(P, W, Q) = \mathrm{ind}_{P(W)}^G\big(e(W) \otimes \mathrm{St}_Q^{P(W)}\big)$$

for the parabolic subgroups Q of G containing P and contained in the maximal parabolic subgroup $P(W)$ where the inflation of W to P extends to a representation $e(W)$, and

$$\mathrm{St}_Q^{P(W)} = (\mathrm{ind}_Q^{P(W)} 1_Q)/ \sum_{Q \subsetneq Q' \subset P(W)} \mathrm{ind}_{Q'}^{P(W)} 1_{Q'}.$$

Any irreducible admissible R-representation V of G is isomorphic to $I_G(P, W, Q)$ for a unique triple (P, W, Q) modulo G-conjugation.

A similar classification holds true for the irreducible admissible genuine mod p representations of the metaplectic double cover of $\mathrm{Sp}_{2n}(F)$ (Koziol–Peskin [124]).

There is a complete description of $\mathrm{ind}_P^G W$ for any irreducible admissible R-representation W of M [106]. As a corollary, one obtains generic irreducibility and for any admissible irreducible R-representation V of G,

$$V \text{ supersingular } \Leftrightarrow V \text{ cuspidal } \Leftrightarrow V \text{ supercuspidal.}$$

When F has characteristic 0, the higher duals $(S^i(I_G(P, W, Q)))_{i \geq 0}$ are computed in terms of $(S^i(W))_{i \geq 0}$ in a few cases (Kohlhaase [117]).

The extensions between R-representations $\mathrm{ind}_P^G W$ of G, parabolically induced from supersingular absolutely irreducible R-representations W of Levi subgroups, are computed in many cases when G is split and R finite (Hauseux [86, 87, 89, 90]).

When $P = B$, the irreducible subquotients of $\mathrm{ind}_B^G W$ are called *ordinary*. An admissible R-representation of G with ordinary irreducible subquotients is called ordinary.

The $\tilde{J}$-invariant functor induces an equivalence between the category of finite length ordinary R-representations of G generated by their $\tilde{J}$-invariant vectors and the category of the finite length ordinary $\mathcal{H}_R(G)$-modules, assuming R algebraically closed (Abe [5]).

The pro-p Iwahori invariant $I_G(P, W, Q)^{\tilde{J}}$ is computed and depends only on the pro-p Iwahori invariant $W^{\tilde{J}_M}$ (Abe–Henniart–Vignéras [9, 10]).

The supersingular admissible irreducible R-representations V of G are not understood, this remains an open crucial question for two decades and a stumbling block for the search of a mod p or p-adic local Langlands correspondence if $G \neq \mathrm{GL}(2, \mathbb{Q}_p)$. The supersingularity can be seen on the pro-p Iwahori invariants (Ollivier–Vignéras [154] for R algebraically closed, but being algebraically closed is not necessary Henniart–Vignéras [106]):

V is supersingular $\Leftrightarrow V^{\tilde{J}}$ is supersingular $\Leftrightarrow$ some nonzero subquotient of $V^{\tilde{J}}$ is supersingular.

The classification of simple supersingular $\mathcal{H}_R(G)$-modules does not help because we do not have enough information on the pro-p Iwahori invariant functor.

When $G = \mathrm{GL}(2, \mathbb{Q}_p)$, Breuil [16] relying on the work of Barthel–Livne classified the supersingular admissible irreducible mod p representations. This was the starting point of the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. There are two main novel features. The mod p local Langlands correspondence involves reducible representations and extends to an exact functor from finite length representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to finite length representations of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ac}}/\mathbb{Q}_p)$.

When $G \neq \mathrm{GL}(2, \mathbb{Q}_p)$, supersingular admissible irreducible mod p representations are classified only for some groups close to $\mathrm{GL}(2, \mathbb{Q}_p)$: for $\mathrm{SL}(2, \mathbb{Q}_p)$ (Abdellatif [1], Cheng [27]), and for the unramified unitary group $U(1,1)(\mathbb{Q}_p^2/\mathbb{Q}_p)$ in two variables (Koziol [118]). When $F \neq \mathbb{Q}_p$, there can be many more supersingular admissible irreducible mod p representations of $\mathrm{GL}(2, F)$ than 2-dimensional irreducible representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ (Breuil–Paskunas [19]); they cannot be described as quotients of a compact induction by a finite number of equations (Hu [115] if $F \supset \mathbb{F}_p((T))$, Schraen [176] if $F/\mathbb{Q}_p$ is quadratic, Wu [204] in general if $F \supsetneq \mathbb{Q}_p$).

When R is a field of characteristic p and $F \supset \mathbb{Q}_p$, Herzig–Koziol–Vignéras [110] proved that any G admits a supersingular admissible irreducible R-representation, using a local method of Paskunas [155] if the semisimple rank r_G of G is 1, and a global method if $r_G > 1$. The existence is not known if $F \supset \mathbb{F}_p((T))$.

There have been recent advances which strongly suggest that the study of mod p representations of G is best done on the derived level. When R is a field of characteristic p, Schneider [172] introduced the unbounded derived category $D_R(G)$ of R-representations of G. When $\tilde{J}$ is torsion free (this forces F to be of characteristic 0), $D_R(G)$ is equivalent to the derived category of differential graded modules over a differential graded version $\mathcal{H}_R(G)^\bullet$ of the pro-p Iwahori Hecke R-algebra of G, by the derived $\tilde{J}$-invariant functor.

The parabolic induction $\mathrm{ind}_P^G : \mathrm{Mod}_R(M) \to \mathrm{Mod}_R(G)$ being exact extends to an exact derived parabolic induction $R\,\mathrm{ind}_P^G : D_R(M) \to D_R(G)$ between the unbounded derived categories. The total derived functor of R_P^G is right adjoint to $R\,\mathrm{ind}_P^G$. The category $D_R(G)$ has arbitrary small direct products and $R\,\mathrm{ind}_P^G$ commutes with arbitrary small direct products (Heyer [112]), hence $R\,\mathrm{ind}_P^G$ has a left adjoint.[52] When $\tilde{J}$ is torsion free, the functor $R\,\mathrm{ind}_P^G$ corresponds to the derived parabolic induction functor on the dg Hecke algebra side, via the derived $\tilde{J}$-invariant functor (Sarah Scherotzke and Schneider [169]).

The Kohlhaase duality functors are related to the derived duality functor $\mathrm{R}\underline{\mathrm{Hom}}(-, R)$ (Schneider–Sorensen [173]).

The cohomology algebra $\mathrm{Ext}^\bullet_{\mathrm{Mod}_R(G)}(R[\tilde{J}\backslash G], R[\tilde{J}\backslash G])$ is simpler than of $\mathcal{H}_R(G)^\bullet$; when $G = \mathrm{SL}(2, \mathbb{Q}_p)$, $p \geq 5$, its structure is explicited by Ollivier and Schneider [152, 153].

52 By Brown representability; Heyer [112] gave another proof.

15. LOCAL LANGLANDS CORRESPONDENCES FOR GL(n, F)

The complex local Langlands correspondence for GL(n, F) is a bijection between the isomorphism classes of irreducible complex representations of GL(n, F) and the isomorphism classes of n-dimensional Weil–Deligne complex representations, given by local class field theory when $n = 1$, and characterized by the requirement that the L and ε factors[53] attached to corresponding pairs of complex representations coincide (Henniart [100]). An n-dimensional Weil–Deligne complex representation is a pair (σ, N) where σ is an n-dimensional Frobenius semisimple complex representation of the Weil group W_F and $N \in \mathrm{End}_{\mathbb{C}}\, \sigma$ is a nilpotent endomorphism satisfying $\sigma(w)N\sigma(w)^{-1} = q^{|w|}N$ for all $w \in W_F$.[54] The supercuspidal irreducible $\mathbb{C}$-representations of GL(n, F) correspond to the n-dimensional irreducible $\mathbb{C}$-representations of W_F.[55]

A twist of the correspondence by an unramified complex character of GL(n, F) is compatible with the automorphisms of $\mathbb{C}$. For a prime r, an isomorphism $\mathbb{C} \simeq \mathbb{Q}_r^{\mathrm{ac}}$ transfers the twisted complex local Langlands correspondence to a local Langlands correspondence for $\mathbb{Q}_r^{\mathrm{ac}}$-representations of GL$(n, F)$. For $r = \ell \neq p$, the nilpotent part is related to the action of the tame inertia group on an ℓ-adic representation of W_F. By reduction modulo ℓ of the ℓ-adic local Langlands correspondence composed with the Zelevinski involution on ℓ-adic representations of GL(n, F), one obtains a *ℓ-modular local Zelevinski correspondence*. The ℓ-modular local Zelevinski correspondence is a parametrization for ℓ-modular irreducible representations of GL(n, F) by n-dimensional Weil–Deligne ℓ-modular representations, defined as above with $\mathbb{F}_\ell^{\mathrm{ac}}$ instead of $\mathbb{C}$. The supercuspidal irreducible ℓ-modular representations of GL(n, F) correspond to the n-dimensional irreducible ℓ-modular representations of W_F. But the nilpotent part N of the Weil–Deligne ℓ-modular representation has no obvious Galois interpretation.

Dat [43–45] obtained a geometric realization of the ℓ-adic local Zelevinski correspondence and of the ℓ-modular local Zelevinski correspondence on the unipotent[56] irreducible $\mathbb{F}_\ell^{\mathrm{ac}}$-representations of GL$(n, F)$ when the order of q in $\mathbb{F}_\ell^*$ is at least n [42],[57] and when $q \equiv 1 \bmod \ell$ and $\ell > n$ [46].[58]

Kurinczuk and Matringe [127–130], extended to ℓ-modular representations the Rankin–Selberg local constants of Jacquet, Piatetski-Shapiro, and Shalika of pairs of complex generic representations of linear groups, and the Artin–Deligne local constants of pairs of complex Weil–Deligne representations. These local constants are preserved by the complex local Langlands correspondence, but not by the ℓ-modular local Zelevinski correspondence. Enlarging the space of ℓ-modular Weil–Deligne representations to representations with not necessarily nilpotent operators, they suggested a *ℓ-modular local Langlands cor-*

53 For a fixed nontrivial $\mathbb{C}$-character of F.

54 $|w|$ is the power of q to which w raises the elements of the residue field k_F.

55 $N = 0$.

56 = in the principal block = subquotients of some $\mathrm{Ind}_B^G(\chi)$ for χ an unramified character of a Borel subgroup B, this is not the definition of Lusztig.

57 The regular case.

58 The limit case.

respondence compatible with the formation of local constants and characterized by a list of natural properties. When R is a noetherian $W(\overline{F}_\ell^{\mathrm{ac}})$-algebra, using the Rankin–Selberg functional equations, Matringe and Moss [138] proved that an R-representation of $\mathrm{GL}(n, F)$ of Whittaker type admits a Kirillov model.

When the characteristic of F is 0, Breuil and Schneider [20] motivated by an hypothetical p-adic extension of the local Langlands correspondence, suggested a *modified local Langlands correspondence* where the complex representations of $\mathrm{GL}(n, F)$ are no more irreducible. The Langlands quotient theorem realizes an irreducible $\mathbb{C}$-representation V of $\mathrm{GL}(n, F)$ as a quotient of a certain parabolically induced representation $\mathrm{ind}_P^G W$. In the modified version, V is replaced by a twist of $\mathrm{ind}_P^G W$ by an unramified character of $\mathrm{GL}(n, F)$.

When the characteristic of F is 0, Emerton and Helm [62] motivated by a local Langlands correspondence in families and by global contexts, introduced the *generic ℓ-adic local Langlands correspondence* which has useful applications to the cohomology of Shimura varieties. For any finite extension $E/\mathbb{Q}_\ell$, it is a map $\rho \mapsto \pi(\rho)$ from n-dimensional continuous E-representations of the Galois group $\mathrm{Gal}(F^{\mathrm{ac}}/F)$ to finite length E-representations of $\mathrm{GL}(n, F)$ with an absolutely irreducible generic socle and no other generic irreducible subquotients.[59] Each $\pi(\rho)$ contains a $\mathrm{GL}(n, F)$-stable O_E-lattice $\pi(\rho)^o$ of reduction having an absolutely irreducible socle and no other generic subquotients, unique modulo homotethy.

The *generic mod ℓ local Langlands correspondence* (Emerton–Helm [62]) is compatible with the generic ℓ-adic local Langlands correspondence by reduction modulo ℓ. Irreducible representations of $\mathrm{GL}(n, F)$ are no longer irrreducible, Weil–Deligne representations are now Galois representations, and the Zelevinski involution does not intervene. For a finite extension $R/\mathbb{F}_\ell$, it is the unique map $\overline{\rho} \mapsto \overline{\pi}(\overline{\rho})$ from n-dimensional R-representations of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$ to finite length R-representations of $\mathrm{GL}(n, F)$ such that

(1) $\overline{\pi}(\overline{\rho})$ has an absolutely irreducible generic socle and no other generic irreducible subquotients,

(2) For all finite extensions $E/\mathbb{Q}_\ell$ of ring of integers O_E and residue field k_E containing R, and continuous representation $\rho : \mathrm{Gal}(F^{\mathrm{ac}}/F) \to \mathrm{GL}(n, O_E)$ lifting $\overline{\rho} \otimes_R k_E$, the reduction of $\pi(\rho)^o$[60] embeds in $\overline{\pi}(\overline{\rho}) \otimes_R k_E$.

(3) $\overline{\pi}(\overline{\rho})$ is minimal with respect to the above two conditions.

For $\mathrm{GL}(2, F)$, the correspondence is fairly concrete and explicit when $\ell \neq 2$ (Helm [95]).

Emerton and Helm [99] introduced also a notion of a *local Langlands correspondence in families*.[61] For any suitable complete local noetherean algebra R with finite residue field k, it is the unique map $\rho \mapsto \pi(\rho)$ from the continuous representations $\rho : \mathrm{Gal}(F^{\mathrm{ac}}/F) \to \mathrm{GL}(n, R)$ to the admissible R-representations of $\mathrm{GL}(n, F)$ that interpolates the generic local

59 It is a slight modification of the Breuil and Schneider correspondence transferred to ℓ-adic representations; the socle of V is the maximal semisimple subrepresentation of V.

60 ρ identifies with a representation $\mathrm{Gal}(F^{\mathrm{ac}}/F) \to \mathrm{GL}(n, E)$.

61 For an example of a local p-adic Langlands correspondence in families for $\mathrm{GL}(2, \mathbb{Q}_p)$, see Ildar Gaisin and Joaquin Rodrigues Jacinto [70].

Langlands correspondences over the points of Spec R and satisfies certain technical hypotheses.

The existence of the map amounts to showing that whenever there is a congruence between two ℓ-adic representations of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$, there is a corresponding congruence on the other side of the ℓ-adic generic local Langlands correspondence. The key idea of the proof is the introduction of the Bernstein center $\mathcal{Z}$ of $\mathrm{Mod}_{\mathbb{Z}_\ell^{\mathrm{ac}}}(\mathrm{GL}(n, F))$ (Helm [96–98]), which encodes deep information about congruences between integral $\mathbb{Q}_\ell^{\mathrm{ac}}$-representations of $\mathrm{GL}(n, F)$. For instance, if two integral irreducible $\mathbb{Q}_\ell^{\mathrm{ac}}$-representations of $\mathrm{GL}(n, F)$ become isomorphic modulo ℓ, then $\mathcal{Z}$ acts on these representations by scalars congruent modulo ℓ.

When G is quasisplit, motivated by a local Langlands correspondence in families, Dat, Helm, Kurinczuk, and Moss [51] studied the scheme of Langlands parameters of G with coefficient the smallest possible ring $R = \mathbb{Z}[1/p]$. In particular, this allows studying a chain of congruences of Langlands parameters modulo several different primes. In a work in progress, they extend the Emerton–Helm–Moss local Langlands correspondence in families to a conjecture which asserts the existence of isomorphisms between

(a) the center of $\mathrm{Mod}_{\mathbb{Z}[q^{-1/2}]}(G)$,

(b) the ring of functions on the moduli stack of Langlands parameters[62] for G over $\mathbb{Z}[q^{-1/2}]$,

(c) the descent to $\mathbb{Z}[q^{-1/2}]$ of the endomorphisms of a Gelfand–Graev representation of G.

They prove the conjecture when G is any classical p-adic group after inverting an integer. The conjecture should follow from a Fargues–Scholze conjecture [65, I.10.2].[63]

The blocks in the category of $\mathbb{Z}^{\mathrm{ac}}[1/p]$-representations of G of depth 0 are in natural bijection with the connected components of the space of tamely ramified Langlands parameters for G over $\mathbb{Z}^{\mathrm{ac}}[1/p]$; there is only one block (the category is indecomposable) if G is semisimple and simply connected, or unramified (Dat–Lanard [53]).

When the characteristic of F is 0, the p-adic local Langlands correspondence for $\mathrm{GL}(n, F)$ is a hypothetical correspondence between continuous unitary E-Banach space representations of $\mathrm{GL}(n, F)$ and n-dimensional continuous E-representations of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$, for any finite extension $E/\mathbb{Q}_p$, given by local class field theory when $n = 1$. Using global methods, Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paskunas, and Sug Woo Shin [26] constructed a candidate when p does not divide $2n$. For $F = \mathbb{Q}_p$ and $n = 2$, it coincides with the p-adic local correspondence envisioned by Breuil 20 years ago, constructed by Pierre Colmez [33], and analyzed by Paskunas [157], Colmez, Dospinescu, Paskunas [35].

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62 Constructions of moduli spaces of Langlands parameters have been also proposed by Fargues and Scholze ([65] over $\mathbb{Z}_\ell$, $\ell \neq p$ using the condensed mathematics of Clausen–Scholze) and by Xinwen Zhu [208]. The local Langlands correspondence is now conjectured to exist at a categorical level (Denis Gaitsgory [71]).

63 Private communication of Dat.

When $n \geq 2$ and D_n is the central division algebra over F of invariant $1/n$, Scholze [175] constructed a candidate for a p-adic and mod p Jacquet–Langlands correspondence from $\mathrm{GL}(n, F)$ to D_n^* in a purely geometric way, using the cohomology of the infinite-level Lubin–Tate space. The mod p Jacquet–Langlands correspondence is a canonical map from the admissible mod p representations of $\mathrm{GL}(n, F)$ to the admissible mod p representations of D_n^* having a continuous action of $\mathrm{Gal}(F^{\mathrm{ac}}/F)$. For $F = \mathbb{Q}_p$ and $n = 2$, it is studied by Dospinescu–Paskunas–Schraen [54].

16. GELFAND–KIRILLOV DIMENSION

Let R be a field and V an irreducible admissible R-representation of G. For any decreasing sequence $(K_i)_{i\geq 1}$ of open compact subgroups of G with limit the trivial group, the dimensions $\dim_R V^{K_i}$ for $i \geq 1$ are finite. If V is finite dimensional, $\dim_R V^{K_i} = \dim_R V$ when i is large enough. Generally, the dimension of V is infinite and the increasing sequence $(\dim_R V^{K_i})_{i\geq 1}$ tends to infinity, but how?

When F has characteristic 0, one can choose an O_F-lattice $\mathfrak{L}$ of the Lie algebra $\mathcal{G}$ of G on which the exponential map exp is defined and such that $K = \exp(\mathfrak{L})$ is a group, and consider the decreasing sequence $(K_i = \exp(p_F^{2i}\mathfrak{L}))_{i\geq 1}$. When $R = \mathbb{C}$, the Harish-Chandra local character expansion of V implies that $\dim_R V^{K_i}$ eventually becomes polynomial[64]

$$\dim_R V^{K_i} = P_{\mathfrak{L},V}(q^i), \quad P_{\mathfrak{L},V}(X) \in \mathbb{Q}[X] \text{ for } i \text{ large enough.}$$

The degree d_V of the polynomial $P_{\mathfrak{L},V}[X]$ does not depend on the choice of $\mathfrak{L}$. It is half the dimension of a unipotent conjugacy class in G,

$$0 \leq d_V \leq \dim_F U,$$

and is 0 if and only if V is finite dimensional. The integer q^{d_V} measures the growth of $(\dim_R V^{K_i})_{i\geq 1}$ for any choice of $\mathfrak{L}$.

For F of either characteristic 0 or p, when $G = \mathrm{GL}(n, F)$, $K_i = 1 + p_F^{i+1} M(n, O_F)$ for $i \geq 1$, if $R = \mathbb{C}$, the Roger Howe local character expansion implies that the dimensions

$$\dim_R V^{K_i} = P_V(q^i), \quad P_V(X) \in \mathbb{Z}[X]$$

are polynomial when i is large, for a polynomial $P_V(X)$ with integral coefficients and degree $d_V \leq n(n-1)/2$. When V is cuspidal (or more generally, generic), $d_V = n(n-1)/2$.

Example. For $\mathrm{GL}(2, F)$, V is infinite dimensional if and only if $d_V = 1$.

Any cuspidal irreducible ℓ-modular representation V of $\mathrm{GL}(n, F)$ lifts to an irreducible cuspidal ℓ-adic representation, implying that the dimensions $\dim_R V^{K_i}$ satisfy the same properties. This is probably true for any irreducible representation of $\mathrm{GL}(n, F)$ over any field R of characteristic ℓ.[65]

64 Harish-Chandra, Notes by Stephen DeBacker and Paul J. Sally, Admissible invariant distributions on reductive p-adic group, University Lecture Series Vol. 16, 1999, 97 pp.

65 Article in preparation.

For a finite field R of characteristic p, $G = \mathrm{GL}(2, \mathbb{Q}_p)$ and V admissible absolutely irreducible, Stefano Morra [144] computed $\dim_R V^{K_i}$ for $i \geq 1$. The dimensions satisfy the above properties.

For F of characteristic 0, R a finite field of characteristic p, K a uniformly powerful open pro-p subgroup of G, K_i the closed subgroup of K generated by $\{k^{p^i}, k \in K\}$ for $i \geq 1$, and V an admissible R-representation of G, there is a positive integer δ_V not depending on the choice of K and positive real numbers $a \leq b$ such that (Calegari–Emerton [24], Emerton–Paskunas [64], Dospinescu–Paskunas–Schraen [54]):

$$a p^{i\delta_V} \leq \dim_R V^{K_i} \leq b p^{i\delta_V}.$$

The integer δ_V which is a sort of Iwasawa dimension of the dual of V, is called the *Gelfand–Kirillov dimension* of V. When $F/\mathbb{Q}_p$ is unramified, the admissible R-representations V of $\mathrm{GL}_2(F)$ studied by Breuil–Herzig–Hu–Morra–Schraen [18] in mod p cohomology satisfy $\delta_V = [F : \mathbb{Q}_p]$. If V is isomorphic to $I_G(P, W, Q)$, we have[66]

$$\delta_{I_G(P,W,Q)} = \delta_W + \dim_{\mathbb{Q}_p} N_Q,$$

where N_Q is the unipotent radical of the parabolic subgroup Q of G.

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66 Article in preparation.

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