GROUPS ACTING ON HYPERBOLIC SPACES—A SURVEY

MLADEN BESTVINA

ABSTRACT

This is a (very subjective) survey paper for nonspecialists, covering group actions on Gromov hyperbolic spaces. The first section is about hyperbolic groups themselves, while the rest of the paper focuses on mapping class groups and $\mathrm{Out}(F_n)$, and the way to understand their large scale geometry using their actions on various hyperbolic spaces constructed using projection complexes. This understanding for $\mathrm{Out}(F_n)$ significantly lags behind that of mapping class groups, and the paper ends with a few open questions.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 20F65; Secondary 20F67, 20F69, 57K20

INTERNATIONAL CONGRESS OF MATHEMATICIANS
2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 2, pp. 678–711
DOI 10.4171/ICM2022/110

Published by EMS Press

1. INTRODUCTION

The goal of this paper is to give a flavor of the developments in geometric group theory in the last 35 years, focusing on groups acting on Gromov hyperbolic spaces. The field of geometric group theory is relatively young and its birth can be attributed to Gromov's paper [71] in 1987, when the subject exploded and attracted many mathematicians. The term itself was coined by Niblo and Roller, who organized and named a very influential conference in 1991 [107,108] (though it was possibly used informally before). Loosely speaking, geometric group theory studies groups by looking at their actions on metric spaces and the geometry and topology of these spaces. Increasingly, methods of other branches of mathematics, such as dynamics and analysis, are also brought to bear.

There were, of course, significant developments that can be comfortably placed within this subject even long before Gromov's paper. Works of Klein, Dehn, Nielsen, Stallings, and others in some sense form the backbone of the subject. The theory of groups acting on trees, i.e., Bass–Serre theory [6,131] and its language, will be used freely in these notes. Gromov's celebrated theorem that groups of polynomial growth are virtually nilpotent [69] appeared in 1981, and Gromov's basic philosophy of viewing groups as metric spaces was eloquently explained in [70]. Of course, the influence on this subject of the work of Thurston cannot be overstated. Perhaps the development of combinatorial group theory, focusing on the combinatorics of the words in a finitely presented group, distracted from a more geometric approach to group theory.

This paper will focus on the part of geometric group theory that studies groups acting on (Gromov) hyperbolic spaces. In the early days, right after Gromov's paper, this meant studying (Gromov) hyperbolic groups. Around 2000, the work of Masur and Minsky [95,96] shifted the focus to groups that are not hyperbolic but admit interesting actions on hyperbolic spaces. The main examples of such groups are mapping class groups of compact surfaces (the subject of the papers by Masur and Minsky) and $\mathrm{Out}(F_n)$, the outer automorphism group of a finite rank free group. This survey will concentrate on these two classes of groups.

The definition of Gromov hyperbolic spaces is modeled on the standard hyperbolic spaces by "coarsification" and captures the fact that geodesic triangles in the hyperbolic plane are "thin." For a wonderful survey of the history of hyperbolic geometry from Lobachevsky to 1980, see Milnor's paper [99]. For much more about this subject, see Bridson–Haefliger [44], Ghys–de la Harpe [68], or Druţu–Kapovich [58]. There are many important topics this survey will not cover, e.g., relative hyperbolicity [61], hyperbolic Dehn filling [74,114], small cancelation [1,112], uniform embeddings in Hilbert spaces [120], the celebrated work of Agol and Wise, see, e.g., [16], random walk [93], Cannon–Thurston maps [100], and many others.

2. HYPERBOLIC GROUPS

Every finitely generated group G can be viewed as a metric space. Fix a finite generating set S which is symmetric, i.e., $S^{-1} = S$. The *word norm* $|g|_S$ of $g \in G$ is the smallest n such that g can be written as $g = s_1 s_2 \cdots s_n$ for $s_i \in S$. Then $d_S(g,h) = |g^{-1}h|_S$ is the *word metric* on G, and left translations $L_x : g \mapsto xg$ are isometries. More geometrically, this is the distance function on the vertices of the Cayley graph Γ_S, with vertex set G, and edges of length 1 between g and gs for $g \in G$ and $s \in S$. If S' is a different finite symmetric generating set for G, the identity map $G \to G$ is bilipschitz with respect to the two word metrics, and are considered equivalent. There is a more general equivalence relation between metric spaces that is very convenient in the subject. Let (X, d_X) and (Y, d_Y) be metric spaces. A (not necessarily continuous) function $f : X \to Y$ is a *quasiisometry* if there is a number $A > 0$ such that

$$\frac{1}{A} d_X(a,b) - A \le d_Y\big(f(a), f(b)\big) \le A d_X(a,b) + A$$

for all $a, b \in X$, and every metric ball of radius A in Y intersects the image of f. Without the second condition, f is a *quasiisometric embedding* (when we want to refer to the constant A, we say A-quasiisometric embedding). Two metric spaces are *quasiisometric* if there is a quasiisometry between them, and this is an equivalence relation. For example, inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasiisometry, as is any bilipschitz homeomorphism or a finite index inclusion between finitely generated groups equipped with word metrics. More generally, the following is considered to be the Fundamental Theorem of Geometric Group Theory.

Theorem 2.1 (Milnor [98], Švarc [135]). *Suppose a group G acts properly and cocompactly by isometries on a proper geodesic metric space X. Then G is finitely generated and any orbit map $G \to X$ is a quasiisometry.*

A metric space is proper if closed metric balls are compact, and it is *geodesic* if any two distinct points a, b are joined by a subset isometric to the closed interval $[0, d(a,b)]$. For example, cocompact lattices in a simple Lie group are quasiisometric to each other. The "Gromov program" is to classify groups, at least in a given class, up to quasiisometry.

According to Gromov, the following definition was given by Rips. There are several other definitions, all of which are equivalent up to changing the value of δ, see [44, 58].

Definition 2.2. Let $\delta \ge 0$. A geodesic metric space X is *δ-hyperbolic* if in any geodesic triangle each side is contained in the δ-neighborhood of the other two sides. We say X is *hyperbolic* if it is δ-hyperbolic for some $\delta \ge 0$. See Figure 1.

For example, trees are 0-hyperbolic and so are complete simply-connected Riemannian manifolds of sectional curvature $\le -\varepsilon < 0$. A fundamental property of hyperbolic spaces is the Morse Lemma, proved by Morse [105], Busemann [48], and Gromov [71] in increasing generality.

Lemma 2.3 (Morse Lemma). *There is a number $D = D(\delta, A)$ such that for any δ-hyperbolic space X and any A-quasiisometric embedding $f : [a,b] \to X$ the image of f is contained in the D-neighborhood of any geodesic from $f(a)$ to $f(b)$.*

It then quickly follows that if two geodesic spaces are quasiisometric and one is hyperbolic, so is the other. In particular, groups that act properly and cocompactly by isometries on proper hyperbolic spaces are hyperbolic.

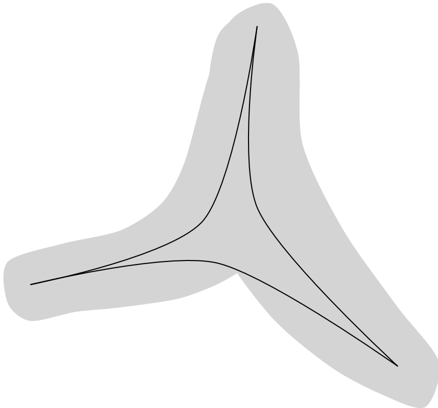

FIGURE 1
The union of the δ-neighborhoods of two sides contains the third.

Hyperbolic groups are well behaved, both topologically and geometrically, and they are generic, so they form a model class of groups in geometric group theory. We now elaborate.

2.1. Classification of elements

Let G be a hyperbolic group. If $g \in G$ has finite order, then there is a coset $\langle g \rangle x$ that has diameter $\leq 4\delta + 2$, so in particular there is an a priori bound on the order in terms of δ and the number of generators. This is proved by a coarse version of the standard argument that a bounded set in $\mathbb{R}^n$ (or any Hadamard manifold) is contained in a unique closed ball of smallest radius. If g has infinite order, then $k \mapsto g^k x$ is a quasiisometric embedding for every $x \in G$, and g is *loxodromic*.

2.2. The Rips complex

The classical Cartan–Hadamard theorem states that closed manifolds of nonpositive sectional curvature have contractible universal cover. In a similar way, every hyperbolic group G acts properly and cocompactly on a contractible simplicial complex, called the *Rips complex*. It is constructed as follows. Fix a number $d > 0$ and form the complex $P_d(G)$: the set of vertices is G, and a set $\{v_0, v_1, \ldots, v_n\}$ of distinct vertices forms a simplex if $d(v_i, v_j) \leq d$ for all i, j. This is a version of the Vietoris approximation of a metric space by a simplicial complex, except here we think of d as being large.

Theorem 2.4. *For $d > 4\delta + 6$, $P_d(G)$ is contractible.*

So, for example, if G is torsion-free, the quotient $P_d(G)/G$ is a finite classifying space for G, and in any case G is finitely presented, and has a classifying space with finitely many cells in each dimension. Every finite subgroup of G fixes a point of $P_d(G)$ (for d large), so it follows that G has finitely many conjugacy classes of finite subgroups. Interestingly, it is not known whether every infinite hyperbolic group is virtually torsion-free, or even if it always has a proper subgroup of finite index.

2.3. Subgroups

If $g \in G$ has infinite order, there is a unique maximal virtually cyclic subgroup $E(g)$ of G that contains g, and $E(g)$ also contains the normalizer of g. It follows that G cannot contain $\mathbb{Z}^2$ as a subgroup. Translation length considerations show that G cannot contain any Baumslag–Solitar groups $B(m,n) = \{a, t \mid ta^m t^{-1} = a^n\}$, $m, n \neq 0$, as subgroups. The long standing open question whether every group with finite classifying space and not containing any $B(m,n)$ is necessarily hyperbolic was recently answered in the negative [86].

2.4. Boundary

Inspired by the visual boundary of Hadamard manifolds, Gromov defined a boundary ∂G of a hyperbolic group (or a proper geodesic metric space which is hyperbolic). It is a compact metrizable space and a point is represented by a quasigeodesic ray $\mathbb{Z}_+ \to G$, with two rays representing the same boundary point if their images stay a bounded distance apart. The topology is based on the principle that rays issuing from a basepoint and with fixed quasigeodesic constants will stay longer together if they represent points that are closer together. If G is infinite and virtually cyclic then ∂G consists of two points, and if G is not virtually cyclic (termed "nonelementary") ∂G has no isolated points.

There is also a natural topology on the union

$$\overline{X} = P_d(G) \sqcup \partial G$$

of the Rips complex and the Gromov boundary that makes it into a compact metrizable space, and G acts naturally by homeomorphisms. Loxodromic elements act by north–south dynamics on $\overline{X}$. The most important property of the boundary, used, for example, in the proof of Mostow rigidity [106], is the following:

Theorem 2.5. *Let $f : X \to Y$ be a quasiisometry between two hyperbolic proper geodesic mmetric spaces. Then f extends to a homeomorphism $\partial X \to \partial Y$.*

Theorem 2.6 ([38]). *$\overline{X}$ is a Euclidean retract, i.e., it is contractible, locally contractible, and finite-dimensional. The covering dimension of ∂G can be computed from the cohomology of G and, in particular, if G is torsion-free,* $\dim \partial G$ *equals the cohomological dimension of G minus 1, and in any case the rational cohomological dimension of ∂G equals the rational cohomological dimension of G minus 1.*

2.5. Asymptotic dimension

In [72] Gromov introduced many quasiisometric invariants of groups and spaces. Here we focus on *asymptotic dimension*. Let X be any metric space. For an integer $n \geq 0$, we write $\operatorname{asdim}(X) \leq n$ provided that for every $R > 0$ there exists a cover of X by uniformly bounded sets such that every ball of radius R in X intersects at most $n+1$ elements of the cover. This is the "large scale" analog of the usual covering dimension. For example, asdim($\mathbb{R}^n$) $= n$ and asdim$(T) \leq 1$ for a tree T with the geodesic metric. This is a quasiisometric invariant, so it is well defined for finitely generated groups as well. See [12] for the basic properties of asdim. There are many groups that contain $\mathbb{Z}^n$ for every n, and they will have infinite asymptotic dimension. However, Gromov proved:

Theorem 2.7 ([72]). *Every hyperbolic group has finite asymptotic dimension.*

One can hardly make a claim that one understands the large-scale geometry of a group if its asymptotic dimension is not known to be finite or infinite. However, the significance of the theorem became particularly clear with the work of Guoliang Yu [143] (see also [57]), who proved that groups with finite asdim and finite classifying space satisfy the Novikov conjecture (this predicts the possible placement of Pontrjagin classes in the cohomology ring of a closed oriented manifold with the given fundamental group).

An even stronger conjecture in manifold topology is the Farrell–Jones conjecture. If it holds for a (torsion-free) group G then one can in principle compute the set of closed manifolds homotopy equivalent to a given closed manifold of dimension ≥ 5 and fundamental group G. Following the work of Farrell and Jones, there has been a great progress in proving the Farrell–Jones conjecture for many groups. For hyperbolic groups, this was done by Bartel, Lück, and Reich [5], see also [3] for a proof using coarse methods that generalize to other groups.

2.6. JSJ decomposition

For simplicity, we now assume that G is a torsion-free hyperbolic group. By Grushko's theorem [75,132], G can be decomposed as a free product $G = G_1 * G_2 * \cdots * G_k * F_r$ where each G_i is noncyclic and freely indecomposable and F_r is a free group. Each G_i is a 1-ended group by the celebrated theorem of Stallings [133], meaning that the Cayley graph of G_i has one end (every finite subgraph has only one unbounded complementary component). Quite unexpectedly, Rips–Sela [119] discovered a further structure theorem for 1-ended hyperbolic groups (the theorem applies to many groups that are not hyperbolic as well). The theorem is motivated by the Jaco–Shalen–Johanssen torus decomposition theorem for 3-manifolds, which provides a canonical decomposition of an aspherical closed orientable 3-manifold by cutting along pairwise disjoint tori so that each piece either has many tori (it is Seifert fibered), or it is not an I-bundle and has no essential tori (except on the boundary, and then by Thurston's hyperbolization theorem it is hyperbolic), or it is an I-bundle. The Rips–Sela theorem can be stated as follows:

Theorem 2.8. *Let G be a 1-ended torsion-free hyperbolic group. Then G is a finite graph of groups with all edge group infinite cyclic, and with vertex groups V coming in three types:*

- (QH) *V is the fundamental group of a compact surface (with a pair of intersecting 2-sided simple closed curves) and the incident edge groups correspond exactly to the boundary components,* (rigid) *V is not cyclic and does not admit a nontrivial splitting over a cyclic group such that all incident edge groups are elliptic, and*

(cyclic) *V is cyclic.*

See also [59,64,76] for different proofs and generalizations, and [40] for how to read off the JSJ decomposition purely from the boundary of G. For example, a splitting over $\mathbb{Z}$ gives a pair of points in ∂G that together separate ∂G, and Bowditch shows how to go in the other direction. Thus the QH vertices give rise to many splittings of G over cyclic groups (one for every simple closed curve), while rigid vertices give rise to none.

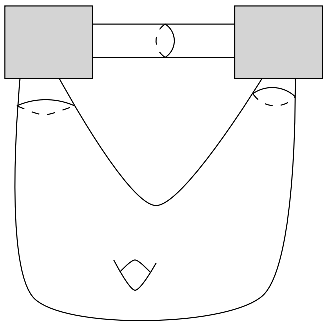

FIGURE 2
A possible JSJ decomposition of a group G, with two rigid vertices and one QH vertex.

We can picture G as the fundamental group of the space obtained from a disjoint union of compact surfaces, "black boxes" and circles by attaching cylinders according to the graph of groups. See Figure 2. The JSJ decomposition is not quite unique, but there are standard moves that transform one such decomposition to another. For example, sometimes one can slide one cylinder over another if they meet at a common circle. The main feature of a JSJ decomposition is that splittings over cyclic groups can be "read off," at least up the standard moves, just like all essential tori in a 3-manifold can be read off from its JSJ decomposition.

2.7. The combination theorem

This is also motivated by 3-manifold theory. The classical Klein–Maskit combination theorem gives conditions under which two discrete groups A, B of isometries of hyperbolic space $\mathbb{H}^3$ with intersection $C = A \cap B$ generate the amalgam $A *_C B$. Thurston's Hyperbolization Theorem [101,138] is proved by cutting the 3-manifold into pieces, and then inductively constructing a hyperbolic structure when gluing the pieces together. There are two opposite extremes in the kinds of gluings, when the intersection of the pieces is quasi-isometrically embedded on both sides, and when it is exponentially distorted. The latter arises when the 3-manifold fibers over the circle and the monodromy is pseudo-Anosov. The following is the hyperbolic group analog.

Theorem 2.9 ([25,26]). *Let G be the fundamental group of a finite graph of hyperbolic groups so that each edge group is quasiisometrically embedded in both vertex groups (but not necessarily in G). Assume the "annuli flare" condition. Then G is a hyperbolic group.*

The precise definition of the annuli flare condition is a bit technical, but let us mention two special cases. The first is when the graph of groups is *acylindrical*, that is, for some $M > 0$ the stabilizer of every segment of length M in the associated Bass–Serre tree is finite. In this case there are no (long) annuli at all. The other case is that of a hyperbolic automorphism $\phi : H \to H$ of a hyperbolic group H. This means that there is $M > 0$ such that for every element $h \in H$ of sufficiently large word length $|h|$ we have

$$\max\{|\phi^M(h)|, |\phi^{-M}(h)|\} \geq 2|h|,$$

so in this case the induced infinite annulus defined on $S^1 \times \mathbb{R}$ sending $S^1 \times \{K\}$ to the loop determined by $\phi^K(h)$ flares exponentially. Aside from automorphisms of closed surface groups induced by pseudo-Anosov homeomorphisms, there are many examples (in fact, they are generic in the sense of random walk [87]) of hyperbolic automorphisms of free groups coming from train track theory [34]. The combination theorem then implies that the mapping torus $H \rtimes_\phi \mathbb{Z}$ is hyperbolic.

The combination theorem has also been used to study hyperbolicity of extensions of free or surface groups in terms of the monodromy homomorphism from the quotient group to the mapping class group or $\mathrm{Out}(F_n)$, giving rise to *convex cocompact subgroups* of these groups [56,63,78,89].

2.8. Random groups are hyperbolic

The most straightforward way to talk about "random groups" is the following model. Fix integers $k \geq 2$ and $m \geq 1$, and for integers $n_1, \ldots, n_m$ consider the finite set

$$N(k, m; n_1, \ldots, n_m)$$

of all group presentations with k generators and m relators of lengths $n_1, \ldots, n_m$. We say that a *random group has property P* if the fraction of groups in $N(k, m; n_1, \ldots, n_m)$ that have P goes to 1 as $\min\{n_1, \ldots, n_m\} \to \infty$.

Theorem 2.10 ([50,110]). *A random group is hyperbolic and its boundary is the Menger curve.*

Thus a random group has rational cohomological dimension 2 and does not split over a finite or a 2-ended group.

Gromov [73] introduced a more sophisticated random model for groups, called the *density model*, that depends on a parameter $d \in (0, 1)$ and properties of random group depend on the chosen range of d. For more information, see [67,111].

2.9. $\mathbb{R}$-trees and applications

$\mathbb{R}$-trees are metric spaces such that any two distinct points x, y are contained in a unique subspace homeomorphic to a closed interval in $\mathbb{R}$ with x, y corresponding to the endpoints, and this subspace is isometric to a closed interval. Simplicial trees with the length metric induced by identifying edges with closed intervals are examples of $\mathbb{R}$-trees. More generally, $\mathbb{R}$-trees can have a dense set of "vertices" (points whose complement has more than two components). For example, let $T = \mathbb{R}^2$ as the underlying set, and define the metric d as follows: $d(x, y) = |x - y|$ is the Euclidean distance if x, y are on the same vertical line, and otherwise if $x = (x_1, x_2)$, $y = (y_1, y_2)$, then $d(x, y) = |y_1| + |y_2| + |x_1 - x_2|$. Thus one imagines train lines running along all vertical lines and along the x-axis, with the distance function being the shortest train trip.

$\mathbb{R}$-trees were put to good use by Morgan and Shalen [102–104] in their work on hyperbolization of 3-manifolds following Thurston's work.

The importance of $\mathbb{R}$-trees in geometric group theory comes from two principles that we briefly review. Let X be a proper hyperbolic space with the isometry group of X acting with coarsely dense orbits.

(1) A sequence of actions of a finitely generated group G on X either, after taking a subsequence, converges (after conjugations) to an isometric action on X, or else it converges to an isometric action on an $\mathbb{R}$-tree.

(2) There is a theory analogous to the Bass–Serre theory, called the "Rips machine," that explains the structure of a group acting isometrically on an $\mathbb{R}$-tree from the stabilizers of the action (under some technical conditions).

2.10. Hyperbolic spaces degenerate to $\mathbb{R}$-trees

This construction is due to F. Paulin [116] and the author [13]. See also [14]. We fix a group G and a finite generating set $a_1, \dots, a_n$. Suppose we are given an isometric action $\rho : g \mapsto \rho(g) : X \to X$ of G on a proper δ-hyperbolic space X, defined up to conjugation by an isometry of X. We impose the mild assumption that the action is *nonelementary*, i.e., the function

$$x \mapsto \max_j \bigl\{ d_X\bigl(x, a_j(x)\bigr) \bigr\}$$

is a proper function $X \to [0, \infty)$. We then choose a basepoint $x_\rho \in X$ where the minimum is attained. Identifying G with the orbit of x_ρ, this induces a left-invariant (pseudo)metric on G via

$$d_\rho(g, h) = d_X\bigl(g(x_\rho), h(x_\rho)\bigr).$$

This metric is "hyperbolic," although G as a discrete set is not a geodesic metric space. To make this precise, it is convenient to give Gromov's reformulation of δ-hyperbolicity, in terms of the "4-point condition." For $a, b \in X$, define the "Gromov product"

$$(a \cdot b) = \frac{1}{2}\bigl(d_X(x_\rho, a) + d_X(x_\rho, b) - d_X(a, b)\bigr).$$

Thus, when X is a tree, $(a \cdot b)$ is the distance between x_ρ and $[a, b]$, and in general it is within 2δ of it. If $a, b, c \in X$ then consider the 3 numbers $(a \cdot b)$, $(b \cdot c)$, and $(c \cdot a)$. When X is a tree, the two smaller numbers are equal. Gromov's 4-point condition is that the two smaller numbers are within δ of each other. Up to changing the value of δ, a geodesic metric space is hyperbolic if and only if it satisfies the 4-point condition. Moreover, if the 4-point condition holds with $\delta = 0$, then the space can be isometrically embedded in an $\mathbb{R}$-tree.

Returning to our setup, assume now that ρ_i is a sequence of isometric actions of G on X, x_{ρ_i} are the corresponding basepoints, and d_{ρ_i} the induced metrics on G. They all satisfy the 4-point condition with a fixed δ. There are now two cases, up to passing to a subsequence. Define $D_i = \max_j \{x_{\rho_i}, a_j(x_{\rho_i})\}$.

Case 1. $D_i \to \infty$. Then rescale the metrics d_{ρ_i} by D_i, i.e., consider d_{ρ_i}/D_i. After a subsequence, this will converge to a (pseudo)metric on G which will now satisfy the 4-point condition with $\delta = 0$. Thus (G, d) can be isometrically embedded into a (unique) $\mathbb{R}$-tree T and there will be an induced isometric action of G on T. Thanks to the careful choice of basepoints, this action will not have a global fixed point.

Case 2. D_i stays bounded. Under the mild condition that the isometry group of X acts with coarsely dense orbits, we can conjugate the given actions so that all x_{ρ_i} belong to a fixed bounded set. Since X is proper, there is a further subsequence so that ρ_i converge to an isometric action ρ of G on X.

2.11. The Rips machine

If a group acts freely on a simplicial tree, it is necessarily free. This simple instance of Bass–Serre theory follows quickly from covering space theory. However, this is not true for $\mathbb{R}$-trees. For example, $\mathbb{Z}^n$ acts freely on $\mathbb{R}$ by letting basis elements act by n rationally independent translations. More interestingly, closed surfaces of Euler characteristic < -1 admit measured foliations with simple singularities and with all leaves being trees (and all but finitely many are lines), see [140]. Lifting to the universal cover, the transverse measure turns the leaf space to an $\mathbb{R}$-tree and the deck group induces a free action of the fundamental group of the surface on this $\mathbb{R}$-tree.

Suppose now we are given an isometric action of a finitely presented group G on an $\mathbb{R}$-tree T. We make a technical condition that the action is *stable* meaning that for every arc $I \subset T$ there is a subarc $J \subset I$ such that the stabilizer of J is equal to the stabilizer of any further subarc of J. This property is frequently satisfied for actions on $\mathbb{R}$-trees obtained by degenerating δ-hyperbolic spaces described above. We then fix a finite simplicial 2-complex K with $G = \pi_1(K)$ and construct a G-equivariant map $\tilde{K} \to T$, called a *resolution* of T. Point inverses form a foliation of $\tilde{K}$ (with certain standard singularities) which descends to K. The Rips machine transforms K with this foliation, changing neither the fundamental group nor the fact that the universal cover resolves T, and puts it in a certain "normal form." The pieces of this normal form are foliated subcomplexes that occur, very surprisingly, in only the following four types:

(simplicial) leaves are compact and the piece resolves a simplicial tree,

(surface) the piece is a surface (perhaps with boundary) and the nonboundary leaves are trees as above, (axial) the piece resolves the tree which is a line, and

(Levitt) the piece is of *Levitt type*.

Levitt-type foliations were first constructed by G. Levitt [91]. Generic leaves are 1-ended graphs, and in fact they are quasiisometric to 1-ended trees with finite graphs attached. In addition to proving this classification, the Rips machine also provides the structure of the group corresponding to these cases, and particularly in the Levitt case. It turns out that if there is a Levitt piece then G always splits along a subgroup which fixes an arc in T. The other three cases are classical, with the simplicial case amounting to Bass–Serre theory. As an example, Rips proved the conjecture of Morgan and Shalen that any finitely generated group acting freely on an $\mathbb{R}$-tree is isomorphic to the free product of surface groups and free abelian groups. For more details, see [28,66].

2.12. Applications

We mention some of the applications of $\mathbb{R}$-trees; for more see [14]. They are a basic tool in the theory of $\mathrm{Out}(F_n)$. Zlil Sela used them extensively in his seminal work on the Tarski problems [124–130].

2.12.1. Automorphisms of hyperbolic groups

Let G be a 1-ended hyperbolic group, and for simplicity assume it is torsion-free. Combining Paulin's construction [117] with the Rips machine, we get

Theorem 2.11. *If G does not split over $\mathbb{Z}$ then* $\mathrm{Out}(G)$ *is finite.*

This is analogous to a consequence of Mostow Rigidity that $\mathrm{Out}(G)$ is finite when G is the fundamental group of a closed hyperbolic n-manifold with $n \geq 3$.

The proof goes like this. Assuming $\mathrm{Out}(G)$ is infinite, choose a sequence f_i of automorphisms in distinct classes and consider isometric actions ρ_i of G on itself given by left translations twisted by f_i, i.e., $g \mapsto (h \mapsto f_i(g)h)$. Since f_i are distinct in $\mathrm{Out}(G)$, we see that we are in Case 2 of the construction outlined above and we obtain an isometric action of G on an $\mathbb{R}$-tree and with arc stabilizers cyclic (or trivial). The Rips machine now yields a splitting of G over a cyclic group.

A proper generalization of this theorem was given by Z. Sela. Fix a JSJ decomposition of G. There are now "visible" automorphisms of G realized as compositions of powers of Dehn twists in the cylinders and homeomorphisms of the QH vertices, which are surfaces.

Theorem 2.12 ([118]). *The subgroup of visible automorphisms has finite index in* $\mathrm{Out}(G)$.

The proof is quite a bit harder. The idea is that if the index is infinite, one can choose a sequence of automorphisms f_i in distinct cosets of the visible subgroup. In addition, one chooses the f_i's to be the "shortest" in their cosets. Then one argues that the action in the limit produces a "new" splitting of G, one not explained by the JSJ, or else the f_i could be shortened for large i.

Recall that a group G is *Hopfian* if every surjective endomorphism of G is an automorphism and it is co-Hopfian if every injective endomorphism is an automorphism. For example, nontrivial free groups are not co-Hopfian. By adapting the above methods to endomorphisms, Sela proved:

Theorem 2.13 ([122, 123]). *Let G be torsion-free hyperbolic. Then G is Hopfian. If G is 1-ended it is also co-Hopfian.*

In 1911 Max Dehn proposed three algorithmic problems about groups: the word problem (decide if a word in the generators represents the trivial element), the conjugacy problem (decide if two words in the generators represent conjugate elements), and the isomorphism problem (decide if two groups given by presentations are isomorphic). Dehn solved the word problem for surface groups and his solution generalizes to hyperbolic groups. There is also a similar solution of the conjugacy problem for hyperbolic groups, see [71]. The isomorphism problem takes more work and uses $\mathbb{R}$-trees. For torsion-free hyperbolic groups that do not split over cyclic subgroups, the isomorphism problem was solved by Sela [121], and for general hyperbolic groups by Dahmani–Guirardel [54].

Even though hyperbolic groups are generally very well behaved, they also contain a certain amount of pathologies, see, e.g., [46].

2.12.2. Local connectivity of ∂G

The use of $\mathbb{R}$-trees completed the proof of the following theorem.

Theorem 2.14. *If G is a 1-ended hyperbolic group, then ∂G is locally connected (as well as connected).*

There are several ingredients in the proof. First, [38] shows that if ∂G is not locally connected then it has (many) cut points. Bowditch [40] then shows that G acts on an $\mathbb{R}$-tree constructed as a kind of a "dual" tree, which does not come with a metric but can be endowed with one using [92]. The Rips machine then yields a splitting of G over a 2-ended group, finishing the proof if such splittings do not exist. Swarup [136] finished the proof in the general case by showing how to continue refining these splittings (in the presence of cut points in ∂G) until the full JSJ decomposition is obtained, at which point a contradiction arises with any further splitting.

2.12.3. Thurston's compactness theorem

With the machinery of $\mathbb{R}$-trees one can give a quick proof of the following theorem.

Theorem 2.15 ([139]). *Let M be a compact aspherical 3-manifold whose fundamental group does not split over a cyclic group. Then the space of hyperbolic structures $H(M)$ on M is compact.*

The space $H(M)$ is the space of discrete and faithful representations of $G = \pi_1(M)$ into the orientation isometry group $\mathrm{PSL}_2(\mathcal{C})$ of hyperbolic 3-space $\mathbb{H}^3$, up to conjugacy (it takes some work to see that the quotient of $\mathbb{H}^3$ by such a group is homeomorphic to the interior of M). Indeed, to rule out Case 2 above, one shows that the limiting action on an $\mathbb{R}$-tree is stable and has abelian arc stabilizers (which follows from discreteness and faithfulness).

3. MAPPING CLASS GROUPS

A fundamental shift in the subject occurred after the work of Masur and Minsky [95,96] on mapping class groups, the work that set the foundations for an eventual understanding of the large scale geometry of these groups. Mapping class groups are not hyperbolic (except for some sporadic surfaces) but naturally act on hyperbolic spaces.

We start by recalling some definitions. Let S be an orientable surface of finite type, i.e., one obtained from a closed orientable surface by removing finitely many points (called punctures). The group $\mathrm{Homeo}_+(S)$ of orientation preserving homeomorphisms of S has the natural compact-open topology which makes it locally path-connected, and the mapping class group (or the Teichmüller modular group) $\mathrm{Mod}(S)$ is the discrete group of (path) components of $\mathrm{Homeo}_+(S)$. Classically, this group has been studied since the early 20th century. A very nice introduction to the subject is the book [62], and we will freely use the standard concepts. For example, the subgroup $\mathrm{PMod}(S)$ of "pure" mapping classes (those that fix the punctures) is generated by finitely many Dehn twists and the group will not be hyperbolic if S is big enough to contain two essential (not bounding a disk or a punctured disk) nonparallel (not cobounding an annulus) disjoint simple closed curves.

To the surface S Harvey [83] associates a simplicial complex $\mathcal{C} = \mathcal{C}(S)$, called the *curve complex* of S. A vertex is an isotopy class of essential simple closed curves. A collection of distinct vertices spans a simplex if each pair can be represented by curves that intersect minimally (most of the time this means "disjointly," but in a torus punctured at most once it means "once" and in a four times punctured sphere it means "twice"). For the purposes of this discussion, we restrict to the 1-skeleton (called the *curve graph*), which we equip with the length metric with all edges of length 1. The group $\mathrm{Mod}(S)$ acts naturally on $\mathcal{C}(S)$. For some very small surfaces, like a 3 times punctured sphere, the curve complex is empty, but otherwise it is infinite, and even locally infinite, a big contrast with Cayley graphs of hyperbolic groups. In a similar way, one can define the arc complex of a surface with punctures.

Theorem 3.1 ([95]). *$\mathcal{C}(S)$ is hyperbolic. An element of* $\mathrm{Mod}(S)$ *acts loxodromically if and only if it is pseudo-Anosov.*

Here are some ideas in the original proof, which uses Teichmüller theory. Let $\mathcal{T} = \mathcal{T}(S)$ be the Teichmüller space of S, i.e., the space of all (marked) hyperbolic structures on S. There is a natural coarse map $\pi : \mathcal{T} \to \mathcal{C}$ that to a hyperbolic metric on S assigns (the isotopy class of) a shortest simple closed geodesic. Any two points in $\mathcal{T}$ are joined by a unique Teichmüller geodesic, and their images under π form a family of coarse paths in $\mathcal{C}$ satisfying (and this needs proof):

- any two points in $\mathcal{C}$ are connected by some such path,
- the family is closed under taking subpaths,
- any two paths in the collection starting at nearby points are contained in each other's uniform Hausdorff neighborhood (i.e., they *fellow travel*), and
- triangles formed by these paths are uniformly thin.

Thus the collection behaves like the collection of geodesics in a hyperbolic space. Remarkably, the existence of such a collection of paths implies that the space is hyperbolic and the paths are (reparametrized) quasigeodesics with uniform constants. See [97], which proves that arc complexes are hyperbolic, and [42].

Since the original proof of hyperbolicity of $\mathcal{C}(S)$, there have been others, the simplest being [84], not using Teichmüller theory at all but constructing a family of paths in $\mathcal{C}(S)$ directly using surgeries on curves. Perhaps surprisingly, the more recent proofs also show that curve graphs are *uniformly* hyperbolic, i.e., δ can be taken independently of the surface.

3.1. The boundary of the curve complex

If X is a hyperbolic space which is not proper, its boundary ∂X may not be compact. For example, the boundary of the wedge of countably many rays joined at the initial point is a discrete countable set, and the boundary of a tree all of whose vertices have countable valence is homeomorphic to the irrationals.

In [90] E. Klarreich identified the boundary $\partial\mathcal{C}$ of the curve complex as a proper quotient of a subspace of Thurston's boundary of Teichmüller space $\mathcal{T}$. This description serves as a model for boundaries of other hyperbolic complexes.

3.2. WPD, acylindrically hyperbolic groups, quasimorphisms

In the absence of properness of the action, one needs some kind of a substitute. The property WPD (for "weak proper discontinuity") was introduced in [32].

Definition 3.2. Suppose a group G acts by isometries on a hyperbolic space X. A loxodromic element $g \in G$ is WPD if for every $x \in X$ and $C > 0$ there is $N > 0$ such that the set

$$\{h \in G \mid d(x, h(x)) \leq C, d(g^N(x), hg^N(x)) \leq C\}$$

is finite. The action of G on X is WPD if G is not virtually cyclic and every loxodromic element is WPD.

The WPD condition says that the collection of translates of an axis (or an orbit) of a loxodromic element is discrete: any two translates are either parallel or else they are in a bounded Hausdorff neighborhood of each other only along a bounded length interval.

Theorem 3.3 ([32]). *The action of* $\mathrm{Mod}(S)$ *on* $\mathcal{C}(S)$ *is WPD. If a nonvirtually cyclic group acts isometrically on a hyperbolic space with a WPD element then the space* $\widetilde{\mathrm{QH}}(G)$ *of (reduced) quasimorphisms on* G *is infinite-dimensional.*

A *quasimorphism* is a function $f : G \to \mathbb{R}$ such that

$$\sup_{a,b\in G} \left| f(ab) - f(a) - f(b) \right| < \infty.$$

Denote by $\mathrm{QH}(G)$ the vector space of all quasimorphisms on G and note the vector subspaces $\mathrm{Hom}(G, \mathbb{R})$ of homomorphisms $G \to \mathbb{R}$ and $B(G)$ of bounded functions on G. Then the space $\widetilde{\mathrm{QH}}(G)$ is defined as the quotient

$$\widetilde{\mathrm{QH}}(G) = \mathrm{QH}(G)/\big(\mathrm{Hom}(G, \mathbb{R}) + B(G)\big)$$

and it can also be identified with the kernel of the natural homomorphism $H^2_b(G; \mathbb{R}) \to H^2(G; \mathbb{R})$ from bounded cohomology of G. For more on bounded cohomology, see [49].

The basic method for showing $\widetilde{\mathrm{QH}}(G)$ is infinite-dimensional is due to Brooks [47] in the case of free groups. Fix a free group F with a basis $a_1, a_2, \dots$ Let w be any cyclically reduced word in the basis. Define $f_w : F \to \mathbb{Z} \subset \mathbb{R}$ as $f_w(x) = C_w(x) - C_{w^{-1}}(x)$, where $C_w(x)$ is the number of occurrences of w as a subword of x, written as a reduced word. That f_w is a quasimorphism can be seen by considering the tripod in the Cayley tree of F spanned by $1, a$, and ab, and marking all occurrences of $w^{\pm 1}$ along it. All such occurrences that do not contain the central vertex will be counted twice, with opposite signs, in the expression $f(ab) - f(a) - f(b)$, and, of course, the number occurrences that do contain the central vertex is uniformly bounded. With a bit more work, one can show that for a suitable choice of w_i's the quasimorphisms f_{w_i} will yield linearly independent elements of $\widetilde{\mathrm{QH}}(F)$. The proof of the second half of Theorem 3.3 is a coarse version of this method, where w is replaced by a long segment along an axis of a WPD element, and the discreteness of the set of translates guarantees that the counting function is finite.

A quick application is the following statement, suggesting that pseudo-Anosov elements of $\mathrm{Mod}(S)$ are "generic."

Corollary 3.4. *Fix a finite generating set and the corresponding word metric on* $\mathrm{Mod}(S)$. *For any* $R > 0$, *there exists* $M > 0$ *such that every ball of radius* M *contains a ball of radius* R *that consist entirely of pseudo-Anosov mapping classes.*

This follows quickly from the feature of the quasimorphisms on $\mathrm{Mod}(S)$ constructed above that they are uniformly bounded on all elements of $\mathrm{Mod}(S)$ which are not pseudo-Anosov.

Bowditch noticed that the action of $\mathrm{Mod}(S)$ on $\mathcal{C}(S)$ satisfies a property stronger than WPD.

Definition 3.5. An isometric action of G on a hyperbolic space X is *acylindrical* if for all $r > 0$ there exist $R, N > 0$ so that whenever $a, b \in X$ with $d(a, b) \geq R$, then there are at most N elements h of G such that $d(a, h(a)) \leq r$ and $d(b, h(b)) \leq r$.

Thus acylindricity gives control in all directions, not only along axes of loxodromic elements.

Theorem 3.6 ([41]). *The action of* $\mathrm{Mod}(S)$ *on* $\mathcal{C}(S)$ *is acylindrical.*

These results motivated Denis Osin to propose acylindrically hyperbolic groups as a generalization of hyperbolic groups. A group is *acylindrically hyperbolic* if it is not virtually cyclic and admits an acylindrical action on a hyperbolic space with unbounded orbits. This class contains many groups of interest (e.g., mapping class groups and $\mathrm{Out}(F_n)$) and many constructions on hyperbolic groups carry over to this larger class, e.g., small cancelation theory, or quasimorphisms indicated above; see [113,115].

3.3. Subsurface projections

The main drawback of acylindrically hyperbolic groups is that in general one does not have access to elements that do not act loxodromically. In the case of mapping class groups, this problem is resolved through *subsurface projections* of Masur and Minsky [95,96].

Let S be a surface as before and $X \subset S$ a connected π_1-injective subsurface which is closed as a subset. Let α be a simple closed curve in S which cannot be homotoped in the complement of X and which is in minimal position with respect to ∂X. Then the intersection $\alpha \cap X$ consists of finitely many disjoint arcs (or just α if $\alpha \subset X$). For each such arc J, consider one or two curves obtained as follows. If the endpoints of J are contained in the same boundary component b of X, there are two ways of closing up J to a closed curve by adding an arc in b; take both of these curves. If the endpoints of J are on distinct boundary components b, b' then form a curve by taking two parallel copies of J and connect them by adding "long" arcs in b and b'. It is not hard to see that taking the collection of all these curves for all arcs J produces a uniformly bounded set $\pi_X(\alpha) \subset \mathcal{C}(X)$ (we collapse all boundary components of X to punctures). This construction makes sense whenever $\mathcal{C}(X)$ is defined (so notably a pair of pants is excluded). It also makes sense when X is an annulus, in which case the curve complex is formed by arcs joining the boundary components, but we will not describe this case in detail. If α is disjoint from X then $\pi_X(\alpha)$ is not defined and we set it to be empty.

Now fix a finite collection of curves $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ in S that "fill" the surface, i.e., every (essential) curve intersects at least one of them. By the classical fact that the distance in the curve complex is bounded by a function of the intersection number, if $\pi_X(\alpha_i)$ and $\pi_X(\alpha_j)$ are both defined then their union has uniformly bounded diameter (with the bound depending on the intersection number between α_i and α_j). We then define

$$\pi_X(\vec{\alpha}) = \bigcup_i \pi_X(\alpha_i).$$

This is always a nonempty, uniformly bounded subset of $\mathcal{C}(X)$.

The following is the fundamental result of Masur and Minsky, expressing (coarsely) the word metric in $\mathrm{Mod}(S)$ in terms of subsurface projections. For $K > 0$ and $x \geq 0$, define $\{\{x\}\}_K$ as 0 if $x < K$ and as x if $x \geq K$.

Theorem 3.7 (The distance formula, [96]). *For all sufficiently large K (depending on $\vec{\alpha}$) and for all $g, h \in \mathrm{Mod}(S)$, we have*

$$d(g,h) \asymp \sum_X \{\{d_X(g(\vec{\alpha}), h(\vec{\alpha}))\}\}_K.$$

The left-hand side is the distance in the word metric. The summation is over all (isotopy classes of) connected, π_1-injective subsurfaces X with $\mathcal{C}(X) \neq \emptyset$, and the displayed summand is the diameter of the set $\pi_X(g(\vec{\alpha})) \cup \pi_X(h(\vec{\alpha}))$. The symbol $\asymp$ means that there is a linear function (depending on K and the finite generating set of $\mathrm{Mod}(S)$) $f(x) = Ax + B$ such that the left-hand side is bounded by the f-value of the right-hand side, and vice versa. In particular, only finitely many terms are $\geq K$.

The distance formula is a powerful tool in the study of large-scale geometry of mapping class groups. It is used in an essential way in the following remarkable theorem, establishing quasiisometric rigidity of mapping class groups. To state the theorem, let $\mathrm{Mod}^{\pm}(S)$ denote the *extended* mapping class group, i.e., allowing orientation-reversing homeomorphisms (this is an index 2 extension of $\mathrm{Mod}(S)$). If G is a finitely generated group with a word metric, denote by $\mathrm{QI}(G)$ the group of quasiisometries $G \to G$ with the equivalence relation $f_1 \sim f_2$ if $\sup_g d(f_1(g), f_2(g)) < \infty$. There is a natural homomorphism $G \to \mathrm{QI}(G)$ sending g to the left translation by g.

Theorem 3.8 ([10,79]). *Let S be a surface of finite type. Except for a small number of sporadic surfaces, the natural homomorphism $\mathrm{Mod}^{\pm}(S) \to \mathrm{QI}(\mathrm{Mod}^{\pm}(S))$ is an isomorphism. In particular, if G is any group quasiisometric to $\mathrm{Mod}(S)$, then there is a homomorphism $G \to \mathrm{Mod}^{\pm}(S)$ with finite kernel and finite index image.*

4. PROJECTION COMPLEXES

It is tempting to view the distance formula as saying that the coarse map

$$\mathrm{Mod}(S) \to \prod_X \mathcal{C}(X)$$

defined by $g \mapsto \pi_X(g(\vec{\alpha}))$ is a quasiisometric embedding, where we equip the right-hand side with the ℓ_1-metric. The trouble is that this is not really a metric, and "cutting off" at K in each coordinate would not satisfy the triangle inequality. Up to modifying each coordinate a bounded amount, the image of this map was identified in [7,10]. The main restriction on the image is the following inequality.

Theorem 4.1 (Behrstock inequality, [7]). *There is a $\theta \geq 0$ such that the following holds. Suppose $X, Y \subset S$ are two subsurfaces such that the boundary of each intersects the other. Then at least one of $d_X(\partial Y, \vec{\alpha})$ and $d_Y(\partial X, \vec{\alpha})$ is $\leq \theta$.*

There is a simple proof of the Behrstock inequality, due to Chris Leininger, see [94]. If we focus on the two coordinates $\mathcal{C}(X) \times \mathcal{C}(Y)$, the inequality says that the image is contained in a Hausdorff neighborhood of the "wedge" of $\mathcal{C}(X) \times \{y\} \cup \{x\} \times \mathcal{C}(Y)$ where

$x = \pi_X(\partial Y)$ and $y = \pi_Y(\partial X)$. This suggests taking wedges instead of products for the right-hand side in order to fix the metrizability problem, and leads to the following construction that can be axiomatized.

Let $\mathcal{Y}$ be a collection of metric spaces (technically we allow the distance to be infinite, for example, we might have disconnected graphs with the path metric). Suppose that for distinct $X, Y \in \mathcal{Y}$ we are given a subset $\pi_X(Y) \subset X$. If $Z \in \mathcal{Y}$, $Z \neq X$, we define

$$d_X(Y, Z) = \operatorname{diam}\bigl(\pi_X(Y) \cup \pi_X(Z)\bigr).$$

We will assume that the following axioms hold for some fixed $\theta \geq 0$:

(P1) $d_X(Y, Y) \leq \theta$,

(P2) if $d_X(Y, Z) > \theta$ then $d_Y(X, Z) \leq \theta$, and

(P3) for $X \neq Z$, the set

$$\bigl\{Y \in \mathcal{Y} \mid d_Y(X, Z) > \theta\bigr\}$$

is finite.

There are many natural situations where these axioms hold.

Examples 4.2. (1) Let T be a simplicial tree and $\mathcal{Y}$ a collection of pairwise disjoint simplicial subtrees. The projection $\pi_X(Y)$ is the point of X nearest to Y. The axioms hold with $\theta = 0$. See Figure 3.

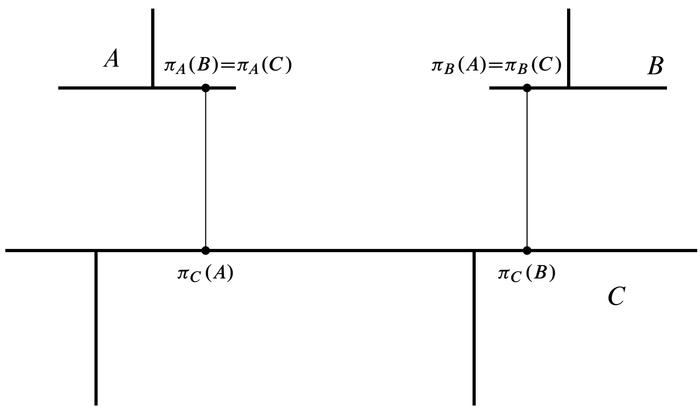

FIGURE 3
The situation of Example 4.2(1), $d_C(A, B) > 0$ while $d_A(B, C) = d_B(A, C) = 0$.

(2) Let S be a closed hyperbolic surface and γ an immersed closed geodesic which is not a multiple. In the universal cover $\tilde{S} = \mathbb{H}^2$ consider the set $\mathcal{Y}$ of all lifts of γ, and define projections as nearest point projections. A similar construction

can be performed with a group acting on a hyperbolic space and a maximal virtually cyclic subgroup that contains a WPD element.

(3) Let S be a complete hyperbolic surface of finite area and a cusp. In the universal cover $\tilde{S} = \mathbb{H}^2$, consider the set $\mathcal{Y}$ of all lifts of a fixed horocyclic curve in the cusp (with either the intrinsic or the induced metric). Again the projection is the nearest point projection. A similar construction can be performed with relatively hyperbolic groups.

(4) Let G be a group acting on a simplicial hyperbolic graph X and let H be the stabilizer of a vertex $v \in X$. Assume that H acts simply transitively on the edges incident to v, and that the metric on the link $\mathrm{Lk}(v, X)$ (which can be identified with H) induced by the path metric on $X \smallsetminus \{v\}$ is proper (finite radius balls contain finitely many points; here we allow distances to be infinite). Let $\mathcal{Y}$ be the collection of links of vertices in the orbit of v with this proper metric on each. If u, w are two distinct vertices in the orbit of v the projection of $\mathrm{Lk}(u, X)$ to $\mathrm{Lk}(w, X)$ is the set of points in $\mathrm{Lk}(w, X)$ that belong to a geodesic between u and w. If (G, H) admit such an action, H is said to be *hyperbolically embedded* in G; see [55]. For example, parabolic subgroups of hyperbolic groups, or maximal virtually cyclic subgroups containing a WPD element as in (2) are hyperbolically embedded, as can be seen by building the projection complex below.

(5) Let S be an orientable surface of finite type and let $\mathcal{Y}$ be a collection of isotopy classes of π_1-injective subsurfaces where subsurface projections are defined, and assume that if $X, Y \in \mathcal{Y}$ and $X \neq Y$ then ∂X is not disjoint from Y (up to isotopy). Define $\pi_Y(X) = \pi_Y(\partial X)$.

The construction of a projection complex $\mathcal{P}(\mathcal{Y})$ (and the blow-up version $\mathcal{C}(\mathcal{Y})$) is kind of a converse to Example 4.2(2) above, where one tries to "reconstruct" the ambient space from the projection data (though usually one gets a different ambient space).

Theorem 4.3 ([19], for a simpler construction see [22]). *Suppose the projection data*

$$\big(\mathcal{Y}, \pi_X(Y), \theta\big)$$

satisfy (P1)–(P3). There is a metric space $\mathcal{C}(\mathcal{Y})$ containing metric spaces in $\mathcal{Y}$ as pairwise disjoint isometrically embedded subspaces and so that $\pi_X(Y)$ agrees, up to a bounded error, with the nearest point projection of Y to X within $\mathcal{C}(\mathcal{Y})$. Moreover,

- *If each $Y \in \mathcal{Y}$ is δ-hyperbolic for some $\delta \geq 0$ then $\mathcal{C}(\mathcal{Y})$ is hyperbolic.*
- *If each $Y \in \mathcal{Y}$ is quasiisometric to a tree (a "quasitree") with fixed QI constants, then $\mathcal{C}(\mathcal{Y})$ is also a quasitree.*
- *If the collection $\mathcal{Y}$ consists of finitely many isometry types of metric spaces and they all have asymptotic dimension $\leq n$ then* $\operatorname{asdim} \mathcal{C}(\mathcal{Y}) \leq n + 1$.

- *The space $\mathcal{P}(\mathcal{Y})$ obtained from $\mathcal{C}(\mathcal{Y})$ by collapsing all embedded copies of spaces in $\mathcal{Y}$ is a quasitree.*
- *If a group G acts by isometries on $\bigsqcup_{Y \in \mathcal{Y}} Y$ preserving the projections (i.e., $g(\pi_X(Z) = \pi_{g(X)}(g(Z))$ for all $g \in G$) then G acts by isometries on $\mathcal{C}(\mathcal{Y})$ extending the action on $\bigsqcup_{Y \in \mathcal{Y}} Y$, and it also acts isometrically on $\mathcal{P}(\mathcal{Y})$.*

We briefly outline the construction. As indicated above, the idea is to start with the disjoint union of all $Y \in \mathcal{Y}$ and then for certain pairs (X, Z) add edges joining points in $\pi_X(Z)$ to points in $\pi_Z(X)$.

Step 1 is to promote (P2) to a stronger property (P2++):

(P2++) If $d_Y(X, Z) > \theta$ then $\pi_Y(X) = \pi_Y(Z)$.

This can be done by modifying the projection $\pi_X(Y)$ by a bounded amount and replacing θ by a larger constant. This modification preserves group equivariance.

In step 2, assuming (P1), (P2++), and (P3), one chooses a constant $K \geq 2\theta$ and posits that X and Z are connected by edges as above provided $d_Y(X, Z) \leq K$ for all $Y \neq X, Z$. The key property that makes the proof of Theorem 4.3 possible is that the set

$$\{X\} \cup \big\{Y \mid d_Y(X, Z) > K\big\} \cup \{Z\}$$

is finite (by (P3)) and is naturally linearly ordered giving a path from X to Z, called a *standard path*, in $\mathcal{P}_K(\mathcal{Y})$. These standard paths are quasigeodesics and behave very nicely. The construction depends on the choice of the constant K: when K is enlarged, there will be more edges attached.

We mention a few applications of this construction to mapping class groups.

Theorem 4.4 ([19]). $\operatorname{asdim}(\operatorname{Mod}(S)) < \infty$.

The basic idea is to replace the infinite product of curve complexes by a smaller space. The collection of all subsurfaces $\mathcal{Y}$ does not satisfy the assumptions of Example 4.2(5) above since subsurfaces can be disjoint or nested. However, one shows that there is a way to write $\mathcal{Y}$ equivariantly as a finite disjoint union $\sqcup \mathcal{Y}_i$ so that each collection $\mathcal{Y}_i$ satisfies Example 4.2(5). Thus one gets the spaces $\mathcal{C}(\mathcal{Y}_i)$. These are all hyperbolic, and crucially, have finite asymptotic dimension by Theorem 4.3 and the theorem of Bell–Fujiwara [11] that curve complexes have finite asymptotic dimension. Then we have a quasiisometric embedding

$$\operatorname{Mod}(S) \to \prod_i \mathcal{C}(\mathcal{Y}_i)$$

which finishes the proof since passing to finite products and subspaces preserves finiteness of asymptotic dimension.

There is quite a bit of inefficiency when we take the product of the blown-up projection complexes over the families $\mathcal{Y}_i$. There is a more involved system of axioms that keeps track of pairs of surfaces that are disjoint or nested leading to the notion of a *hierarchically hyperbolic group*, due to J. Behrstock, M. Hagen, and A. Sisto. For example, in [8] they derive a bound on asdim(Mod(S)), using [18], which is quadratic in the complexity of the surface. There are other applications of this theory, for example, in [9] they show how to understand quasiflats in mapping class groups and how to approximate a "hull" of a finite set by a CAT(0) cube complex.

Theorem 4.5 ([21]). *There is a classification, in terms of the Nielsen–Thurston normal form, of those elements g of* $\mathrm{Mod}(S)$ *that have stable commutator length* $\mathrm{scl}(g) = 0$.

Recall that for $g \in [G, G]$ $\mathrm{cl}(g)$ is the smallest k such that g can be written as a product of k commutators, and $\mathrm{scl}(g) = \lim_n \frac{\mathrm{cl}(g^n)}{n}$. By Bavard duality (see [49]), $\mathrm{scl}(g) > 0$ is equivalent to having a quasimorphism $G \to \mathbb{R}$ which is unbounded on the powers of g. Projection complexes are used to construct actions of finite index subgroups of $\mathrm{Mod}(S)$ on hyperbolic spaces with a power of a given element acting loxodromically, and then the Brooks method can be used to construct such quasimorphisms. It is worth stating this fact:

Theorem 4.6 ([21]). *Let S be a finite type surface. There is a torsion-free finite index subgroup $G <$* $\mathrm{Mod}(S)$ *such that for every element $g \in G$ of infinite order there an action of G on a hyperbolic space such that g is loxodromic.*

For example, this applies to (powers of) Dehn twists. By contrast, a theorem of Bridson [43] says that whenever $\mathrm{Mod}(S)$ (with S of genus ≥ 3) acts on a CAT(0) space, Dehn twists have translation length 0.

Projection complexes are useful more generally for constructing quasicocycles on groups G with coefficients in orthogonal representations on strictly convex Banach spaces (such as $l^p(G)$ for $1 < p < \infty$); see [20].

Theorem 4.7 (Balasubramanya [2]). *If a group G acts on a hyperbolic space with a WPD element, then it admits a cobounded acylindrical action on a quasitree.*

Another proof of Balasubramanya's theorem is given in [22]. The quasitree is the projection complex applied to Example 4.2(2) and acylindricity is proved using the geometry of standard paths.

F. Dahmani, V. Guirardel, and D. Osin solved a long standing open problem when they proved the following.

Theorem 4.8 ([55]). *Let $\phi \in$* $\mathrm{Mod}(S)$ *be a pseudo-Anosov mapping class. Then for a suitable power ϕ^n with $n > 0$ the subgroup normally generated by ϕ^n is free.*

They derive this theorem using the method of *rotating families*.

Theorem 4.9 ([55]). *For every $\delta \geq 0$ there is $R > 0$ such that the following holds. Let X be a δ-hyperbolic space and G a group of isometries of X. Let $C \subset X$ be a G-invariant set which is R-separated (meaning that $d(c, c') > R$ if $c, c' \in C$ are distinct). Suppose for every $c \in C$ we are given a subgroup G_c of the stabilizer* $\mathrm{Stab}_G(c)$ *such that*

(i) $G_{g(c)} = gG_c g^{-1}$ *for* $c \in C$ *and* $g \in G$, *and*

(ii) *if* $g \in G_c \setminus \{1\}$, $c' \in C$ *and* $c' \neq c$ *then every geodesic from* c' *to* $g(c')$ *passes through* c.

Then the subgroup of G *generated by* $\cup_{c \in C} G_c$ *is the free product of a subcollection of the family* $\{G_c\}_{c \in C}$.

To prove Theorem 4.8, they apply this theorem to the space obtained from the curve complex $\mathcal{C}(S)$ by equivariantly coning off an orbit of the elementary closure $\mathrm{EC}(\phi)$. Pretending that this orbit is in an isometrically embedded line, one would attach the universal cover of a disk of large radius in $\mathbb{H}^2$ punctured at the center, and then completed to add the cone point back in. The set of these cone points is the set C from the theorem, and G_c is the cyclic group generated by (a conjugate of) a suitable power ϕ^n.

More recently, M. Clay, J. Mangahas, and D. Margalit proved a version of Theorem 4.9 that applies to projection complexes. Rotating families are replaced by *spinning families*.

Theorem 4.10 ([51]). *For every* θ *and* K, *there is* L *so that the following holds. Suppose a group* G *acts on the projection data and on the associated projection complex* $\mathcal{P} = \mathcal{P}_K(\mathcal{Y})$. *Suppose for every vertex* $v \in \mathcal{P}$ *we are given a subgroup* G_v *of the stabilizer* $Stab_G(v)$ *such that*

(i) $G_{g(v)} = gG_v g^{-1}$ *for any vertex* v *and* $g \in G$, *and*

(ii) *if* v, v' *are distinct vertices and* $g \in G_v \setminus \{1\}$ *then* $d_v(v', g(v')) > L$.

Then the subgroup of G *generated by* $\bigcup_v G_v$ *is the free product of a subcollection of the family* $\{G_v\}_{v \in \mathcal{P}^{(0)}}$.

They derive Theorem 4.8 directly from Theorem 4.10 using the projection complex as in Example 4.2(2). They also prove several statements about normal closures of powers of other kinds of elements, or collections of elements, in $\mathrm{Mod}(S)$. One extreme behavior is that the normal closure is free, another that it is the whole $\mathrm{Mod}(S)$, but surprisingly there are examples when the normal closure turns out to be a certain kind of (infinitely generated) right angled Artin groups.

In [24] the two theorems above are revisited, and in particular the paper shows how to derive Theorem 4.9 from Theorem 4.10.

Here are two more applications of projection complexes to mapping class groups, though we will not comment on the proofs.

Theorem 4.11 ([4]). *Mapping class groups satisfy the Farrell–Jones conjecture.*

Theorem 4.12 ([60]). *Mapping class groups are semihyperbolic.*

This means that one can equivariantly choose uniform quasigeodesics connecting any pair of points in $\mathrm{Mod}(S)$ so that they fellow-travel, i.e., if the endpoints are at distance ≤ 1 then each is in the other's uniform Hausdorff neighborhood.

5. GROUP $\mathrm{Out}(F_n)$

Let F_n be the free group of rank $n \geq 2$, $\mathrm{Aut}(F_n)$ its automorphism group, and $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/F_n$ the *outer automorphism group* of F_n, obtained by quotienting out the inner automorphisms. This group has been studied for over a century, see Nielsen's paper [109] where he proves that $\mathrm{Out}(F_n)$ is generated by $n+1$ involutions. A big impediment in the study of $\mathrm{Out}(F_n)$, and free groups in general, was the tendency to think of elements of free groups as words in a basis. A much more flexible approach is to think of a free group as the fundamental group of a graph, which is not necessarily a rose R_n (a wedge of n circles). For example, the proof that subgroups of free groups are free is essentially trivial using covering spaces and general graphs, while the more algebraic proof is much less transparent. In [134] J. Stallings introduced the operation of *folding* graphs and used it to show that many standard algorithmic problems about free groups have easy solutions.

5.1. Outer space

Given this philosophy, the definition of Culler–Vogtmann's Outer space CV_n should seem very natural. Fix the rose R_n. A point in CV_n is represented by a homotopy equivalence $h : R_n \to \Gamma$, called *marking*, where Γ is a finite graph with all vertices of valence > 2 equipped with a *metric* of volume 1, i.e., an assignment of positive numbers to its edges that add to 1. Two such markings $h : R_n \to \Gamma$ and $h' : R_n \to \Gamma'$ represent the same point in CV_n if there is an isometry $\phi : \Gamma \to \Gamma'$ such that ϕh is homotopic to h'. Formally, the definition is analogous to the definition of Teichmüller space, where metric graphs are replaced by hyperbolic surfaces. There are many useful analogies between mapping class groups and $\mathrm{Out}(F_n)$, perhaps stemming from the classical theorem of Dehn–Nielsen–Baer (see [62]) that when G is the fundamental group of a closed orientable surface S then $\mathrm{Out}(G) \cong \mathrm{Mod}^{\pm}(S)$. While Teichmüller space is diffeomorphic to Euclidean space, Outer space is a contractible polyhedron and the study of $\mathrm{Out}(F_n)$ is decidedly more combinatorial compared to the study of mapping class groups. The group $\mathrm{Out}(F_n)$ acts naturally on CV_n by changing the marking. The action is proper. For more on Outer space and the consequences to the structure of $\mathrm{Out}(F_n)$, see the original paper [53], as well as the excellent survey [141], and also [15].

5.2. The boundary of Outer space

By taking universal covers, another way to think about a point $h : R_n \to \Gamma$ in CV_n is as a free action of F_n on a simplicial metric tree. The construction in Section 2.10 then yields a compactification of CV_n with the points in the ideal boundary $\partial\,\mathrm{CV}_n$ represented by actions of F_n on $\mathbb{R}$-trees (which are either nonsimplicial or non-free). This construction was carried out in [52]. Exactly which trees arise in $\partial\,\mathrm{CV}_n$ was identified in [27,85].

5.3. Lipschitz metric and train-track maps

There is a natural notion of a *Lipschitz distance* between two points $h_i : R_n \to \Gamma_i$, $i = 1, 2$. It is defined by

$$d(\Gamma_1, \Gamma_2) = \log \lambda$$

where $\lambda \geq 1$ is the smallest possible Lipschitz constant of all maps $f : \Gamma_1 \to \Gamma_2$ that commute with markings, i.e., $h_2 f$ is homotopic to h_1 (and Γ_i are viewed as geodesic metric spaces). This "metric" is not symmetric, but satisfies the triangle inequality $d(\Gamma_1, \Gamma_3) \leq d(\Gamma_1, \Gamma_2) + d(\Gamma_2, \Gamma_3)$, and $d(\Gamma, \Gamma') \geq 0$ with equality only for $\Gamma = \Gamma'$. This metric has interesting properties and displays a mixture of behaviors of the well-studied metrics on Teichmüller space (Teichmüller, Weil–Petersson, and Thurston metrics). It can be used in the $\mathrm{Out}(F_n)$ setting in a way similar to the Bers' proof of the Nielsen–Thurston classification of mapping classes (see [62]) to give a proof of the following train-track theorem; see [17].

Theorem 5.1 ([35]). *Every irreducible automorphism $\phi \in \mathrm{Out}(F_n)$ can be represented by a train-track map $f : \Gamma \to \Gamma$ for some $\Gamma \in \mathrm{CV}_n$.*

A marking gives an identification between $\pi_1(\Gamma)$ and F_n and $f : \Gamma \to \Gamma$ "represents" ϕ if the induced endomorphism on $\pi_1(\Gamma)$ is ϕ. We say that ϕ is *irreducible* if it cannot be represented by some $f : \Gamma \to \Gamma$ that leaves a proper subgraph with nontrivial π_1 invariant. The map f is a *train-track map* if all positive powers of f are locally injective on all edges of Γ. It is easy to control the growth of lengths of loops under iteration by train-track maps, which makes them important in the study of the dynamics of an automorphism. More generally, when ϕ is not irreducible, there are *relative* train-track representatives.

The Lipschitz metric admits geodesic paths, called *folding paths*, which are induced, in the spirit of Stallings, by identifying segments of the same length and issuing from the same vertex. For more on this, see [17].

5.4. Hyperbolic complexes

By analogy with the arc and curve complexes, there are several complexes where $\mathrm{Out}(F_n)$ acts.

5.4.1. The free splitting complex FS_n

This one is analogous to the arc complex. A k-simplex is a $(k+1)$-edge free splitting of F_n, i.e., it is a minimal action of F_n on a simplicial tree with vertices of valence > 2, with trivial edge stabilizers and with $(k+1)$ orbits of edges. Passing to a face is induced by equivariantly collapsing an orbit of edges. Outer space CV_n is naturally a subset of FS_n, which can be viewed as a "simplicial completion" of CV_n.

5.4.2. The cyclic splitting complex FZ_n

This is defined the same way, except that the edge stabilizers can be cyclic subgroups. It is analogous to the curve complex.

5.4.3. The free factor complex FF_n

This one is different from FZ_n but can also be viewed as an analog of the curve complex. A vertex of FF_n is a *proper free factor* $A < F_n$, i.e., a subgroup such that $F_n = A * B$ for some $A \neq 1 \neq B$, defined up to conjugation. A k-simplex is a k-tuple of distinct conjugacy classes of proper free factors that are nested after suitable conjugation.

There are natural coarse equivariant maps

$$\mathrm{CV}_n \to \mathrm{FS}_n \to \mathrm{FZ}_n \to \mathrm{FF}_n .$$

For example, $\mathrm{FS}_n \to \mathrm{FF}_n$ sends a free splitting to a nontrivial vertex group (or if they are all trivial, to a free factor represented by a subgraph of the quotient graph).

Now, it turns out that all three of these complexes are hyperbolic, and there are several others that this survey is not mentioning. The first hyperbolic $\mathrm{Out}(F_n)$-complex was constructed in [29], though it is not canonical. The hyperbolicity of FF_n was established in [30] along the lines of the Masur–Minsky's argument for the curve complex, by projecting folding paths from CV_n to FF_n. A novel argument by Handel–Mosher [80] established hyperbolicity of FS_n, by considering folding paths directly in FS_n. Kapovich–Rafi [88] found a general criterion that a Lipschitz map $X \to Y$ has to satisfy in order for the hyperbolicity of X to imply the hyperbolicity of Y. Essentially, Lipschitz images of thin triangles are thin triangles. The maps $\mathrm{FS}_n \to \mathrm{FZ}_n \to \mathrm{FF}_n$ satisfy the Kapovich–Rafi criterion, so the hyperbolicity of FS_n implies the hyperbolicity of the other two. Loxodromic elements in FF_n are precisely the *fully irreducible automorphisms* (those whose positive powers are irreducible) and they are all WPD (in FS_n there are more loxodromic elements and they are not all WPD). Thus the space of quasimorphisms on $\mathrm{Out}(F_n)$ is infinite-dimensional and $\mathrm{Out}(F_n)$ is acylindrically hyperbolic. Handel and Mosher [81, 82] extended this and proved the H_b^2-alternative: any subgroup of $\mathrm{Out}(F_n)$ which is not virtually abelian has an infinite-dimensional space of quasimorphisms. This recovers the theorem of Bridson and Wade [45] that no higher rank lattice embeds as a subgroup of $\mathrm{Out}(F_n)$. The proof is much more involved than the H_b^2-alternative for mapping class groups [32].

The boundary of FF_n was identified with a proper quotient of a subspace of $\partial\,\mathrm{CV}_n$ in [39] and in [77].

5.5. Subfactor projections

By analogy with the Masur–Minsky subsurface projections, there are *subfactor projections*, see [31, 137]. Let A, B be two proper free factors in F_n. Our goal is to define $\pi_A(B) \in \mathrm{FS}(A)$, the projection of B to the free splitting complex of A. Choose $\Gamma \in \mathrm{CV}_n$ so that B is represented by a subgraph Γ_B of Γ. Then represent A by an immersion $\Gamma_A \to \Gamma$. Thus Γ_A determines a simplex in Outer space for A, and can be projected to $\mathrm{FS}(A)$ (or $\mathrm{FF}(A)$). It takes some work to show that coarsely this projection does not depend on the choice of Γ, at least when A and B are sufficiently far apart in FF_n. Moreover, the set $\mathcal{Y}$ of all free factors can be equivariantly and finitely partitioned into $\sqcup \mathcal{Y}_i$ so that projection is defined within each $\mathcal{Y}_i$, and this projection satisfies the projection axioms. One then gets a map

$$\mathrm{Out}(F_n) \to \prod_i \mathcal{C}(\mathcal{Y}_i)$$

in the same way as for mapping class groups (see the discussion after Theorem 4.4). However, here the map is *not* a quasiisometric embedding. The main issue is that there is no analog of annulus projections: when A has rank 1, the corresponding complex $\mathrm{FS}(A)$ is a single point. For example, the orbits on the right-hand side under the powers of any polynomially growing automorphism are bounded. For more on this, see [142].

5.6. Questions

The following is the key question, if one hopes to understand $\mathrm{Out}(F_n)$ using hyperbolic methods. The other questions reiterate the state of affairs that the large scale geometry of $\mathrm{Out}(F_n)$ is lagging behind the one of mapping class groups.

(1) Given $\phi \in \mathrm{Out}(F_n)$ of infinite order, is there a finite index subgroup $G < \mathrm{Out}(F_n)$ and an isometric action of G on a hyperbolic space so that a positive power of ϕ that belongs to G acts loxodromically?

This is true for mapping class groups (see Theorem 4.6), and it is also true for automorphisms ϕ that grow exponentially.

(2) Do any of hyperbolic $\mathrm{Out}(F_n)$-complexes admit *tight (quasi-)geodesics*?

These were defined for curve complexes by Masur and Minsky, and a very strong finiteness property was established by Bowditch [41]. Thus the question is asking for an equivariant collection of uniform quasigeodesics so that any two are connected by at least one, but only finitely many of these.

Bowditch used his strong finiteness of tight geodesics to show that translation lengths in the curve complex are rational, and Bell–Fujiwara [11] used it to show that curve complexes have finite asymptotic dimension.

(3) Do the hyperbolic $\mathrm{Out}(F_n)$-complexes $\mathrm{FS}_n, \mathrm{FZ}_n, \mathrm{FF}_n$ have finite asymptotic dimension? Are the translation lengths always rational? Does $\mathrm{Out}(F_n)$ have finite asymptotic dimension?

We remark that the Novikov conjecture is known for $\mathrm{Out}(F_n)$ [33].

The following seems out of reach with the present methods, although [36] is a promising start:

(4) Does $\mathrm{Out}(F_n)$ satisfy the Farrell–Jones conjecture?

(5) Does the local and global connectivity of $\partial\,\mathrm{FF}_n$ go to infinity as $n \to \infty$?

By the work of Gabai [65], the answer is yes for the boundary of the curve complex. Each $\partial\,\mathrm{FF}_n$ is finite0dimensional [37], and [23] is a start. Of course, the same question can be asked about the boundaries of FZ_n and FS_n.

ACKNOWLEDGMENTS

I would like to thank all my collaborators over the years, and particularly Ken Bromberg, Mark Feighn, and Koji Fujiwara with whom I wrote many papers. I had a lot of fun and learned many things from you. Here is to the future papers!

FUNDING

This work was partially supported by the National Science Foundation, DMS-1905720.

REFERENCES

[1] G. N. Arzhantseva, C. H. Cashen, D. Gruber, and D. Hume, Negative curvature in graphical small cancellation groups. *Groups Geom. Dyn.* **13** (2019), no. 2, 579–632.

[2] S. H. Balasubramanya, Acylindrical group actions on quasi-trees. *Algebr. Geom. Topol.* **17** (2017), no. 4, 2145–2176.

[3] A. Bartels, Coarse flow spaces for relatively hyperbolic groups. *Compos. Math.* **153** (2017), no. 4, 745–779.

[4] A. Bartels and M. Bestvina, The Farrell–Jones conjecture for mapping class groups. *Invent. Math.* **215** (2019), no. 2, 651–712.

[5] A. Bartels, W. Lück, and H. Reich, The K-theoretic Farrell–Jones conjecture for hyperbolic groups. *Invent. Math.* **172** (2008), no. 1, 29–70.

[6] H. Bass, Covering theory for graphs of groups. *J. Pure Appl. Algebra* **89** (1993), no. 1–2, 3–47.

[7] J. A. Behrstock, Asymptotic geometry of the mapping class group and Teichmüller space. *Geom. Topol.* **10** (2006), 1523–1578.

[8] J. Behrstock, M. F. Hagen, and A. Sisto, Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups. *Proc. Lond. Math. Soc. (3)* **114** (2017), no. 5, 890–926.

[9] J. Behrstock, M. F. Hagen, and A. Sisto, Quasiflats in hierarchically hyperbolic spaces. *Duke Math. J.* **170** (2021), no. 5, 909–996.

[10] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, Geometry and rigidity of mapping class groups. *Geom. Topol.* **16** (2012), no. 2, 781–888.

[11] G. C. Bell and K. Fujiwara, The asymptotic dimension of a curve graph is finite. *J. Lond. Math. Soc. (2)* **77** (2008), no. 1, 33–50.

[12] G. Bell and A. Dranishnikov, Asymptotic dimension. *Topology Appl.* **155** (2008), no. 12, 1265–1296.

[13] M. Bestvina, Degenerations of the hyperbolic space. *Duke Math. J.* **56** (1988), no. 1, 143–161.

[14] M. Bestvina, $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pp. 55–91, North-Holland, Amsterdam, 2002.

[15] M. Bestvina, The topology of Out(F_n). In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pp. 373–384, Higher Ed. Press, Beijing, 2002.

[16] M. Bestvina, Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston's vision. *Bull. Amer. Math. Soc. (N.S.)* **51** (2014), no. 1, 53–70.

[17] M. Bestvina, Geometry of outer space. In *Geometric group theory*, pp. 173–206, IAS/Park City Math. Ser. 21, Amer. Math. Soc., Providence, RI, 2014.

[18] M. Bestvina and K. Bromberg, On the asymptotic dimension of the curve complex. *Geom. Topol.* **23** (2019), no. 5, 2227–2276.

[19] M. Bestvina, K. Bromberg, and K. Fujiwara, Constructing group actions on quasi-trees and applications to mapping class groups. *Publ. Math. Inst. Hautes Études Sci.* **122** (2015), 1–64.

[20] M. Bestvina, K. Bromberg, and K. Fujiwara, Bounded cohomology with coefficients in uniformly convex Banach spaces. *Comment. Math. Helv.* **91** (2016), no. 2, 203–218.

[21] M. Bestvina, K. Bromberg, and K. Fujiwara, Stable commutator length on mapping class groups. *Ann. Inst. Fourier (Grenoble)* **66** (2016), no. 3, 871–898.

[22] M. Bestvina, K. Bromberg, K. Fujiwara, and A. Sisto, Acylindrical actions on projection complexes. *Enseign. Math.* **65** (2019), no. 1–2, 1–32.

[23] M. Bestvina, J. Chaika, and S. Hensel, Connectivity of the Gromov boundary of the free factor complex. 2021, arXiv:2105.01537v2.

[24] M. Bestvina, R. Dickmann, G. Domat, S. Kwak, P. Patel, and E. Stark, Free products from spinning and rotating families. 2020, arXiv:2010.10735v2.

[25] M. Bestvina and M. Feighn, Addendum and correction to: "A combination theorem for negatively curved groups" [*J. Differential Geom.* **35** (1992), no. 1, 85–101]. *J. Differential Geom.* **43** (1996), no. 4, 783–788.

[26] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups. *J. Differential Geom.* **35** (1992), no. 1, 85–101.

[27] M. Bestvina and M. Feighn, Outer limits, 1992, revised 1994, preprint, http://www.math.utah.edu/~bestvina/research.html.

[28] M. Bestvina and M. Feighn, Stable actions of groups on real trees. *Invent. Math.* **121** (1995), no. 2, 287–321.

[29] M. Bestvina and M. Feighn, A hyperbolic Out(F_n)-complex. *Groups Geom. Dyn.* **4** (2010), no. 1, 31–58.

[30] M. Bestvina and M. Feighn, Hyperbolicity of the complex of free factors. *Adv. Math.* **256** (2014), 104–155.

[31] M. Bestvina and M. Feighn, Subfactor projections. *J. Topol.* **7** (2014), no. 3, 771–804.

[32] M. Bestvina and K. Fujiwara, Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.* **6** (2002), 69–89.

[33] M. Bestvina, V. Guirardel, and C. Horbez, Boundary amenability of Out(F_N). 2021, arXiv:1705.07017v2.

[34] M. Bestvina and M. Handel, Train tracks and automorphisms of free groups. *Ann. of Math. (2)* **135** (1992), no. 1, 1–51.

[35] M. Bestvina and M. Handel, Train tracks and automorphisms of free groups. *Ann. of Math. (2)* **135** (1992), no. 1, 1–51.

[36] M. Bestvina and C. Horbez, A compactification of outer space which is an absolute retract. *Ann. Inst. Fourier (Grenoble)* **69** (2019), no. 6, 2395–2437.

[37] M. Bestvina, C. Horbez, and R. D. Wade, On the topological dimension of the Gromov boundaries of some hyperbolic Out(F_N)-graphs. *Pacific J. Math.* **308** (2020), no. 1, 1–40.

[38] M. Bestvina and G. Mess, The boundary of negatively curved groups. *J. Amer. Math. Soc.* **4** (1991), no. 3, 469–481.

[39] M. Bestvina and P. Reynolds, The boundary of the complex of free factors. *Duke Math. J.* **164** (2015), no. 11, 2213–2251.

[40] B. H. Bowditch, Cut points and canonical splittings of hyperbolic groups. *Acta Math.* **180** (1998), no. 2, 145–186.

[41] B. H. Bowditch, Tight geodesics in the curve complex. *Invent. Math.* **171** (2008), no. 2, 281–300.

[42] B. H. Bowditch, Uniform hyperbolicity of the curve graphs. *Pacific J. Math.* **269** (2014), no. 2, 269–280.

[43] M. R. Bridson, Semisimple actions of mapping class groups on CAT(0) spaces. In *Geometry of Riemann surfaces*, pp. 1–14, London Math. Soc. Lecture Note Ser. 368, Cambridge Univ. Press, Cambridge, 2010.

[44] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren Math. Wiss. [Fundam. Principles Math. Sci.] 319, Springer, Berlin, 1999.

[45] M. R. Bridson and R. D. Wade, Actions of higher-rank lattices on free groups. *Compos. Math.* **147** (2011), no. 5, 1573–1580.

[46] M. R. Bridson and D. T. Wise, Malnormality is undecidable in hyperbolic groups. *Israel J. Math.* **124** (2001), 313–316.

[47] R. Brooks, Some remarks on bounded cohomology. In *Riemann surfaces and related topics: proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, NY, 1978)*, pp. 53–63, Ann. of Math. Stud. 97, Princeton Univ. Press, Princeton, NJ, 1981.

[48] H. Busemann, Extremals on closed hyperbolic space forms. *Tensor (N.S.)* **16** (1965), 313–318.

[49] D. Calegari, *scl*. MSJ Mem. 20, Mathematical Society of Japan, Tokyo, 2009.

[50] C. Champetier, Propriétés statistiques des groupes de présentation finie. *Adv. Math.* **116** (1995), no. 2, 197–262.

[51] M. Clay, J. Mangahas, and D. Margalit, Right-angled Artin groups as normal subgroups of mapping class groups. *Compos. Math.* **157** (2021), no. 8, 1807–1852.

[52] M. Culler and J. W. Morgan, Group actions on $\mathbb{R}$-trees. *Proc. Lond. Math. Soc. (3)* **55** (1987), no. 3, 571–604.

[53] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups. *Invent. Math.* **84** (1986), no. 1, 91–119.

[54] F. Dahmani and V. Guirardel, The isomorphism problem for all hyperbolic groups. *Geom. Funct. Anal.* **21** (2011), no. 2, 223–300.

[55] F. Dahmani, V. Guirardel, and D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.* **245** (2017), no. 1156, v+152 pp.

[56] S. Dowdall and S. J. Taylor, Hyperbolic extensions of free groups. *Geom. Topol.* **22** (2018), no. 1, 517–570.

[57] A. N. Dranishnikov, S. C. Ferry, and S. A. Weinberger, An etale approach to the Novikov conjecture. *Comm. Pure Appl. Math.* **61** (2008), no. 2, 139–155.

[58] C. Druţu and M. Kapovich, *Geometric group theory*. Amer. Math. Soc. Colloq. Publ. 63, American Mathematical Society, Providence, RI, 2018.

[59] M. J. Dunwoody and M. E. Sageev, JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.* **135** (1999), no. 1, 25–44.

[60] M. G. Durham, Y. N. Minsky, and A. Sisto, Stable cubulations, bicombings, and barycenters. 2020, arXiv:2009.13647.

[61] B. Farb, Relatively hyperbolic groups. *Geom. Funct. Anal.* **8** (1998), no. 5, 810–840.

[62] B. Farb and D. Margalit, *A primer on mapping class groups*. Princeton Math. Ser. 49, Princeton University Press, Princeton, NJ, 2012.

[63] B. Farb and L. Mosher, Convex cocompact subgroups of mapping class groups. *Geom. Topol.* **6** (2002), 91–152.

[64] K. Fujiwara and P. Papasoglu, JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.* **16** (2006), no. 1, 70–125.

[65] D. Gabai, On the topology of ending lamination space. *Geom. Topol.* **18** (2014), no. 5, 2683–2745.

[66] D. Gaboriau, G. Levitt, and F. Paulin, Pseudogroups of isometries of R and Rips' theorem on free actions on R-trees. *Israel J. Math.* **87** (1994), no. 1–3, 403–428.

[67] E. Ghys, Groupes aléatoires (d'après Misha Gromov, . . .). *Astérisque* (2004), no. 294, viii, 173–204.

[68] E. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math. 83, Birkhäuser Boston, Inc., Boston, MA, 1990.

[69] M. Gromov, Groups of polynomial growth and expanding maps. *Publ. Math. Inst. Hautes Études Sci.* **53** (1981), 53–73.

[70] M. Gromov, Infinite groups as geometric objects. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pp. 385–392, PWN, Warsaw, 1984.

[71] M. Gromov, Hyperbolic groups. In *Essays in group theory*, pp. 75–263, Math. Sci. Res. Inst. Publ. 8, Springer, New York, 1987.

[72] M. Gromov, Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pp. 1–295, London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press, Cambridge, 1993.

[73] M. Gromov, Random walk in random groups. *Geom. Funct. Anal.* **13** (2003), no. 1, 73–146.

[74] D. Groves and J. F. Manning, Dehn filling in relatively hyperbolic groups. *Israel J. Math.* **168** (2008), 317–429.

[75] I. Gruschko, Über die Basen eines freien Produktes von Gruppen. *Rec. Math. [Mat. Sb.] N.S.* **8** (1940), no. 50, 169–182.

[76] V. Guirardel and G. Levitt, JSJ decompositions of groups. *Astérisque* (2017), no. 395, vii+165 pp.

[77] U. Hamenstädt, The boundary of the free splitting graph and the free factor graph. 2014, arXiv:1211.1630v5.

[78] U. Hamenstaedt, Word hyperbolic extensions of surface groups. 2005, arXiv:math/0505244v2.

[79] U. Hamenstaedt, Geometry of the mapping class groups III: quasi-isometric rigidity. 2007, arXiv:math/0512429v2.

[80] M. Handel and L. Mosher, The free splitting complex of a free group, I: hyperbolicity. *Geom. Topol.* **17** (2013), no. 3, 1581–1672.

[81] M. Handel and L. Mosher, Hyperbolic actions and 2nd bounded cohomology of subgroups of $\mathrm{Out}(F_n)$. Part I: infinite lamination subgroups. 2019, arXiv:1511.06913v6.

[82] M. Handel and L. Mosher, Hyperbolic actions and 2nd bounded cohomology of subgroups of $\mathrm{Out}(F_n)$. Part II: finite lamination subgroups. 2019, arXiv:1702.08050v5.

[83] W. J. Harvey, Boundary structure of the modular group. In *Riemann surfaces and related topics: proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, NY, 1978)*, pp. 245–251, Ann. of Math. Stud. 97, State Univ., New York, Stony Brook, NY, 1981.

[84] S. Hensel, P. Przytycki, and R. C. H. Webb, 1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 4, 755–762.

[85] C. Horbez, The boundary of the outer space of a free product. *Israel J. Math.* **221** (2017), no. 1, 179–234.

[86] G. Italiano, B. Martelli, and M. Migliorini, Hyperbolic 5-manifolds that fiber over S^1. 2021, arXiv:2105.14795v3.

[87] I. Kapovich, J. Maher, C. Pfaff, and S. J. Taylor, Random trees in the boundary of Outer space. 2021, arXiv:1904.10026v3.

[88] I. Kapovich and K. Rafi, On hyperbolicity of free splitting and free factor complexes. *Groups Geom. Dyn.* **8** (2014), no. 2, 391–414.

[89] A. E. Kent and C. J. Leininger, Shadows of mapping class groups: capturing convex cocompactness. *Geom. Funct. Anal.* **18** (2008), no. 4, 1270–1325.

[90] E. Klarreich, The boundary at infinity of the curve complex and the relative Teichmüller space. 1999, arXiv:1803.10339. To appear in *Groups Geom. Dyn.*

[91] G. Levitt, Constructing free actions on ℝ-trees. *Duke Math. J.* **69** (1993), no. 3, 615–633.

[92] G. Levitt, Non-nesting actions on real trees. *Bull. Lond. Math. Soc.* **30** (1998), no. 1, 46–54.

[93] J. Maher and G. Tiozzo, Random walks on weakly hyperbolic groups. *J. Reine Angew. Math.* **742** (2018), 187–239.

[94] J. Mangahas, A recipe for short-word pseudo-Anosovs. *Amer. J. Math.* **135** (2013), no. 4, 1087–1116.

[95] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.* **138** (1999), no. 1, 103–149.

[96] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.* **10** (2000), no. 4, 902–974.

[97] H. Masur and S. Schleimer, The geometry of the disk complex. *J. Amer. Math. Soc.* **26** (2013), no. 1, 1–62.

[98] J. Milnor, A note on curvature and fundamental group. *J. Differential Geom.* **2** (1968), 1–7.

[99] J. Milnor, Hyperbolic geometry: the first 150 years. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), no. 1, 9–24.

[100] M. Mj, Cannon–Thurston maps. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 885–917, World Sci. Publ., Hackensack, NJ, 2018.

[101] J. W. Morgan, On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture (New York, 1979)*, pp. 37–125, Pure Appl. Math. 112, Academic Press, New York, 1979.

[102] J. W. Morgan and P. B. Shalen, Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)* **120** (1984), no. 3, 401–476.

[103] J. W. Morgan and P. B. Shalen, Degenerations of hyperbolic structures. II. Measured laminations in 3-manifolds. *Ann. of Math. (2)* **127** (1988), no. 2, 403–456.

[104] J. W. Morgan and P. B. Shalen, Degenerations of hyperbolic structures. III. Actions of 3-manifold groups on trees and Thurston's compactness theorem. *Ann. of Math. (2)* **127** (1988), no. 3, 457–519.

[105] H. M. Morse, A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.* **26** (1924), no. 1, 25–60.

[106] G. D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *Publ. Math. Inst. Hautes Études Sci.* **34** (1968), 53–104.

[107] G. A. Niblo and M. A. Roller (eds.), *Geometric group theory, Vol. 1*. London Math. Soc. Lecture Note Ser. 181, Cambridge University Press, Cambridge, 1993.

[108] G. A. Niblo and M. A. Roller (eds.), *Geometric group theory, Vol. 2*. London Math. Soc. Lecture Note Ser. 182, Cambridge University Press, Cambridge, 1993.

[109] J. Nielsen, Über die Isomorphismen unendlicher Gruppen ohne Relation. *Math. Ann.* **79** (1918), no. 3, 269–272.

[110] A. Y. Ol'shanskiĭ, Almost every group is hyperbolic. *Internat. J. Algebra Comput.* **2** (1992), no. 1, 1–17.

[111] Y. Ollivier, *A January 2005 invitation to random groups*. Ensaios Mat. [Math. Surveys] 10, Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.

[112] D. Osajda, Small cancellation labellings of some infinite graphs and applications. *Acta Math.* **225** (2020), no. 1, 159–191.

[113] D. Osin, Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.* **368** (2016), no. 2, 851–888.

[114] D. V. Osin, Peripheral fillings of relatively hyperbolic groups. *Invent. Math.* **167** (2007), no. 2, 295–326.

[115] D. V. Osin, Groups acting acylindrically on hyperbolic spaces. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 919–939, World Sci. Publ., Hackensack, NJ, 2018.

[116] F. Paulin, Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.* **94** (1988), no. 1, 53–80.

[117] F. Paulin, Outer automorphisms of hyperbolic groups and small actions on **R**-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, pp. 331–343, Math. Sci. Res. Inst. Publ. 19, Springer, New York, 1991.

[118] E. Rips and Z. Sela, Structure and rigidity in hyperbolic groups. I. *Geom. Funct. Anal.* **4** (1994), no. 3, 337–371.

[119] E. Rips and Z. Sela, Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)* **146** (1997), no. 1, 53–109.

[120] Z. Sela, Uniform embeddings of hyperbolic groups in Hilbert spaces. *Israel J. Math.* **80** (1992), no. 1–2, 171–181.

[121] Z. Sela, The isomorphism problem for hyperbolic groups. I. *Ann. of Math. (2)* **141** (1995), no. 2, 217–283.

[122] Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.* **7** (1997), no. 3, 561–593.

[123] Z. Sela, Endomorphisms of hyperbolic groups. I. The Hopf property. *Topology* **38** (1999), no. 2, 301–321.

[124] Z. Sela, Diophantine geometry over groups. I. Makanin–Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.* **93** (2001), 31–105.

[125] Z. Sela, Diophantine geometry over groups. II. Completions, closures and formal solutions. *Israel J. Math.* **134** (2003), 173–254.

[126] Z. Sela, Diophantine geometry over groups. IV. An iterative procedure for validation of a sentence. *Israel J. Math.* **143** (2004), 1–130.

[127] Z. Sela, Diophantine geometry over groups. V_1. Quantifier elimination. I. *Israel J. Math.* **150** (2005), 1–197.
[128] Z. Sela, Diophantine geometry over groups. V_2. Quantifier elimination. II. *Geom. Funct. Anal.* **16** (2006), no. 3, 537–706.
[129] Z. Sela, Diophantine geometry over groups. VII. The elementary theory of a hyperbolic group. *Proc. Lond. Math. Soc. (3)* **99** (2009), no. 1, 217–273.
[130] Z. Sela, Diophantine geometry over groups VIII: stability. *Ann. of Math. (2)* **177** (2013), no. 3, 787–868.
[131] J.-P. Serre, *Trees*. Springer, Berlin–New York, 1980.
[132] J. R. Stallings, A topological proof of Grushko's theorem on free products. *Math. Z.* **90** (1965), 1–8.
[133] J. R. Stallings, On torsion-free groups with infinitely many ends. *Ann. of Math. (2)* **88** (1968), 312–334.
[134] J. R. Stallings, Topology of finite graphs. *Invent. Math.* **71** (1983), no. 3, 551–565.
[135] A. S. Švarc, A volume invariant of coverings. *Dokl. Akad. Nauk SSSR* **105** (1955), 32–34.
[136] G. A. Swarup, On the cut point conjecture. *Electron. Res. Announc. Am. Math. Soc.* **2** (1996), no. 2, 98–100.
[137] S. J. Taylor, A note on subfactor projections. *Algebr. Geom. Topol.* **14** (2014), no. 2, 805–821.
[138] W. P. Thurston, The geometry and topology of 3-manifolds, 1980, electronic version http://library.msri.org/books/gt3m/.
[139] W. P. Thurston, Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. *Ann. of Math. (2)* **124** (1986), no. 2, 203–246.
[140] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)* **19** (1988), no. 2, 417–431.
[141] K. Vogtmann, Automorphisms of free groups and outer space. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000)*, Geometriae Dedicata, pp. 1–31, 2002.
[142] K. Vogtmann, On the geometry of outer space. *Bull. Amer. Math. Soc. (N.S.)* **52** (2015), no. 1, 27–46.
[143] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)* **147** (1998), no. 2, 325–355.

MLADEN BESTVINA

Department of Mathematics, University of Utah, 155 S 1400 E, RM 233, Salt Lake City, UT 84112, USA, bestvina@math.utah.edu