

AUTOMORPHIC FUNCTIONS ON MODULI SPACES OF BUNDLES ON CURVES OVER LOCAL FIELDS: A SURVEY

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ABSTRACT

This paper is the written version of D. Kazhdan's plenary talk at ICM 2022. It is dedicated to an exposition of recent results and (mostly) conjectures attempting to construct an analog of the theory of automorphic functions on moduli spaces of bundles on curves over local fields (both archimedean and non-archimedean). The talk is based on joint works of D. Kazhdan with A. Braverman, P. Etingof, E. Frenkel, and A. Polishchuk.

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1. INTRODUCTION

1.1. Langlands correspondence over functional fields

Let \mathcal{C} be a smooth projective irreducible curve over a finite field \mathbb{F}_q . One can consider the global field $\mathbb{F} = \mathbb{F}_q(\mathcal{C})$ of rational functions on \mathcal{C} and its adèle ring \mathbb{A} . Given a split semisimple group G one can study automorphic forms on the adelic group $G(\mathbb{A})$ – these are (by definition) irreducible representations of $G(\mathbb{A})$ which appear in the space of \mathbb{C} -valued functions on $G(\mathbb{A})/G(\mathbb{F})$. For many purposes, it is important to consider discrete automorphic representations – these are automorphic representations appearing in $L^2(G(\mathbb{A}_F)/G(\mathbb{F}))$.

In this introduction we restrict our attention to unramified automorphic representations, i.e., those which have a $G(\mathcal{O}_{\mathbb{F}})$ -invariant vector where $\mathcal{O}_{\mathbb{F}} \subset \mathbb{A}$ is the ring of integral adèles. In other words, we consider functions on $G(\mathcal{O}_{\mathbb{F}})\backslash G(\mathbb{A})/G(\mathbb{F})$ which are eigenfunctions of certain commuting family of linear operators, called Hecke operators; for every place c of \mathbb{F} (which is the same as a point of $\mathcal{C}(\overline{\mathbb{F}_q})$ up to the action of Frobenius), one constructs the algebra of Hecke operators which is isomorphic to the complexified Grothendieck ring of finite-dimensional representations of the Langlands dual group G^\vee (and for different c these algebras commute with each other). The weak form of the Langlands conjecture (now proved by V. Lafforgue for global fields of positive characteristic) asserts that (after the replacement of the coefficient field \mathbb{C} by $\overline{\mathbb{Q}_\ell}$) the common eigenvalues of all the Hecke operators come from ℓ -adic G^\vee -local systems on \mathcal{C} .

The quotient $G(\mathcal{O}_{\mathbb{F}})\backslash G(\mathbb{A})/G(\mathbb{F})$ is canonically isomorphic to the set of \mathbb{F}_q -points of the moduli stack $\text{Bun}_G(\mathcal{C})$ of principal G -bundles on \mathcal{C} . Thus Hecke eigenfunctions are functions on $\text{Bun}_G(\mathcal{C})(\mathbb{F}_q)$ and unramified discrete automorphic forms correspond to Hecke eigenfunction lying in $L^2(\text{Bun}_G(\mathcal{C})(\mathbb{F}_q))$ (with respect to the Tamagawa measure).

We fix a curve \mathcal{C} and a group G , and will write Bun instead of $\text{Bun}_G(\mathcal{C})$ when it does not lead to a confusion.

1.2. Hecke eigenfunctions on moduli spaces of bundles over local fields

This survey reports on an attempt to extend the above constructions and results to the case when instead of a curve over \mathbb{F}_q we start with a curve over a local field F . The idea to consider Hecke eigenfunctions in this case was first formulated by Langlands in the case $F = \mathbb{C}$ (cf. [25] and also [18]) several years ago. A systematic study of this question was started in [9] in a slightly different framework. To simplify notations we often assume that G is semisimple and the genus g of \mathcal{C} is ≥ 2 .

Here several difficulties are present. First, since Bun is a stack, it is not clear what space of functions on $\text{Bun}(F)$ to consider. In fact, a big part of this paper is devoted to a discussion of three different spaces with actions of Hecke algebras one can attach to stacks over local fields and the relation between them (cf. Sections 2 and 3). In the first approach (which follows the papers of P. Etingof, E. Frenkel, and D. Kazhdan), the action of the Hecke algebra is defined on the Hilbert space $L^2(\text{Bun}_{\text{st}}(F))$ of half-measures where $\text{Bun}_{\text{st}} \subset \text{Bun}$ is the open Deligne–Mumford substack of stable bundles. In this case the space is familiar, but

one has to justify the convergence of the integrals defining Hecke operators. In the second and third approaches, the action of Hecke operators is well defined, but it is not easy to describe spaces on which they act. Some of our conjectures are on the relation between these different realizations.

In all three approaches the definition of Hecke operators, in fact, comes from [6] where some version of Satake isomorphism for Hecke algebras over a local field F is studied (formally, [6] only deals with non-archimedean fields, but the extension to archimedean case is straightforward).

Remark 1.1. In [9] (which deals with the case $F = \mathbb{C}$), the role of Hecke operators is played by the algebra \mathcal{D} of global differential operators on $\text{Bun}(F)$ (and their complex conjugate). In fact, as was observed in [3] there is no nontrivial regular differential operators acting on functions, but there is a large algebra of differential operators on half-forms. This algebra \mathcal{D} is commutative and is equal to algebra of functions on the moduli space of certain special G^\vee -local systems on \mathcal{C} called opers. This is another reason why half-forms are better suited for this problem. One of the main purposes of [9] is a formulation of a conjectural description of eigenvalues of the algebra $\mathcal{A} = \mathcal{D} \otimes \overline{\mathcal{D}}$ in terms of certain G^\vee -local systems on \mathcal{C} (opers with real monodromy). For $G = \text{SL}(2)$, a very close conjecture was formulated by J. Teschner in [27].

A systematic study of Hecke operators as self-adjoint operators acting on a Hilbert space started in [10] (in the case $F = \mathbb{C}$). As was mentioned above, the definition of Hecke operators is based on [6], and it again follows from [6] that in order to define Hecke operators one must work with half-forms; in this case Hecke operators are given by certain integrals (which are not guaranteed to converge). In [10] the authors conjectured that these integrals, in fact, define compact self-adjoint operators on $L^2(\text{Bun})(F)$ for any local field F (in particular, contrary to the case of finite fields, their common spectrum on $L^2(\text{Bun}(F))$ is discrete); in the case $F = \mathbb{C}$, it is expected that their eigenvectors are essentially the same as the eigenvectors for the algebra \mathcal{A} (we shall give a precise formulation in Section 6). It is also explained in [10] (in the case $F = \mathbb{C}$) how to produce Hecke eigenvalues from opers with real monodromy (again, this is reviewed in Section 6). For non-archimedean fields F and $G = \text{SL}_2$, analogous conjectures were formulated earlier by M. Kontsevich in [22].

In Section 5 we propose two other constructions of modules over the Hecke algebra – the last one only in the non-archimedean case. As was mentioned before, the space $L^2(\text{Bun})$ is not the only choice of functional space one can work with. One can define another functional space (still having to do with half-forms) on which the Hecke operators will automatically act. The relationship between this space and $L^2(\text{Bun})$, in the case when $G = \text{SL}_2$, is the subject of a forthcoming paper by A. Braverman, D. Kazhdan, and A. Polishchuk. We review the relevant definitions and statements in Sections 2 and 3.

As a byproduct, when F is non-archimedean and the curve \mathcal{C} is defined over its ring of integers \mathcal{O}_F (and has good reduction), we give a conjectural construction of finite-dimensional spaces of *cuspidal* functions with an action of Hecke operators generalizing the

space of cuspidal functions on $\text{Bun}(\mathbb{F}_q)$, where \mathbb{F}_q is the residue field of F (but in this way one gets only a very small portion of Hecke eigen-functions).¹ This is reviewed in Section 5.

In Sections 6 and 7, we formulate in the archimedean case a precise conjecture on the interpretation of the spectrum of Hecke operators on $L^2(\text{Bun}(F))$ in terms of some kind of Galois data (involving the dual group G^\vee). It would be extremely interesting to find an interpretation of the spectrum of Hecke operators to for the non-archimedean case.

1.3. Relation of the archimedean case to geometric Langlands correspondence and conformal field theory

In the case when the field F is archimedean, our program is related to the quantum gauge theory (see [19]).

In this case Beilinson and Drinfeld associate to every G^\vee -oper o a certain algebraic \mathbb{D} -module M_o on Bun which is a *Hecke eigenmodule* which is equipped with a canonical generator (here \mathbb{D} stands for the sheaf of differential operators on Bun acting on half-forms). This is an important part of a general geometric Langlands conjecture. The \mathbb{D} -module M_o can be thought of as a system of linear differential equations on Bun_{st} . The corresponding Hecke eigen-half-form (in the case when o has real monodromy) is a solution of both this system of equations and its complex conjugate.

The difference between the traditional categorical Langlands correspondence and the analytic Langlands correspondence for complex curves can be illustrated by an analogy with the two-dimensional conformal field theory (CFT). In CFT, there are two types of correlation functions. The first is chiral correlation functions, also known as conformal blocks. They form a vector space for fixed values of the parameters of the CFT, so we obtain a vector bundle of conformal blocks on the space of parameters, equipped with a projectively flat connection (or more generally, a twisted \mathbb{D} -module). Conformal blocks are its *multivalued* horizontal sections. The second type is the “physical” correlation functions. They can be expressed as sesquilinear combinations of conformal blocks and their complex conjugates (anticonformal blocks), which is a *single-valued* function of the parameters.

The Hecke eigensheaves on Bun constructed in the categorical Langlands correspondence may be viewed as sheaves of conformal blocks of a certain CFT. They are parametrized by all G^\vee -opers on the curve. It turns out that for special G^\vee -opers (namely, the real ones) there exists a sesquilinear linear combinations of these conformal blocks and their complex conjugates which are single-valued functions (more precisely, $1/2$ -measures) on Bun . These are the automorphic forms of the analytic theory. Thus, the objects of the analytic theory of automorphic forms on Bun can be constructed from the objects of the categorical theory in roughly the same way as the correlation functions of CFT are obtained from conformal blocks (see [16] and the references therein for more details). An important difference with traditional CFT is that while usually in CFT the monodromy of conformal blocks is typically unitary, here the monodromy is expected to be in a split real group.

1 The construction itself is, in fact, not conjectural – we can do it rigorously. But at the moment, we cannot prove that the resulting eigenfunctions are not equal to 0.

1.4. Notations

We shall use the letter k to denote an arbitrary field (which could be finite) and the letter F for local fields. For a variety (or stack) X over k , we denote by $X(k)$ the set of k -points (for a stack we consider isomorphism classes of points). If F is non-archimedean, we denote by \mathcal{O}_F its ring of integers. We shall also consider the field $\mathcal{K} = k((t))$ (or $F((t))$) with ring of integers which we denote just by \mathcal{O} .

For a split semisimple group G , we denote by G^\vee the Langlands dual group of G considered as a group over \mathbb{C} . We fix a Borel subgroup $B = TU$ of G , where T is a maximal torus and U is a maximal unipotent subgroup; similarly we have a Borel subgroup $B^\vee = T^\vee U^\vee \subset G^\vee$.

We denote by Λ and Λ^\vee the lattices of coweights and of weights of T (so Λ is also the lattice of weights of T^\vee) and by $\Lambda^+ \subset \Lambda$ the subset of dominant coweights.

1.5. Organization of the paper

In Section 2 we review some basic information about varieties and stacks over local fields and various spaces of functions on them. In Section 3 we begin the discussion of the moduli stack Bun of G -bundles on a curve \mathcal{C} over a local field F and formulate some conjectures about the relation between various function spaces one attaches to Bun . In Section 4 we review the definition of Hecke operators and the formulation of the unramified Langlands correspondence for curves over \mathbb{F}_q . In Section 5 we explain the definition of Hecke operators in the case of local fields, formulate our main conjectures and also discuss some constructions specific for the non-archimedean case. Section 6 is dedicated to the case $F = \mathbb{C}$ and Section 7 to the case $F = \mathbb{R}$.

2. SMOOTH SECTIONS OF LINE BUNDLES ON VARIETIES AND STACKS

2.1. Smooth sections on varieties

If X is an algebraic variety over a local field F (archimedean or not), the set $X(F)$ is endowed with a natural topology.

Definition 2.1. A function $f : X(F) \rightarrow \mathbb{C}$ is smooth if

- (a) F is non-archimedean and f is locally constant;
- (b) F is archimedean and (locally) there exists a closed embedding $X \hookrightarrow Y$ where Y is a smooth variety over F and a C^∞ -function $\bar{f} : Y(F) \rightarrow \mathbb{C}$ such that $f = \bar{f}|_{X(F)}$.

We denote by $C^\infty(X)$ the space of smooth functions on $X(F)$ and by $\mathcal{S}(X)$ its subspace of functions with compact support.

For a line bundle \mathcal{L} over X , we denote by $\mathcal{L}^0 := \mathcal{L} \setminus X$ the corresponding \mathbb{G}_m -torsor over X and set

$$|\mathcal{L}|^\kappa = \mathcal{L}^0(F) \times_{F^*} \mathbb{C}_\kappa,$$

where \mathbb{C}_κ denotes the 1-dimensional space \mathbb{C} on which F^* acts by $|\cdot|^\kappa$. Then $|\mathcal{L}|^\kappa$ is a complex line bundle over $X(F)$. Since the bundle $|\mathcal{L}|^\kappa$ is locally trivial with respect to the natural topology, we can define its space of smooth sections which we denote by $C^\infty(X, |\mathcal{L}|^\kappa)$. Similarly, we denote by $\mathcal{S}(X, |\mathcal{L}|^\kappa) \subset C^\infty(X, |\mathcal{L}|^\kappa)$ the subspace of sections with compact support.

In the case when X is smooth we shall often take $\mathcal{L} = \omega_X$, where ω_X is the line bundle of differential forms of top degree and write $\mathcal{S}_\kappa(X)$ instead of $\mathcal{S}(X, |\omega|^\kappa)$. The case $\kappa = 1/2$ is of special interest since the space $\mathcal{S}_{1/2}(X)$ is endowed with a natural Hermitian product. We denote by $L^2(X)$ its Hilbert space completion.

Remark 2.2. (1) If $U \subset X$ is an open subset and $Z = X \setminus U$ then we have a short exact sequence

$$0 \rightarrow \mathcal{S}(U, |\mathcal{L}|_U^\kappa) \rightarrow \mathcal{S}(X, |\mathcal{L}|^\kappa) \rightarrow \mathcal{S}(Z, |\mathcal{L}|_Z^\kappa) \rightarrow 0.$$

(2) More generally, instead of choosing $\mathcal{L} \in \text{Pic}(X)$ and $\kappa \in \mathbb{C}$ we can start with any element of $\text{Pic}(X) \otimes \mathbb{C}$ – all the above definitions make sense in this context.

2.2. Smooth sections on stacks

In this subsection we extend the above definitions to a class of algebraic stacks.

Definition 2.3. An algebraic stack \mathcal{Y} is *admissible* if locally there exists a presentation of \mathcal{Y} as a quotient stack X/G where X is a smooth variety and G is an affine algebraic group. We denote by $p : X \rightarrow \mathcal{Y}$ the projection.

A presentation of \mathcal{Y} as a quotient $\mathcal{Y} = X/\text{GL}_n$ is called an *admissible presentation*.²

Remark 2.4. (1) Any smooth admissible stack of finite type can be presented as a quotient X/GL_n for a smooth variety X (see [21]). As follows from the Hilbert's 90, we have $\mathcal{Y}(F) = X(F)/\text{GL}_n(F)$.

(2) Any admissible stack is automatically locally of finite type.

(3) A line bundle on a quotient X/G is a G -equivariant line bundle on X .

Definition 2.5. (1) Assume that F is non-archimedean, \mathcal{Y} is an admissible stack of finite type over F . Choose an admissible presentation $\mathcal{Y} = X/\text{GL}_n$ for some variety X and set

$$\mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa) = \mathcal{S}(X, |\mathcal{L}_X|^\kappa)_{\text{GL}(n, F)},$$

² The definition of admissibility that we use here is close to the one introduced in [21] but slightly different. It is easy to see that every admissible stack locally has an admissible presentation.

where the latter space stands for the space of $\mathrm{GL}(n, F)$ -coinvariants on $\mathcal{S}(X, |\mathcal{L}_X|^\kappa)$.

- (2) If F is non-archimedean and \mathcal{Y} is only locally of finite type, then we can write \mathcal{Y} as a direct limit of open substacks \mathcal{Y}_i of finite type over F and define $\mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa) := \varinjlim \mathcal{S}(\mathcal{Y}_i, |\mathcal{L}|^\kappa)$.
- (3) In the case when F is archimedean we make an analogous definition but take coinvariants $\mathcal{S}(X, |\mathcal{L}_X|^\kappa)_{\mathrm{GL}(n, F)}$ in the category of topological spaces where $\mathcal{S}(X, |\mathcal{L}_X|^\kappa)$ is endowed with Fréchet topology.³

The above definition makes sense because of the following

Claim 2.6. *If \mathcal{Y} is an admissible stack of finite type then the space $\mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa)$ does not depend on a choice of an admissible presentation $\mathcal{Y} = X/\mathrm{GL}_n$.*

Remark 2.7. In the case when F is non-archimedean, $\mathcal{L} = \omega_X$, and $\kappa = 1$, this claim is proven in [21, SECTION 6]. The same arguments work in the general case.

2.3. Functoriality

If \mathcal{Y} is an admissible stack and \mathcal{U} is an open substack, we have a natural map $\mathcal{S}(\mathcal{U}, |\mathcal{L}|^\kappa) \rightarrow \mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa)$, which is not injective in general.

More generally, let $f : \mathcal{Z} \rightarrow \mathcal{Y}$ be a smooth representable map of admissible stacks and $\omega_{\mathcal{Z}/\mathcal{Y}}$ be the relative canonical bundle. Then we have a natural (“integration over the fibers”) map

$$\mathcal{S}(\mathcal{Z}, |\mathcal{L}|^\kappa \otimes |\omega_{\mathcal{Z}/\mathcal{Y}}|) \rightarrow \mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa).$$

2.4. An example: stacks over \mathcal{O}_F

In this subsection we consider the case when the field F is non-archimedean (with residue field k) and construct some explicit elements in $\mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa)$. Assume that $\mathcal{Y} = X/G$ where both X and G are defined over \mathcal{O}_F and that $X_{\mathcal{O}_F}$ is a regular scheme over \mathcal{O}_F such that $\mathcal{Y}(F) = X(F)/G(F)$. Assume also that the line bundle \mathcal{L} is defined over \mathcal{O}_F . Then in the same way as before we can define $\mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}, |\mathcal{L}|^\kappa)$ with an obvious map $\mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}, |\mathcal{L}|^\kappa) \rightarrow \mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa)$.

Consider now the case when $\mathcal{L} = \omega_{\mathcal{Y}}$. Then the complex line bundle $|\mathcal{L}|$ has a canonical trivialization on $\mathcal{Y}(\mathcal{O}_F)$. Let $\mathcal{S}(\mathcal{Y}(k))$ denote the space of \mathbb{C} -valued functions with finite support on $\mathcal{Y}(k)$. Then the above trivialization gives rise to a map $\mathcal{S}(\mathcal{Y}(k)) \rightarrow \mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}, |\mathcal{L}|^\kappa)$. Composing it with the map $\mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}, |\mathcal{L}|^\kappa) \rightarrow \mathcal{S}(\mathcal{Y}, |\mathcal{L}|^\kappa)$, we get a map $E_{\mathcal{Y}, \kappa} : \mathcal{S}(\mathcal{Y}(k)) \rightarrow \mathcal{S}_\kappa(\mathcal{Y})$.

Remark 2.8. (1) This map is often not injective.

3 We define the space $\mathcal{S}(X, |\mathcal{L}_X|^\kappa)_{\mathrm{GL}(n, F)}$ as the quotient of $\mathcal{S}(X, |\mathcal{L}_X|^\kappa)$ by the closure of the subset generated by elements of the form $g(s) - s$ where $g \in \mathrm{GL}_n(F)$ and $s \in \mathcal{S}(X, |\mathcal{L}_X|^\kappa)$.

- (2) We will be mostly interested in the space $\mathcal{S}_{1/2}(\mathcal{Y})$ (for a particular choice of \mathcal{Y}). In the case when \mathcal{Y} was a smooth scheme, this space had a canonical Hermitian product. We do not expect to see a Hermitian product on $\mathcal{S}_{1/2}(\mathcal{Y})$ for general admissible stacks \mathcal{Y} but we define a class of *excellent stacks* when such a product exists.
- (3) We write $\mathcal{M}(\mathcal{Y}) := \mathcal{S}_{1/2}(\mathcal{Y})$.

2.5. Nice and excellent stacks

In this subsection we assume that \mathcal{Y} is an admissible stack which contains an open substack $\mathcal{Y}_{vs} \subset \mathcal{Y}$ such that $\mathcal{Y}_{vs} = Y_{vs}/Z$ where Y_{vs} is a smooth scheme and Z is a finite group acting trivially on Y_{vs} .⁴

- Remark 2.9.** (1) To simplify the notations, let us assume that $Z = \{e\}$ (but generalization to arbitrary Z is straightforward).
- (2) A choice of this open substack is not unique, and some of the definitions below depend on this choice.
- (3) Let $L^2(\mathcal{Y}_{vs})$ be the Hilbert space completion of the space of smooth half-measures on $\mathcal{Y}_{vs}(F)$ with compact support. It is easy to see that this space is in fact independent of the choice of \mathcal{Y}_{vs} .

If \mathcal{Y} is of finite type over F , we choose a presentation $\mathcal{Y} = X/\mathrm{GL}_n$, denote by U the preimage of \mathcal{Y}_{vs} in X and by $p : U \rightarrow \mathcal{Y}_{vs}$ the quotient map.

Let s be a smooth section with compact support of the complex line bundle $p^*|\omega_{\mathcal{Y}_{vs}}|^\kappa \otimes |\omega_{X/\mathcal{Y}}|$. Then $s|_U$ is a section of $p^*|\omega_{\mathcal{Y}_{vs}}|^\kappa \otimes |\omega_{U/\mathcal{Y}}|$. We can try to integrate it over the fibers of p to get a section of $|\omega_{\mathcal{Y}_{vs}}|^\kappa$ on \mathcal{Y}_{vs} . The problem is that these integrals might not converge since the intersection of the support of s with the fibers of the map p might not be compact.

- Definition 2.10.** (1) The stack \mathcal{Y} is κ -bounded if there exists an open substack of finite type $\mathcal{Y}_0 \subset \mathcal{Y}$ such that the map $\mathcal{S}_\kappa(\mathcal{Y}_0) \rightarrow \mathcal{S}_\kappa(\mathcal{Y})$ is an isomorphism.
- (2) A pair $(\mathcal{Y}, \mathcal{Y}_{vs})$ is κ -nice if \mathcal{Y} is κ -bounded and for every s as above supported on the preimage of \mathcal{Y}_{vs} the push-forward $p_*(s)$ is well-defined (i.e., it is absolutely convergent) and defines a smooth section of $|\omega_{\mathcal{Y}_{vs}}|^\kappa$ on \mathcal{Y}_{vs} .
- (3) A pair as above is *excellent* if it is nice for all $\kappa \geq 1/2$ and for $\kappa = 1/2$ we have $p_*(s) \in L^2(\mathcal{Y}_{vs})$ for every smooth section s with compact support.

When the substack $\mathcal{Y}_{vs} \subset \mathcal{Y}$ is fixed we refer to the stack \mathcal{Y} as “nice” or “excellent.”

⁴ The subscript *vs* stands for “very stable.” The reason for this notation is that later when we work with the stack Bun of G -bundles on a curve, we define $\mathrm{Bun}_{vs} \subset \mathrm{Bun}$ as the open subset of very stable bundles.

Remark 2.11. The convergence in the definition of κ -niceness is automatically true for $\kappa \geq 1$.

If \mathcal{Y} is κ -nice, then the map $s \mapsto p_*(s)$ descends to a map $\mathcal{S}_\kappa(\mathcal{Y}) \rightarrow C^\infty(\mathcal{Y}_{\text{vs}})$. If \mathcal{Y} is excellent we get a map $\mathcal{M}(\mathcal{Y}) = \mathcal{S}_{1/2}(\mathcal{Y}) \rightarrow L^2(\mathcal{Y}_{\text{vs}}) = L^2(\mathcal{Y})$.

Example 2.12. Let $X = (\mathbb{P}^1)^3$, $G = \text{PGL}_2$, and $\mathcal{Y} = X/G$, where G acts diagonally; we take U to be the complement to all diagonals in $(\mathbb{P}^1)^3$. Then G acts freely on U and we set $\mathcal{Y}_{\text{vs}} = U/G$ (note that \mathcal{Y}_{vs} is just $\text{Spec } F$). In this case one can check that \mathcal{Y} is nice for $\kappa > 1/3$ and the stack \mathcal{Y} is excellent.

3. THE CASE OF Bun_G : PRELIMINARIES

We fix a split connected semisimple group G and denote by Z its center.

Let \mathcal{C} be a smooth complete irreducible curve over a field k .

Definition 3.1. (1) Bun_G is the stack of the principal G -bundles on \mathcal{C} and $\text{Bun}_{G,\text{st}} \subset \text{Bun}_G$ is the open substack of stable bundles.

(2) For a G -bundle \mathcal{F} on \mathcal{C} we denote by $\text{Ad}_{\mathcal{F}}$ the adjoint bundle to \mathcal{F} associated with the adjoint action of G on \mathfrak{g} .

(3) A G -bundle \mathcal{F} is *very stable* if there is no nonzero section of $\Gamma(\mathcal{C}, \text{Ad}_{\mathcal{F}}) \otimes \omega_{\mathcal{C}}$ whose values at all points of \mathcal{C} are nilpotent.

(4) We denote by $\text{Bun}_{G,\text{vs}} \subset \text{Bun}_G$ the substack of very stable bundles.

Remark 3.2. If \mathcal{C} is of genus ≥ 2 then:

(1) Every very stable bundle is stable.

(2) Bun_{st} is a dense open subset of Bun of the form Y/Z where Y is a smooth scheme of finite type over F and Z acts trivially on Y .

(3) Bun_{vs} is a dense open subset of Bun_{st} .

(4) When it does not lead to a confusion we shall drop the subscript G from the notation (e.g., we shall write Bun for Bun_G).

Claim 3.3. *The stack Bun is κ -bounded for all κ .*

Remark 3.4. This statement is inspired by the proof of the main result of [7].

Conjecture 3.5. Assume that the genus g of \mathcal{C} is ≥ 2 .

(1) Bun is κ -nice for $\text{Re}(\kappa) \geq 1/2$. In particular, for $\kappa \geq 1/2$, we get a map $\iota_\kappa : \mathcal{S}_\kappa(\text{Bun}) \rightarrow C_\kappa^\infty(\text{Bun}_{\text{vs}})$.

(2) For $\kappa \geq 1/2$ any section in the image of the map ι_κ extends to a continuous section of $|\omega_{\text{Bun}}|^\kappa$ on Bun_{st} .

(3) Bun is excellent.

For $G = \mathrm{PGL}_2$, the first assertion of Conjecture 3.5 (as well as some special cases of the second and third assertions) will appear in a forthcoming paper of A. Braverman, D. Kazhdan, and A. Polishchuk. Let us note that (again for $G = \mathrm{PGL}_2$) the second assertion can be reduced to the following purely algebro-geometric statement using [2] (we can prove the conjecture for curves of genus 2 and 3).

Conjecture 3.6. Let \mathcal{E} be a stable bundle on \mathcal{C} of degree $2g - 1$. Let $F_{\mathcal{E}}$ denote the scheme of pairs (\mathcal{L}, s) where $\mathcal{L} \in \mathrm{Pic}^0(\mathcal{C})$ and $s \in \mathbb{P}(H^0(\mathcal{C}, \mathcal{L} \otimes \mathcal{E}))$. Then

- (1) $F_{\mathcal{E}}$ is irreducible.
- (2) $\dim F_{\mathcal{E}} = g$.
- (3) $F_{\mathcal{E}}$ has rational singularities.

4. AFFINE GRASSMANNIAN AND HECKE OPERATORS: THE CASE OF FINITE FIELD

In this section we collect some facts about the canonical class of certain Schubert varieties that we shall need in the future. All the results of this section follow easily from [12] and [3]. In what follows we a ground field k and set denote by \mathcal{O} the ring functions on the formal one-dimensional disc D over k and by \mathcal{K} the field of functions on the punctured disc D^* . So $\mathcal{O} \sim k[[t]]$ and $\mathcal{K} \sim k((t))$. We denote by ω_D the canonical bundle on D and fix a square root $\omega_D^{1/2}$ (unique up to an isomorphism; the isomorphism is unique up to ± 1).

4.1. The affine Grassmannian

Let G be a split semisimple group over k and $\mathrm{Gr}_G := G(\mathcal{K})/G(\mathcal{O})$. It is known that Gr_G has a natural structure of a proper ind-scheme over k and the orbits of the group $G(\mathcal{O})$ on Gr_G are parameterized by the elements of Λ_+ .

For each $\lambda \in \Lambda^+$, we shall denote by Gr_G^λ the corresponding orbit and by $\overline{\mathrm{Gr}}_G^\lambda$ the closure of Gr_G^λ .

4.2. Satake isomorphism

In the rest of this section we assume that k is a finite field.

Let $\mathcal{H}(G, k)$ be the algebra of compactly supported $G(\mathcal{O})$ -biinvariant distributions on $G(\mathcal{K})$ (by choosing a Haar measure on $G(\mathcal{K})$ such that $G(\mathcal{O})$ has volume 1, we can identify these distributions with functions). Let G^\vee be the Langlands dual group, considered as a group over \mathbb{C} . The Satake isomorphism identifies $\mathcal{H}(G, k)$ with the complexified Grothendieck ring of the category $\mathrm{Rep}(G^\vee)$ of finite-dimensional representations of G^\vee . It can also be identified with the algebra $\mathbb{C}[T^\vee]^W$ of W -invariant polynomial functions on T^\vee .

4.3. Hecke operators

Let now \mathcal{C} be a smooth projective irreducible curve over k . As before we consider the stack $\mathrm{Bun} := \mathrm{Bun}_G$ of principal G -bundles on \mathcal{C} . Let $c \in \mathcal{C}$ be a closed point with

residue field k' which is a finite extension of k . Choose of a local parameter near c ,⁵ (in the end nothing will depend on this choice) and consider the stack Hecke_c classifying triples $(\mathcal{E}_1, \mathcal{E}_2, \eta)$ where every \mathcal{E}_i is a principal G -bundle on \mathcal{C} and η is an isomorphism between \mathcal{E}_1 and \mathcal{E}_2 on $\mathcal{C} \setminus \{c\}$. We have canonical projections

$$\begin{array}{ccc} \text{Hecke}_c & \xrightarrow{\text{pr}_2} & \text{Bun} \\ \text{pr}_1 \downarrow & & \\ \text{Bun} & & \end{array} \quad (1)$$

Every fiber of the map pr_2 is isomorphic to Gr_G and this isomorphism is canonical up to the action of $G(\mathcal{O})$. Thus every $h \in \mathcal{H}(G, k')$ defines a canonical function \tilde{h} on Hecke_c . We can use it as a correspondence, and set

$$T_{h,c}(f) = \text{pr}_{2,*}(\text{pr}_1^*(f) \cdot \tilde{h})$$

for any $f : \text{Bun}(\mathbb{F}_q) \rightarrow \mathbb{C}$. This construction defines an action of the algebra $\mathcal{H}(G, k')$ on the space of all functions on $\text{Bun}(k)$ (given a choice of c as above). For different choices of c , these operators commute.

Claim 4.1. (1) *The operators $T_{h,c}$ preserve the space $\mathcal{S}(\text{Bun})$ of functions with finite support on $\text{Bun}(k)$.*

(2) *Let $L^2(\text{Bun}(k))$ be the L^2 -completion of the space $\mathcal{S}(\text{Bun}(k))$ with respect to the standard L^2 -norm given by the measure on the (discrete) set $\text{Bun}(k)$ where the volume of every \mathcal{E} is equal to $\frac{1}{\#\text{Aut}(\mathcal{E})}$. Then for every c the action of $\mathcal{H}(G, k)$ extends to an action on $L^2(\text{Bun}(k))$ by bounded operators. If h is real-valued, the operator $T_{h,c}$ is self-adjoint.*

4.4. Langlands conjectures

In the theory of automorphic forms, we are usually interested in eigenfunctions of all the operators $T_{h,c}$. Let us replace the field of coefficients \mathbb{C} by $\overline{\mathbb{Q}}_\ell$ where ℓ is a prime number different from the characteristic of \mathbb{F}_q . Then (the weak form) of the Langlands conjecture states that if f is such an eigenfunction, then the eigenvalues of all the operators $T_{h,c}$ come from a homomorphism $\rho : \mathbb{W}(\mathcal{C}) \rightarrow G^\vee(\overline{\mathbb{Q}}_\ell)$ where $\mathbb{W}(\mathcal{C})$ is the Weil group of \mathcal{C} (a close cousin of the fundamental group of \mathcal{C}). In fact, in this form the Langlands conjecture has been proved by V. Lafforgue (cf. [24]).

Let us recall the connection between Hecke-eigenvalues and homomorphisms ρ as above. First of all, any c defines a conjugacy class $\text{Fr}_c \subset \mathbb{W}(\mathcal{C})$. For any $V \in \text{Rep}(G^\vee)$ the by Satake isomorphism associates to V an element in $\mathcal{H}(G, k)$, which we denote by h_V . We denote $T_{V,c}$ the corresponding Hecke operator. We say that the eigenvalue of an eigenfunction f comes from ρ if

$$T_{V,c}(f) = \text{Tr}(\rho(\text{Fr}_c), V) \cdot f \quad (2)$$

for all c and V .

⁵ That is an identification of the formal neighborhood of c with $\text{Spec } k'[[t]]$.

In general, Hecke eigenfunctions lie neither in $\mathcal{S}(\text{Bun})$ nor in $L^2(\text{Bun})$ (here we come back to considering \mathbb{C} -coefficients). Those which lie in the former are called *cuspidal*, and those which lie in the latter are called *discrete*. The fact that not all eigenfunctions are discrete is related to the fact that the operators T_h have both discrete and continuous spectrum.

Remark 4.2. Note that the operators $T_{h,c}$ would be compact, if the stack Bun were of finite type over k (in fact, $L^2(\text{Bun}(k))$ would be finite-dimensional in this case), and so in that their common spectrum would be discrete. So, the existence of continuous spectrum of Hecke operators is related to the fact that Bun is not of finite type over k .

5. THE AFFINE GRASSMANNIAN AND HECKE OPERATORS: THE CASE OF LOCAL FIELD

5.1. More on formal discs

We are going to make a very mild change of notation (compared to the previous section). Namely, let F be a field (very soon we shall assume that F is a local field). In what follows we denote by \mathcal{O} some discrete valuation ring over F which (as a discrete valuation ring) is isomorphic to $F[[t]]$ (the point is that we do not want to fix this isomorphism). We let \mathcal{K} be the field of fractions of \mathcal{O} . We set $D = \text{Spec}(\mathcal{O})$, $D^* = \text{Spec}(\mathcal{K})$. We shall denote by 0 the canonical F -point of D .

We let ω_D be canonical sheaf of D and let $\omega_{D,0}$ be its fiber at 0 . This is a vector space over F .

5.2. Line bundles on Gr_G

It is well-known (cf. [3] and [12]) that every finite-dimensional representation V of G gives rise to a (determinant) line bundle \mathcal{L}_V on Gr_G ; the fiber of this bundle over a point $g \in G(\mathcal{K})/G(\mathcal{O})$ is equal to the determinant of the vector space $g(V(\mathcal{O}))/g(V(\mathcal{O})) \cap V(\mathcal{O})$. In particular, we let \mathcal{L}_g denote the line bundle corresponding to the adjoint representation of G . The line bundle \mathcal{L}_g^{-1} has a square root (unique up to isomorphism) which we denote by $\mathcal{L}_{\text{crit}}$.

The following result from [3] is crucial for us:

Theorem 5.1. *For every $\lambda \in \Lambda_+$, there is a canonical isomorphism*

$$\mathcal{L}_{\text{crit}}|_{\text{Gr}_G^\lambda} \simeq \omega_{\text{Gr}_G^\lambda} \otimes \omega_{D,0}^{-\langle \lambda, \rho^\vee \rangle}.$$

(Here, as before, $\omega_{\text{Gr}_G^\lambda}$ denotes the canonical bundle of on Gr_G^λ).⁶

⁶ The formulation of the theorem requires a clarification when G is not simply connected, since in this case $\langle \lambda, \rho^\vee \rangle$ might be a half-integer (and not an integer). It is sufficient for our purposes to say that we choose a square root of $\omega_{D,0}$ and that the isomorphism above is canonical up to ± 1 . This potential sign will disappear when we apply $|\cdot|$ to both sides which we shall do in applications.

We need more information about the structure of the varieties $\overline{\text{Gr}}_G^\lambda$. The following result is proved in [12] (cf. also [23] and [26] for the corresponding result in characteristic 0).

Theorem 5.2. (1) Each $\overline{\text{Gr}}_G^\lambda$ is a normal and Cohen–Macaulay projective variety over F .

(2) Each $\overline{\text{Gr}}_G^\lambda$ has a resolution of singularities⁷ and, for every such resolution $\pi^\lambda : \widetilde{\text{Gr}}_G^\lambda \rightarrow \overline{\text{Gr}}_G^\lambda$, one has

$$R\pi_*^\lambda(\mathcal{O}_{\widetilde{\text{Gr}}_G^\lambda}) = \mathcal{O}_{\overline{\text{Gr}}_G^\lambda}.$$

(in other words, $\overline{\text{Gr}}_G^\lambda$ has rational singularities).

The next result is an easy corollary of Theorems 5.2 and 5.1 (cf. [6] for a proof):

Theorem 5.3. (1) For every $\lambda \in \Lambda_+$, the variety $\overline{\text{Gr}}_G^\lambda$ is Gorenstein. Moreover, the canonical sheaf of $\overline{\text{Gr}}_G^\lambda$ is isomorphic to $\mathcal{L}_{\text{crit}}|_{\overline{\text{Gr}}_G^\lambda} \otimes \omega_{D,0}^{(\lambda,\rho^\vee)}$. Abusing the notation, we shall denote this sheaf by $\omega_{\overline{\text{Gr}}_G^\lambda}$.

(2) For any $\lambda \in \Lambda_+$, let $\pi^\lambda : \widetilde{\text{Gr}}_G^\lambda \rightarrow \overline{\text{Gr}}_G^\lambda$ be any resolution of singularities. Then the identification between $(\pi^\lambda)^* \omega_{\overline{\text{Gr}}_G^\lambda}$ and $\omega_{\widetilde{\text{Gr}}_G^\lambda}$ that one has at the generic point of $\widetilde{\text{Gr}}_G^\lambda$ comes from an embedding

$$(\pi^\lambda)^* \omega_{\overline{\text{Gr}}_G^\lambda} \hookrightarrow \omega_{\widetilde{\text{Gr}}_G^\lambda}.$$

(In the case $\text{char } k = 0$, this implies that $\overline{\text{Gr}}_G^\lambda$ has canonical singularities).

5.3. Hecke algebra over local field

In this subsection F can be any local field.

Let us now work over a local field F instead of k with the corresponding ring \mathcal{O} and its field of fractions \mathcal{K} .⁸ Then we would like to define the Hecke algebra $\mathcal{H}(G, F)$. First we consider the space

$$C_{1/2}^\infty(\text{Gr}_G) = \lim_{\leftarrow} \mathcal{S}(\overline{\text{Gr}}^\lambda(F), |\mathcal{L}_{\text{crit}}|).$$

Assume first that F is non-archimedean. Then we define $\mathcal{H}(G, F)$ to be the space of all $G(\mathcal{O})$ -invariant linear functionals on $C_{1/2}^\infty(\text{Gr}_G)$ with compact support. The latter condition means that we consider functionals $\delta : C_{1/2}^\infty(\text{Gr}_G) \rightarrow \mathbb{C}$ which factorize through a map $C_{1/2}^\infty(\text{Gr}_G) \rightarrow \mathcal{S}(\overline{\text{Gr}}^\lambda(F), |\mathcal{L}_{\text{crit}}|)$ for some λ . It is easy to see that $\mathcal{H}(G, F)$ is an algebra with respect to convolution.

It turns out that Theorems 5.1, 5.2, and 5.3 allow one to construct a lot of elements in $\mathcal{H}(G, F)$ (what follows is essentially equivalent to the main result of [6]). Namely, let λ be as above and let $\phi \in C_{1/2}^\infty(\text{Gr}_G)$. Let us first trivialize the space $\omega_{D,0}$. Then $\phi|_{\overline{\text{Gr}}_G^\lambda}$

⁷ Of course, this statement is not a priori clear only if $\text{char } F > 0$.

⁸ In the case when F is non-archimedean the reader should not confuse $\mathcal{O} = F[[t]]$ with \mathcal{O}_F which is the ring of integers of F .

is a distribution on $\text{Gr}^\lambda(F)$ and we can try to consider its integral. A priori it might not be well defined since $\text{Gr}_G^\lambda(F)$ is not compact, but it is explained in [6] that Theorems 5.2 and 5.3 imply that in fact this integral is absolutely convergent and thus defines an element $h_\lambda \in \mathcal{H}(G, F)$. These elements have the property that for any dominant λ and μ we have

$$h_\lambda \star h_\mu = h_{\lambda+\mu}.$$

In other words, we get an embedding $\mathbb{C}[\Lambda_+] \hookrightarrow \mathcal{H}(G, F)$. It is easy to see that it is actually an isomorphism.

If we do not want to trivialize the space $\omega_{D,0}$ then canonically h_λ is a map from $|\omega_{D,0}|^{-(\lambda, \rho^\vee)} \rightarrow \mathcal{H}(G, F)$, and we get an isomorphism

$$\bigoplus_{\lambda \in \Lambda_+} |\omega_{D,0}|^{(\lambda, \rho^\vee)} \simeq \mathcal{H}(G, F)$$

(the left-hand side has an obvious algebra structure).

5.4. Hecke operators for curves over local fields: the first approach

We would like to define Hecke operators in some space of actual functions on Bun (or, rather, sections of $|\omega_{\text{Bun}}|^{1/2}$), or maybe some open subset of it. Let us assume that the genus of \mathcal{C} is ≥ 2 . Then, as we have discussed before, Bun contains a dense open sub-stack Bun_{st} of stable bundles which is a Deligne–Mumford stack. So, one can try to start with a smooth section ϕ of $|\omega_{\text{Bun}}|^{1/2}$ on $\text{Bun}_{\text{st}}(F)$ and apply the operator $\mathbb{T}_{\lambda,c}$ using the diagram (1).⁹

In this case the definition will involve integration over Gr_G^λ , and we are not guaranteed that the corresponding integral is convergent. The trouble is caused by the following: take some $\mathcal{E} \in \text{Bun}(F)$ (which one can assume to be stable or even very stable) and consider $\text{pr}_2^{-1}(\mathcal{E})$. Let us identify it with Gr_G and consider the corresponding $G(\mathcal{O})$ -invariant subset $\overline{\text{Gr}}_G^\lambda$ in it. Let S be a compact subset of $\text{Bun}_{\text{st}}(F)$. Then typically $\text{pr}_1^{-1}(S) \cap \overline{\text{Gr}}_G^\lambda$ is not compact.

We say that $\phi \in C_{1/2}^\infty(\text{Bun}_{\text{vs}})$ is good if the integral defining $\mathbb{T}_{\lambda,c}(\phi)$ is absolutely convergent and the result is again an element of $C_{1/2}^\infty(\text{Bun}_{\text{vs}})$. The following result is easy:

Claim 5.4. *Assume the validity of Conjecture 3.5(1). Then the image of the map $\iota_{1/2}$ consists of good sections and the map $\iota_{1/2}$ commutes with the operator $\mathbb{T}_{\lambda,c}$.*

Note that the image of $\iota_{1/2}$ obviously contains $\mathcal{S}_{1/2}(\text{Bun}_{\text{vs}})$. Thus Conjecture 3.5(1) implies that any $\phi \in \mathcal{S}_{1/2}(\text{Bun}_{\text{vs}})$ is good. On the other hand, without assuming Conjecture 3.5(1) we cannot a priori construct any good element of $C_{1/2}^\infty(\text{Bun}_{\text{vs}})$.

We now proceed to the discussion of the action of the Hecke operators on $L^2(\text{Bun})$. The main expectation is the following:

⁹ We are slightly abusing the notation here: namely, we are going to denote by $\mathbb{T}_{\lambda,c}$ both the operator on $\mathcal{M}(\text{Bun})$ and on some space of sections of $|\omega_{\text{Bun}}|^{1/2}$ which we are going to discuss below. We hope that it does not lead to a confusion.

Conjecture 5.5. The operators $\mathbb{T}_{\lambda,c}$ on $L^2(\text{Bun})$ are bounded, compact, and self-adjoint. In particular, their common spectrum is discrete.

Philosophically, the reason for the fact that in the case of local fields the operators $\mathbb{T}_{\lambda,c}$ have discrete spectrum (as opposed to the case of finite fields) is that in the case of local fields we always work only with some open subset of Bun of finite type (cf. also Claim 3.3), and as was noted in Remark 4.2, the source for noncompactness of the Hecke operators in the case of finite fields has to do with the fact that the stack Bun is not globally of finite type (and in particular, not quasicompact).

5.5. Hecke operators for curves over local fields: the second approach

We now go back to the setup of Section 3. We would like to define Hecke operators in this context. First, we need to decide on what space they are going to act. The first (and the easiest) choice is to work with the space $\mathcal{M}(\text{Bun}) = \mathcal{S}_{1/2}(\text{Bun})$ (another choice is discussed in the next subsection). In what follows it will be convenient (but not necessary) to choose a particular square root $\omega_{\text{Bun}}^{1/2}$ of ω_{Bun} (this is always possible, but the choice is slightly not canonical).

Let us also choose a closed point c of the scheme \mathcal{C} with residue field F' which is a finite unramified extension of F ; we shall take \mathcal{O} to be the local ring of c (so, it is a discrete valuation ring over F' noncanonically isomorphic to $F'[[t]]$). To emphasize the dependence on c , we denote the corresponding Hecke algebra by $\mathcal{H}_c(G)$ (instead of $\mathcal{H}(G, F')$).

Then we again can consider the diagram (1) as in Section 4.3. Then since the line bundle $\mathcal{L}_{\text{crit}}$ on Gr_G is $G(\mathcal{O})$ -equivariant, we can define a line bundle $\widetilde{\mathcal{L}}_{\text{crit}}$ on Hecke_c whose restriction to every fiber of pr_2 is canonically isomorphic to $\mathcal{L}_{\text{crit}}$ (this property makes sense since every fiber is canonically isomorphic to Gr_G up to the action of $G(\mathcal{O})$).

Lemma 5.6. *We have*

$$\text{pr}_1^* \omega_{\text{Bun}}^{1/2} \simeq \text{pr}_2^* \omega_{\text{Bun}}^{1/2} \otimes \widetilde{\mathcal{L}}_{\text{crit}}. \quad (3)$$

The isomorphism (3) easily allows one to define action of $\mathcal{H}(G, F') \simeq \mathbb{C}[\Lambda_+]$ on $\mathcal{M}(\text{Bun})$. We denote by $\mathbb{T}_{\lambda,c}$ the operator corresponding to $h_{\lambda,c}$ (more generally, we denote by $\mathbb{T}_{h,c}$ the operator corresponding to any $h \in \mathcal{H}(G, F')$). For different choices of c , these actions commute. Therefore, one can try to study eigenvectors of all these operators in $\mathcal{M}(\text{Bun})$.

Remark 5.7. Recall that the operators $\mathbb{T}_{\lambda,c}$ are canonically defined only up to a scalar; canonically each $\mathbb{T}_{\lambda,c}$ is an operator from $\mathcal{M}(\text{Bun})$ to $\mathcal{M}(\text{Bun}) \otimes |\omega_{\mathcal{E},c}|^{(\lambda,\rho^\vee)}$. Therefore when we vary c each eigenvalue gives rise to a section of $|\omega_{\mathcal{E}}|^{-(\lambda,\rho^\vee)}$. This will not be important for us until the end of Section 6 (where it will in fact become quite crucial).

Note that $\mathcal{M}(\text{Bun})$ is an analog of the space of functions with finite support on $\text{Bun}(k)$ (where k is a finite field). But unlike in the case of finite fields, we expect the following (some philosophical reasons for this difference are discussed in the next subsection):

Conjecture 5.8. Assume that F is non-archimedean. Then the space $\mathcal{M}(\text{Bun})$ has a basis of Hecke eigenvectors. Similarly, in the archimedean case, the space $\mathcal{M}(\text{Bun})$ has a topological basis of Hecke eigenvectors.

Before we try to say something about the eigenvalues, let us discuss a slightly different version of Hecke operators.

5.6. Example

We now want to explain how to produce some Hecke eigenfunction using the construction of Section 2.4.

In the case when F is non-archimedean and that \mathcal{C} is defined over \mathcal{O}_F , i.e., we choose a model $\mathcal{C}_{\mathcal{O}_F}$ of \mathcal{C} over \mathcal{O}_F . We assume that $\mathcal{C}_{\mathcal{O}_F}$ is a regular scheme and we denote by \mathcal{C}_k the corresponding curve over k . Then the stack Bun is canonically defined over \mathcal{O}_F , and we have the map $E_{\text{Bun},1/2} : \mathcal{S}(\text{Bun}(k)) \rightarrow \mathcal{M}(\text{Bun})$ (see Section 2.4).

We claim that this map commutes with the Hecke operators in the appropriate sense. Namely, let F' be a finite Galois extension of F with ring of integers $\mathcal{O}_{F'}$ and residue field k' . Then one can construct a homomorphism $\gamma_{F'} : \mathcal{H}(G, F') \rightarrow \mathcal{H}(G, k')$ with the following property. Let c be a closed point of \mathcal{C} whose residue field is F' . Note that $\mathcal{C}(F') = \mathcal{C}(\mathcal{O}_{F'})$, so c has canonical reduction \bar{c} which is a closed point of \mathcal{C}_k with residue field k' . Then for any $h \in \mathcal{H}(F, F')$ and for any $\phi \in \mathcal{S}(\text{Bun}(k))$ we have

$$E_{\text{Bun},1/2}(T_{\gamma_{F'}(h)}(\phi)) = \mathbb{T}_h(E_{\text{Bun},1/2}(\phi)). \quad (4)$$

Remark 5.9. We do not know how to describe the map $\gamma_{F'}$ in general. It is easy to see that $\gamma_{F'}(h_\lambda)$ is supported on $\overline{\text{Gr}}_G^\lambda(k')$ (when viewed as a function on $\text{Gr}_G(k')$). But this information is sufficient only in the case when $G = \text{PGL}_n$ when minuscule coweights generate Λ .

Equation (4) implies that $E_{\text{Bun},1/2}$ sends Hecke eigenfunctions to Hecke eigenfunctions. This operator is certainly not injective, but we expect it to be injective on cuspidal functions. More precisely (assuming the validity of Conjecture 3.5), we formulate the following

Conjecture 5.10. Assume the validity of Conjecture 3.5. Then the composition of $\iota_{1/2} \circ E_{\text{Bun},1/2}$ is unitary on cuspidal functions.

Conjecture 5.10 implies that we can attach a nonzero Hecke eigenvector in $L^2(\text{Bun})$ to any *cuspidal* Hecke eigenfunction in $\mathcal{S}(\text{Bun}(k))$. On the other hand, we expect that the map $E_{\text{Bun},1/2}$ is highly noninjective on noncuspidal functions. For example, let $G = \text{PGL}_2$ and let $\mathcal{S}(\text{Bun}(k))_{\text{cusp}}^\perp$ denote the space of functions with finite support which are orthogonal to all cuspidal functions (with respect to the standard Hermitian product). This space is infinite-dimensional, but we expect that

$$\dim E_{\text{Bun},1/2}(\mathcal{S}(\text{Bun}(k))_{\text{cusp}}^\perp) = 1.$$

Note that equation (4) implies that the action of any $\mathbb{T}_{\lambda,c}$ on any section in the image of $E_{\text{Bun},1/2}$ depends only on \bar{c} (and not on c). This is certainly a very restrictive condition.

Also, one should think about $E_{\text{Bun},1/2}$ as some kind of Eisenstein series operator between the group $G(k)$ and the group $G(F)$ (with $G(\mathcal{O}_F)$ playing the role of a parabolic subgroup). This is in fact the source for our notation.

5.7. Parabolic bundles

We would like to introduce a generalization of the above setup, which allows in particular, to consider the case of curves of genus ≤ 1 when we may analyze some explicit nontrivial examples.

- Definition 5.11.**
- (1) Let us denote by Fl the variety of Borel subgroups of G .
 - (2) For a G -bundles \mathcal{F} on \mathcal{C} , we denote by $\text{Fl}_{\mathcal{F}}$ the associated Fl-bundle over \mathcal{C} .
 - (3) For a divisor $D \subset \mathcal{C}$ defined over k , we denote by Bun^D the stack of G -bundles \mathcal{F} on \mathcal{C} with a section a of $\text{Fl}_{\mathcal{F}}$ over D .

It is easy to extend the definition of the Hecke operators $\mathbb{T}_{\lambda,c}$ for $c \notin D$. All our constructions and conjectures can be extended to this case. As was noted above, considering parabolic points allows one to consider explicit examples. For example, in the case when $\mathcal{C} = \mathbb{P}^1$, D consists of at least 3 points and G is of rank 1, Conjecture 5.5 is proved in [8] (Proposition 3.13).

5.8. More spaces with Hecke action

5.8.1. The map $E_{\mathcal{Y},\kappa,n}$

Here we would like to discuss how to generalize the construction of Sections 2.4 and 5.6. Namely, let \mathcal{Y} be as in Section 2.4. Let $A_n = \mathcal{O}_F/\mathfrak{m}_F^n$. Let \mathcal{Y}_n denote the reduction of $\mathcal{Y}_{\mathcal{O}_F}$ modulo \mathfrak{m}_F^n . This is a regular stack over A_n . We consider the set $\mathcal{Y}_n(A_n)$ and we set $\mathcal{S}(\mathcal{Y}_n(A_n))$ to be the vector space of \mathbb{C} -valued functions on $\mathcal{Y}_n(A_n)$ with finite support. Then for any $\kappa \in \mathbb{C}$, we have the obvious map

$$\mathcal{S}(\mathcal{Y}_n(A_n)) \rightarrow \mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}) \simeq \mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}, |\omega_{\mathcal{Y}}^{\kappa}|).$$

Composing it with the natural map $\mathcal{S}(\mathcal{Y}_{\mathcal{O}_F}, |\omega_{\mathcal{Y}}^{\kappa}|) \rightarrow \mathcal{S}_{\kappa}(\mathcal{Y})$, we get a map

$$E_{\mathcal{Y},\kappa,n} : \mathcal{S}(\mathcal{Y}_n(A_n)) \rightarrow \mathcal{S}_{\kappa}(\mathcal{Y}).$$

It is easy to see that this map is surjective if the map $\mathcal{Y}(\mathcal{O}) \rightarrow \mathcal{Y}(F)$ is surjective; in particular, this is true for $\mathcal{Y} = \text{Bun}_G$ for a reductive group G .

In fact, when G is a reductive but not semisimple group, we also need the following variant of the definition of $\mathcal{S}_{\kappa}(Y)$ for $Y = \text{Bun}_G$.

Definition 5.12. Let G be a reductive group and Z the connected component of the center of G (so Z is a torus). For a character $\chi_n : \text{Bun}_{Z,n}(A_n) \rightarrow \mathbb{C}^{\times}$, we denote by $\mathcal{S}_{\chi_n}(\text{Bun}_{G,n}(A_n))$ the vector the space of $(\text{Bun}_{Z,n}(A_n), \chi_n)$ -coinvariants in $\mathcal{S}(\text{Bun}_{G,n}(A_n))$. Similarly, for $\chi : \text{Bun}_Z(F) = \text{Bun}_Z(\mathcal{O}) \rightarrow \mathbb{C}^{\times}$, we denote by $\mathcal{S}_{\kappa}(\text{Bun}_G)_{\chi}$ the space of $(\text{Bun}_Z(F), \chi)$ -coinvariants in $\mathcal{S}_{\kappa}(\text{Bun}_G)$.

As before we have a map $E_{\text{Bun}_G, \kappa, n, \chi} : \mathcal{S}_{\chi_n}(\text{Bun}_{G,n}(A_n)) \rightarrow \mathcal{S}_{\kappa, \chi}(\text{Bun}_G)$ provided that χ is equal to the pullback of κ_n under the natural map $\text{Bun}_Z(F) = \text{Bun}_Z(\mathcal{O}) \rightarrow \text{Bun}_{Z,n}(A_n)$.

5.8.2. Commutation with Hecke operators

We now want to specialize to the case $\mathcal{Y} = \text{Bun}_G$ and $\kappa = 1/2$. We claim that in this case the map $E_{\text{Bun}, 1/2, n}$ commutes with the Hecke operators in the sense similar to (4). To explain the formulation we first need to discuss an analog of the homomorphism γ_F ; this is a local question.

Namely, let us consider the ring $\mathcal{K}_n = A_n((t))$. This is a locally compact topological ring; its subring $\mathcal{O}_n = A_n[[t]]$ is open and compact. Thus the group $G(\mathcal{K}_n)$ is a totally disconnected locally compact topological group with an open compact subgroup $G(\mathcal{O}_n)$. Hence, we may consider the corresponding Hecke algebra

$$\mathcal{H}_n(G) = \mathcal{H}(G(\mathcal{K}_n), G(\mathcal{O}_n)).$$

Here is a variant. Let \mathcal{C} be a smooth projective curve over \mathcal{O}_F . We denote by \mathcal{C}_n its reduction mod \mathfrak{m}_F^n . Let F' be a finite unramified extension of F and let c be an F' -point of \mathcal{C} . As before we can also view it as an $\mathcal{O}_{F'}$ -point of \mathcal{C} and we denote by c_n its reduction modulo $\mathfrak{m}_{F'}^n$. This is an A'_n -point of \mathcal{C}_n . Then we might consider the corresponding Hecke algebra $\mathcal{H}_{c_n}(G)$. It is noncanonically isomorphic to $\mathcal{H}_n(G)$.

This Hecke algebra is quite bad: it is not commutative for $n > 1$ and apparently it does not have any reasonable description. However, it has the following two important features:

- (1) Let \mathcal{C} above and let c be a point of \mathcal{C} defined over a finite unramified extension F' of F . Then the (noncommutative) algebra $\mathcal{H}_{c_n}(G)$ acts on $\mathcal{S}(\text{Bun}_n)$. Given $h \in \mathcal{H}_{c_n}(G)$, we denote by $T_{h,n}$ the corresponding operator on $\mathcal{S}(\text{Bun}_n)$.
- (2) We have a canonical homomorphism $\gamma_{c,n} : \mathcal{H}_c(G) \rightarrow \mathcal{H}_{c_n}(G)$.
- (3) For any $h \in \mathcal{H}_c(G)$ and $\phi \in \mathcal{S}(\text{Bun}_n)$, we have

$$E_{\text{Bun}, 1/2, n}(\gamma_{c,n}(\mathbb{T}_{h,c})(\phi)) = \mathbb{T}_{h,c}(E_{\text{Bun}, 1/2, n}(\phi)). \quad (5)$$

5.8.3. Eigenfunctions and cuspidal functions: the idea

Definition 5.13. Let χ be a unitary character of $\text{Bun}_Z(A_n)$.

- (1) A function $\phi \in \mathcal{S}(\text{Bun}_{G,n})_\chi$ is cuspidal if the span of $\{T_{h,n}(\phi)\}$, $h \in \mathcal{H}_c(G)$, $c \in \mathcal{C}_n$ is finite dimensional.
- (2) $\mathcal{S}_{\text{cusp}}(\text{Bun}_{G,n})_\chi \subset \mathcal{S}(\text{Bun}_n)_\chi$ is the subspace of cuspidal functions.

Conjecture 5.14. $\mathcal{S}_{\text{cusp}}(\text{Bun}_n)_\chi$ is finite dimensional for any n and $\dim(\mathcal{S}_{\text{cusp}}(\text{Bun}_n)_\chi) \sim q^n \dim(\text{Bun}_{G/Z})$ for $q \gg 1$.

Remark 5.15. Such that Hecke operators $\gamma_{c,n}(\mathbb{T}_{h,c})$ are self-adjoint with respect to the natural Hermitian form on $\mathcal{S}_{\text{cusp}}(\text{Bun}_n)_\chi$.

Since $\text{Bun}_G(\mathcal{O})$ maps surjectively to $\text{Bun}_G(F)$, it follows from the statement at the end of Section 5.8.1 that

$$\mathcal{S}_\kappa(\text{Bun}_G) = \bigcup_n E_{\text{Bun}_G, \kappa, n}(\mathcal{S}(\text{Bun}_{G,n}(A_n)))$$

(and a similar statement holds for the space $\mathcal{S}_{\kappa, \chi}(\text{Bun}_G)$). We can now define

$$\mathcal{S}_{\kappa, \text{cusp}}(\text{Bun}_G) = \bigcup_n E_{\text{Bun}_G, \kappa, n}(\mathcal{S}_{\text{cusp}}(\text{Bun}_{G,n}(A_n)))$$

(and again similarly for $\mathcal{S}_{\kappa, \chi, \text{cusp}}(\text{Bun}_G)$). Note that for $\kappa = 1/2$ this space is locally finite dimensional with respect to the Hecke operators.

5.9. The case of $G = \text{GL}_2$

The proof of Conjecture 5.14 in the case $G = \text{GL}_2$ will appear in a forthcoming publication by A. Braverman, D. Kazhdan, and A. Polishchuk. In this subsection we outline a notion of the *constant term* used in our proof Conjecture 5.14 (again in the case when $G = \text{GL}_2$; for simplicity, we shall also restrict ourselves to the case $n = 2$). This notion is used for a different (but equivalent) definition of cuspidality.

5.9.1. The constant term in the usual case

Recall that the usual constant term operator (for $n = 1$) is defined as follows. Let P be a parabolic subgroup of G ; it has a natural homomorphism to M – the Levi factor. Consider the diagram

$$\begin{array}{ccc} \text{Bun}_P(k) & \xrightarrow{p} & \text{Bun}_G(k) \\ q \downarrow & & \\ \text{Bun}_M(k) & & \end{array} \tag{6}$$

Then the constant term $\mathbf{c}_{G,P}$ is equal to $q! \circ p^*$ (when k is a finite field).

Claim 5.16. A function ϕ on $\text{Bun}_G(k)$ is cuspidal in the sense of Definition 5.13 if and only if $\mathbf{c}_{G,P}(\phi) = 0$ for all parabolic subgroups P of G ; let $\mathcal{S}_{\text{cusp}}(\text{Bun}_G(k))$ be the space of all cuspidal functions.

The following facts are well known and are easy to prove:

- (1) $\mathcal{S}_{\text{cusp}}(\text{Bun}_G(k))$ is invariant under the Hecke operators.
- (2) $\mathcal{S}_{\text{cusp}}(\text{Bun}_G(k))$ consists of functions with finite support.
- (3) $\dim \mathcal{S}_{\text{cusp}}(\text{Bun}_G(k)) < \infty$ if G is semisimple. More generally, $\dim \mathcal{S}_{\text{cusp}, \chi}(\text{Bun}_G(k)) < \infty$ if G is reductive and χ is a character of Bun_{Z^0} where Z^0 is the connected component of identity of the center of G .

We expect that for a proof of Conjecture 5.14, one has to extend the definition of a *constant term* onto the space $\mathcal{S}(\mathrm{Bun}_n)_\chi$.

The definition is not completely straightforward; as was mentioned above, we shall only discuss the case $G = \mathrm{GL}_2$ and $n = 2$. So, we shall now assume that $G = \mathrm{GL}_2$ and again just write Bun instead of $\mathrm{Bun}_{\mathrm{GL}_2}$. Also in this case the only proper parabolic subgroup up to conjugacy is the Borel subgroup; we shall also denote the corresponding constant term operator (that we are going to define) simply by $\mathbf{c}^{(2)}$.

In this case $T = \mathbb{G}_m \times \mathbb{G}_m$, so $\mathrm{Bun}_T(\mathcal{C}_2) = \mathrm{Pic}(\mathcal{C}_2) \times \mathrm{Pic}(\mathcal{C}_2)$, so it would be natural to expect that our constant term operator $\mathbf{c}^{(2)}$ maps functions on $\mathrm{Bun}(\mathcal{C}_2)$ to functions on $\mathrm{Pic}(\mathcal{C}_2) \times \mathrm{Pic}(\mathcal{C}_2)$. However, we do not know how to define such an operator if we want it to commute with the Hecke operators in some reasonable sense. Instead, let us do the following. Consider the semigroup Pic'_2 (which contains the Picard group Pic_2 of \mathcal{C}_2). By definition, Pic'_2 consists of coherent sheaves \mathcal{M} on \mathcal{C} such that $t\mathcal{M} \neq 0$ and there exists an imbedding $\mathcal{M} \hookrightarrow \mathcal{L}$ where \mathcal{L} is a line bundle on \mathcal{C} . The tensor product defines the semigroup structure on Pic'_2 .

Example 5.17. Let $\mathcal{C} = \mathrm{Spec}(A_2[x])$, $J \subset A_2[x]$ be the maximal ideal generated by (x, t) , and \mathcal{J} the corresponding sheaf on \mathcal{C} . Then $\mathcal{J} \otimes \mathcal{J} = \mathcal{I}$ where $I \subset xA_2[x]$ is generated by (x^2, tx) .

We would like now to define an analog of the diagram (6). Namely, we consider the diagram

$$\begin{array}{ccc} \mathrm{Bun}'_B(\mathcal{C}_2) & \xrightarrow{p_2} & \mathrm{Bun}_2 \\ q_2 \downarrow & & \\ \mathrm{Pic}'_2 \times \mathrm{Pic}'_2 & & \end{array} \quad (7)$$

where $\mathrm{Bun}'_B(\mathcal{C}_2)$ consists of all short exact sequences

$$0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{L}_2 \rightarrow 0,$$

where $\mathcal{F} \in \mathrm{Bun}_2$, $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}'_2$. It is easy to see that in this case we have $\mathcal{L}_i = \underline{\mathrm{Hom}}(\mathcal{L}_j, \det(\mathcal{F}))$ for $i, j = 1, 2$ and $i \neq j$. So if we fix $\det(\mathcal{F})$ and one of the bundles \mathcal{L}_1 of \mathcal{L}_2 , this determines the isomorphism class of the other.

Theorem 5.18. (1) *The space $\mathcal{S}_{\mathrm{cusp}}(\mathrm{Bun}_2)$ of Conjecture 5.14 is the space of functions $\phi \in \mathcal{S}(\mathrm{Bun}_2)$ such that $(q_2)_! p_2^*(\phi) = 0$.*

(2) *$\dim \mathcal{S}_{\mathrm{cusp}, \chi}(\mathrm{Bun}_2) < \infty$ for any unitary character $\chi : \mathrm{Pic}_2 \rightarrow \mathbb{C}^*$. In fact, Conjecture 5.14 holds in this case.*

The proof will be discussed in another publication. Let us note that it is not difficult to deduce the second assertion of Theorem 5.18 from the first.

5.10. Main question

Assuming the above conjectures, one can ask how to describe the Hecke eigenvalues. It would be extremely interesting to relate them to some kind of Galois data (involving

the dual group G^\vee). At the moment, we do not know how to do it in the non-archimedean case even for $G = \mathrm{GL}_2$ when we defined an action of these operators of finite-dimensional spaces $E_{\mathrm{Bun}, 1/2, n}(\mathcal{S}_{\mathrm{cusp}, \chi}(\mathrm{Bun}_n))$.

In the archimedean case, a precise conjecture of this sort is formulated in [9] and [10]. We discuss it in the next section.

6. THE CASE $F = \mathbb{C}$

6.1. From Hecke operators to differential operators: the idea

In this section we specialize to the case $F = \mathbb{C}$. In this case, in addition to Hecke operators, one can introduce another player, namely the algebra of twisted (polynomial) differential operators on Bun , which will, roughly speaking, act on the same space as the Hecke operators and the two actions will commute. This will allow us to formulate a variant of Langlands conjecture in this case. More precisely, we are going to relate the Hecke eigenvalues to some particular G^\vee -local systems on \mathcal{C} – opers with real monodromy. Let us begin by recalling basic information about opers and differential operators on Bun .

6.2. Opers

For a principal G^\vee -bundle \mathcal{G} on \mathcal{C} , we denote by $\mathrm{Fl}_{\mathcal{G}}$ the associated Fl -bundle on \mathcal{C} where Fl is the variety of Borel subgroups of G^\vee .

Definition 6.1. (1) A G^\vee -oper on \mathcal{C} is a triple (\mathcal{G}, ∇, s) , where \mathcal{G} is a principal G^\vee -bundle on \mathcal{C} , ∇ is a connection on \mathcal{G} , and s is a section of $\mathrm{Fl}_{\mathcal{G}}$ satisfying an analog of the Griffiths-type condition with respect to ∇ (see [4]). We denote by $\mathrm{Oper}_{G^\vee}(\mathcal{C})$ the variety of opers.¹⁰

(2) For an oper $o = (\mathcal{F}, \nabla, s)$, we denote by $\zeta_o : \pi_1(\mathcal{C}) \rightarrow G^\vee(\mathbb{C})$ the morphism defined by the connection ∇ . We denote by $\mathrm{Oper}_G^\vee(\mathcal{C})^{\mathbb{R}} \subset \mathrm{Oper}_G^\vee(\mathcal{C})$ the subset of opers o such that the homomorphisms ζ_o and $\bar{\zeta}_o$ are conjugate, where $\bar{\cdot} : G^\vee(\mathbb{C}) \rightarrow G^\vee(\mathbb{C})$ is the complex conjugation corresponding to a choice of a split real form of G^\vee .

Let us make several comments. First, it is known (cf. [3]) that given just a pair (\mathcal{G}, ∇) , the B structure s is unique if it exists. Thus $\mathrm{Oper}_{G^\vee}(\mathcal{C})$ is actually a closed subset of the moduli stack of G^\vee -bundles with a connection (in other words, for such a local system to be an oper is a property rather than a structure). Second, let us comment on the reality condition in (2). Obviously, one way to guarantee this condition is to require that the monodromy representation of $\pi_1(\mathcal{C})$ corresponding to (\mathcal{G}, ∇) is conjugate to a homomorphism going

10 If G^\vee is adjoint then the moduli stack of opers is, in fact, an algebraic variety (which is isomorphic to an affine space of dimension $\mathrm{rk}(G)$). If G^\vee is not adjoint then formally one needs to consider the coarse moduli space here, since the center Z of G^\vee is equal to the group of automorphisms of every oper. We shall ignore this subtlety for the rest of this section.

into $G^\vee(\mathbb{R})$ for a real split form of G^\vee . We expect that the converse is also true, and this is proved for $G^\vee = \mathrm{SL}(2)$ in [10] (Remark 1.8), but we do not know how to prove this in general. However, it is not hard to see (cf. again [10]) that up to conjugation the image of the monodromy homomorphism $\pi_1(\mathcal{C}) \rightarrow G^\vee$ lies in some inner form of the split real form of G^\vee . When we are in the setup of Section 5.7 and $|D| \geq 1$ it is also shown in [10] that the monodromy lies in the split real form of G^\vee .

6.3. Opers and differential operators

Let \mathcal{D} be the algebra of global sections of the sheaf $\mathbb{D}_{1/2}(\mathrm{Bun})$ of regular differential operators on $\omega_{\mathrm{Bun}}^{1/2}$. We denote by $\tau : \mathcal{D} \rightarrow \mathcal{D}$ the involution on \mathcal{D} induced by the Cartan involution of G .

The following statement is one of the main results of [3] (a local version of this result appears in [13]).

Theorem 6.2. (1) *The algebra \mathcal{D} is commutative.*

(2) $\mathrm{Spec}(\mathcal{D}) = \mathrm{Oper}_{G^\vee}(\mathcal{C})$.

(3) *Let $o \in \mathrm{Oper}_{G^\vee}(\mathcal{C})$ and let $\chi_o : \mathcal{D} \rightarrow \mathbb{C}$ be the corresponding homomorphism. Let also $I_o \subset \mathbb{D}_{1/2}(\mathrm{Bun})$ be the sheaf of ideals of $\mathbb{D}_{1/2}(\mathrm{Bun})$ generated elements of the form $d - \chi_o(d)$ where $d \in \mathcal{D}$. Then the $\mathbb{D}_{1/2}(\mathrm{Bun})$ -module $M_o := \mathbb{D}_{1/2}(\mathrm{Bun})/I_o$ is $\mathcal{O}_{\mathrm{Bun}}$ -coherent when restricted to $\mathrm{Bun}_{\mathrm{vs}}$.¹¹*

6.4. Differential operators and Hecke operators

Recall that we denote by $C_{1/2}^\infty(\mathrm{Bun}_{\mathrm{vs}})$ the space of smooth 1/2-forms on $\mathrm{Bun}_{\mathrm{vs}}$. The algebra $\mathcal{A} := \mathcal{D} \otimes \overline{\mathcal{D}}$ acts naturally on $C_{1/2}^\infty(\mathrm{Bun}_{\mathrm{vs}})$. We denote by $\hat{\tau}$ the involution on \mathcal{A} such that $\hat{\tau}(d_1 \otimes \bar{d}_2) = d_2^\tau \otimes \bar{d}_1^\tau$ and define $\mathcal{A}^{\mathbb{R}} \subset \mathcal{A}$ as the subalgebra of $\hat{\tau}$ -fixed points.

We would like to claim that the action of the algebra \mathcal{A} on 1/2-forms commutes with the action of the Hecke operators. Here we must be careful, a priori it is not clear how to construct one vector space on which both algebras will act. For this, we need to formulate one more definition.

Let us define a space $\mathcal{S}ch(\mathrm{Bun})$ – “the Schwartz space of Bun .” Namely, we set

$$\mathcal{S}ch(\mathrm{Bun}) = \{\phi \in C_{1/2}^\infty(\mathrm{Bun}_{\mathrm{vs}}) \mid a(\phi) \in L^2(\mathrm{Bun}) \text{ for any } a \in \mathcal{A}\}. \quad (8)$$

For $a \in \mathcal{A}$, we denote by \hat{a} the induced endomorphism of $\mathcal{S}ch(\mathrm{Bun})$.

Note that by definition $\mathcal{S}ch(\mathrm{Bun}) \subset L^2(\mathrm{Bun})$ and also $\mathcal{S}(\mathrm{Bun}_{\mathrm{vs}}) \subset \mathcal{S}ch(\mathrm{Bun})$. The reader might ask why we start with C^∞ -forms on $\mathrm{Bun}_{\mathrm{vs}}$ rather than on $\mathrm{Bun}_{\mathrm{st}}$. The reason is that below we want to study eigenvectors of \mathcal{A} on $\mathcal{S}ch(\mathrm{Bun})$, and it follows easily from Theorem 6.2(3) that any such eigenvector is automatically smooth on $\mathrm{Bun}_{\mathrm{vs}}$ (but there is no reason for it to be smooth on $\mathrm{Bun}_{\mathrm{st}}$).

¹¹ Part (3) of this theorem explains the reason for our belief in Conjecture 3.5(2).

- Conjecture 6.3.** (1) Any $a \in \mathcal{A}^{\mathbb{R}}$ extends to an (unbounded) self-adjoint operator on $L^2(\text{Bun})$.
- (2) The space $\mathcal{S}ch(\text{Bun})$ is stable under the action of all Hecke operators.
- (3) $\mathcal{S}ch(\text{Bun}) = \iota_{1/2}(\mathcal{S}_{1/2}(\text{Bun}))$.¹²
- (4) The action of \mathcal{A} on $\mathcal{S}ch(\text{Bun})$ commutes with the action of Hecke operators.
- (5) There exists a dense (in the L^2 -sense) subspace $\mathcal{S}ch(\text{Bun})_0$ of $\mathcal{S}ch(\text{Bun})$ which is stable under \mathcal{A} and the Hecke operators and such that $\mathcal{S}ch(\text{Bun})_0$ is a direct sum of 1-dimensional eigenspaces for \mathcal{A} (in other words, the space $\mathcal{S}ch(\text{Bun})_0$ is locally finite dimensional for \mathcal{A} and every generalized eigenvalue has multiplicity 1).

Let us comment on the multiplicity 1 statement. A $1/2$ -form is actually an eigenvector if it satisfies a certain system of linear differential equations. Locally on $\text{Bun}_{\text{vs}}(\mathbb{C})$, the space of solutions is finite dimensional but certainly not one dimensional (this has to do with the fact the \mathbb{D} -module M_o has high rank on Bun_{vs} ; for example, for SL_2 this rank is 2^{3g-3}). However, globally most of these solutions become multivalued, so the multiplicity-one conjecture says that only one-dimensional space of solutions is single-valued globally. This, in fact, would follow if we knew that the \mathbb{D} -module M_o was irreducible and had regular singularities. For $G = \text{PGL}_2$, this can be deduced from [20] (and probably similar analysis can be carried over for PGL_n).

Conjecture 6.3 implies that $L^2(\text{Bun})$ is a (completed) direct sum of eigenspaces for \mathcal{A} and eigenvalues have multiplicity 1. A priori any such eigenvalue is given by a pair of opers (o, o') , but part (1) of Conjecture 6.3 implies that $o' = \overline{o}$, so we are supposed to attach an eigenspace to a single oper o . It is also not difficult to see that $o \in \text{Oper}_{G^\vee}(\mathcal{C})^{\mathbb{R}}$. We denote the corresponding eigenspace by $L^2(\text{Bun})_o$. Note that Conjecture 6.3 implies that $L^2(\text{Bun})_o \subset \mathcal{S}ch(\text{Bun})$.

Conjecture 6.4. We have $L^2(\text{Bun})_o \neq 0$ if and only if $o \in \text{Oper}_{G^\vee}(\mathcal{C})^{\mathbb{R}}$.

Remark 6.5. As was remarked above, the “only if” direction is easy. What is hard is to prove existence of eigenvectors for \mathcal{A} which lie in L^2 .

Note that Conjectures 5.5, 6.3, and 6.4 together imply the following

Corollary 6.6. *Let \mathcal{W} denote the set of Hecke eigenvalues on $L^2(\text{Bun})$. Then there exists a surjective map $\eta : \text{Oper}_{G^\vee}(\mathcal{C})^{\mathbb{R}} \rightarrow \mathcal{W}$ such that for any $c \in \mathcal{C}$ and any $h \in \mathcal{H}(G, \mathbb{C})$ the operator $\mathbb{T}_{h,c}$ acts on $L^2(\text{Bun})_0$ by $\eta(o)(h)$.*

Let us comment on the connection between Corollary 6.6 and Conjecture 5.5. We actually expect the map η to be finite-to-one (and in many cases it should be an isomorphism), so Conjecture 5.5 should imply that $\text{Oper}_{G^\vee}(\mathcal{C})^{\mathbb{R}}$ should be a discrete subset of $\text{Oper}_{G^\vee}(\mathcal{C})$.

¹² Note that if we assume the validity of Conjecture 3.5 for $F = \mathbb{C}$, then (3) implies (2).

This assertion is not obvious, and at the moment we do not know how to prove it in general, but let us note that for $G^\vee = \mathrm{PGL}_2$ it was proven by G. Faltings in [11].

6.5. Eigenvalues of Hecke operators

We conclude this section by describing a conjectural formula for the map η (the contents of this subsection is described in more detail in [10]). More precisely, we are going to do the following. We would like to understand the scalar by which the operator $\mathbb{T}_{\lambda,c}$ acts in $L^2(\mathrm{Bun})_o$. We can actually regard c as a variable here. In view of Remark 5.7, this eigenvalue is, in fact, a section $\Phi_{\lambda,o}$ of $|\omega_{\mathcal{C},c}|^{-\langle \lambda, \rho^\vee \rangle}$ (recall that ρ^\vee denotes the half-sum of positive coroots of G).

For $\lambda \in \Lambda^+$, let V_λ be the corresponding irreducible finite-dimensional representation of G^\vee . Choose an $o = (\mathcal{F}, \nabla, s) \in \mathrm{Oper}_{G^\vee}(\mathcal{C})$. Moreover, the Griffiths transversality condition implies that the T^\vee -bundle induced from the B^\vee -structure s by means of the homomorphism $B^\vee \rightarrow T^\vee$ is induced from $\omega_{\mathcal{C}}$ by means of the cocharacter $\rho^\vee : \mathbb{G}_m \rightarrow T^\vee$.¹³ Therefore if we denote by $(\mathcal{V}_{o,\lambda}, \nabla_{o,\lambda})$ the vector bundle on \mathcal{C} associated to \mathcal{F} via the representation V_λ (with the corresponding flat connection), then s defines an embedding

$$\omega_{\mathcal{C}}^{\langle \lambda, \rho^\vee \rangle} \hookrightarrow \mathcal{V}_{o,\lambda}$$

and hence a morphism

$$\mathcal{O}_{\mathcal{C}} \hookrightarrow \omega_{\mathcal{C}}^{-\langle \lambda, \rho^\vee \rangle} \otimes \mathcal{V}_{o,\lambda}.$$

We let σ_λ be the image of 1 under this morphism.

Let now $o \in \mathrm{Oper}_{G^\vee}(C)^\mathbb{R}$. Then we have isomorphism of $\mathcal{V}_{o,\lambda}$ and $\overline{\mathcal{V}}_{o,\lambda}$ as flat C^∞ -bundles (and this isomorphism is canonical up to the action of the center of G^\vee). Since $V_\lambda^* \simeq V_{-w_0(\lambda)}$, we get a pairing $(\cdot, \cdot)_\lambda$ between C^∞ -sections of $\mathcal{V}_{o,\lambda}$ and of $\overline{\mathcal{V}}_{o,\lambda}$. Since $\langle -w_0(\lambda), \rho^\vee \rangle = \langle \lambda, \rho^\vee \rangle$, we can regard $\overline{\sigma}_{-w_0(\lambda)}$ as a section of $\overline{\omega}_{\mathcal{C}}^{\langle \lambda, \rho^\vee \rangle} \otimes \overline{\mathcal{V}}_{\lambda}^*$. Since $\omega_{\mathcal{C}}^{-\langle \lambda, \rho^\vee \rangle} \otimes \overline{\omega}_{\mathcal{C}}^{-\langle \lambda, \rho^\vee \rangle} = |\omega_{\mathcal{C}}|^{-\langle \lambda, \rho^\vee \rangle}$, we can formulate the following Conjecture (cf. [10]):

Conjecture 6.7.

$$\Phi_{\lambda,o} = (\sigma_\lambda, \overline{\sigma}_{-w_0(\lambda)})_\lambda \in C^\infty(\mathcal{C}, |\omega_{\mathcal{C}}|^{-\langle \lambda, \rho^\vee \rangle}).$$

6.6. Parabolic bundles: results

All the conjectures of this section can be easily generalized to the setup of Section 5.7. In the case when \mathcal{C} is \mathbb{P}^1 and the cardinality of the divisor D is 3, 4, or 5, they are proven in [8] (and most of them are proven in [8] even for $|D| > 5$).

¹³ Strictly speaking, this makes sense only if G^\vee is simply connected since ρ^\vee is a well-defined cocharacter of T^\vee only in that case. For general G , the corresponding T^\vee -bundle is induced from $\omega_{\mathcal{C}}^{1/2}$ by the character $2\rho^\vee$ for some choice of $\omega_{\mathcal{C}}^{1/2}$. To simplify the notation, we are going to write the answer in the case when G^\vee is simply connected – the generalization to any G is straightforward.

7. THE CASE $F = \mathbb{R}$

In this section we would like to describe the conjectural picture of the analytic Langlands correspondence in the case $F = \mathbb{R}$. This picture has been developed by P. Etingof, E. Frenkel, D. Gaiotto, D. Kazhdan, and E. Witten, and is discussed in [19, SECTION 6].

Warning. Some of the letters used in the previous section (such as σ or τ) will have a different meaning in this section.

7.1. Real groups, L -groups, and all that

Let G be a connected complex semisimple group. Recall that a *real structure* on G is defined by an antiholomorphic involution $\sigma : G \rightarrow G$. The corresponding *group of real points* is G^σ (it may be disconnected). The inner class of σ gives rise to a based root datum involution $s = s_\sigma$ for G which is also one for G^\vee . If G is semisimple, this is just a Dynkin diagram automorphism.

Recall [1] that to G, s we may attach the *Langlands L -group* ${}^L G = {}^L G_s$, the semidirect product of $\mathbb{Z}/2 = \text{Gal}(\mathbb{C}/\mathbb{R})$ by G^\vee , with the action of $\mathbb{Z}/2$ defined by $\gamma \circ s$, where γ is the Cartan involution.

7.2. L -systems

Let \mathcal{C} be a compact complex Riemann surface of genus $g \geq 2$. Let $\tau : \mathcal{C} \rightarrow \mathcal{C}$ be an antiholomorphic involution. Given a holomorphic principal G -bundle \mathcal{E} on \mathcal{C} , we can define the antiholomorphic bundle $\tau(\mathcal{E})$, hence a holomorphic bundle $\sigma\tau(\mathcal{E})$. Let us say that \mathcal{E} is *real* under σ if there exists an isomorphism $A : \mathcal{E} \rightarrow \sigma\tau(\mathcal{E})$ such that

$$\sigma\tau(A) \circ A = 1. \tag{9}$$

This isomorphism A is unique if it exists, and (9) is automatic if $\text{Aut}(\mathcal{E}) = 1$, which happens generically for stable bundles if G is adjoint. In this case, $gA : \mathcal{E} \rightarrow g\sigma\tau(\mathcal{E})$ has the same property for $\sigma' = g\sigma$, where $g \in G$ and $g\sigma(g) = 1$. Thus the moduli space of such stable bundles depends only on s [5, PROPOSITION 3.8]. We will denote it by $\text{Bun}_{G,s}$.

Consider the simplest case when τ has no fixed points, i.e., $\mathcal{C}(\mathbb{R}) = \emptyset$. Let ζ be a local system on the nonorientable surface \mathcal{C}/τ with structure group ${}^L G$. Let us say that ζ is an *L -system* if it attaches to every orientation-reversing path in \mathcal{C}/τ a conjugacy class in ${}^L G$ that maps to the nontrivial element in $\mathbb{Z}/2$. The following conjecture is formulated in [19, SECTION 6] (in the case of the compact inner class).

Conjecture 7.1. The spectrum of Hecke operators on $L^2(\text{Bun})$ is parametrized by L -systems on \mathcal{C}/τ with values in ${}^L G = {}^L G_s$ whose pullback to \mathcal{C} has a structure of a G -oper.

Example 7.2. Let $s = \gamma$. Then ${}^L G = \mathbb{Z}/2 \times G^\vee$, so an L -system is the same thing as a G^\vee -local system on \mathcal{C}/τ . So in this case the condition on the G^\vee -local system on \mathcal{C} to occur in the spectrum is (conjecturally) that it extends to the 3-manifold $M := (\mathcal{C} \times [-1, 1]) / (\tau, -\text{Id})$ whose boundary is \mathcal{C} (and this extension is a part of the data).

Namely, in this case the spectral local systems are ζ which are isomorphic to ζ^τ and such that ζ is an oper (hence also an anti-oper), so ζ is a real oper “with real coefficients.” But among these we should only choose those that extend to \mathcal{C}/τ (and then the multiplicity of eigenvalue may be related to the number of such extensions). This agrees with the picture [19, SECTION 6.2] coming from 4-dimensional supersymmetric gauge theory.¹⁴ More precisely, recall that by a result of Beilinson and Drinfeld [3], opers for adjoint groups have no nontrivial automorphisms. So for any connected semisimple G , we get an obstruction for such ζ to extend to \mathcal{C}/τ which lies in $Z/Z^2 = H^2(\mathbb{Z}/2, Z)$, where Z is the center of G^\vee .¹⁵ Moreover, if this obstruction vanishes then the freedom for choosing the extension is in a torsor over $H^1(\mathbb{Z}/2, Z) = Z_2$, the 2-torsion subgroup in Z .

Example 7.3. Let $G = K \times K$ for some complex group K , and s be the permutation of components (the only real form in this inner class is K regarded as a real group). This is equivalent to the case $F = \mathbb{C}$ considered above (for \mathcal{C} defined over \mathbb{R}). Then ${}^L G_s = {}^L G_{\gamma \circ s} = \mathbb{Z}/2 \ltimes (K^\vee \times K^\vee)$, where $\mathbb{Z}/2$ acts by permutation. So an L -system is a $K^\vee \times K^\vee$ local system on \mathcal{C} of the form (ρ, ρ^τ) . Thus the spectrum is parametrized by ζ such that both ζ and ζ^τ are opers, i.e., ζ is both an oper and an anti-oper, i.e. a real oper, which agrees with the conjecture for $F = \mathbb{C}$. (Note that in this case $H^i(\mathbb{Z}/2, Z) = 1$ so there is no obstructions or freedom for extensions).

Remark 7.4. If $\mathcal{C}(\mathbb{R}) \neq \emptyset$, the story gets more complicated, and we will not discuss the details here. Let us just indicate that, as explained in [19, SECTION 6], to define the appropriate moduli space and the spectral problem on it, we need to fix a real form G_i of G in the inner class s for each component (oval) C_i of $\mathcal{C}(\mathbb{R})$, and the eigenvalues of Hecke operators are conjecturally parametrized by a certain kind of “real” opers corresponding to this data, i.e., opers with real coefficients satisfying appropriate reality conditions on the monodromy of the corresponding G^\vee -connection. Furthermore, in the tamely ramified case, when we also have a collection of marked points D on \mathcal{C} defined over \mathbb{R} , to define the most general version of our spectral problem, we need to fix a unitary representation π_i of the real group G_i for every marked point $c \in D$ on C_i and a unitary representation of the complex group $G_{\mathbb{C}}$ for every pair of complex conjugate marked points $c, \bar{c} \in D$ not belonging to $\mathcal{C}(\mathbb{R})$. For example, the case of parabolic structures corresponds to taking s to be the split inner class, G_i the split forms, and π_i the unitary principal series representations. In the genus zero case,

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- 14 More precisely, as was explained to us by E. Witten, what comes from ordinary gauge theory is this picture for the compact inner class s . To obtain other inner classes, one needs to consider twisted gauge theory where the twisting is by a Dynkin diagram automorphism of G . Namely, gauge fields in this theory are invariant under complex conjugation τ up to such an automorphism.
 - 15 Indeed, $\pi_1(\mathcal{C}/\tau)$ is generated by $\pi_1(\mathcal{C})$ and an element t such that $tbt^{-1} = \beta(b)$ for some automorphism β of $\pi_1(\mathcal{C})$, and $t^2 = c \in \pi_1(\mathcal{C})$, so that $\beta^2(b) = cbc^{-1}$. So given a representation $\zeta : \pi_1(\mathcal{C}) \rightarrow G^\vee$, an L -system would be given by an assignment $\zeta(t) = T \in G^\vee$ such that (1) $T^2 = \zeta(c)$ and (2) $T\zeta(a)T^{-1} = \zeta(\beta(a))$. If $\zeta \cong \rho \circ \beta$ then T satisfying (2) is unique up to multiplying by $u \in Z$, and $T^2 = \zeta(c)z$, $z \in Z$. Moreover, if T is replaced by Tu then z is replaced by zu^2 , hence the obstruction to satisfying (1) lies in Z/Z^2 .

this was discussed in detail in [8], and it was shown that this problem leads to appearance of T -systems.

7.3. Connection to Gaudin model

Recall that the *Gaudin model* for a simple complex Lie algebra \mathfrak{g} is the problem of diagonalization of the *Gaudin hamiltonians*

$$H_i := \sum_{1 \leq j \leq N, j \neq i} \frac{\Omega_{ij}}{z_i - z_j}$$

on the space $(V_1 \otimes \cdots \otimes V_N)^\mathfrak{g}$, where V_i are finite-dimensional \mathfrak{g} -modules, $z_i \in \mathbb{C}$ are distinct points, $\Omega \in (S^2 \mathfrak{g})^\mathfrak{g}$ is the Casimir tensor dual to the Killing form, and Ω_{ij} denotes the action of Ω in the i th and j th factor. These operators commute, and if $\mathfrak{g} \neq \mathfrak{sl}_2$ then there are also higher Gaudin hamiltonians associated to the Feigin–Frenkel higher Sugawara central elements at the critical level (see [14]), which commute with each other and with H_i , and the problem is to simultaneously diagonalize all these operators.

It turns out that this problem (for real z_i) is a special case of the spectral problem considered in this paper, in the case $F = \mathbb{R}$. Namely, let us take $\mathcal{C} = \mathbb{P}^1$ with the usual real structure and fix the compact inner class s of the complex simply connected group G with $\text{Lie}(G) = \mathfrak{g}$. As explained in the previous remark, on the real locus $\mathbb{P}^1(\mathbb{R})$, we are supposed to fix a real form of G in this inner class, and we fix the compact form G_c . Further, consider marked points z_1, \dots, z_N on the real locus (the tamely ramified case). Then we are supposed to fix a unitary representation of G_c at every z_i , and we take it to be V_i . Then the Hilbert space of the analytic Langlands theory is $\mathcal{H} = (V_1 \otimes \cdots \otimes V_n)^{G_c}$ (so in this case it is finite dimensional), and the quantum Hitchin system comprises the Gaudin hamiltonians (including the higher ones), cf. [14].

As is explained in [15, 17], the Bethe ansatz method shows that the eigenvectors of the Gaudin hamiltonians are labeled by monodromy-free G^\vee -opers on \mathbb{P}^1 with first-order poles at z_i and residues in the conjugacy class of $-\lambda_i - \rho$, where λ_i is the highest weight of V_i . These are exactly the “real opers” for this situation. Thus the results of [15, 17] may be considered as a finite-dimensional instance of the tamely ramified analytic Langlands correspondence for genus zero and $F = \mathbb{R}$.

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