EVOLUTION OF FORM AND SHAPE

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ABSTRACT

The evolution of form and shape can be described by differential equations. Many of these equations originate in various branches of science and engineering. They are fundamental and in a sense canonical. The fact that they make sense geometrically means that they are relevant everywhere and have fundamental properties that appear over and over in many settings. Understanding them requires simultaneous insight into analysis and geometry and the interplay between these.

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1. INTRODUCTION

The evolution of form and shape can be described by differential equations. These equations are classical, and those we will consider are variants of the heat equation that governs how heat distributes over time. The questions and equations, many of which originate in various branches of science and engineering, are fundamental and in a sense canonical, and as a consequence come up in many areas. The Laplace equation, for example, is the canonical linear second order partial differential equation once we have a metric structure. The Laplace operator appears classically in the physics of gravity, electricity and magnetism, fluid mechanics, and quantum mechanics, it has played a central role in many areas of mathematics, and its study in increasing generality played a central role in the development of the theory of PDEs. The fact that the equations make sense geometrically means that they are relevant everywhere in physical settings, and they have certain fundamental properties that appear over and over. Understanding them requires simultaneous insight into analysis and geometry, and the interplay between these. The new ideas and techniques to deal with these questions apply to many different situations. Recent years have seen dramatic progress on many of these questions thanks to the combined efforts of many people with different points of views and techniques. The goal here is to give a flavor of some of these results.

The first equation we will consider is mean curvature flow of hypersurfaces. Surface tension is the tendency of fluid surfaces to shrink into the smallest surface area possible. Mathematically, the force of surface tension is described by the mean curvature. In equilibrium the mean curvature is zero and one gets minimal surfaces. Minimal surfaces date back to Euler and Lagrange and the beginning of the calculus of variations. Many of the techniques developed have played key roles in geometry and partial differential equations. Examples include monotonicity and tangent cone analysis originating in the regularity theory for minimal surfaces, estimates for nonlinear equations based on the maximum principle arising in Bernstein's classical work, and even Lebesgue's definition of the integral that he developed in his thesis on the Plateau problem for minimal surfaces.

Under mean curvature flow, the surface moves to decrease surface area as fast as possible. If we think of the hypersurface as the level set of a function and insist that all level sets move by mean curvature flow, then this gives rise to a nonlinear degenerate parabolic PDE on a Euclidean space. This is the level set formulation of the equation. The level set method has been intensively studied in many pure and applied fields over the last 35 years. One of the first questions that comes up is the regularity of solutions. The equation is degenerate and a priori solutions are only defined weakly. We will see that the regularity of solutions is equivalent to a question that has been widely studied in geometry over the last 40 years, namely, the question of uniqueness of blowups. This is very much in the spirit of the simple fact that a function is differentiable at a point if, at all sufficiently small scales, it not only looks like a linear function but the same linear function independent of scale.

As growth of solutions to PDEs plays an important role in many different areas, we will discuss the growth of some classical and basic equations on manifolds. These include harmonic and caloric functions. That is, functions that are either solutions to the Laplace equation or the heat equation. We will also discuss more general eigenfunctions of drift equations. Drift Laplacians are ubiquitous in many areas, including quantum field theory, stochastic PDE, and anywhere the heat equation or Gaussian appears, such as functional inequalities, parabolic PDEs, geometric flows, and probability. The drift term arises in two different ways. One is whenever there is a natural scaling or, more generally, a gradient flow. A second way it arises is when there is a natural measure, in which case the drift operator is the canonical self-adjoint second order operator. There is a long history of studying the growth of solutions to differential equations, inequalities, and systems. These new growth estimates have direct application to longstanding open questions.

Analysis of noncompact manifolds almost always requires some controlled behavior at infinity. Without such, one can neither show nor expect strong properties. On the other hand, such assumptions restrict the possible applications and often too severely. In a wide range of areas, noncompact spaces come with a Gaussian weight and a drift Laplacian. Eigenfunctions are L^2 in the weighted space allowing for extremely rapid growth. Rapid growth would be disastrous for many applications. Surprisingly, for very general tensors, manifolds, and weights, we will show the same polynomial growth bounds that Laplace and Hermite observed for functions on a Euclidean space for the standard Gaussian. This covers all shrinkers for Ricci and mean curvature flows.

These new growth estimates for the PDEs open a door to study delicate analytical questions on a wide class of non-compact manifolds without assuming any asymptotic decay at infinity. They provide an analytic framework for investigating nonlinear PDE on Gaussian spaces where previously the Gaussian weight allowed wild growth that made it impossible to approximate nonlinear by linear. They are key to bound the growth of diffeomorphisms of noncompact manifolds and to solving the "gauge problem." Many key problems are defined intrinsically without a canonical coordinate system. In those problems, the infinite-dimensional diffeomorphism group (gauge group) becomes a major issue and dealing with it a major obstacle. Ricci flow is such an example. There are many problems where this degeneracy under diffeomorphisms plays a central role, but most techniques rely on compactness or rapid decay which we do not have in the situations we consider.

Another common feature for all of these problems is that they are dynamical and can be thought of as infinite-dimensional dynamical systems. Classical results from dynamics do not apply directly, but they do give some guiding principles, [85,88,92]. In mathematics, structural stability is a fundamental property of a dynamical system, which means that the qualitative behavior of the trajectories is unaffected by small perturbations. Given a smooth function f on a finite-dimensional space, the gradient ∇f points in the direction of the steepest ascent. The critical points of f are the points where ∇f vanishes. If p is a local minimum of f, then the second derivative test tells us that the Hessian matrix of f at p is nonnegative. More generally, the number of negative eigenvalues of the Hessian is called the index of the critical point. A fundamental method to find the minimum of f is the method of gradient descent. Here, we make an initial guess p_0 and then iteratively move in the negative gradient direction, the direction of the steepest descent, by setting $p_{i+1} = p_i - \nabla f(p_i)$. The function $f(x(t))$ decreases as efficiently as possible as $x(t)$ heads towards the minimum.

The dynamics near a nondegenerate critical point are determined by the index. If the index is zero, then the critical point is attracting and the entire neighborhood flows towards the critical point. However, when the index is positive, a generic point will flow out of the neighborhood, missing the critical point. In the final part we will discuss stable structures in geometry.

Part 1. Optimal regularity of PDEs. In mean curvature flow, the velocity vector field is the mean curvature vector and the evolving front is the level set of a function that satisfies a nonlinear degenerate parabolic equation. Solutions are defined in a weak, so-called "viscosity" sense; in general, they may not even be differentiable (let alone twice differentiable). However, it turns out that for a monotonically advancing front viscosity solutions are in fact twice differentiable and satisfy the equation in the classical sense. Moreover, the situation becomes very rigid when the second derivative is continuous.

Suppose $\Sigma \subset \mathbf{R}^{n+1}$ is an embedded hypersurface and $\mathbf{n}$ is the unit normal of Σ. The *mean curvature* is given by $H = \operatorname{div}_\Sigma(\mathbf{n})$. Here

$$\operatorname{div}_\Sigma(\mathbf{n}) = \sum_{i=1}^{n} \langle \nabla_{e_i} \mathbf{n}, e_i \rangle,$$

where e_i is an orthonormal basis for the tangent space of Σ. For example, at a point where $\mathbf{n}$ points in the x_{n+1} direction and the principal directions are in the other axis directions,

$$\operatorname{div}_\Sigma(\mathbf{n}) = \sum_{i=1}^{n} \frac{\partial \mathbf{n}_i}{\partial x_i}$$

is the sum (n times the mean) of the principal curvatures. If $\Sigma = u^{-1}(s)$ is the level set of a function u on $\mathbf{R}^{n+1}$ and s is a regular value, then $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ and

$$H = \sum_{i=1}^{n} \langle \nabla_{e_i} \mathbf{n}, e_i \rangle = \operatorname{div}_{\mathbf{R}^{n+1}} \left(\frac{\nabla u}{|\nabla u|} \right).$$

The last equality used that $\langle \nabla_{\mathbf{n}} \mathbf{n}, \mathbf{n} \rangle$ is automatically 0 because $\mathbf{n}$ is a unit vector.

A one-parameter family of smooth hypersurfaces $M_t \subset \mathbf{R}^{n+1}$ flows by the *mean curvature flow* if the speed is equal to the mean curvature and points inward:

$$x_t = -H\mathbf{n},$$

where H and $\mathbf{n}$ are the mean curvature and unit normal of M_t at the point x. Our flows will always start at a smooth embedded connected hypersurface, even if it becomes disconnected and nonsmooth at later times. The earliest reference to the mean curvature flow we know of is in the work of Birkhoff from the 1910s, where he used a discrete version of this, and independently in the material science literature of the 1920s.

Two key properties.

- H is the gradient of area, so the mean curvature flow is the negative gradient flow for volume (Vol M_t decreases most efficiently).
- (Avoidance property) If M_0 and N_0 are disjoint, then M_t and N_t remain disjoint.

The avoidance principle is simply a geometric formulation of the maximum principle. An application of it shows that if one closed hypersurface encloses another, then the outer one can never catch up with the inner. The reason for this is that if it did there would be a first point of contact, and right before that the inner one would contract faster than the outer, contradicting that the outer was catching up.

Curve shortening flow. When $n = 1$ and the hypersurface is a curve, the flow is the curve shortening flow. Under the curve shortening flow, a round circle shrinks through round circles to a point in finite time. A remarkable result of Grayson [103] from 1987 (using earlier work of Gage and Hamilton [100]) shows that any simple closed curve in the plane remains smooth under the flow until it disappears in finite time in a point. Right before it disappears, the curve will be an almost round circle.

Level set flow. The analytical formulation of the flow is the level set equation that can be deduced as follows. Given a closed embedded hypersurface $\Sigma \subset \mathbf{R}^{n+1}$, choose a function $v_0 : \mathbf{R}^{n+1} \to \mathbf{R}$ that is zero on Σ, positive inside the domain bounded by Σ, and negative outside. (Alternatively, choose a function that is negative inside and positive outside.)

- If we simultaneously flow $\{v_0 = s_1\}$ and $\{v_0 = s_2\}$ for $s_1 \neq s_2$, then avoidance implies they stay disjoint.
- In the level set flow, we look for $v : \mathbf{R}^{n+1} \times [0, \infty) \to \mathbf{R}$ so that each level set $t \to \{v(\cdot, t) = s\}$ flows by mean curvature and $v(\cdot, 0) = v_0$.
- If $\nabla v \neq 0$ and the level sets of v flow by mean curvature, then

$$v_t = |\nabla v| \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right).$$

This is degenerate parabolic and undefined when $\nabla v = 0$. It may not have classical solutions.

In a paper from 1988, Osher and Sethian [159] studied this equation numerically. The analytical foundation was provided by Evans and Spruck [98] in a series of four papers in the early 1990s and, independently and at the same time, by Chen, Giga, and Goto [41]; see also [5]. Both of these two groups constructed (continuous) viscosity solutions and showed uniqueness. The notion of viscosity solutions had been developed by Lions and Crandall in the early 1980s. The work of these two groups on the level set flow was one of the significant applications of this theory.

Examples of singularities. Under mean curvature flow, a round sphere remains round but shrinks and eventually becomes extinct in a point. A round cylinder remains round and eventually becomes extinct in a line. The marriage ring is the example of a thin torus of revolution in $\mathbf{R}^3$. Under the flow, the marriage ring shrinks to a circle then disappears.

Dumbbell. If the neck is sufficiently thin, then under the evolution the neck of a rotationally symmetric mean convex dumbbell in $\mathbf{R}^3$ pinches off first and the surface disconnects into two components. Later each component (bell) shrinks to a round point. This example falls into a larger category of surfaces that are rotationally symmetric around an axis. Because of the symmetry, then the solution reduces to a one-dimensional heat equation. This was analyzed already in the early 1990s by Angenent, Altschuler, and Giga [4]; cf. also the work of Soner and Souganidis from around the same time. A key tool in the arguments of Angenent–Altschuler–Giga was a parabolic Sturm–Liouville theorem of Angenent that holds in one spatial dimension.

Singular set. Under mean curvature flow, closed hypersurfaces contract, develop singularities, and eventually become extinct. The *singular set* $\mathcal{S}$ is the set of points in space and time where the flow is not smooth.

In the first three examples—the sphere, cylinder, and marriage ring—$\mathcal{S}$ is a point, line, and closed curve, respectively. In each case, the singularities occur only at a single time. In contrast, the dumbbell has two singular times with one singular point at the first time and two at the second.

Mean convex flows. A hypersurface is convex if every principal curvature is positive. It is mean convex if $H > 0$, i.e., if the sum of the principal curvatures is positive at every point. Under the mean curvature flow, a mean convex hypersurface moves inward and, since mean convexity is preserved, it will continue to move inward and eventually sweep out the entire compact domain bounded by the initial hypersurface.

Monotone movement can be modeled particularly efficiently numerically by the Fast Marching Method of Sethian.

Level set flow for mean convex hypersurfaces. When the hypersurfaces are mean convex, the equation can be rewritten as a degenerate elliptic equation for a function u defined by

$$u(x) = \{t \mid x \in M_t\}.$$

We say that u is the *arrival time* since it is the time the hypersurfaces M_t arrive at x as the front sweeps through the compact domain bounded by the initial hypersurface. Kohn and Serfaty [131] provided a game theoretic interpretation of the arrival time. It follows easily that if we set $v(x,t) = u(x) - t$, then v satisfies the level set flow. Now the level set equation $v_t = |\nabla v| \operatorname{div}(\nabla v/|\nabla v|)$ becomes

$$-1 = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

This is a degenerate elliptic equation that is undefined when $\nabla u = 0$. Note that if u satisfies this equation, then so does u plus a constant. This just corresponds to shifting the time when the flow arrives by a constant. A particular example of a solution to this equation is the function $u = -\frac{1}{2}(x_1^2 + x_2^2)$, that is, the arrival time for shrinking round cylinders in $\mathbf{R}^3$. In general, Evans–Spruck (cf. Chen–Giga–Goto) constructed Lipschitz solutions to this equation.

Singular set of mean convex level set flow. The singular set of the flow is the critical set of u. Namely, $(x, u(x))$ is singular if and only if $\nabla_x u = 0$. For instance, in the example of the shrinking round cylinders in $\mathbf{R}^3$, the arrival time is given by $u = -\frac{1}{2}(x_1^2 + x_2^2)$ and the flow is singular in the line $x_1 = x_2 = 0$; that is, exactly where $\nabla u = 0$.

We will next see that even though the arrival time was only a solution to the level set equation in a weak sense, it always turns out to be a twice differentiable classical solution.

Differentiability [79,80].

- u is twice differentiable everywhere, with bounded second derivatives, and smooth away from the critical set.
- u satisfies the equation everywhere in the classical sense.
- At each critical point, the Hessian is symmetric and has only two eigenvalues 0 and $-\frac{1}{k}$; $-\frac{1}{k}$ has multiplicity $k+1$.

This result is equivalent to saying that at a critical point, say $x=0$ and $u(x)=0$, the function u is (after possibly a rotation of $\mathbf{R}^{n+1}$) up to higher order terms equal to the quadratic polynomial

$$-\frac{1}{k}\left(x_1^2+\cdots+x_{k+1}^2\right).$$

This second-order approximation is simply the arrival time of the shrinking round cylinders. It suggests that the level sets of u right before the critical value and near the origin should be approximately cylinders (with an $(n-k)$-dimensional axis). This has indeed been known for a long time and is due to Huisken [114–116], White [182–184], Huisken–Sinestrari [117,118], Andrews [8], and Haslhofer–Kleiner [109]. It also suggests that those cylinders should be nearly the same (after rescaling to unit size). That is, the axis of the cylinders should not depend on the value of the level set. This last property, however, was only very recently established in [78] (cf. [90]) and is the key to proving that the function is twice differentiable.[1] The proof that the axis is unique, independent on the level set, relies on a key new inequality that draws its inspiration from real algebraic geometry although the proof is entirely new. This kind of uniqueness is a famously difficult problem in geometric analysis and no general case had previously been known.

Regularity of solutions. We have seen that the arrival time is always twice differentiable, and one may wonder whether there is even more regularity. Huisken [116] showed already in 1990 that the arrival time is C^2 for *convex* M_0. However, in 1992 Ilmanen gave an example of a rotationally symmetric *mean convex* M_0 in $\mathbf{R}^3$ where u is not C^2. This result of Ilmanen [120] shows that the above theorem about differentiability cannot be improved to C^2. We will see later that in fact one can entirely characterize when the arrival time is C^2. In the plane, Kohn and Serfaty [131] showed that u is C^3, and for $n>1$ Sesum [168] gave an example of a *convex* M_0 where u is not C^3. Thus Huisken's result is optimal for $n>1$.

The next result shows that one can entirely characterize when the arrival time is C^2.

Continuous differentiability [82]. u is C^2 if and only if:

1 Uniqueness of the axis is parallel to the fact that a function is differentiable at a point precisely if on all sufficiently small scales at that point it looks like the *same* linear function.

- There is exactly one singular time (where the flow becomes extinct).
- The singular set $\mathcal{S}$ is a k-dimensional, closed, connected, embedded, C^1 submanifold of cylindrical singularities.

Moreover, the axis of each cylinder is the tangent plane to $\mathcal{S}$.

When u is C^2 in $\mathbf{R}^3$, the singular set $\mathcal{S}$ is either:

(1) A single point with a spherical singularity, or

(2) A simple closed C^1 curve of cylindrical singularities.

The examples of the sphere and marriage ring show that each of these phenomena can happen, whereas the example of the dumbbell does not fall into either, showing that in that case the arrival time is not C^2.

We can restate this result for $\mathbf{R}^3$ in terms of the structure of the critical set and Hessian: u is C^2 if and only if u has exactly one critical value and the critical set is either:

(1) A single point where Hess_u is $-\frac{1}{2}$ times the identity, or

(2) A simple closed C^1 curve where Hess_u has eigenvalues 0 and -1 with multiplicities 1 and 2, respectively.

In case (2), the kernel of Hess_u is tangent to the curve, in fact, more is true, see [84].

2. UNIQUENESS OF BLOWUPS IN GEOMETRY

We saw that the key for optimal regularity for the level set equation was to show that the second-order approximation to a solution is independent of scale. The level sets of the second-order approximation are cylinders, and the key was that the axis of the cylinders was independent of scales.

This, independence of scale, is part of a larger question about uniqueness of blowups that has been widely studied whenever singularities occur. Indeed, once singularities occur, one naturally wonders what the singularities are like. A standard technique for analyzing singularities is to magnify around them. Unfortunately, singularities in many of the interesting problems in geometric PDEs looked at under a microscope will resemble one blowup, but under higher magnification, it might (as far as anyone knows) resemble a completely different blowup. Whether this ever happens is perhaps the most fundamental question about singularities; see, e.g., [171] and [108]. By general principles, the set of blowups is connected and, thus, the difficulty for uniqueness is when the blowups are not isolated in the space of blowups.

One of the first major results on uniqueness was by Allard–Almgren in 1981 [3], where uniqueness of tangent cones with smooth cross-section for minimal varieties is proven under an additional integrability assumption on the cross-section. The integrability condition applies in a number of important cases, but it is difficult to check and is not satisfied in many other important cases.

The next breakthrough on uniqueness was inspired by some old results in real algebraic geometry. Perhaps surprisingly, blowups for a number of important geometric PDEs can essentially be reformulated as infinite-dimensional gradient flows of analytic functionals. Thus, the uniqueness question would follow from an infinite-dimensional version of Lojasiewicz's theorem for gradient flows of analytic functionals. In real algebraic geometry, Lojasiewicz's theorem asserts that any integral curve of the gradient flow of an analytic function that has an accumulation point has a unique limit. Lojasiewicz proved this result in the early 1960s as a consequence of his gradient inequality. Infinite-dimensional versions of Lojasiewicz's theorem and the underlying Lojasiewicz inequalities were proven in a celebrated work of Simon [170] for the area, energy, and related functionals, and used, in particular, to prove a fundamental result about uniqueness of tangent cones with smooth cross-section of minimal surfaces. This holds, for instance, at all singular points of an area-minimizing hypersurface in $\mathbf{R}^8$. It also holds for singularities with smooth compact tangent flows for mean curvature flow by Schulze [174].

These method are very powerful and have had a major impact, but they do not apply when the blowups are noncompact. Indeed, in the most important examples, for essentially all of the natural flows the most common singularities are products with nontrivial Euclidean factors and thus are noncompact.

We will say that a singular point is *cylindrical* if at least one tangent flow is a multiplicity-one cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$. We will later see that these are the most common and most important singularities. In [78] we showed that at each cylindrical singular point of a mean curvature flow the blowup is unique, that is, it does not depend on the sequence of rescalings.

Theorem 2.1. *Let M_t be an MCF in $\mathbf{R}^{n+1}$. At each cylindrical singular point, the tangent flow is unique. That is, any other tangent flow is also a cylinder with the same $\mathbf{R}^k$ factor that points in the same direction.*

This settled a major open problem that was open even in the case of mean convex hypersurfaces where it was known that all singularities are cylindrical. Moreover, this was the first general uniqueness theorem for blowups to a geometric PDE at a noncompact singularity.

To prove our uniqueness result, we established two completely new infinite-dimensional Lojasiewicz-type inequalities. Infinite-dimensional Lojasiewicz inequalities were pioneered 30 years ago by Simon [170]. However, unlike all other infinite-dimensional Lojasiewicz inequalities we know of, ours do not follow from a reduction to the classical finite-dimensional Lojasiewicz inequalities from the 1960s from algebraic geometry, rather we prove our inequalities directly and do not rely on Lojasiewicz's arguments or results.

This is only a brief introduction to a very central and active area, see [37, 39, 47, 52, 74, 76, 78, 95, 101, 112, 154, 155, 174].

3. REGULARITY OF SINGULAR SET

A major theme in PDEs over the last 50 years has been understanding singularities and the set where singularities occur. In the presence of a scale-invariant monotone quantity, blowup arguments can often be used to bound the dimension of the singular set; see, e.g., [3]. Unfortunately, these dimension bounds say little about the structure of the set. The key to get more structure is uniqueness of blowups. Uniqueness of tangents has important applications to regularity of the singular set; see, e.g., [171]. We will see in this section that the results of the previous sections lead to a rather complete description of the singular set for MCF with cylindrical singularities:

Theorem 3.1 ([81]). *Let $M_t \subset \mathbf{R}^{n+1}$ be an MCF of closed embedded hypersurfaces with only cylindrical singularities, then the space-time singular set is contained in finitely many (compact) embedded C^1 submanifolds each of dimension at most $(n-1)$ together with a set of dimension at most $(n-2)$.*

In fact, [81] proves considerably more than what is stated in Theorem 3.1; see Theorem 4.18 there. For instance, instead of just proving the first claim of the theorem, the entire stratification of the space-time singular set is Lipschitz of the appropriate dimension. Moreover, this holds without ever discarding *any* subset of measure zero of any dimension as is always implicit in any definition of rectifiable. To illustrate the much stronger version, consider the case of evolution of surfaces in $\mathbf{R}^3$. In that case, this gives that the space-time singular set is contained in finitely many (compact) embedded Lipschitz curves with cylinder singularities together with a countable set of spherical singularities. In higher dimensions, the direct generalization of this is proven.

Theorem 3.1 has the following corollaries:

Corollary 3.2 ([81]). *Let $M_t \subset \mathbf{R}^{n+1}$ be an MCF of closed embedded mean convex hypersurfaces or an MCF with only cylindrical singularities, then the conclusion of Theorem 3.1 holds.*

More can be said in dimensions three and four:

Corollary 3.3 ([81]). *If M_t is as in Theorem 3.1 and $n = 2$ or 3, then the evolving hypersurface is completely smooth (i.e., has no singularities) at almost all times. In particular, any connected subset of the space-time singular set is completely contained in a time-slice.*

A key technical point in [81] is to prove a strong parabolic Reifenberg property for MCF with generic singularities. In fact, the space-time singular set is proven to be (parabolically) Reifenberg vanishing. In analysis, a subset of a Euclidean space is said to be Reifenberg (or Reifenberg flat) if on all sufficiently small scales it is, after rescaling to unit size close, to a k-dimensional plane. The dimension of the plane is always the same but the plane itself may change from scale to scale. Many snowflakes, like the Koch snowflake, are Reifenberg with Hausdorff dimension strictly larger than one. A set is said to be Reifenberg vanishing if the closeness to a k-plane goes to zero as the scale goes to zero. It is said to have the strong

Reifenberg property if the k-dimensional plane depends only on the point but not on the scale.

Using the uniqueness of tangent flows, [81] shows that the singular set in space-time is strong (half) Reifenberg vanishing with respect to the parabolic Hausdorff distance. This is done in two steps, showing first that nearby singularities sit inside a parabolic cone (i.e., between two oppositely oriented space-time paraboloids that are tangent to the time-slice through the singularity). In fact, this parabolic cone property holds with vanishing constant. Next, in the complementary region of the parabolic cone in space-time (that is essentially space-like), the parabolic Reifenberg essentially follows from the space Reifenberg that the uniqueness of tangent flows implies.

An immediate consequence, of independent interest, of the parabolic cone property with vanishing constant is that nearby a generic singularity in space-time (nearby is with respect to the parabolic distance) all other singularities happen at almost the same time.

These results should be contrasted with a result of Altschuler–Angenent–Giga [4] showing that in $\mathbf{R}^3$ the evolution of any rotationally symmetric surface obtained by rotating the graph of a function $r = u(x)$, $a < x < b$ around the x-axis is smooth except at finitely many singular times where either a cylindrical or spherical singularity forms. For more general rotationally symmetric surfaces (even mean convex), the singularities can consist of nontrivial curves. For instance, consider a torus of revolution bounding a region Ω. If the torus is thin enough, it will be mean convex. Since the symmetry is preserved and because the surface always remains in Ω, it can only collapse to a circle. Thus at the time of collapse, the singular set is a simple closed curve.

White showed that a mean convex surface evolving by MCF in $\mathbf{R}^3$ must be smooth at almost all times, and at no time can the singular set be more than 1-dimensional. In fact, White's general dimension reducing argument [180, 181] gives that the singular set of any MCF with only cylindrical singularities has dimension at most $(n - 1)$.

These results motivate the following conjecture:

Conjecture 3.4 ([81]). *Let M_t be an MCF of closed embedded hypersurfaces in $\mathbf{R}^{n+1}$ with only cylindrical singularities. Then the space-time singular set has only finitely many components.*

If this conjecture was true, then it would follow that in $\mathbf{R}^3$ and $\mathbf{R}^4$ MCF with only generic singularities is smooth except at finitely many times; cf. the three-dimensional conjecture at the end of Section 5 in [183].

Part 2. Growth of solutions to differential equations. On a Riemannian manifold M with metric $\langle \cdot, \cdot \rangle$ and Levi-Civita connection ∇, the gradient of a function f is defined by

$$V(f) = \langle \nabla f, V \rangle \quad \text{for all vectors fields } V. \tag{3.5}$$

The Laplacian of f is the trace of the Hessian. That is, if e_i is an orthonormal frame for M, then

$$\Delta f = \operatorname{Tr}\operatorname{Hess}_f = \sum_i \operatorname{Hess}_f(e_i, e_i) = \sum_i \langle \nabla_{e_i} V, e_i \rangle. \tag{3.6}$$

The Laplace operator is the canonical linear second order partial differential equation once we have a metric structure.

4. HARMONIC FUNCTIONS WITH POLYNOMIAL GROWTH

The classical Liouville theorem, named after Joseph Liouville (1809–1882), states that a bounded (or even just positive) harmonic function on all of $\mathbf{R}^n$ must be constant. There is a very short proof of this for bounded functions using the mean value property:

> *Given two points, choose two balls with the given points as centers and of equal radius. If the radius is large enough, the two balls will coincide except for an arbitrarily small proportion of their volume. Since the function is bounded, the averages of it over the two balls are arbitrarily close, and so the function assumes the same value at any two points.*

The Liouville theorem has had a huge impact across many fields, such as complex analysis, partial differential equations, geometry, probability, discrete mathematics, and complex and algebraic geometry, as well as many applied areas. The impact of the Liouville theorem has been even larger as the starting point of many further developments.

On manifolds with nonnegative Ricci curvature, mean values inequalities hold, but are no longer equalities, and the above proof does not give a Liouville type property. However, in the 1970s, S. T. Yau [187] showed that the Liouville theorem holds for such manifolds. Later, in the mid 1970s, Yau together with S. Y. Cheng [42] showed a gradient estimate on these manifolds giving an effective version of the Liouville theorem; see also Schoen [165].

The situation is very different for negatively curved manifolds such as hyperbolic space. This is easiest seen in two dimensions where being harmonic is conformally invariant, so each harmonic function on the Euclidean disk is also harmonic in the hyperbolic metric. In particular, each continuous function on the circle extends to a harmonic function on the disk and the space of bounded harmonic functions is infinite dimensional; cf. Anderson [6], Sullivan [173], and Anderson–Schoen [7].

On a Euclidean space, as soon as one allows a polynomial rate of growth, there are lots of harmonic functions. In fact, on a Euclidean space the harmonic functions with polynomial growth are the harmonic polynomials which play a central role in analysis. On a general manifold, the situation is much more complicated, and one does not expect an explicit representation. Given a manifold M and a constant d, $\mathcal{H}_d(M)$ is the linear space of harmonic functions of polynomial growth at most d. Namely, $u \in \mathcal{H}_d(M)$ if $\Delta u = 0$ and for some $p \in M$ and a constant C_u depending on u

$$\sup_{B_R(p)} |u| \leq C_u(1+R)^d \quad \text{for all } R. \tag{4.1}$$

In 1974, S. T. Yau conjectured that manifolds with nonnegative Ricci curvature should have a strong Liouville property, namely that $\mathcal{H}_d(M)$ is finite dimensional for each d when $\mathrm{Ric}_M \geq 0$. The conjecture was settled in [59]; see [86] for more results.[2] In fact, [59, 62, 63] proved finite dimensionality under much weaker assumptions of:

(1) A volume doubling bound,

(2) A scale-invariant Poincaré inequality or mean value inequality.

Both (1) and (2) hold for $\mathrm{Ric} \geq 0$ by the Bishop–Gromov volume comparison and work of Buser. However, these properties do not require much regularity of the space and are quite flexible. In particular, they make sense for more general metric-measure spaces and are preserved by bi-Lipschitz changes of the metric. Moreover, properties (1) and (2) make sense also for discrete spaces, vastly extending the theory and methods out of the continuous world. This extension opens up applications to geometric group theory and discrete mathematics, some of which we will touch upon later.

An interesting feature of these dimension estimates is that they follow from "rough" properties of M and are therefore surprisingly stable under perturbation. Unlike a Ricci curvature bound, these properties are stable under bi-Lipschitz transformations, cf. [134]. Moreover, these properties make sense also for discrete spaces, vastly extending the theory and methods out of the continuous world. Kleiner [128] (see also Shalom–Tao [169, 175, 176]) used, in part, this in his new proof of an important and foundational result in geometric group theory, originally due to Gromov [104]. Harmonic functions also play a central role in complex geometry, [136, 142, 157].

5. ANCIENT CALORIC FUNCTIONS WITH POLYNOMIAL GROWTH

Harmonic functions are functions that are in equilibrium for the Laplace equation. For the heat equation, equilibrium is reached when solutions have existed for all prior times. This naturally leads to the question of whether there is a generalization of the results in the previous section to ancient solutions of the heat equation with polynomial growth. Ancient solutions are those that are defined for all negative t. Many solutions of the heat equation, including the fundamental solution, cannot be extended to all negative t. Given $d > 0$, $u \in \mathcal{P}_d(M)$ if u is ancient (defined for all negative t), $\partial_t u = \Delta u$ and for some $p \in M$ and a constant C_u,

$$\sup_{B_R(p)\times[-R^2,0]} |u| \leq C_u(1+R)^d \quad \text{for all } R. \tag{5.1}$$

[2] For Yau's 1974 conjecture, see: page 117 in [188], problem 48 in [189], Conjecture 2.5 in [97, 124–126, 165], Conjecture 1 in [137], and problem (1) in [138], amongst others.

On $\mathbf{R}^n$, these functions are the classical caloric polynomials that include the spherical harmonics and generalize the Hermite polynomials.

A manifold has polynomial volume growth if there are constants C and d_V so that $\mathrm{Vol}(B_R(p)) \leq C(1+R)^{d_V}$ for some $p \in M$, and all $R > 0$.[3] In [89] the following sharp inequality, which is an equality on $\mathbf{R}^n$, was shown:

Theorem 5.2. *If M has polynomial volume growth and k is a nonnegative integer, then*

$$\dim \mathcal{P}_{2k}(M) \leq \sum_{i=0}^{k} \dim \mathcal{H}_{2i}(M). \tag{5.3}$$

Since $\mathcal{H}_{d_1} \subset \mathcal{H}_{d_2}$ for $d_1 \leq d_2$, Theorem 5.2 implies:

Corollary 5.4. *If M has polynomial volume growth, then for all $k \geq 1$,*

$$\dim \mathcal{P}_{2k}(M) \leq (k+1) \dim \mathcal{H}_{2k}(M). \tag{5.5}$$

Combining this with the bound $\dim \mathcal{H}_d(M) \leq C d^{n-1}$ when $\mathrm{Ric}_{M^n} \geq 0$ from [59] gives:

Corollary 5.6. *There exists $C = C(n)$ so that if $\mathrm{Ric}_{M^n} \geq 0$, then for $d \geq 1$,*

$$\dim \mathcal{P}_d(M) \leq C d^n. \tag{5.7}$$

The exponent n in (5.7) is sharp: There is a constant c depending on n so that for $d \geq 1$,

$$c^{-1} d^n \leq \dim \mathcal{P}_d\big(\mathbf{R}^n\big) \leq c d^n. \tag{5.8}$$

Recently, Lin and Zhang [141] proved very interesting related results, adapting the methods of [59,62,63] to get the bound d^{n+1}.

An immediate corollary of the parabolic gradient estimate of Li and Yau [139] is that if $d < 2$ and $\mathrm{Ric} \geq 0$, then $\mathcal{P}_d(M) = \mathcal{H}_d(M)$ consists only of harmonic functions of polynomial growth. In particular, $\mathcal{P}_d(M) = \{\text{constant functions}\}$ for $d < 1$ and, moreover, $\dim \mathcal{P}_1(M) \leq n+1$, by Li and Tam [138], with equality if and only if $M = \mathbf{R}^n$ by [38].

The exponent $n-1$ is also sharp in the bound for $\dim \mathcal{H}_d$ when $\mathrm{Ric}_{M^n} \geq 0$. However, as in Weyl's asymptotic formula, the coefficient of d^{n-1} can be related to the volume [63]:

$$\dim \mathcal{H}_d(M) \leq C_n \, \mathrm{V}_M \, d^{n-1} + o\big(d^{n-1}\big), \tag{5.9}$$

where

- V_M is the "asymptotic volume ratio" $\lim_{r\to\infty} \mathrm{Vol}(B_r)/r^n$.
- $o(d^{n-1})$ is a function of d with $\lim_{d\to\infty} o(d^{n-1})/d^{n-1} = 0$.

Combining (5.9) with Corollary 5.4 gives $\dim \mathcal{P}_d(M) \leq C_n \, \mathrm{V}_M \, d^n + o(d^n)$ when $\mathrm{Ric}_{M^n} \geq 0$.

3 A volume-doubling space with doubling constant C_D has polynomial volume growth of degree $\log_2 C_D$.

6. GROWTH OF DRIFT EQUATIONS

The Laplacian Δ is self-adjoint with respect to the ordinary L^2 inner product. However, if we instead use a weighted L^2 inner product, then the Laplacian may not be self-adjoint but there is a natural self-adjoint elliptic operator known as the drift Laplacian. Drift Laplacians are ubiquitous in many areas, including quantum field theory, stochastic PDEs, and anywhere the heat equation or Gaussian appear, such as functional inequalities, parabolic PDEs, geometric flows, and probability. The drift term arises whenever there is natural measure or a natural scaling or, more generally, a gradient flow.

To make the drift Laplacian precise, fix a function ϕ and define the weighted L^2-norm $\|\cdot\|_\phi$ by

$$\|u\|_\phi^2 \equiv \int_M u^2 \, e^{-\phi} \, . \tag{6.1}$$

Similarly, we will define the weighted inner product by

$$\langle u, v\rangle_\phi \equiv \int_M uv \, e^{-\phi} \, . \tag{6.2}$$

The drift Laplacian $\mathcal{L}_\phi$ is defined by

$$\mathcal{L}_\phi u = \Delta u - \langle \nabla\phi, \nabla u\rangle = e^{\phi} \operatorname{div}\big(e^{-\phi} \, \nabla u\big) \tag{6.3}$$

and

$$\langle \mathcal{L}_\phi u, v\rangle_\phi = -\int_M \langle \nabla u, \nabla v\rangle \, e^{-\phi} = \langle u, \mathcal{L}_\phi v\rangle_\phi . \tag{6.4}$$

The operator is self adjoint and under reasonable hypothesis has discrete eigenvalues going to infinity, see, for instance, [11,43,111,144]. The best-known example is the Ornstein–Uhlenbeck operator on $\mathbf{R}^n$,

$$\mathcal{L} = \Delta - \frac{1}{2}\nabla_x , \tag{6.5}$$

where $\phi = \frac{|x|^2}{4}$ and $\|\cdot\|_\phi$ is the Gaussian L^2-norm.

Drift Laplacians were considered very early on. Laplace discovered that on the line eigenfunctions of $\mathcal{L}u = u'' - \frac{x}{2}u'$ in the Gaussian L^2 space are polynomials whose degree is exactly twice the eigenvalue. These polynomials were later rediscovered twice. First by Chebyshev and a few years later by Hermite. They are now known as the Hermite polynomials and the eigenvalue equation as the Hermite equation. The first few eigenfunctions are: constants with eigenvalue 0, the linear function x with eigenvalue $\frac{1}{2}$, and the quadratic polynomial $x^2 - 2$ with eigenvalue 1. The Hermite polynomials and their higher-dimensional analogues play an important role in diverse fields. We will describe a vast generalization of these results that has many applications.

6.1. Growth of drift equations

We will next describe optimal polynomial growth bounds for eigenfunctions of drift Laplacians in a general setting that includes all shrinking solitons for both Ricci and mean curvature flows (or MCF). These bounds are sharp for the Ornstein–Uhlenbeck operator on Euclidean space.

There is a long history of studying the growth of solutions to differential equations, inequalities, and systems. At a very rough level, there are two main techniques. The first, exemplified in the work of Carleman and Hörmander, is to consider weighted L^2-norms with growing weights. The second, seen, for instance, in the work of Hadamard and Almgren, is to study the growth of spherical maxima or averages. The second is an extreme version of the first where the weight is a measure concentrated on a lower-dimensional set. As such, the second method typically gives stronger information and requires greater structure, such as invariance under dilations. However, general manifolds do not come with any dilation structure.

The growth estimates that we describe here hold in remarkable generality and without any assumptions on asymptotic decay. This is surprising and in contrast to most other situations, like unique continuation, that require very strong geometric assumptions on the space. A typical starting point for growth estimates is a Pohozaev identity or commutator estimate that come from a dilation, or approximate dilation, structure. We have none of these here in this general setting. In contrast, we rely on a miraculous cancelation for just the right quantity. A consequence of the generality is that the growth estimates hold for all singularities which is key for applications.

In many settings, one has an n-dimensional Riemannian manifold (M, g), that could even be flat Euclidean space, with two nonnegative functions f and S satisfying

$$\Delta f + S = \frac{n}{2}, \tag{6.6}$$

$$|\nabla f|^2 + S = f, \tag{6.7}$$

and where f is proper and C^n. The weight e^{-f} gives a drift Laplacian $\mathcal{L}$ on tensors u

$$\mathcal{L}u = \mathrm{e}^f \operatorname{div}\big(\mathrm{e}^{-f}\, \nabla u\big) = \Delta u - \nabla_{\nabla f} u \tag{6.8}$$

that is self-adjoint with respect to the L^2-norm $\|u\|_{L^2}^2 = \int |u|^2 \mathrm{e}^{-f}$. Using the function f, we can define a very natural exhaustion function b that will share many of the same properties that the distance function has on a Euclidean space with the standard Gaussian measure. Since $|\nabla \sqrt{f}| \le \frac{1}{2}$ by (6.7), $b = 2\sqrt{f}$ satisfies $|\nabla b| \le 1$ as in [35]. On $\mathbf{R}^n$, $f = \frac{|x|^2}{4}$ and $S = 0$ satisfy (6.6), (6.7) with $\mathcal{L} = \Delta - \frac{1}{2}\nabla_x$ the Ornstein–Uhlenbeck operator and $b = |x|$. In a Ricci flow, singularities are gradient shrinking solitons, f is the potential, and S is scalar curvature.[4] In an MCF, singularities are shrinkers $\Sigma \subset \mathbf{R}^N$, $f = \frac{|x|^2}{4}$, and $S = |\mathbf{H}|^2$, where $\mathbf{H}$ is the mean curvature vector.[5]

Throughout, $\lambda > 0$ is a constant and u is a tensor on M. We will often assume that

$$\langle \mathcal{L}u, u\rangle \ge -\lambda |u|^2; \tag{6.9}$$

4 See [32, 40, 49–51, 107, 129, 160, 178].

5 See, e.g., [72, 78, 115].

this includes eigentensors with $\mathcal{L}u = -\lambda u$. To understand the growth of u, we will study a weighted average of $|u|^2$ on level sets of b,

$$I(r) = r^{1-n} \int_{b=r} |u|^2 |\nabla b|. \tag{6.10}$$

This is defined at regular values of b, but extends continuously to all values to be differentiable a.e. and absolutely continuous. The weight $|\nabla b|$ will play a crucial role (cf. [1, 53, 60, 61, 75, 105]). The growth of I will be bounded above in terms of the solid integral

$$D(r) = r^{2-n} \mathrm{e}^{\frac{r^2}{4}} \int_{b<r} \big(|\nabla u|^2 + \langle \mathcal{L}u, u\rangle\big) \mathrm{e}^{-f}. \tag{6.11}$$

The frequency $U = \frac{D}{I}$ is defined when I is positive and will measure the growth of $\log I$.

The frequency U describes the rate of growth of the function u. To illustrate this, when u is a degree m Hermite polynomial, so $\lambda = \frac{m}{2}$, it is easy to see that

$$U(r) = m\big(1 + O(r^{-2})\big) = 2\lambda\big(1 + O(r^{-2})\big). \tag{6.12}$$

The next theorem from [91] shows that an L^2 tensor satisfying (6.9) has frequency bounded by 2λ and, accordingly, it grows at most polynomially at this rate. This may seem surprising since the weight e^{-f} decays rapidly, so the L^2 condition a priori allows extremely rapid growth.

Theorem 6.13. *Suppose $u, \mathcal{L}u \in L^2$, (6.6), (6.7), (6.9) hold, and u does not vanish identically outside a compact set. Given $\varepsilon > 0$, there exists $R = R(n, \lambda, \varepsilon)$ such that if $r > R$, then*

$$U(r) \le 2\lambda + \varepsilon, \tag{6.14}$$

and for all $r_2 > r_1 > R$,

$$I(r_2) \le I(r_1)\left(\frac{r_2}{r_1}\right)^{2(2\lambda+\varepsilon)}. \tag{6.15}$$

This is sharp for the Ornstein–Uhlenbeck operator on $\mathbf{R}^n$ where the L^2 eigenfunctions are Hermite polynomials with degree twice the eigenvalue. Note that u cannot vanish on an open set if u has unique continuation, e.g., if $\mathcal{L}u = -\lambda u$.

Our results give that polynomially growing "special functions" are dense in L^2. This gives manifold versions of some very classical problems in analysis. Whereas Weierstrass's approximation theorem shows that polynomials are dense among continuous functions on any compact interval, the classical Bernstein problem [145], dating back to 1924, asks if polynomials are dense on $\mathbf{R}$ in the weighted $L^p(\mathrm{e}^{-f}\,dx)$ space if f is assumed to grow sufficiently fast at infinity. On the line, the Hermite polynomials are dense in $L^2(\mathrm{e}^{-\frac{|x|^2}{4}}\,dx)$ and Lennart Carleson (and implicitly Izumi–Kawata) showed that polynomials are dense in $L^p(\mathrm{e}^{-|x|^\alpha}\,dx)$ if and only if $\alpha \ge 1$. A similar problem in several complex variables is the *completeness problem*, going back to Carleman in 1923, about the density of polynomials in weighted L^2 spaces of holomorphic functions [22].

Almgren's frequency has been used to show unique continuation [102] and structure of the nodal sets [143]; prior to this, the main tool in unique continuation was Carleman estimates that still is the primary technique. Almgren's frequency bounds relied on scaling for $\mathbf{R}^n$; cf. [60, 61]. The papers [18] (cf. [179]), [83] developed frequencies for conical and cylindrical MCF shrinkers and did not involve a weight like $|\nabla b|$. Theorem 6.13, in contrast, holds very generally, including for all shrinkers in both Ricci flow and MCF. A much weaker version of Theorem 6.13, that was not relative, was proven in [83] in the special case of MCF.

Part 3. Stable structures. In mathematics, structural stability is a fundamental property of a dynamical system which means that the qualitative behavior of the trajectories is unaffected by small perturbations. Given a smooth function f on a finite-dimensional space, the gradient ∇f points in the direction of the steepest ascent. The critical points of f are the points where ∇f vanishes. If p is a local minimum of f, then the second derivative test tells us that the Hessian matrix of f at p is nonnegative. More generally, the number of negative eigenvalues of the Hessian is called the index of the critical point. A fundamental method to find the minimum of f is the method of gradient descent. Here, we make an initial guess p_0 and then iteratively move in the negative gradient direction, the direction of the steepest descent, by setting $p_{i+1} = p_i - \nabla f(p_i)$. This can also be done continuously by defining a negative gradient flow

$$\frac{dx}{dt} = -\nabla f\big(x(t)\big). \tag{6.16}$$

The function $f(x(t))$ decreases as efficiently as possible as $x(t)$ heads towards the minimum. The dynamics near a nondegenerate critical point are determined by the index. If the index is zero, then the critical point is attracting and the entire neighborhood flows towards the critical point. However, when the index is positive, a generic point will flow out of the neighborhood, missing the critical point.

Many of the fundamental problems in geometry can be understood as problems about dynamical systems on an infinite-dimensional space. Sometimes this is immediate. For instance, in the case of geodesics or minimal surfaces. Geodesics are critical points for energy, whereas minimal surfaces are critical points for area. Another example where the connection to dynamical systems is immediate is the mean curvature flow that is the negative gradient flow for area. In other cases the connection is hidden, but no less fundamental. An example of this is uniqueness of blowups, that we discussed earlier. Uniqueness can be thought of as the question of whether a related recurrent flow has a limit or is wandering. One of the most basic and fundamental questions about a dynamical system is the question of equilibria: which equilibria are stable (generic) and which are not. For a nongeneric equilibrium, a nearby flow line passes by the equilibria and thus the nongeneric ones can typically be ignored.

We will look for stable structures in four situations and discuss what is known and unknown, see [58]. Those four are: (1) minimal hypersurfaces; (2) minimal submanifolds of higher codimension; (3) singularities that are stable or generic, and cannot be perturbed away, for motion by mean curvature of hypersurfaces; and, finally, (4) singularities for motion by mean curvature in higher codimension.

7. MINIMAL SURFACES

Let $\Sigma^n \subset \mathbf{R}^N$ be a smooth submanifold (possibly with boundary). Given an infinitely differentiable (i.e., smooth), compactly supported, normal (orthogonal to Σ) vector field V on Σ, consider the one-parameter variation

$$\Sigma_{s,V} = \left\{x + sV(x) \mid x \in \Sigma\right\}. \tag{7.1}$$

This gives a path $s \to \Sigma_{s,V}$ in the space of submanifolds with $\Sigma_{0,V} = \Sigma$. The so-called first variation formula of area or volume is the equation (integration is with respect to d Vol)

$$\frac{d}{ds}_{s=0} \operatorname{Vol}(\Sigma_{s,V}) = \int_\Sigma \langle V, \mathbf{H}\rangle, \tag{7.2}$$

where $\mathbf{H}$ is the mean curvature vector. When Σ is a hypersurface, $\mathbf{H}$ is the unit normal times the sum of the principal curvatures. In general, $\mathbf{H} = -\sum_i A(e_i, e_i)$ where A is the second fundamental form and e_i is an orthonormal frame for the tangent space of Σ; $A(e_i, e_j) = A_{ij} = \nabla^{\perp}_{e_i} e_j$ where ∇ is the Euclidean derivative and "$\perp$" is the component orthogonal to the submanifold. When Σ is noncompact, $\Sigma_{s,V}$ is replaced by $\Gamma_{s,V} = \{x + sV(x) \mid x \in \Gamma\}$ where Γ is a compact subset of Σ containing the support of V.

The submanifold Σ is said to be a *minimal* if

$$\frac{d}{ds}_{s=0} \operatorname{Vol}(\Sigma_{s,V}) = 0 \quad \text{for all } V, \tag{7.3}$$

or, equivalently, by (7.2), if $\mathbf{H}$ is identically zero. Thus Σ is minimal if and only if it is a critical point for the volume functional. Since a critical point is not necessarily a minimum, the term minimal is misleading but time-honored. It is easy to see that being minimal is equivalent to all the coordinate functions of $\mathbf{R}^N$ restricted to the submanifold are harmonic with respect to the Laplacian, Δ_Σ, on the submanifold. In higher codimension, the minimal surface equation is a complicated system.

A computation shows that if Σ is minimal, then the second derivative of volume is

$$\frac{d^2}{ds^2}_{s=0} \operatorname{Vol}(\Sigma_{s,V}) = -\int_\Sigma \langle V, LV\rangle, \tag{7.4}$$

where $LV = \Delta_\Sigma V + \langle A_{ij}, \mathrm{V}\rangle A_{ij}$ is the so-called second variational (or Jacobi) operator. This is an operator on the normal bundle of Σ and is the Laplacian plus a zeroth-order term. When the submanifold is a hypersurface, this simplifies and becomes $LV = \Delta_\Sigma V + |A|^2 V$, where $|A|^2$ is the sum of the squares of the principal curvatures. It simplifies further if one identifies V with its projection $\phi = \langle V, \mathbf{n}\rangle$ onto the unit normal $\mathbf{n}$. Then $L\phi = \Delta_\Sigma \phi + |A|^2 \phi$.

A minimal submanifold is *stable* if it passes the second derivative test

$$\frac{d^2}{ds^2} \operatorname{Vol}(\Sigma_{s,V}) \geq 0 \quad \text{for all } V. \tag{7.5}$$

Obviously, if a minimal surface is area or volume minimizing among competitors with the same boundary, then it is stable as well. However, stability is much more general than being minimizing. Stability becomes a question about whether the Jacobi operator L is nonnegative or not. The operator L is much simpler for hypersurfaces and, in particular, it is easy to see that a minimal graph is stable. In higher codimension, the question of stability becomes much more complicated because of the vector-valued nature of L and the curvature of the normal bundle. For example, minimal graphs are not necessarily stable in higher codimension.[6]

A classical theorem of Bernstein from 1916 shows that entire (that is, where the domain of definition is all of $\mathbf{R}^2$) minimal graphs in $\mathbf{R}^3$ are planes. Whether this is true in higher dimensions became known as the Bernstein problem. This problem played an important role in the field for decades and is closely related to regularity for area minimizers. In 1965 and 1966, De Giorgi and Almgren proved the Bernstein theorem for graphs in $\mathbf{R}^4$ and $\mathbf{R}^5$. In 1968, Simons extended the Bernstein theorem to $\mathbf{R}^6$, $\mathbf{R}^7$, and $\mathbf{R}^8$, which was shown to be sharp the next year by Bombieri, De Giorgi, and Giusti. Simons' influential paper introduced the second variation operator and stability to minimal surface theory. Stability of hypersurfaces was studied by Schoen–Simon–Yau [166], who showed that, as long as the dimension of the hypersurface is at most six and the volumes of balls are up to a constant the same as Euclidean balls of the same radius and dimension, all stable minimal hypersurfaces are planes, cf. [186] and references there. In $\mathbf{R}^3$ Fischer-Colbrie and Schoen [99] showed the same, but without assuming area bounds. This was also proved independently by Do Carmo and Peng. Schoen [164] (see also [46,57]) later showed a local version of this that has had a huge influence on the development of minimal surfaces in three dimensions. Stable minimal surfaces can be constructed variationally, see, for instance, [152]. These estimates can also be applied to low index minimal surfaces, [146,147,172]. See [64–71] and [161] for more about minimal surfaces.

The situation is much more complicated in higher codimension where there is no analog of the Bernstein theorem, cf. [96,163]. A simple argument of Wirtinger from the 1930s, using Stokes' formula, shows that any complex submanifold of $\mathbf{C}^N$ is volume minimizing among things with the same boundary and, thus, a stable minimal submanifold. This gives a plethora of area-minimizing, and thus also stable, minimal submanifolds once the codimension is at least two. Moreover, these examples can have arbitrarily large areas. Remarkably, Micallef [153] proved a converse in $\mathbf{R}^4$. Namely, he showed that a stable oriented, parabolic minimal surface in $\mathbf{R}^4$ is complex for some orthogonal complex structure. Being parabolic is a conformal property that holds, for instance, if the volume of balls grows at most quadratically. Examples of Arezzo and Micallef show that this converse does not hold for surfaces in codimension larger than two.

6 By [153], Osserman's minimal graph $x_3 = \frac{1}{2} \cos \frac{x_2}{2} (e^{x_1} - 3 e^{-x_1})$ and $x_4 = -\frac{1}{2} \sin \frac{x_2}{2} (e^{x_1} - 3 e^{-x_1})$ in $\mathbf{R}^4$ is not stable.

8. MOTION BY MEAN CURVATURE

Surface tension is the tendency of fluid surfaces to shrink into the minimum surface area possible. Mathematically, the force of surface tension is described by the mean curvature.

A one-parameter family of n-dimensional submanifolds $M_t \subset \mathbf{R}^N$ is said to move by motion by mean curvature, see, for instance, [9, 92], if the time derivative of the position vector x moves by minus the mean curvature. That is,

$$\frac{\partial x}{\partial t} = -\mathbf{H}. \tag{8.1}$$

It follows from the first variation formula that the mean curvature flow is the negative gradient flow for area. That is, the mean curvature flow moves the submanifold in the direction where the area or volume decreases as fast as possible.

We can view the mean curvature flow as a type of heat equation. This is exemplified by that the coordinate functions of the ambient Euclidean space restricted to the evolving submanifolds satisfy the heat equation

$$\frac{\partial x}{\partial t} = \Delta_{M_t} x. \tag{8.2}$$

This equation is nonlinear since the Laplacian Δ_{M_t} depends on M_t. Moreover, since the submanifolds are evolving, the induced metric is time-varying so the Laplacian Δ_{M_t} is also time-varying. From the first variation formula (7.2), it follows easily that the mean curvature flow moves in the direction where the volume decreases as fast as possible; thus, the mean curvature flow is the negative gradient flow of volume. The motion is by surface tension. In higher codimension, (8.1) and (8.2) are complicated parabolic systems where much less is known.

Since the coordinate functions on the evolving submanifolds satisfy the heat equation, it follows from the parabolic maximum principle that the evolving submanifolds remain inside the convex hull of the initial submanifold. A straightforward computation shows that also the function $|x|^2 - 2nt$ satisfies the heat equation on the evolving submanifolds. At the initial time $t = 0$, this is nonnegative and therefore, by the parabolic maximum principle, it remains nonnegative as long as the flow exists. Since we have already seen that $\max_{M_t} |x|^2$ remains bounded under the evolution, it follows that the flow must become extinct in finite time and, thus, singularities occur. There are two approaches either considering a weak flow through singularities or considering flow with surgery through singularites; see, [17, 30, 110, 119, 130] for surgery.

For a fixed constant $c > 0$, rescaling the flow parabolically

$$t \to c M_{c^{-2}} = M_{c,t} \tag{8.3}$$

gives a new solution to motion by mean curvature that has the effect that the submanifolds are magnified by the constant c. If we simultaneously with rescaling also reparametrize time, then we get a rescaled mean curvature flow. It is easy to see that such a one-parameter family satisfies the rescaled mean curvature flow equation

$$\frac{\partial x}{\partial t} = \frac{x^\perp}{2} - \mathbf{H}. \tag{8.4}$$

The rescaled mean curvature flow, which is so critical for understanding the mean curvature flow, can itself be interpreted as the negative gradient flow of a functional that we call the Gaussian surface area.

8.1. Gaussian surface area and entropy

The Gaussian surface area F of an n-dimensional submanifold $\Sigma^n \subset \mathbf{R}^N$ is

$$F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_\Sigma \mathrm{e}^{-\frac{|x|^2}{4}} . \tag{8.5}$$

The constant $(4\pi)^{-\frac{n}{2}}$ is a normalization that makes the Gaussian area equal to one for an n-plane through the origin. Following [72], the entropy λ is the supremum of F over all translations and dilations

$$\lambda(\Sigma) = \sup_{c,x_0} F(c\Sigma + x_0). \tag{8.6}$$

By considering all centers and scales and taking the supremum over these, we get some rough low-regularity measure of the complexity of the submanifold. In particular, it is easy to see that the entropy is always at least 1 and achieved only on a n-dimensional plane.

It follows easily from Huisken's monotonicity formula that the entropy is monotone under mean curvature flow and, moreover, the entropy at the initial time gives an upper bound for the entropy of any future singularity; see [72].

Prior to the entropy, many results focused on either convexity conditions or graphical restrictions as these were preserved under the flow by the maximum principle. These properties, however, are pretty strong and heavily restrict the types of singularities that can occur. The entropy now plays a central role in mean curvature flow and a great deal is now known about low entropy flows, [20, 21, 45, 48, 55].

If V is a normal vector field and $\Sigma_{s,V}$, as before, is the variation $\Sigma_{s,V} = \{x + sV(x) \mid x \in \Sigma\}$, then an easy computation shows that

$$\frac{d}{ds}_{s=0} F(\Sigma_{s,V}) = (4\pi)^{-\frac{n}{2}} \int_\Sigma \left\langle V, \mathbf{H} - \frac{x^\perp}{2} \right\rangle \mathrm{e}^{-\frac{|x|^2}{4}} . \tag{8.7}$$

It follows that the Gaussian surface area F is monotone nonincreasing under the rescaled mean curvature flow and constant if and only if

$$\mathbf{H} = \frac{x^\perp}{2}. \tag{8.8}$$

This equation is the shrinker equation and is equivalent to the rescaled flow is static. Or, equivalently, the evolution under the mean curvature flow is by rescaling. That is, a later time slice is exactly like an earlier, just scaled down. That Gaussian surface area is monotone under the rescaled flow corresponds to Huisken's celebrated monotonicity formula [115]. From this, it follows also that the entropy is a Lyapunov function for both the mean curvature flow and the rescaled mean curvature flow.

From Huisken's monotonicity [115], as well as work of Ilmanen [121] and White [180], one knows that every sequence $c_i \to \infty$ has a subsequence (also denoted by c_i) such that $M_{c_i,t}$ converges to a shrinker $M_{\infty,t}$ (so $M_{\infty,t} = \sqrt{-t}\, M_{\infty,-1}$) with $\sup_t \lambda(M_{\infty,t}) \leq$

$\sup_t \lambda(M_t)$. Such a limit is said to be a tangent flow at the origin. Similarly, one can magnify (blow up) around any other point in space time. If one does not fix the point around where one blows up, but still looks at limits of a sequence of blowups, then the limiting flows are not shrinkers, but even then the limiting flows will exist for all negative times and are said to be ancient flows.

The shrinker equation (8.8) is a second order nonlinear elliptic equation that is closely related to the classical minimal surface equation. In fact, shrinkers are minimal surfaces for a conformally changed metric that is not particularly well-behaved: it is not complete and the curvature is unbounded. This perspective has limited utility for global questions, but it is very useful for local regularity (e.g., any tangent cone is a minimal cone); cf. [55,72,73].

8.2. Second variation and stability

We have already seen that shrinkers are critical points for the Gaussian area. The critical points for the Gaussian surface area are the fixed points for the rescaled flow. To understand the dynamics of the flow, we would like to understand which fixed points can be avoided and, more generally, the dynamics near any fixed point.

When Σ is a shrinker, we therefore look at the second derivative. A calculation (see [72]) gives

$$\frac{d^2}{ds^2}_{s=0} F(\Sigma_{s,V}) = -(4\pi)^{-\frac{n}{2}} \int_\Sigma \langle V, LV\rangle \, \mathrm{e}^{-\frac{|x|^2}{4}} \, . \tag{8.9}$$

Here $LV = \mathcal{L}V + \langle A_{ij}, V\rangle A_{ij} + \frac{1}{2}V$ is the second variation operator, and $\mathcal{L}V = \Delta_\Sigma V - \frac{1}{2}\nabla^\perp_{x^T} V$ is the Ornstein–Uhlenbeck operator on the normal bundle. For hypersurfaces, there is a similar simplification of the operator L, as we saw for the second derivative of volume; cf. [10,14,135] for higher codimension.

For any shrinker, translations and scaling give directions where the Gaussian area decreases [72], so there are no stable shrinkers in the usual sense. Translation of a submanifold in the direction $E \in \mathbf{R}^N$ is infinitesimally given by the normal part $E^\perp$ of E. Similarly, rescaling is given by the normal vector field $\frac{x^\perp}{2}$. This corresponds to $E^\perp$ (with $E \in \mathbf{R}^N$) and $\mathbf{H} = \frac{x^\perp}{2}$ being eigenvectors of L with eigenvalues $-\frac{1}{2}$ and -1, respectively. Perturbing by either translation or scaling has the effect of moving the same singularity to a different point in space or time. However, the singularity is not avoided; it just occurs at another time or place for the flow. For this reason, we say [72] that a shrinker is *F-stable* if

$$\frac{d^2}{ds^2}_{s=0} F(\Sigma_{s,V}) \geq 0 \quad \text{for all } V \text{ orthogonal to } \mathbf{H} \text{ and to all } E^\perp. \tag{8.10}$$

Here orthogonal means with respect to the Gaussian inner product on the space of normal vector fields. It is easy to see that spheres and planes are F-stable in any codimension. In [72] the F-stable hypersurfaces were classified.

Theorem 8.11. *The only F-stable hypersurfaces are the planes and the round sphere.*

At first it may seem surprising that round cylinders are not F-stable. Indeed, for nonompact shrinkers, it turns out that the right notion of stability is that of entropy stability, however, for compact singularities those two notions of stability are the same [72]. A shrinker is *entropy-stable* if it is a local minimum for the entropy λ. Entropy-unstable shrinkers are singularities that can be perturbed away, whereas entropy-stable ones cannot; see [72].

Even for hypersurfaces, examples show that singularities of the mean curvature flow are too numerous to classify. The hope is that the generic ones that cannot be perturbed away are much simpler. Indeed, in all dimensions, generic singularities (that is, entropy-stable shrinkers) of hypersurfaces moving by mean curvature flow have been classified in [72].

Theorem 8.12. *In all dimensions, generic singularities (that is, entropy-stable shrinkers) of hypersurfaces are round generalized cylinders* $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$.

The generic singularities in $\mathbf{R}^3$ are the sphere $\mathbf{S}^2_2$, cylinder $\mathbf{S}^1_{\sqrt{2}} \times \mathbf{R}$, and plane $\mathbf{R}^2$. In contrast to the Bernstein theorems for minimal hypersurfaces, this classification of generic singularities holds in every dimension.

The paper [55] showed that for hypersurfaces round spheres are the shrinkers with the smallest entropy. The authors of [55] conjectured further that round spheres had the least entropy for any closed hypersurface; this was proven by Bernstein–Wang [20] up to dimension 7 and extended by Zhu [190] to higher dimensions; cf. also [21,24,127,185]. For surfaces embedded shrinkers with genus zero has been classified by Brendle, [28].

8.3. Higher codimension

For the mean curvature flow in higher codimension, we search again for the stable singularities. Recall that stable singularities are those that are entropy stable, which is equivalent to being F-stable for closed shrinkers. In higher codimension, [87] gave the following bound for the entropy:

Theorem 8.13. *If* $\Sigma^2 \subset \mathbf{R}^N$ *is an* F*-stable shrinker diffeomorphic to a two-sphere, then*

$$\lambda(\Sigma) < 4 = \mathrm{e}\,\lambda\big(\mathbf{S}^2_2\big). \tag{8.14}$$

The sharp constant is unknown, but (8.14) is at most off by a factor of e. By [87], similar bounds also hold for other closed shrinking surfaces of any finite index where the entropy bound depends on the genus and index. This implies that any such F-stable shrinker, that, a priori, lies in a high-dimensional Euclidean space, in fact, lies in a linear subspace of some fixed small dimension. The sharp bound for the dimension of the linear space is unknown, though [87] provides sharp dimension bounds in various other important situations.

There is no analog of (8.14) for minimal surfaces in $\mathbf{R}^4$. Namely, viewing $\mathbf{R}^4$ as $\mathbf{C}^2$, one sees that the parametrized complex submanifold $z \to (z, z^m)$ is a stable minimal variety that is topologically a plane for each integer m. It has $\mathrm{Area}(B_r \cap \Sigma) \geq C m r^2$ for $r \geq 1$. In contrast, [87] implies that $\mathrm{Area}(B_r \cap \Sigma) \leq C(1+\gamma) r^2$ for a closed stable 2-dimensional shrinker Σ of genus γ. Similarly, there is no analog of the codimension bound for minimal surfaces. Indeed, for each m, the parametrized surface $z \to (z, z^2, z^3, \ldots, z^{m+1})$ is a stable minimal variety that is topologically a plane. Its real codimension is $2m$ and it is not contained in a proper subspace.

Once one has the entropy bound in (8.14), to conclude that stable singularities have low codimension, one needs a result about the number of linearly independent coordinate functions. The coordinate functions on a mean curvature flow produce a linear space of caloric functions, i.e., solutions of the heat equation, that grow at most linearly. The bound on the codimension is a consequence of a much more general result about polynomial growth caloric functions on an ancient mean curvature flow that has a variety of other useful applications.

Let $M_t^n \subset \mathbf{R}^N$ be an ancient mean curvature flow of n-dimensional submanifolds with entropies $\lambda(M_t) \leq \lambda_0 < \infty$. Recall that ancient flows are solutions that exist for all negative times. The space $\mathcal{P}_d$ of polynomial growth caloric functions consists of $u(x,t)$ on $\bigcup_t M_t \times \{t\}$ so that $(\partial_t - \Delta_{M_t})u = 0$ and there exists C depending on u with

$$\left|u(x,t)\right| \leq C\left(1 + |x|^d + |t|^{\frac{d}{2}}\right) \quad \text{for all } (x,t) \text{ with } x \in M_t,\ t < 0. \tag{8.15}$$

The simplest example is when the flow consists of a static (constant in time) hyperplane $\mathbf{R}^n$. In this case, $\mathcal{P}_d(\mathbf{R}^n)$ consists of polynomials in $(t, x_1, \dots, x_n)$ known as the caloric polynomials and, using the special structure in this case, it is easy to see that $\dim \mathcal{P}_d(\mathbf{R}^n) \approx c_n d^n$. The paper [87] showed sharp bounds for $\dim \mathcal{P}_d$ for all $d \geq 1$ for an ancient flow with $\lambda(M_t) \leq \lambda_0$,

$$\dim \mathcal{P}_d \leq C_n \lambda_0 d^n. \tag{8.16}$$

One remarkable consequence when $d = 1$ is a bound for the codimension. Namely, the flow sits inside a linear subspace of dimension at most $\dim \mathcal{P}_1$, since a linear relation for coordinate functions specifies a hyperplane containing the flow.

The next result we will describe gives sharp bounds for codimension in arguably some of the most important situations for ancient flows. The bounds mentioned above were sharp in the exponent of d and, thus, asymptotically sharp as $d \to \infty$. The next result is more delicate and obtains sharp constants for d fixed.

Suppose that $M_t^n \subset \mathbf{R}^N$ is an ancient MCF with $\sup_t \lambda(M_t) < \infty$. For each constant $c > 0$ define the flow $M_{c,t}$ by $M_{c,t} = \frac{1}{c} M_{c^2 t}$. It follows that $M_{c,t}$ is an ancient MCF as well. Since $\sup_t \lambda(M_t) < \infty$, it follows from Huisken's monotonicity [115], as well as work of Ilmanen [121] and White [180], that every sequence $c_i \to \infty$ has a subsequence (also denoted by c_i) such that $M_{c_i,t}$ converges to a shrinker $M_{\infty,t}$ (so $M_{\infty,t} = \sqrt{-t}\, M_{\infty,-1}$) with $\sup_t \lambda(M_{\infty,t}) \leq \sup_t \lambda(M_t)$. We will say that such an $M_{\infty,t}$ is a tangent flow at $-\infty$ of the original flow. In [87] the following sharp bound for the codimension was shown:

Theorem 8.17. *If $M_t^n \subset \mathbf{R}^N$ is an ancient MCF and one tangent flow at $-\infty$ is a cylinder $\mathbf{S}^k_{\sqrt{2k}} \times \mathbf{R}^{n-k}$, then M_t is a flow of hypersurfaces in a Euclidean subspace.*

Combining this result with results of Angenent–Daskalopoulos–Sesum [12, 13], Brendle–Choi [29], and Choi–Haslhofer–Hershkovits [48] gives uniqueness for ancient flows of surfaces in higher codimension.

Part 4. The gauge group. Comparing and recognizing metrics can be extraordinarily difficult because of the group of diffeomorphisms. Two metrics, which could even be the same, could look completely different in different coordinates. Many key problems are defined intrinsically without a canonical coordinate system. In those problems, the infinite-dimensional diffeomorphism group (gauge group) becomes a major issue and dealing with it a major obstacle. Ricci flow is such an example.

> *"Gauge theory is a term which has connotations of being a fearsomely complicated part of mathematics—for instance, playing an important role in quantum field theory, general relativity, geometric PDEs, and so forth. But the underlying concept is really quite simple: a gauge is nothing more than a coordinate system that varies depending on location …By fixing a gauge (thus breaking or spending the gauge symmetry), the model becomes something easier to analyse mathematically …Deciding exactly how to fix a gauge (or whether one should spend the gauge symmetry at all) is a key question in the analysis of gauge theories, and one that often requires the input of geometric ideas and intuition into that analysis."* [177]

One of the most interesting results of transformation groups is the existence of slices. A slice for the action of a group on a manifold is a submanifold which is transverse to the orbits near a given point.[7] Ebin and Palais proved the existence of a slice for the infinite-dimensional diffeomorphism group of a *compact* manifold acting on the space of all Riemannian metrics. However, here we will be interested in when the manifolds are not compact.

8.4. A new approach to dealing with the gauge group

We describe a new way of dealing with the diffeomorphism group from [91] that should be useful in a broad range of applications, and explain how it can be used to solve a well-known problem in Ricci flow. A key new tool is a detailed analysis of a natural second-order system operator $\mathcal{P}$. The operator will be used to "fix the gauge." The analysis applies to all noncompact singularities. This makes it particularly useful, but also delicate. At each scale, a diffeomorphism is applied to fix the gauge, requiring precise and delicate estimates for the growth of the diffeomorphism. The gauge-fixing diffeomorphism satisfies a nonlinear system of PDEs, where $\mathcal{P}$ is the linearization. We will need, and show, strong bounds for the displacement function of the gauge-fixing diffeomorphism.

Suppose we have two weighted manifolds. Assume that on a large, but compact set, the manifolds, metrics, and weights almost agree after identification by a diffeomorphism.

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7 If the group is compact and Lie and the space is completely regular, Mostow proved, as a generalization of works of Gleason, Koszul, Montgomery, Yang, and others, that there is a slice through every point. If the group is not compact but Lie and if the space is a Cartan space, then Palais proves the same result.

On this set, in these coordinates, we write the metric on one as g and on the other as $g+h$, where h is small, and the weights as e^{-f} and e^{-f-k}, where k is small. We would like to mod out by the diffeomorphism group, by adjusting by a diffeomorphism to put the equation in an appropriate gauge so that the difference h in the metrics is orthogonal to the action of the group. Orthogonality corresponds to making $\operatorname{div}_f h = 0$,[8] which means finding a diffeomorphism Φ so that

$$\operatorname{div}_f\big(\Phi^*(g+h)-g\big)=0. \tag{8.18}$$

The pullback metric is quadratic in the differential of Φ, so this is a second-order nonlinear system of PDEs for Φ. This is the PDE that is in the spirit of the slice theorem for group actions and a solution Φ gives the desired "gauge-fixing." Terms involving $\operatorname{div}_f h$ come up again and again, so many quantities simplify in this gauge and things become easier.

In **[91]** we construct the diffeomorphism solving (8.18) using an iteration scheme for the linearized operator $\mathcal{P}$ on vector fields Y. We show first sharp polynomial bounds on $\mathcal{P}$ and then use them to show sharp polynomial bounds for the displacement function of Φ

$$x \to \operatorname{dist}_g\big(x, \Phi(x)\big). \tag{8.19}$$

The bounds are relative, meaning that better initial bounds give better bounds further out. These optimal bounds hold on all singularities and give a key new tool for dealing with the gauge group of all noncompact singularities.

The linearization of (8.18) is to find a vector field whose Lie derivative of the metric has div_f equal to $-\operatorname{div}_f h$. The Lie derivative in a direction Y can also be written as $-2\operatorname{div}_f^* Y$, where div_f^* is the operator adjoint of div_f with respect to the weighted measure. Therefore, the linearization of (8.18) is $\mathcal{P}Y = \frac{1}{2}\operatorname{div}_f h$, where

$$\mathcal{P}Y = \operatorname{div}_f \circ \operatorname{div}_f^* Y. \tag{8.20}$$

Solutions of $\mathcal{P}Y = \frac{1}{2}\operatorname{div}_f h$ are unique once we require that Y is orthogonal to the kernel of $\mathcal{P}$. The kernel is the Killing fields. We will solve $\mathcal{P}Y = \frac{1}{2}\operatorname{div}_f h$ on any shrinker and show via L^2-methods that $\|Y\|_{W^{1,2}} \le \|\operatorname{div}_f h\|_{L^2}$. Given the noncompactness, the L^2-estimates are not sufficient to implement the iteration scheme, and we need stronger polynomial estimates. The problems are magnified by that initial closeness is only on a given compact set. As one builds out to get closeness on larger sets, one needs at each step to adjust the entire diffeomorphism so that the normalization is zero on larger and larger sets. Understanding $\mathcal{P}$ and proving growth estimates is a major point.

The L^2-theory for $\mathcal{P}$ shares formal similarities with Hörmander's influential L^2 $\bar\partial$-method in several complex variables. In the L^2 $\bar\partial$-method, one solves the Poisson equation $\bar\partial u = F$, with estimates, where $\bar\partial F = 0$. To do so, one introduces the adjoint of $\bar\partial$ with respect to a weight. Hörmander's idea for the weight came from Carleman's method for proving unique continuation of a PDE. Here we solve $\mathcal{P}Y = F$, where $F = \frac{1}{2}\operatorname{div}_f h$ is

8 For a symmetric two-tensor h, the f-divergence is $\operatorname{div}_f(h) = e^{f}\operatorname{div}(e^{-f} h) = \operatorname{div}(h) - h(\nabla f, \cdot)$.

orthogonal to the kernel of div_f^*. Hörmander's method gives weighted L^2-bounds for $\bar{\partial}$ similar to our weighted bounds for $\mathcal{P}$. To introduce a second weight to capture the growth à la Carleman and Hörmander is less natural here. Instead, we go a different route to prove stronger bounds.

8.5. Bounding the growth of gauge transformations

We need to control the growth of Y to control the metric in the new coordinates, but Y will be constructed using weighted L^2-methods and, thus, a priori could grow rapidly. The next theorem from [91] shows that an L^2 eigenvector field with eigenvalue λ for $\mathcal{P}$ grows polynomially of degree at most $4\lambda + 1$. A Poisson version is used to control Y with $\mathcal{P}Y = \frac{1}{2}\operatorname{div}_f h$. We set $b = 2\sqrt{f}$ and measure the growth of Y by the weighted average

$$I_Y(r) = r^{1-n} \int_{b=r} |Y|^2 |\nabla b|. \tag{8.21}$$

A one-parameter family of smooth manifolds, [15–17, 25, 106, 129, 130], is said to flow by the Ricci flow if

$$g_t = -2\operatorname{Ric}.$$

The triple (M, g, f) is a gradient shrinking soliton, or shrinker for short, if

$$\operatorname{Ric} + \operatorname{Hess}_f = \frac{1}{2} g;$$

shrinkers are the singularities in Ricci flow, [33, 36, 107, 122, 158, 162].

Theorem 8.22. *For any shrinker (M, g, f), if $Y \in L^2$, $\mathcal{P}Y = \lambda Y$ and*

$$Z = Y + \frac{2}{2\lambda + 1} \nabla \operatorname{div}_f(Y),$$

then $\operatorname{div}_f(Z) = 0$ *and for any* $\delta > 0$ *and* $r_2 > r_1 > R = R(\lambda, n, \delta)$,

$$I_{\nabla \operatorname{div}_f(Y)}(r_2) \leq \left(\frac{r_2}{r_1}\right)^{4\lambda+\delta} I_{\nabla \operatorname{div}_f(Y)}(r_1), \tag{8.23}$$

$$I_Z(r_2) \leq \left(\frac{r_2}{r_1}\right)^{8\lambda+2+\delta} I_Z(r_1). \tag{8.24}$$

Each of these growth bounds is sharp and so is the requirement that $Y \in L^2$. Combining them bounds Y. As a corollary, L^2 Killing fields on a shrinker grow at most linearly.

Corollary 8.25. *On any shrinker, for any L^2 Killing field Y, $\nabla \operatorname{div}_f(Y)$ is parallel and if $Z = Y + 2\nabla \operatorname{div}_f(Y)$, then* $\operatorname{div}_f(Z) = 0$ *and for any* $\delta > 0$ *and* $r_2 > r_1 > R = R(n, \delta)$,

$$I_Z(r_2) \leq \left(\frac{r_2}{r_1}\right)^{2+\delta} I_Z(r_1). \tag{8.26}$$

It is easy to see that this is sharp; on the two-dimensional Gaussian soliton, $Y = x_2 e_1 - x_1 e_2$ is a Killing field with $\operatorname{div}_f(Y) = 0$ that grows linearly.

On a shrinker, the operator $\mathcal{P}$ relates to the manifold version of the much studied Ornstein–Uhlenbeck operator $\mathcal{L}$ on vector fields Y by the formula

$$-2\mathcal{P}Y = \nabla \operatorname{div}_f Y + \mathcal{L}Y + \frac{1}{2}Y. \tag{8.27}$$

Whereas $\mathcal{P}$ is a true system operator, $\mathcal{L}$ is not, and for that reason, $\mathcal{P}$ is more complicated. On the other hand, on solitons, $\mathcal{P}$ has many nice properties: it commutes with $\mathcal{L}$ and if Y is an eigenvector field of $\mathcal{P}$ with eigenvalue $-\lambda$, then $\nabla \operatorname{div}_f(Y)$ is an eigenvector field of $\mathcal{L}$ with eigenvalue λ. The unweighted version of $\mathcal{P}$ was used implicitly by Bochner to show that closed manifolds with negative Ricci curvature have no Killing fields. Building on this the unweighted operator was later used by Bochner and Yano to show that the isometry group of such manifolds is finite. The unweighted operator also arises in general relativity. The relationship between $\mathcal{P}$ and the unweighted version, used by Bochner, mirrors the relationship between the Ornstein–Uhlenbeck operator and the Laplacian.

8.6. Applications

This new understanding of the "gauge group" can be used to settle a well-known problem in Ricci flow. Namely, using it one can show, see [91], a strong rigidity for cylinders, quotients of cylinders, and more general shrinking solitons; [23, 34], cf. [140].

Theorem 8.28. *Let Σ be the round cylinder $\mathbf{S}^\ell \times \mathbf{R}^{n-\ell}$ (or quotient of such) as a shrinker with potential $f_\Sigma = \frac{|x|^2}{4} + \frac{\ell}{2}$. There exists an $R = R(n)$ such that if (M^n, g, f) is another shrinker and $\{f_\Sigma \leq R\} \cap \Sigma$ is close to $\{f \leq R\} \subset M$ in the smooth topology and f_Σ and f are close on this set, then (M, g, f) is a round shrinking cylinder (or quotient of such).*

Since blowups only converge on compact subsets, rather than globally, the most useful characterizations involve only a compact subset as in Theorem 8.28. An important difficulty is that there are nontrivial infinitesimal variations, i.e., variations in the kernel of the linearized operator (not generated by diffeomorphisms). One consequence of Theorem 8.28 is that these infinitesimal variations are not integrable; cf. also [54].

The principle behind Theorem 8.28 is that closeness to a large enough piece of Σ propagates outwards, becoming even closer on larger scales. We will explain some of the ideas behind this shortly. A much weaker extension will follow from pseudolocality [160], which says that flatness propagates forward in time; accordingly, flatness propagates outward in space for shrinkers. This gives a priori curvature estimates on a slightly larger scale. However, it gives little control over the metric itself because of the gauge invariance and, second, there is a loss in the estimates that makes it impossible to iterate. There are three major ingredients in the proof of Theorem 8.28; we loosely refer to these as propagation of almost splitting, gauge fixing, and quadratic rigidity in the right gauge. These are of independent interest and will described in order next.

"Propagation of almost splitting" shows that if a shrinker is close to a product $N \times \mathbf{R}^{n-\ell}$ on a large scale, then it remains close on a fixed larger scale. The closeness on the first scale is used to get $n - \ell$ eigenvalues that are exponentially close to $\frac{1}{2}$, which is a lower bound for any shrinker that is only achieved by linear functions on products. The corresponding eigenfunctions will have exponentially small L^2-bounds for their Hessians, which forces the gradients to be virtually parallel on small sets but says little on large balls because of the Gaussian weight. It is here that the growth bounds from [91] first play a crucial role, showing that the Hessian bounds can only grow polynomially so the initial exponential smallness gives control on larger scales. These almost parallel vector fields are then used to construct a diffeomorphism to Σ on the larger scale, giving vastly more control than what followed from pseudolocality. This is very much a Ricci flow fact that does not have an MCF analogue where we do not have a corresponding description of the bottom of the spectrum.

The almost splitting gives considerable control on the larger scale, but does not fix the gauge—the difference in metrics is small, but is not orthogonal to the action of the gauge group. Moreover, even when the two metrics are the same, the difference between the potentials could be a linear function, corresponding to a translation along the axis.

There are many other important uniqueness results in Ricci flow, see, for instance, [16, 26, 27, 132, 133].

Part 5. Minimal surfaces. Surfaces that locally minimize area have been extensively used to model physical phenomena, including soap films, black holes, compound polymers, protein folding, etc. The mathematical field dates to the 1740s.

Minimal surfaces with uniform curvature or area bounds are well understood, yet essentially nothing was known without such bounds. We discuss here the theory of embedded (i.e., without self-intersections) minimal surfaces in Euclidean space $\mathbf{R}^3$ without a priori bounds; see [64–70, 77, 161] for more. The study is divided into three cases, depending on the topology of the surface. In case one the surface is a disk, in case two the surface is a planar domain (genus zero), and the third case is that of finite (nonzero) genus. The complete understanding of the disk case is applied in both cases two and three. In all three cases the surface is allowed to have a boundary. This is an essential point and makes the results particularly useful. For instance, given any minimal surface, independent of its topology, if a component of the intersection of the surface with a Euclidean ball is a disk, then case one applies and gives a good description of that component. Similarly, for cases two and three. The surface itself may then be thought of as built out of these snapshots (or building blocks). We will here mostly only discuss the case of disks.

The helicoid, which is a double spiral staircase, was discovered to be a minimal surface by Meusnier in 1776. As we will see, the helicoid is the most important example of an embedded minimal disk. In fact, we will see that every such disk is either a graph of a function or part of a double spiral staircase. For planar domains the fundamental examples are the catenoid, also discovered by Meusnier in 1776, and the Riemann examples discovered by Riemann in the beginning of the 1860s.[9] Finally, for general fixed genus, an important example is the recent example by Hoffman–Weber–Wolf of a genus-one helicoid. The genus-one helicoid is a complete minimal surface that on a large scale, away from the genus, looks essentially like an ordinary helicoid. This illustrates that the helicoid is one of the basic building blocks of general minimal surfaces. This is also true for the Riemann examples. The Riemann examples are a two-parameter family of complete minimal surfaces. As the

9 Riemann worked on minimal surfaces in the period 1860–1861. He died in 1866. The Riemann example was published post-mortem in 1867 in an article edited by Poggendorf.

parameters degenerate, the Riemann examples looks like either a collection of catenoids stacked on top of each other or two oppositely oriented helicoids (with parallel axes) glued together.

In the last section we discuss why (complete) embedded minimal surfaces are automatically proper (i.e., why divergent sequences of points on the surface diverge in Euclidean space). This question is known as the Calabi–Yau conjectures for embedded surfaces. For immersed (but not embedded) surfaces, there are counterexamples by Jorge-Xavier and Nadirashvili.

8.7. Minimal graphs and the helicoid

The derivation of the equation for a minimal graph goes back to Lagrange's 1762 memoir. There are questions of existence of solutions, uniqueness of equilibria, and the global structure of the space (or spaces) of examples. At the intersection of all of these questions is the question of what the (shape of the) natural building blocks are. In a broad sense, graphs and helicoids are in a fundamental way the key building blocks of embedded minimal surfaces.

There are two local models for embedded minimal *disks*. One model is the plane (or, more generally, a minimal graph) and the other is a piece of a helicoid.

Minimal graphs over proper simply connected domains in $\mathbf{R}^2$ gives a large class of embedded minimal disks, however, by a classical theorem of Bernstein from 1916 entire (i.e., where $\Omega = \mathbf{R}^2$) minimal graphs are planes.

The second model comes from the helicoid which was discovered by Meusnier in 1776.[10] The helicoid is a "double spiral staircase" given by sweeping out a horizontal line rotating at a constant rate as it moves up a vertical axis at a constant rate. Each half-line traces out a spiral staircase and together the two half-lines trace out (up to scaling) the double spiral staircase $(s\cos t, s\sin t, t)$, where $s, t \in \mathbf{R}$.

For the results about embedded minimal disks, it will be important to understand a sequence of helicoids obtained from a single helicoid by rescaling as follows:

Consider the sequence $\Sigma_i = a_i\Sigma$ of rescaled helicoids where $a_i \to 0$. (That is, rescale $\mathbf{R}^3$ by a_i, so points that used to be distance d apart will in the rescaled $\mathbf{R}^3$ be distance $a_i d$ apart.) The curvatures of this sequence of rescaled helicoids are blowing up (i.e., the curvatures go to infinity) along the vertical axis. The sequence converges (away from the vertical axis) to a foliation by flat parallel planes; that is, it converges to the collection of planes $x_3 =$ constant. The singular set (the axis) then consists of removable singularities.

10 Meusnier had been a student of Monge. He also discovered that the catenoid is minimal in the sense of Lagrange, and he was the first to characterize a minimal surface as a surface with vanishing mean curvature. Unlike the helicoid, the catenoid is not topologically a plane but rather a cylinder.

8.8. Multivalued graphs, spiral staircases, double spiral staircases

To be able to give a precise meaning to the statement that the helicoid is a double spiral staircase, we will need the notion of a multivalued graph, each staircase will be a multivalued graph. Intuitively, a multivalued graph is a surface covering an annulus, such that over a neighborhood of each point of the annulus, the surface consists of N graphs. To make this notion precise, let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbf{R}$. An *N-valued graph* on the annulus $D_s \setminus D_r$ is a single valued graph of a function u over $\{(\rho, \theta) \mid r < \rho \leq s, |\theta| \leq N\pi\}$. For working purposes, we generally think of the intuitive picture of a multisheeted surface in $\mathbf{R}^3$, and we identify the single-valued graph over the universal cover with its multivalued image in $\mathbf{R}^3$.

The multivalued graphs that we will consider will all be embedded, which corresponds to a nonvanishing separation between the sheets (or the floors). If Σ is the helicoid, then $\Sigma \setminus \{x_3 - \text{axis}\} = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are ∞-valued graphs on $\mathbf{C} \setminus \{0\}$; Σ_1 is the graph of the function $u_1(\rho, \theta) = \theta$ and Σ_2 is the graph of the function $u_2(\rho, \theta) = \theta + \pi$. (Further, Σ_1 is the subset where $s > 0$ in the parametrization of the helicoid and Σ_2 the subset where $s < 0$.) In either case the separation between the sheets is constant, equal to 2π. A *multivalued minimal graph*, see chapter 1 in [71], is a multivalued graph of a function u satisfying the minimal surface equation.

8.9. Structure of embedded minimal disks

All of our results for disks, as well as for other topological types, require only a piece of a minimal surface. In particular, the surfaces may well have boundaries and when we, for instance, say in the next theorem "Any embedded minimal disk in $\mathbf{R}^3$ is *either* a graph of a function *or* part of a double spiral staircase", then we mean that if the surface is contained in a Euclidean ball of radius r_0 and the boundary is contained in the boundary of that ball, then in a concentric Euclidean ball with radius a fixed (small) fraction of r_0, any component of the surface is *either* a graph of a function *or* part of a double spiral staircase. That the surfaces are allowed to have boundaries is a major point and makes the results particularly useful. Note also that as the conclusion is for a "fixed fraction of the surface" this is an interior estimate.

The following is the main structure theorem for embedded minimal disks:

Theorem 8.29. *Any embedded minimal disk in* $\mathbf{R}^3$ *is* either *a graph of a function* or *part of a double spiral staircase. In particular, if for some point the curvature is sufficiently large, then the surface is part of a double spiral staircase (it can be approximated by a piece of a rescaled helicoid). On the other hand, if the curvature is below a certain threshold everywhere, then the surface is a graph of a function.*

As a consequence of this structure theorem we get the following compactness result:

Corollary 8.30. *A sequence of embedded minimal disks with curvatures blowing up (i.e., going to infinity*[11]*) at a point mimics the behavior of a sequence of rescaled helicoids with curvature going to infinity.*

8.10. Two key ideas behind the proof of the structure theorem for disks

The first of these key ideas says that if the curvature of such a disk Σ is large at some point $x \in \Sigma$, then near x a multivalued graph forms (in Σ), and this extends (in Σ) almost all the way to the boundary[12] of Σ. Moreover, the inner radius, r_x, of the annulus where the multivalued graph is defined is inversely proportional to $|A|(x)$, and the initial separation between the sheets is bounded by a constant times the inner radius.

An important ingredient in the proof of Theorem 8.29 is that general embedded minimal disks with large curvature at some interior point can be built out of N-valued graphs. In other words, any embedded minimal disk can be divided into pieces each of which is an N-valued graph. Thus the disk itself should be thought of as being obtained by stacking these pieces (graphs) on top of each other.

The second key result (Theorem 8.31) is a curvature estimate for embedded minimal disks in a half-space (in this theorem r_0 is a scaling factor, which after rescaling can be taken to be one):

Theorem 8.31. *There exists $\varepsilon > 0$ such that for all $r_0 > 0$, if $\Sigma \subset B_{2r_0} \cap \{x_3 > 0\} \subset \mathbf{R}^3$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2r_0}$, then for all components Σ' of $B_{r_0} \cap \Sigma$ which intersect $B_{\varepsilon r_0}$,*

$$\sup_{x \in \Sigma'} \left|A_\Sigma(x)\right|^2 \leq r_0^{-2}. \tag{8.32}$$

This theorem has an equivalent formulation that may be easier to appreciate. Namely, for $\varepsilon > 0$ sufficiently small, (8.32) is equivalent to the statement that Σ' is a graph over (a domain in) the plane $\{x_3 = 0\}$.

Theorem 8.31 is an interior estimate where the curvature bound, (8.32), is on the ball B_{r_0} of one-half of the radius of the ball B_{2r_0} containing Σ. This is just like a gradient estimate for a harmonic function where the gradient bound is on one-half of the ball where the function is defined. Theorem 8.31 is often referred to as *the one-sided curvature estimate* (since Σ is assumed to lie on one side of a plane). The assumption in Theorem 8.31 that Σ is simply connected (i.e., that Σ is a disk) is crucial, as can be seen from the example of a rescaled catenoid. Rescaled catenoids converge (with multiplicity two) to the flat plane. Likewise, by considering the universal cover of the catenoid, one sees that Theorem 8.31 requires the disk to be embedded, and not just immersed.

The one-sided curvature estimate has strong implications for embedded minimal surfaces. We will return to some of these applications later, but note here that it can be

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11 A minimal surface in $\mathbf{R}^3$ the curvature $K = -\frac{1}{2}|A|^2$ is nonpositive; so that by the curvatures of a sequence is going to infinity we mean that $K \to -\infty$ or, equivalently, $|A|^2 \to \infty$.

12 Our results require only that we have a piece of a minimal surface and thus it may have boundary.

applied even to ends of embedded minimal surfaces with finite topology to give a different of a conjecture of Nitsche, see [56, 93].

8.11. Uniqueness theorems

There is a long history of uniqueness theorems for properly embedded minimal surfaces, but all of those made very strong assumptions. A typical example is Catalan's theorem. Catalan proved in 1842 that any complete ruled minimal surface is either a plane or a helicoid. A surface is said to be *ruled* if it has the parametrization $X(s,t) = \beta(t) + s\delta(t)$, where $s, t \in \mathbf{R}$, and β and δ are curves in $\mathbf{R}^3$. The curve $\beta(t)$ is called the *directrix* of the surface, and a line having $\delta(t)$ as direction vector is called a *ruling*. For the helicoid, the x_3-axis is a directrix, and for each fixed t the line $s \to (s\cos t, s\sin t, t)$ is a ruling. More recent uniqueness results (for instance, by Lopez, Meeks, Nirenberg, Nitsche, Osserman, Perez, Ros, Schoen, Shiffman, and Simon) assumed either finite total curvature or periodicity. The structure theorems in [65–68] opened up the possibility of showing uniqueness theorems in complete generality.

To give a flavor of some of the results that led to spetacular development in the theory of minimal surfaces, we will mention just a few highlights. Using the above structure theorem for disks, Meeks–Rosenberg [150] proved, cf. [19], that the plane and the helicoid are the only complete properly embedded simply-connected minimal surfaces in $\mathbf{R}^3$. The Riemann examples were shown to be unique by Meeks–Perez–Ros [148]. In addition to the structure theory for disks, they also used the structure theory of all finite-genus embedded minimal surfaces from [70]. The paper [148] also introduced two very interesting new techniques into the subject: the KdV equation and a careful analysis of the Shiffman function.

9. EMBEDDED MINIMAL SURFACES ARE AUTOMATICALLY PROPER

Implicit in all of the results mentioned above was an assumption that the minimal surfaces were proper. However, as we will see next, it turns out that embedded minimal surfaces are, in fact, automatically proper. This was the content of the Calabi–Yau conjectures which were proven to be true for embedded surfaces in [66].

9.1. Proper embeddings

An immersed surface in $\mathbf{R}^3$ is *proper* if the preimage of any compact subset of $\mathbf{R}^3$ is compact in the surface. For instance, a line is proper whereas a curve that spiral infinitely into a circle is not.

9.2. The Calabi–Yau conjectures; the statements and examples

The Calabi–Yau conjectures about surfaces date back to the 1960s. Their original form was given in 1965 where Calabi [31] made the following two conjectures about minimal surfaces[13]:

Conjecture 9.1. *Prove that a complete minimal surface in* $\mathbf{R}^3$ *must be unbounded.*

Calabi continued: "It is known that there are no compact minimal surfaces in $\mathbf{R}^3$ (or of any simply connected complete Riemannian 3-dimensional manifold with sectional curvature ≤ 0). A more ambitious conjecture is":

Conjecture 9.2. *A complete [non-flat] minimal surface in* $\mathbf{R}^3$ *has an unbounded projection in every line.*

The *immersed* versions of these conjectures turned out to be false. Namely, Jorge and Xavier **[123]** constructed non-flat minimal immersions contained between two parallel planes in 1980, giving a counterexample to the immersed version of the more ambitious Conjecture 9.2. Another significant development came in 1996, when Nadirashvili **[156]** constructed a complete immersion of a minimal disk into the unit ball in $\mathbf{R}^3$, showing that Conjecture 9.1 also failed for immersed surfaces; cf. **[2]**.

The main result in **[70]** is an effective version of properness for disks, giving a chord–arc bound.[14] Obviously, intrinsic distances are larger than extrinsic distances, so the significance of a chord–arc bound is the reverse inequality, i.e., a bound on intrinsic distances from above by extrinsic distances. Given such a chord–arc bound, one has that as intrinsic distances go to infinity, so do extrinsic distances. Thus as an immediate consequence:

Theorem 9.3. *A complete embedded minimal disk in* $\mathbf{R}^3$ *must be proper.*

Theorem 9.3 gives immediately that the first of Calabi's conjectures is true for *embedded* minimal disks. Another immediate consequence of the chord–arc bound together with the one-sided curvature estimate (i.e., Theorem 8.31) is a version of that estimate for intrinsic balls. As a corollary of this intrinsic one-sided curvature estimate, we get that the second, and more ambitious, of Calabi's conjectures is also true for *embedded* minimal disks. The second Calabi conjecture (for embedded disks) is an immediate consequence of the following half-space theorem:

Theorem 9.4. *The plane is the only complete embedded minimal disk in* $\mathbf{R}^3$ *in a half-space.*

Theorem 9.4 is a byproduct of the proof of Theorem 9.3. However, given Theorem 9.3, Theorem 9.4 follows from the half-space theorem of **[113]**.

The results for disks imply both of Calabi's conjectures and properness also for embedded surfaces with finite topology. A surface Σ is said to have finite topology if it is homeomorphic to a closed Riemann surface with a finite set of points removed or "punctures." Each puncture corresponds to an end of Σ.

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13 S. S. Chern **[44]** also promoted these conjectures at roughly the same time and they were revisited several times by S. T. Yau.

14 A chord–arc bound is a bound above and below for the ratio of intrinsic to extrinsic distances.

See [94,149,151] for related results and further references.

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REFERENCES

[1] V. Agostiniani, M. Fogagnolo, and L. Mazzieri, Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature. *Invent. Math.* **222** (2020), no. 3, 1033–1101.

[2] A. Alarcon and F. Forstneric, New complex analytic methods in the theory of minimal surfaces: a survey. *J. Aust. Math. Soc.* **106** (2019), no. 3, 287–341.

[3] W. K. Allard and F. J. Almgren Jr., On the radial behavior of minimal surfaces and the uniqueness of their tangent cones. *Ann. of Math. (2)* **113** (1981), no. 2, 215–265.

[4] S. Altschuler, S. B. Angenent, and Y. Giga, Mean curvature flow through singularities for surfaces of rotation. *J. Geom. Anal.* **5** (1995), no. 3, 293–358.

[5] L. Ambrosio and H. M. Soner, Level set approach to mean curvature flow in arbitrary codimension. *J. Differential Geom.* **43** (1996), no. 4, 693–737.

[6] M. T. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature. *J. Differential Geom.* **18** (1983), no. 4, 701–721.

[7] M. T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math. (2)* **121** (1985), no. 2, 429–446.

[8] B. Andrews, Noncollapsing in mean-convex mean curvature flow. *Geom. Topol.* **16** (2012), no. 3, 1413–1418.

[9] B. Andrews, B. Chow, C. Guenther, and M. Langford, *Extrinsic geometric flows*. Grad. Stud. Math. 206, American Mathematical Society, Providence, RI, 2020.

[10] B. Andrews, H. Li, and Y. Wei, $\mathscr{F}$-stability for self-shrinking solutions to mean curvature flow. *Asian J. Math.* **18** (2014), no. 5, 757–777.

[11] B. Andrews and L. Ni, Eigenvalue comparison on Bakry–Emery manifolds. *Comm. Partial Differential Equations* **37** (2012), no. 11, 2081–2092.

[12] S. Angenent, P. Daskalopoulos, and N. Sesum, Unique asymptotics of ancient convex mean curvature flow solutions. *J. Differential Geom.* **111** (2019), no. 3, 381–455.

[13] S. Angenent, P. Daskalopoulos, and N. Sesum, Uniqueness of two-convex closed ancient solutions to the mean curvature flow. *Ann. of Math. (2)* **192** (2020), no. 2, 353–436.

[14] C. Arezzo and J. Sun, Self-shrinkers for the mean curvature flow in arbitrary codimension. *Math. Z.* **274** (2013), no. 3–4, 993–1027.

[15] R. Bamler, Structure theory of singular spaces. *J. Funct. Anal.* **272** (2017), no. 6, 2504–2627.

[16] R. Bamler, Recent developments in Ricci flows. *Not. Amer. Math. Soc.* **68**, (2021), no. 9, 1486–1498.

[17] R. Bamler and B. Kleiner, Uniqueness and stability of Ricci flow through singularities. *Acta Math.*, to appear.

[18] J. Bernstein, Asymptotic structure of almost eigenfunctions of drift Laplacians on conical ends. *Amer. J. Math.* **142** (2020), no. 6, 1897–1929.

[19] J. Bernstein and C. Breiner, Conformal Structure of Minimal Surfaces with Finite Topology. *J. Reine Angew. Math.* **655** (2011), 129–146.

[20] J. Bernstein and L. Wang, A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. *Invent. Math.* **206** (2016), no. 3, 601–627.

[21] J. Bernstein and L. Wang, A topological property of asymptotically conical self-shrinkers of small entropy. *Duke Math. J.* **166** (2017), no. 3, 403–435.

[22] S. Biard, J. Fornaess, and J. Wu, Weighted L^2 version of Mergelyan and Carleman approximation. *J. Geom. Anal.* **31** (2021), 3889–3914.

[23] C. Böhm, Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. *Invent. Math.* **134** (1998), no. 1, 145–176.

[24] K. Brakke, *The motion of a surface by its mean curvature*. Math. Notes 20, Princeton University Press, Princeton, 1978.

[25] S. Brendle, *Ricci flow and the sphere theorem*. Grad. Stud. Math. 111, American Mathematical Society, Providence, RI, 2010.

[26] S. Brendle, Rotational symmetry of self-similar solutions to the Ricci flow. *Invent. Math.* **194** (2013), no. 3, 731–764.

[27] S. Brendle, Rotational symmetry of Ricci solitons in higher dimensions. *J. Differential Geom.* **97** (2014), no. 2, 191–214.

[28] S. Brendle, Embedded self-similar shrinkers of genus 0. *Ann. of Math. (2)* **183** (2016), no. 2, 715–728.

[29] S. Brendle and K. Choi, Uniqueness of convex ancient solutions to mean curvature flow in $\mathbf{R}^3$. *Invent. Math.* **217** (2019), no. 1, 35–76.

[30] S. Brendle and G. Huisken, Mean curvature flow with surgery of mean convex surfaces in $\mathbf{R}^3$. *Invent. Math.* **203** (2016), no. 2, 615–654.

[31] E. Calabi, Final chapter in Problems in differential geometry. In *Proceedings of the United States-Japan seminar in differential geometry, Kyoto, Japan, 1965*, edited by S. Kobayashi and J. Eells Jr. Nippon Hyoronsha Co., Ltd., Tokyo, 1966.

[32] H. D. Cao, Recent progress on Ricci solitons. Recent advances in geometric analysis. In *Recent advances in geometric analysis*, pp. 1–38, Adv. Lect. Math. (ALM) 11, Int. Press, Somerville, MA, 2010.

[33] H. D. Cao, R. Hamilton, and T. Ilmanen, Gaussian densities and stability for some Ricci solitons. 2004, arXiv:math/0404165v1.

[34] H. D. Cao and C. He, Linear stability of Perelman's ν-entropy on symmetric spaces of compact type. *J. Reine Angew. Math.* **709** (2015), 229–246.

[35] H. D. Cao and D. Zhou, On complete gradient shrinking Ricci solitons. *J. Differential Geom.* **85** (2010), 175–186.

[36] J. Carrillo and L. Ni, Sharp logarithmic Sobolev inequalities on gradient solitons and applications. *Comm. Anal. Geom.* **17** (2009), no. 4, 721–753.

[37] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)* **144** (1996), no. 1, 189–237.

[38] J. Cheeger, T. H. Colding, and W. P. Minicozzi II, Linear growth harmonic functions on complete manifolds with nonnegative Ricci curvature. *Geom. Funct. Anal.* **5** (1995), no. 6, 948–954.

[39] J. Cheeger and G. Tian, On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay. *Invent. Math.* **118** (1994), no. 3, 493–571.

[40] B.-L. Chen, Strong uniqueness of the Ricci flow. *J. Differential Geom.* **82** (2009), no. 2, 363–382.

[41] Y. G. Chen, Y. Giga, and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* **33** (1991), no. 3, 749–786.

[42] S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.* **28** (1975), 333–354.

[43] X. Cheng and D. Zhou, Eigenvalues of the drifted Laplacian on complete metric measure spaces. *Commun. Contemp. Math.* **19** (2017), 1650001.

[44] S. S. Chern, The geometry of G-structures. *Bull. Amer. Math. Soc.* **72** (1966), 167–219.

[45] O. Chodosh, K. Choi, C. Mantoulidis, and F. Schulze, Mean curvature flow with generic low-entropy initial data. Preprint.

[46] O. Chodosh and C. Li, Stable minimal hypersurfaces in $\mathbf{R}^4$. Preprint.

[47] O. Chodosh and F. Schulze, Uniqueness of asymptotically conical tangent flows. *Duke Math. J.* 170 (2021), no. 16, 3601–3657.

[48] K. Choi, R. Haslhofer, and O. Hershkovits, Ancient low entropy flows, mean convex neighborhoods, and uniqueness. *Acta Math.* (to appear).

[49] B. Chow and P. Lu, On κ-noncollapsed complete noncompact shrinking gradient Ricci solitons which split at infinity. *Math. Ann.* **366** (2016), no. 3–4, 1195–1206.

[50] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*. Grad. Stud. Math. 77, AMS, Providence, RI, 2006.

[51] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications. Part I. Geometric aspects*. Math. Surveys Monogr. 135, American Mathematical Society, Providence, RI, 2007.

[52] T. H. Colding, Spaces with Ricci curvature bounds. In *Proceedings of the International Congress of Mathematicians*, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 299–308.

[53] T. H. Colding, New monotonicity formulas for Ricci curvature and applications, I. *Acta Math.* **209** (2012), no. 2, 229–263.

[54] T. H. Colding, T. Ilmanen, and W. P. Minicozzi II, Rigidity of generic singularities of mean curvature flow. *Publ. Math. Inst. Hautes Études Sci.* **121** (2015), 363–382.

[55] T. H. Colding, T. Ilmanen, W. P. Minicozzi II, and B. White, The round sphere minimizes entropy among closed self-shrinkers. *J. Differential Geom.* **95** (2013), 53–69.

[56] T. H. Colding and W. P. Minicozzi II, Complete properly embedded minimal surfaces in $\mathbf{R}^3$. *Duke Math. J.* 107 (2001), no. 2, 421–426.

[57] T. H. Colding and W. P. Minicozzi II, Estimates for parametric elliptic integrands. *Int. Math. Res. Not.* **6** (2002), 291–297.

[58] T. H. Colding and W. P. Minicozzi II, In search of stable geometric structures. *Notices Amer. Math. Soc.* **66** (2019), no. 11, 1785–1791.

[59] T. H. Colding and W. P. Minicozzi II, Harmonic functions on manifolds. *Ann. of Math. (2)* **146** (1997), no. 3, 725–747.

[60] T. H. Colding and W. P. Minicozzi II, Harmonic functions with polynomial growth. *J. Differential Geom.* **46** (1997), no. 1, 1–77.

[61] T. H. Colding and W. P. Minicozzi II, Large scale behavior of kernels of Schrödinger operators. *Amer. J. Math.* **119** (1997), no. 6, 1355–1398.

[62] T. H. Colding and W. P. Minicozzi II, Liouville theorems for harmonic sections and applications. *Comm. Pure Appl. Math.* **51** (1998), no. 2, 113–138.

[63] T. H. Colding and W. P. Minicozzi II, Weyl type bounds for harmonic functions. *Invent. Math.* **131** (1998), no. 2, 257–298.

[64] T. H. Colding and W. P. Minicozzi II, Disks that are double spiral staircases. *Notices Amer. Math. Soc.* **50** (2003), no. 3, 327–339.

[65] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks. *Ann. of Math. (2)* **160** (2004), no. 1, 27–68.

[66] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks. *Ann. of Math. (2)* **160** (2004), no. 1, 69–92.

[67] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains. *Ann. of Math. (2)* **160** (2004), no. 2, 523–572.

[68] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply connected. *Ann. of Math. (2)* **160** (2004), no. 2, 573–615.

[69] T. H. Colding and W. P. Minicozzi II, Shapes of embedded minimal surfaces. *Proc. Natl. Acad. Sci. USA* **103** (2006), no. 30, 11106–11111.

[70] T. H. Colding and W. P. Minicozzi II, The Calabi–Yau conjectures for embedded surfaces. *Ann. of Math. (2)* **167** (2008), no. 1, 211–243.

[71] T. H. Colding and W. P. Minicozzi II, *A course in minimal surfaces*. Grad. Stud. Math. 121, AMS, Providence, RI, 2011.

[72] T. H. Colding and W. P. Minicozzi II, Generic mean curvature flow I; generic singularities. *Ann. of Math.* **175** (2012), no. 2, 755–833.

[73] T. H. Colding and W. P. Minicozzi II, Smooth compactness of self-shrinkers. *Comment. Math. Helv.* **87** (2012), no. 2, 463–475.

[74] T. H. Colding and W. P. Minicozzi II, On uniqueness of tangent cones for Einstein manifolds. *Invent. Math.* **196** (2014), no. 3, 515–588.

[75] T. H. Colding and W. P. Minicozzi II, Ricci curvature and monotonicity for harmonic functions. *Calc. Var. Partial Differential Equations* **49** (2014), no. 3–4, 1045–1059.

[76] T. H. Colding and W. P. Minicozzi II, Łojasiewicz inequalities and applications. In *Regularity and evolution of nonlinear equations, Essays dedicated to Richard Hamilton, Leon Simon, and Karen Uhlenbeck*, pp. 63–82, Surv. Differ. Geom. 19, International Press, 2015.

[77] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold V; fixed genus. *Ann. of Math. (2)* **181** (2015), no. 1, 1–153.

[78] T. H. Colding and W. P. Minicozzi II, Uniqueness of blowups and Łojasiewicz inequalities. *Ann. of Math.* **182** (2015), no. 1, 221–285.

[79] T. H. Colding and W. P. Minicozzi II, Differentiability of the arrival time. *Comm. Pure Appl. Math.* **LXIX** (2016), 2349–2363.

[80] T. H. Colding and W. P. Minicozzi II, Level set method for motion by mean curvature. *Notices Amer. Math. Soc.* **63** (2016), no. 10, 1148–1153.

[81] T. H. Colding and W. P. Minicozzi II, The singular set of mean curvature flow with generic singularities. *Invent. Math.* **204** (2016), no. 2, 443–471.

[82] T. H. Colding and W. P. Minicozzi II, Regularity of the level set flow. *Comm. Pure Appl. Math.* **71** (2018), no. 4, 814–824.

[83] T. H. Colding and W. P. Minicozzi II, Sharp frequency bounds for eigenfunctions of the Ornstein-Uhlenbeck operator. *Calc. Var. Partial Differential Equations* **57** (2018), no. 5, 138.

[84] T. H. Colding and W. P. Minicozzi II, Arnold–Thom gradient conjecture for the arrival time. *Comm. Pure Appl. Math.* **72** (2019), no. 7, 1548–1577.

[85] T. H. Colding and W. P. Minicozzi II, Dynamics of closed singularities. *Ann. Inst. Fourier (Grenoble)* **69** (2019), no. 7, 2973–3016.

[86] T. H. Colding and W. P. Minicozzi II, Liouville properties. *Notices ICCM* **7** (2019), no. 1, 16–26.

[87] T. H. Colding and W. P. Minicozzi II, Complexity of parabolic systems. *Publ. Math. Inst. Hautes Études Sci.* (2020), 83–135.

[88] T. H. Colding and W. P. Minicozzi II, Wandering singularities. *J. Differential Geom.* **119** (2021), 403–420.

[89] T. H. Colding and W. P. Minicozzi II, Optimal bounds for ancient caloric functions. *Duke Math. J.* 170 (2021), no. 18, 4171-4182.

[90] T. H. Colding and W. P. Minicozzi II, Regularity of elliptic and parabolic systems. Preprint.

[91] T. H. Colding and W. P. Minicozzi II, Singularities of Ricci flow and diffeomorphisms. Preprint.

[92] T. H. Colding, W. P. Minicozzi II, and E. K. Pedersen, Mean curvature flow. *Bull. Amer. Math. Soc. (N.S.)* **52** (2015), no. 2, 297–333.

[93] P. Collin, Topologie et courbure des surfaces minimales proprement plongees de $\mathbf{R}^3$. *Ann. of Math. (2)* **145** (1997), no. 1, 1–31.

[94] B. Coskunuzer, W. H. Meeks III, and G. Tinaglia, Non-properly embedded H-planes in $\mathbf{H}$. *J. Differential Geom.* **105** (2017), no. 3, 405–425.

[95] C. De Lellis, The regularity theory for the area functional (in geometric measure theory). Preprint.

[96] C. De Lellis, E. Spadaro, and L. Spolaor, Uniqueness of tangent cones for two-dimensional almost-minimizing currents. *Comm. Pure Appl. Math.* **70** (2017), no. 7, 1402–1421.

[97] H. Donnelly and C. Fefferman, Nodal domains and growth of harmonic functions on noncompact manifolds. *J. Geom. Anal.* **2** (1992), 79–93.

[98] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I. *J. Differential Geom.* **33** (1991), 635–681.

[99] D. Fischer-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.* **33** (1980), no. 2, 199–211.

[100] M. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves. *J. Differential Geom.* **23** (1986), no. 1, 69–96.

[101] Z. Gang and D. Knopf, Universality in mean curvature flow neckpinches. *Duke Math. J.* **164** (2015), no. 12, 2341–2406.

[102] N. Garofalo and F. H. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.* **35** (1986), no. 2, 245–268.

[103] M. A. Grayson, The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.* **26** (1987), no. 2, 285–314.

[104] M. Gromov, Groups of polynomial growth and expanding maps. *Publ. Math. Inst. Hautes Études Sci.* **53** (1981), 53–73.

[105] M. Gursky and J. Viaclovsky, Rigidity and stability of Einstein metrics for quadratic curvature functionals. *J. Reine Angew. Math.* **700** (2015), 37–91.

[106] R. Hamilton, Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), 255–306.

[107] R. Hamilton, The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pp. 7–136, Int. Press, Cambridge, MA, 1993.

[108] R. M. Hardt, Singularities of harmonic maps. *Bull. Amer. Math. Soc. (N.S.)* **34** (1997), no. 1, 15–34.

[109] R. Haslhofer and B. Kleiner, Mean curvature flow of mean convex hypersurfaces. *Comm. Pure Appl. Math.* **70** (2017), no. 3, 511–546.

[110] R. Haslhofer and B. Kleiner, Mean curvature flow with surgery. *Duke Math. J.* **166** (2017), no. 9, 1591–1626.

[111] H.-J. Hein and A. Naber, New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow. *Comm. Pure Appl. Math.* **67** (2014), no. 9, 1543–1561.

[112] H. J. Hein and S. Sun, Calabi-Yau manifolds with isolated conical singularities. *Publ. Math. Inst. Hautes Études Sci.* 126 (2017), 73–130.

[113] D. Hoffman and W. H. Meeks III, The strong halfspace theorem for minimal surfaces. *Invent. Math.* **101** (1990), no. 2, 373–377.

[114] G. Huisken, Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.* **20** (1984), no. 1, 237–266.

[115] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.* **31** (1990), no. 1, 285–299.

[116] G. Huisken, Local and global behaviour of hypersurfaces moving by mean curvature. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990). Part 1*, pp. 175–191, Proc. Sympos. Pure Math. 54, Amer. Math. Soc., Providence, RI, 1993.

[117] G. Huisken and C. Sinestrari, Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Math.* **183** (1999), no. 1, 45–70.

[118] G. Huisken and C. Sinestrari, Mean curvature flow singularities for mean convex surfaces. *Calc. Var. Partial Differ. Equ.* **8** (1999), 1–14.

[119] G. Huisken and C. Sinestrari, Mean curvature flow with surgeries of two-convex hypersurfaces. *Invent. Math.* **175** (2009), no. 1, 137–221.

[120] T. Ilmanen, Generalized flow of sets by mean curvature on a manifold. *Indiana Univ. Math. J.* **41** (1992), no. 3, 671–705.

[121] T. Ilmanen, Singularities of mean curvature flow of surfaces. Preprint, 1995.

[122] T. Ivey, Ricci solitons on compact three-manifolds. *Differential Geom. Appl.* **3** (1993), no. 4, 301–307.

[123] L. Jorge and F. Xavier, A complete minimal surface in $\mathbf{R}^3$ between two parallel planes. *Ann. of Math. (2)* **112** (1980), 203–206.

[124] A. Kasue, Harmonic functions of polynomial growth on complete manifolds. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990). Part 1*, pp. 281–290, Proc. Sympos. Pure Math. 54, Amer. Math. Soc., Providence, RI, 1993.

[125] A. Kasue, Harmonic functions of polynomial growth on complete manifolds II. *J. Math. Soc. Japan* **47** (1995), 37–65.

[126] J. Kazdan, Parabolicity and the Liouville property on complete Riemannian manifolds. In *Aspects of Math.*, pp. 153–166, Vieweg, Braunschweig, 1987.

[127] D. Ketover and X. Zhou, Entropy of closed surfaces and min–max theory. *J. Differential Geom.* **110** (2018), no. 1, 31–71.

[128] B. Kleiner, A new proof of Gromov's theorem on groups of polynomial growth. *J. Amer. Math. Soc.* **23** (2010), no. 3, 815–829.

[129] B. Kleiner and J. Lott, Notes on Perelman's papers. *Geom. Topol.* **12** (2008), no. 5, 2587–2855.

[130] B. Kleiner and J. Lott, Singular Ricci flows. *Acta Math.* (2017), 65–134.

[131] R. V. Kohn and S. Serfaty, A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.* **59** (2006), no. 3, 344–407.

[132] B. Kotschwar and L. Wang, Rigidity of asymptotically conical shrinking gradient Ricci solitons. *J. Differential Geom.* **100** (2015), no. 1, 55–108.

[133] B. Kotschwar and L. Wang, A uniqueness theorem for asymptotically cylindrical shrinking Ricci solitons. *J. Differential Geom.* (to appear).

[134] P. Kuchment, An overview of periodic elliptic operators. *Bull. Amer. Math. Soc. (N.S.)* **53** (2016), no. 3, 343–414.

[135] Y.-I. Lee and Y.-K. Lue, The stability of self-shrinkers of mean curvature flow in higher co-dimension. *Trans. Amer. Math. Soc.* **367** (2015), no. 4, 2411–2435.

[136] P. Li, Linear growth harmonic functions on Kähler manifolds with non–negative Ricci curvature. *Math. Res. Lett.* **2** (1995), 79–94.

[137] P. Li, The theory of harmonic functions and its relation to geometry. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, pp. 307–315, Proc. Sympos. Pure Math. 54, Part 1, Amer. Math. Soc., Providence, RI, 1993.

[138] P. Li and L. F. Tam, Linear growth harmonic functions on a complete manifold. *J. Differential Geom.* **29** (1989), 421–425.

[139] P. Li and S. T. Yau, On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** (1986), no. 3–4, 153–201.

[140] Y. Li and B. Wang, Rigidity of the round cylinders in Ricci shrinkers. Preprint.

[141] F. H. Lin and Q. S. Zhang, On ancient solutions of the heat equation. *Comm. Pure Appl. Math.* **72** (2019), no. 9, 2006–2028.

[142] G. Liu, Three-circle theorem and dimension estimate for holomorphic functions on Kähler manifolds. *Duke Math. J.* **165** (2016), no. 15, 2899–2919.

[143] A. Logunov, Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. *Ann. of Math. (2)* **187** (2018), no. 1, 221–239.

[144] J. Lott, Some geometric properties of the Bakry–Emery–Ricci tensor. *Comment. Math. Helv.* **78** (2003), no. 4, 865–883.

[145] D. S. Lubinsky, A survey of weighted polynomial approximation with exponential weights. *Surv. Approx. Theory* **3** (2007), 1–105.

[146] F. C. Marques and A. Neves, Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Invent. Math.* **209** (2017), no. 2, 577–616.

[147] F. C. Marques and A. Neves, The space of cycles, a Weyl law for minimal hypersurfaces and Morse index estimates. In *Surveys in differential geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry*, pp. 319–329, Surv. Differ. Geom. 22, Int. Press, Somerville, MA, 2018.

[148] W. Meeks III, J. Perez, and A. Ros, Properly embedded minimal planar domains. *Ann. of Math. (2)* **181** (2015), no. 2, 473–546.

[149] W. Meeks III, J. Perez, and A. Ros, The embedded Calabi-Yau conjecture for finite genus. *Duke Math. J.* 170 (2021), no. 13, 2891–2956.

[150] W. Meeks III and H. Rosenberg, The uniqueness of the helicoid. *Ann. of Math. (2)* **161** (2005), no. 2, 727–758.

[151] W. Meeks III and G. Tinaglia, Limit lamination theorem for H-disks. *Invent. Math.* **226** (2021), 393–420.

[152] W. Meeks and S.-T. Yau, Topology of three-dimensional manifolds and the embedding problems in minimal surface theory. *Ann. of Math. (2)* **112** (1980), no. 3, 441–484.

[153] M. Micallef, Stable minimal surfaces in Euclidean space. *J. Differential Geom.* **19** (1984), no. 1, 57–84.

[154] A. Naber, The geometry of Ricci curvature. In *Proceedings of the International Congress of Mathematicians–Seoul 2014. Vol. II*, pp. 911–937, Kyung Moon Sa, Seoul, 2014.

[155] A. Naber and D. Valtorta, The singular structure and regularity of stationary varifolds. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 10, 3305–3382.

[156] N. Nadirashvili, Hadamard's and Calabi–Yau's conjectures on negatively curved and minimal surfaces. *Invent. Math.* **126** (1996), no. 3, 457–465.

[157] L. Ni, A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature. *J. Amer. Math. Soc.* **17** (2004), no. 4, 909–946.

[158] L. Ni and N. Wallach, On a classification of gradient shrinking solitons. *Math. Res. Lett.* **15** (2008), no. 5, 941–955.

[159] S. Osher and J. A. Sethian, Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations. *J. Comput. Phys.* **79** (1988), no. 1, 12–49.

[160] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. 2002, arXiv:math/0211159.

[161] J. Perez, A new golden age of minimal surfaces. Notices Amer. Math. Soc. **64** (2017), no. 4, 347–358.

[162] P. Petersen and W. Wylie, Rigidity of gradient Ricci solitons. *Pacific J. Math.* **241** (2009), 329–345.

[163] T. Riviere and G. Tian, The singular set of 1–1 integral currents. *Ann. of Math. (2)* **169** (2009), no. 3, 741–794.

[164] R. Schoen, Estimates for stable minimal surfaces in three-dimensional manifolds. Seminar on minimal submanifolds. In *Ann. of Math. Stud.*, pp. 111–126, 103, Princeton Univ. Press, Princeton, NJ, 1983.

[165] R. Schoen, The effect of curvature on the behavior of harmonic functions and mappings. In *Nonlinear partial differential equations in differential geometry (Park City, UT, 1992)*, pp. 127–184, IAS/Park City Math. Ser. 2, 1992, Amer. Math. Soc., Providence, RI, 1996.

[166] R. Schoen L, Simon, and S. T. Yau, Curvature estimates for minimal hypersurfaces. *Acta Math.* **134** (1975), no. 3–4, 275–288.
[167] F. Schulze, Uniqueness of compact tangent flows in mean curvature flow. *J. Reine Angew. Math.* **690** (2014), 163–172.
[168] N. Sesum, Rate of convergence of the mean curvature flow. *Comm. Pure Appl. Math.* **61** (2008), no. 4, 464–485.
[169] Y. Shalom and T. Tao, A finitary version of Gromov's polynomial growth theorem. *Geom. Funct. Anal.* **20** (2010), no. 6, 1502–1547.
[170] L. Simon, Asymptotics for a class of evolution equations, with applications to geometric problems. *Ann. of Math.* **118** (1983), 525–571.
[171] L. Simon, Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps. In *Surveys in differential geometry, Vol. II*, pp. 246–305, Int. Press, Cambridge, MA, 1995.
[172] A. Song, Existence of infinitely many minimal hypersurfaces in closed manifolds. Preprint.
[173] D. Sullivan, The Dirichlet problem at infinity for a negatively curved manifold. *J. Differential Geom.* **18** (1983), no. 4, 723–732.
[174] G. Székelyhidi, Uniqueness of certain cylindrical tangent cones, preprint.
[175] T. Tao, Kleiner's proof of Gromov's theorem. https://terrytao.wordpress.com/2008/02/14/kleiners-proof-of-gromovs-theorem/
[176] T. Tao, A proof of Gromov's theorem. https://terrytao.wordpress.com/2010/02/18/a-proof-of-gromovs-theorem/
[177] T. Tao, "What is a gauge". https://terrytao.wordpress.com/2008/09/27/what-is-a-gauge/.
[178] P. Topping, *Lectures on the Ricci flow*. London Math. Soc. Lecture Note Ser. 325, Cambridge University Press, Cambridge, 2006.
[179] L. Wang, Uniqueness of self-similar shrinkers with asymptotically conical ends. *J. Amer. Math. Soc.* **27** (2014), 613–638.
[180] B. White, Partial regularity of mean-convex hypersurfaces flowing by mean curvature. *Int. Math. Res. Not. IMRN* **4** (1994), 185–192.
[181] B. White, Stratification of minimal surfaces, mean curvature flows, and harmonic maps. *J. Reine Angew. Math.* **488** (1997), 1–35.
[182] B. White, The size of the singular set in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.* **13** (2000), no. 3, 665–695.
[183] B. White, Evolution of curves and surfaces by mean curvature. In *Proceedings of the international congress of mathematicians, Vol. I (Beijing, 2002)*, pp. 525–538, Higher Ed. Press, 2002.
[184] B. White, The nature of singularities in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.* **16** (2003), no. 1, 123–138.
[185] B. White, A local regularity theorem for mean curvature flow. *Ann. of Math. (2)* **161** (2005), no. 3, 1487–1519.

[186] N. Wickramasekera, Regularity and compactness for stable codimension 1 CMC varifolds. In *Current developments in mathematics 2017*, pp. 87–174, Int. Press, Somerville, MA, 2019.

[187] S. T. Yau, Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.* **28** (1975), 201–228.

[188] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana Univ. Math. J.* **25** (1976), no. 7, 659–670.

[189] S. T. Yau, Nonlinear analysis in geometry. *Enseign. Math. (2)* **33** (1987), 109–158.

[190] J. Zhu, On the entropy of closed hypersurfaces and singular self-shrinkers. *J. Differential Geom.* **114** (2020), no. 3, 551–593.

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