ONE-DIMENSIONAL QUASIPERIODIC OPERATORS: GLOBAL THEORY, DUALITY, AND SHARP ANALYSIS OF SMALL DENOMINATORS

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ABSTRACT

Spectral theory of one-dimensional discrete one-frequency Schrödinger operators is a field with the origins in and strong ongoing ties to physics. It features a fascinating competition between randomness (ergodicity) and order (periodicity), which is often resolved on a deep arithmetic level. This leads to an especially rich spectrum of phenomena, many of which we are only beginning to understand. The corresponding analysis involves, in particular, dealing with small denominator problems. It has led to the development of non-KAM methods in this traditionally KAM domain, and to results completely unattainable by the old techniques, also in a number of other settings. This article accompanies the author's lecture at the International Congress of Mathematicians 2022. It covers several related recent developments.

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One-dimensional discrete one-frequency Schrödinger operators

$$(H_{V,\alpha,x}u)_n := u_{n-1} + u_{n+1} + V(x+n\alpha)u_n,$$
$$u \in \ell^2(\mathbb{Z}), \quad \alpha \in \mathbb{T} := \mathbb{R}\backslash\mathbb{Q},\ x \in \mathbb{T},\ V : \mathbb{T} \to \mathbb{R}, \tag{0.1}$$

and related questions of the dynamics of quasiperiodic cocycles have not been underrepresented at the ICMs. As I remember, roughly within the last 25 years, there were sectional lectures by H. Eliasson in 1998, myself in 2002, B. Fayad, R. Krikorian, and J. You in 2018, as well as plenary lectures by A. Avila in 2010 and 2014, devoted either in part or in full to this topic.

The field itself is not at all new. It may be seen as having been originated in physics when Peierls [103] and later his student Harper [61] studied the tight-binding two-dimensional electron in a uniform perpendicular magnetic field (also known as the Harper model) and derived the by now iconic family $H_{2\lambda\cos,\alpha,x}$ that we now, following Barry Simon [105], call the almost Mathieu operator. It remains hugely popular in physics, being directly linked to several remarkable experimental discoveries and Nobel prizes, providing, in particular, the theoretical underpinning of the Quantum Hall Effect, as proposed by D. J. Thouless in 1983 (see, e.g. [18,19]). A Google search for "Harper's model physics" leads to many thousands of hits.

The field may also be seen as having been originated in a numerical experiment, as the interest was picked after Douglas Hofstadter came up with what we now call the Hofstadter's butterfly [64]—a beautiful numerically produced fractal (Figure 1), discovered even before the word "fractal" was coined by Benoit Mandelbrot. Finally, the field may be seen as having been originated from the first application of KAM in the spectral theory—a pioneering work of Dinaburg and Sinai [37], that preceded Hofstadter. The field has consistently attracted top mathematical physicists (e.g., Bellissard, Deift, Simon, Sinai, Spencer), dynamicists (e.g., Avila, Eliasson, Herman, Krikorian, You), and analysts (e.g., Bourgain, Eliott, Sarnak, Schlag). Indeed, it turned out to be a fantastic ever-expanding playground for the analysts and dynamicists alike, leading to strong cross-fertilization of ideas that have a tendency to later expand to other subjects. Jean Bourgain wrote a book [28] devoted to analytic, mostly one-dimensional, quasiperiodic operators that summarized significant new understanding achieved around the turn of the century, where the work of Jean and collaborators was central.

It is therefore all the more surprising that as of the time of this writing it seems that the field is on the verge of further significant breakthroughs, with our current understanding covering just the tip of an exciting iceberg. Given the remarkable current momentum, we will refrain from making an attempt at an overview of the vast past literature, neither even very recent nor a number of important milestones, and will concentrate instead only on two selected topics that enjoyed significant recent advances and hold a particular promise to shape some of the future discourse.

For the review up to about five years ago, see [82], and for various fine issues related to continuity of the Lyapunov exponents, featuring, in particular, very important work by M. Goldstein and W. Schlag, see the recent book by P. Duarte and S. Klein [38]. The 2018

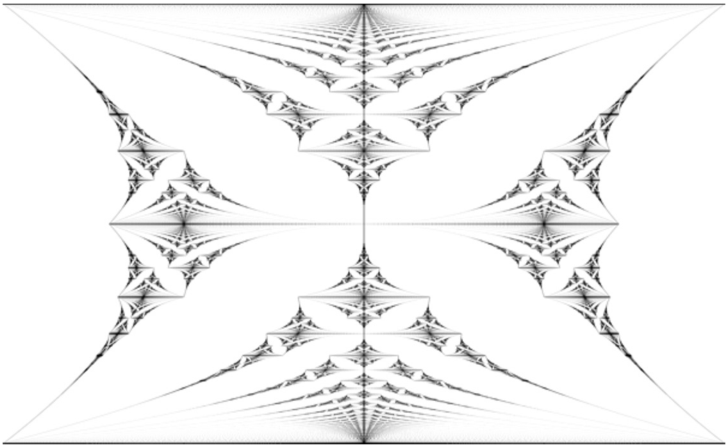

FIGURE 1
Hofstadter's butterfly.

ICM proceedings by J. You **[117]** summarize, among other things, the quantitative reducibility breakthrough developed in his group, that has led to a number of powerful consequences. There are also recent expositions **[68, 80]** that include some further remarkable results of roughly the last decade that could not make it into this article.

1. SPECTRAL THEORY MEETS (DUAL) DYNAMICS

Quasiperiodic operators (0.1) are, of course, a particular case of one-dimensional discrete ergodic Schrödinger operators

$$(H_x u)_n := u_{n-1} + u_{n+1} + V(T^n x)u_n, \quad u \in \ell^2(\mathbb{Z}), \tag{1.1}$$

where $x \in X$, and (X, μ, T) is an ergodic dynamical system. Operators with ergodic potentials (also in the continuum or in a more general multidimensional/covariant setting) always have spectra and closures of the other spectral components constant for μ-a.e. x **[95, 102]**. In case of the minimal underlying dynamics, such as, e.g., the irrational rotation of the circle in (0.1), the spectra **[21]** and absolutely continuous spectra in the one-dimensional case **[97]** are constant for *all* x. In contrast, the point and singular continuous parts (that are constant a.e.) can depend sensitively on x. It is an interesting problem, usually attributed to B. Simon, and open even in the setting of (0.1) whether this still holds when they are combined together (see Problem 6 in **[67]**).

The spectral theory of one-dimensional ergodic Schrödinger operators (1.1) is deeply connected to the study of linear cocycles over corresponding underlying dynamics. By an $\mathrm{SL}(2, \mathbb{R})$ cocycle, we mean a pair (T, A), where $T : X \to X$ is ergodic, A is a measurable 2×2 matrix-valued function on X and $\det A = 1$.

We can regard it as a dynamical system on $X \times \mathbb{R}^2$ with

$$(T, A) : (x, f) \mapsto \big(Tx, A(x)f\big), \quad (x, f) \in X \times \mathbb{R}^2.$$

A one-parameter family of Schrödinger cocycles over (X, μ, T), indexed by the energy $E \in \mathbb{C}$, is given by $(T, A) : (X, \mathbb{R}^2) \mapsto (X, \mathbb{R}^2)$ where $(T, A) : (x, y) \mapsto (Tx, A(x, E)y)$, and $A \in \mathrm{SL}(2, \mathbb{C})$ is the transfer-matrix

$$A(x, E) := \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},$$

with $x \in X$, $y \in \mathbb{R}^2$, and $E \in \mathbb{C}$. The eigenvalue equation $Hu = Eu$ can be rewritten dynamically as

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A\big(T^n x, E\big) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}.$$

The (top) Lyapunov exponent is then defined as $L(E) := \lim_{n\to\infty} \int \frac{1}{n} \ln \|A_n(x, E)\| \, d\mu$, where

$$A_n(x, E) := \prod_{i=n-1}^{0} A(T^i x, E). \tag{1.2}$$

Two classical results link dynamics/Lyapunov exponents to the spectral theory of ergodic operators:

- (Johnson's theorem [91]) For minimal (X, μ, T), the spectrum $\sigma(H)$ (which is constant in $x \in X$) is given by the set of $E \in \mathbb{R}$ such that the Schrödinger cocycle $(T, A(\cdot, E))$ is not uniformly hyperbolic.
- (Kotani theory [94]) The absolutely continuous spectrum $\sigma_{ac}(H)$ (μ- a.e. constant for any ergodic (X, μ, T) and constant for minimal systems [97]) is given by the essential closure of the set $\{E : L(E) = 0\}$.

Therefore, for minimal, and in particular quasiperiodic, underlying dynamics, spectrum and absolutely continuous spectrum of H_x are encoded by the dynamics of the one-parameter family $A(x, E)$ of transfer-matrix cocycles, indexed by the energy E, but, for the spectrum, not by any explicit quantity. One recent surprising development is that for analytic one-frequency quasiperiodic Schrodinger operators, the spectrum (and therefore absence of uniform hyperbolicity of the corresponding cocycles) can be characterized more directly. In [47] we introduce a new object, dual Lyapunov exponent $\hat{L}(E)$, and prove

Theorem 1.1 ([47]). *For quasiperiodic operators* (0.1) *with analytic* V*,*

$$\sigma(H) = \big\{E : L(E)\hat{L}(E) = 0\big\}. \tag{1.3}$$

Exponent $\hat{L}(E)$ is defined as the limit of lowest Lyapunov exponents of dual high-dimensional cocycles (see Sections 2 and 4) which is proved to exist. There are interesting questions of varying levels of difficulty on whether this can be appropriately extended to higher-dimensional analytic one-frequency quasiperiodic Schrodinger cocycles, corresponding to operators on the strips, to multifrequency analytic cocycles, to nonanalytic potentials, or even other underlying dynamics. Perhaps the most natural question is whether one can find an analytic characterization of the absence of uniform hyperbolicity for all analytic one-frequency quasiperiodic cocycles. For the latter, there is a topological obstruction, but one can reduce the question, say, to cocycles homotopic to the identity.

2. AUBRY DUALITY AND HIGHER-DIMENSIONAL COCYCLES

The early work of Dinaburg–Sinai [37] notwithstanding, it is fair to say that the study of the spectral theory of quasiperiodic operators has been largely shaped around and driven by several explicit models, all coming from physics. The most prominent of those is the almost Mathieu family $H_{2\lambda\cos,\alpha,x}$, which can be argued to be the tight-binding analogue of a harmonic oscillator. Besides being the main model in the related physics studies and that featured in the Hofstadter's butterfly, it is also the simplest, in many ways, analytic case, yet it seems to represent most of the nontrivial properties expected to be encountered in the more general situation. In some sense, it plays the same role in the theory of quasiperiodic operators that the Ising model plays in statistical mechanics, and similarly to the latter, it does have an important additional symmetry.

Namely, we define the Aubry dual of the one-frequency Schrödinger operator (0.1) as

$$(\hat{H}_{V,\alpha,\theta}u)_n = \sum_{k=-\infty}^{\infty} V_k u_{n+k} + 2\cos 2\pi(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}, \tag{2.1}$$

where V_k is the kth Fourier coefficient of V.[1] It can be useful to view this as a transformation of the entire family indexed by x for fixed V, α. In this regard, this transform can be viewed as a unitary conjugation on $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{Z})$, via

$$U\psi(x,n) = \hat{\psi}(n, x + \alpha n), \tag{2.2}$$

where $\hat{\psi} : L^2(\mathbb{Z} \times \mathbb{T}) \to L^2(\mathbb{T} \times \mathbb{Z})$ is the Fourier transform. The almost Mathieu family is self-dual with respect to this transformation $\hat{H}_{2\lambda\cos,\alpha,x} = H_{\frac{2}{\lambda}\cos,\alpha,\theta}$, and, in particular, $H_{2\cos,\alpha,x}$, that is, $H_{2\lambda\cos,\alpha,x}$ with $\lambda = 1$, is the self-dual (also called critical) point.

Aubry duality can be explained by the magnetic nature and corresponding gauge invariance of two-dimensional magnetic Laplacians that lead to $H_{V,\alpha,x}$ [101]. In particular, spectra and integrated densities of states of $H_{V,\alpha,x}$ and $\hat{H}_{V,\alpha,x}$ coincide. However, it is not the case for the spectral type, and indeed it is natural to expect that a Fourier-type transform would take localized eigenfunctions (point spectrum!) into extended ones (absolutely continuous spectrum!), and vice versa. That was the basis for several predictions by physicists Aubry and Andre [1] about the almost Mathieu family with irrational α, namely that the spectrum of $H_{2\lambda\cos,\alpha,x}$ is absolutely continuous for $\lambda < 1$ (called subcritical) and pure point for $\lambda > 1$ (called supercritical). This was described in the paper where transformation (2.1)

1 There is a more general, multidimensional definition, but we stick to the one-dimensional case for this exposition.

was introduced in the context of the almost Mathieu family, leading to the name Aubry duality. This problem, along with a few others related to this family, was heavily popularized by Barry Simon in [106,108], fueling an increased interest in the mathematics community.

Aubry duality has been formulated and explored on different levels, e.g., [10,55,101]. It has consistently played a central role in the analysis of quasiperiodic operators, in proving absolutely continuous spectrum and reducibility [10,31], point spectrum [17,24,50,57,70],[2] or its absence [11,69].

In general, operator (2.1) is long-range. If V is a trigonometric polynomial of degree d, the transfer-matrix $A(x,E)$ of the eigenvalue equation $\hat{H}_{V,\alpha,x}\Psi = E\Psi$ gives rise to a $2d$-dimensional cocycle, which has a complex-symplectic structure [60], so we will view it as an $\mathrm{Sp}(2d,\mathbb{C})$ cocycle (α, A), $A \in \mathrm{Sp}(2d,\mathbb{C})$, a linear skew product

$$(\alpha, A) : \left\{ \begin{array}{ccc} \mathbb{T} \times \mathbb{C}^{2d} & \to & \mathbb{T} \times \mathbb{C}^{2d} \\ (x, v) & \mapsto & (x+\alpha, A(x,E)\cdot v) \end{array} \right\}.$$

The Lyapunov exponents $L_1(\alpha,A) \geq L_2(\alpha,A) \geq \cdots \geq L_{2d}(\alpha,A)$, repeated according to their multiplicity, are defined by

$$L_k(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \ln\bigl(\sigma_k\bigl(A_n(x)\bigr)\bigr)dx,$$

where for a matrix $B \in M_m(\mathbb{C})$, $\sigma_1(B) \geq \cdots \geq \sigma_m(B)$ denote its singular values (eigenvalues of $\sqrt{B^*B}$). Since for real E the transfer-matrix $A(x,E)$ of the eigenvalue equation $\hat{H}_{V,\alpha,x}\Psi = E\Psi$ is symplectic, its Lyapunov exponents come in the opposite pairs $\{\pm L_i(\alpha,A)\}_{i=1}^{d}$. We will now denote

$$\hat{L}_i = L_{d-i}(\alpha, A), \tag{2.3}$$

so that $0 \leq \hat{L}_1 \leq \hat{L}_2 \leq \cdots \leq \hat{L}_d$.

In general, Lyapunov exponents are not nicely behaved with respect to parameter changes. They can be (and most likely, typically are) discontinuous in α at $\alpha \in \mathbb{Q}$ (the almost Mathieu cocycle is one example), are generally discontinuous in A in C^0, and can be discontinuous in A even in C^∞ [35,81,113,114]. It is a remarkable fact, enabling much of the related theory, that Lyapunov exponents are continuous in the analytic category.

Theorem 2.1 ([12,29,31,73]). *The functions* $\mathbb{R} \times C^\omega(\mathbb{T}, M_m(\mathbb{C})) \ni (\alpha, A) \mapsto L_k(\alpha, A) \in [-\infty, \infty)$ *are continuous at any* (α', A') *with* $\alpha' \in \mathbb{R}\backslash\mathbb{Q}$.[3]

For the almost Mathieu operator, it leads to the exact formula for the Lyapunov exponent for energies E in the spectrum of $H_{2\lambda\cos,\alpha,x}$. We have $L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}$ [30].

For Diophantine α, this continuity extends to sufficiently smooth Gevrey spaces [35,92], and it is a remarkable recent result [48] that for certain α the transition in the topology

2 Made possible with the development of recent powerful methods [7,14,65,118] to establish nonperturbative reducibility directly and independently of localization for the dual model.

3 In dimension one, it extends to the Lyapunov exponents of multifrequency cocycles $\mathbb{R} \times C^\omega(\mathbb{T}^b, \mathrm{SL}_2(\mathbb{C})) \ni (\alpha, A) \mapsto L(\alpha, A) \in [0, \infty)$.

for continuity of L occurs sharply at the Gevrey space G^2. It should be noted that both the original spectacular counterexample [113] and its refinements [48,114] require α to be a fixed irrational of bounded type, i.e., having a continued fraction expansion with bounded coefficients. This set includes the golden mean but forms a set of zero Lebesgue measure. The authors of all these papers also vary the cocycle, i.e., the potential. This still leaves open the question whether continuous behavior of the Lyapunov exponents at least for Schrödinger cocycles with regularity lower than G^2 is possible if α is not of bounded type. Another open question is whether it is true that for a fixed potential of lower than G^2 regularity, the Lyapunov exponent is necessarily a continuous function of energy.

3. AVILA'S GLOBAL THEORY AND CLASSIFICATION OF ANALYTIC ONE-FREQUENCY COCYCLES

While many results exist in lower regularity, the analyticity of V in (0.1) brings on board powerful ideas related to subharmonicity (leading, in particular, to the crucially important for other developments continuity results) and the technique of semialgebraic sets introduced to the field by J. Bourgain [28]. As a result, a lot more can be said about analytic quasiperiodic operators. Particularly, while Kotani theory based its characterization of the absolutely continuous spectrum on compexifying the energy, for analytic quasiperiodic operators there is one more natural parameter to complexify, namely the phase. This idea goes back to M. Herman [63], and has been fruitfully used to prove positivity (and later continuity) of the Lyapunov exponent in [29,63,110]. Avila [5] discovered a remarkable related structure that has served as a foundation of his global theory (later extended to the high-dimensional cocycles in [12]). Define

$$L_\epsilon(E) := \lim_{n\to\infty} \int \frac{1}{n} \ln \left\| \prod_{j=n-1}^{0} A_j(x + j\alpha + i\epsilon, E) \right\| d\mu.$$

Avila observed that, for a given cocycle, L_ϵ is a convex function of ϵ, and proved that it has quantized derivative in ϵ.

Theorem 3.1 ([5]). *For any complex-analytic one-frequency cocycle,*

$$\omega(A) = \lim_{\epsilon\to 0^+} \frac{L_\epsilon(A) - L_0(A)}{2\pi\epsilon} \in \mathbb{Z}.$$

This was enabled through approximation by the rationals due to the continuity of the Lyapunov exponent in the analytic category [32]. The fact that such continuity does not hold even for higher Gevrey cocycles [48,113,114] complicates potential nonanalytic extensions.

Theorem 3.1 already enables full analytic computation of the Lyapunov exponents for E in the spectrum, as well as of their complexifications L_ϵ and further analysis for several models originating and relevant in physics: the almost Mathieu operator [5], the extended Harper's model [81], recently discovered models with mobility edges [112] and unitary almost Mathieu operator [34], models arising in the study of the quantum graph graphene [23], and others.

Avila classified analytic cocycles $A(x)$ depending on the behavior of the Lyapunov exponent L_ϵ of the complexified cocycle $A(x + i\epsilon)$. Namely, he distinguishes three cases, with the terminology inspired by the almost Mathieu family:

(Subcritical) $L_\epsilon = 0, \epsilon < \delta,\ \delta > 0$, or, alternatively, $L_0 = \omega(A) = 0$.

(Critical) $L_0 = 0, L_\epsilon > 0, \epsilon > 0$, or, alternatively, $L_0 = 0, \omega(A) > 0$.

(Supercritical) $L_0 > 0$.

For the almost Mathieu family, these three regimes are uniform over the spectrum, corresponding to the supercritical ($\lambda > 1$), subcritical ($\lambda < 1$), and critical ($\lambda = 1$) values of the coupling constant. Spectrally, there is purely absolutely continuous spectrum for all x and all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in the subcritical case [3], purely singular continuous spectrum for all x and all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in the critical case [69], and pure point spectrum for a.e. x, α with sharp spectral transitions depending on the arithmetics of both α and x between pure point spectrum and singular continuous spectrum in the supercritical case (see Section 5). Remarkably, the critical almost Mathieu operators appear at the boundary of the two other regimes.

For general quasiperiodic operators, this classification leads to the corresponding division of energies in the spectrum, depending on (sub/super)criticality of the cocycle $A(\cdot, E)$. For convenience we will call the energy in the spectrum (super/sub)critical according to whether the corresponding transfer-matrix cocycle is such. It is expected that the key spectral properties of spectra in the three above regimes follow those of the corresponding almost Mathieu operators.

Indeed, pure point spectrum for a.e. x, α holds through the *supercritical* set of energies, for any analytic potential [30]. It is an important open problem to make this result arithmetic, and it is expected that certain universal features of the transitions and structure of the eigenfunctions discovered in [77,78] will hold globally, throughout the supercritical regime, see Section 6.3.

The *subcritical* regime is subject to the almost reducibility conjecture (ARC) which claims that subcritical cocycles are almost reducible, that is, have constant cocycles in the closure of their analytic conjugacy class (note that since almost reducibility implies subexponential growth of the iterates of the cocycle that is uniform in the (complexified) phase, the converse is obviously true). The idea of reducing nonperturbative (global) to perturbative (local) results originated from an earlier work by Avila and Krikorian [14]. ARC was first formulated in [10], and first established for the almost Mathieu operator [3,10]. It was solved by Avila for the Liouville case in [4], and the solution for the complementary Diophantine case has been announced [5] to appear in [2]. Also, L. Ge has recently found a different proof [46].

Almost reducible (and therefore subcritical) cocycles enjoy all the dynamical and spectral consequences of the Eliasson's perturbative regime [39]. In particular, there is purely absolutely continuous spectrum throughout the subcritical regime. Moreover, reducibility can be made quantitative [117], and even arithmetically so [50], allowing for a wealth of conclusions. However, it remains true that the absolutely continuous spectrum is fully characterized by the subcritical regime, with no delicate dependence, as far as the spectral decomposition goes, on any other parameters.

The *critical* regime is expected (see [11, 82]) to support only singular continuous spectrum (again, no dependence on the other parameters, as long as α is irrational) but fully establishing it even for the critical almost Mathieu operator took decades and was only accomplished recently [69].

On the other hand, the key result of Avila's global theory [5] is that operators with critical energies throughout the spectrum, like the critical almost Mathieu operator, are an anomaly, that does not happen typically. In fact, for prevalent (in a certain measure-theoretic sense) potentials, there are no critical energies, and the spectrum is contained in finitely many intervals, with either only subcritical or only supercritical regime within each.[4] Moreover, the set of potentials and energies (V, E) such that E is critical is contained in a countable union of codimension-one analytic submanifolds of $C^\omega(\mathbb{T};\mathbb{R}) \times \mathbb{R}$. Another remarkable related fact is that Lyapunov exponent enjoys even much stronger regularity when restricted to potentials and energies with a fixed value of acceleration: it becomes real-analytic on this (typically rather irregular) set, in both the energy E and any parameter λ ranging in a real analytic manifold Λ, if V_λ in $C^\omega(\mathbb{T};\mathbb{R})$ is a family real-analytic in parameter λ.

From the point of view of the global theory, it becomes particularly important to study the universal features of the two prevalent regimes, subcritical and supercritical. As mentioned above, the absolutely continuous spectrum is fully characterized by the subcritical regime, with no delicate dependence, as far as the spectral decomposition goes, on any other parameters. The picture for the supercritical regime is a lot more interesting, and is in a certain sense at the beginning of its development.

Going back to the complexified cocycle L_ϵ, quantizatization of acceleration means that as a function of $\epsilon > 0$, L_ϵ is convex, piecewise affine, and thus is fully characterized by $L = L_0$ and monotone increasing sequences of turning points b_i and slopes $n_i \in 2\pi\mathbb{Z}_+$, so that the slope of L_ϵ between b_i and b_{i+1} is n_i. Clearly, sequences b_i and n_i present a very important intrinsic characterization of the cocycle and the corresponding Schrödinger operator. What information do they give us?

4. DUAL LYAPUNOV EXPONENTS OR GLOBAL THEORY DEMYSTIFIED

It turns out that Aubry duality not only provides a new proof of quantization of acceleration, but holds key to the mystery of the global theory. We have

Theorem 4.1 ([47]). *Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $V \in C^\omega(\mathbb{T},\mathbb{R})$. Then there exist nonnegative $\{\hat{L}_i(E)\}$ such that for any $E \in \mathbb{R}$,*

$$\hat{L}_i(E) = \lim_{d\to\infty} \hat{L}_i^d(E),$$

4 A part of this picture was previously established in the semiclassical regime in the continuum in [40].

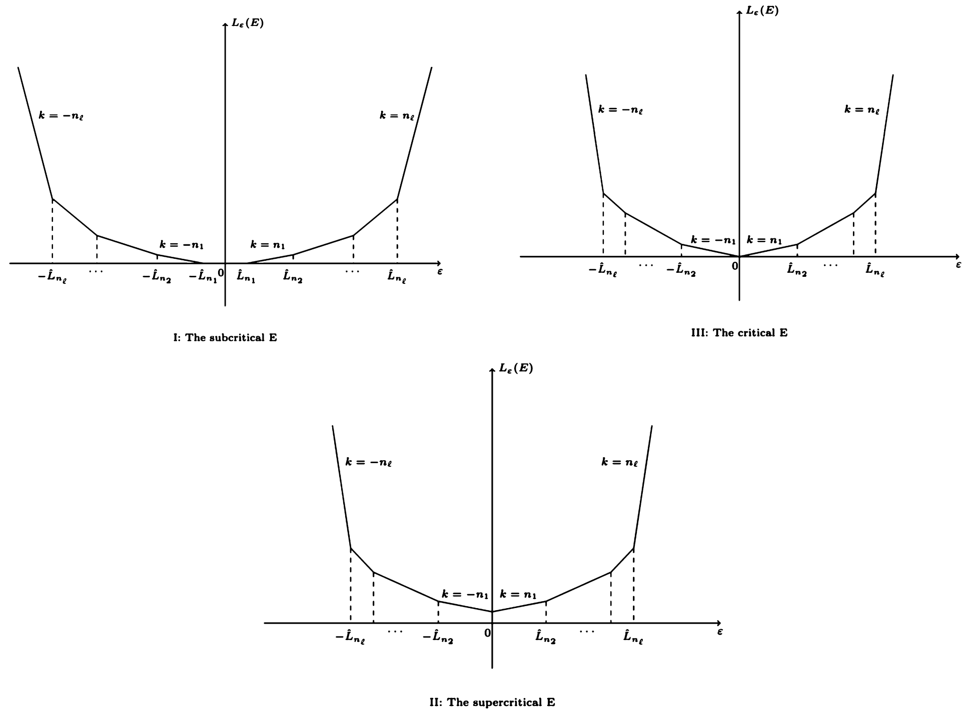

FIGURE 2
The complexified Lyapunov exponent.

where $\hat{L}_i^d(E), i = 1, \ldots, d$*, are the Lyapunov exponents, as defined in* (2.3)*, of the* $\mathrm{Sp}(2d, \mathbb{C})$ *transfer-matrix cocycle of the dual eigenvalue equation* $\hat{H}_{V^d,\alpha,x}\Psi = E\Psi$*, with* $V^d(x) = D_d \star V$ *and* D_d *being the Dirichlet kernel. Moreover,*

$$L_\epsilon(E) = L_0(E) - \sum_{\{i:\hat{L}_i(E)<2\pi|\epsilon|\}} \hat{L}_i(E) + 2\pi\big(\#\{i : \hat{L}_i(E) < 2\pi|\epsilon|\}\big)|\epsilon|$$

In fact, the theorem also holds for $V \in C_h^\omega(\mathbb{T}, \mathbb{R})$ and $|\epsilon| < h$, where $C_h^\omega(\mathbb{T}, \mathbb{R})$ is the space of bounded analytic functions f defined on a strip $\{|\Im z| < h\}$ with the norm $\|f\|_h = \sup_{|\Im z|<h} |f(z)|$. See Fig. 2 for an illustration of the three possible scenarios.

This means that for the trigonometric polynomials V the turning points b_i are given precisely by the Lyapunov exponents $\hat{L}_i(E)$ of the dual cocycle, and increases in the slopes are given by the 2π times their multiplicities; for analytic V, these objects are given by the limits of those quantities for successive trigonometric polynomial cutoffs of V. We call $\hat{L}_i(E)$ the dual Lyapunov exponents, the objects that play a role similar to that of zeros of an analytic function in the Jensen's formula. In particular, the acceleration $\omega(E)$ turns out to be precisely the number of vanishing dual Lyapunov exponents (an analogue of the winding number for an analytic function on $\mathbb{T}$).

Besides unraveling the mystery of the behavior of complexified Lyapunov exponents, this leads to a new understanding of the key statement of Avila's global theory, namely that for prevalent operators (0.1), almost all pairs of potentials and energies are acritical. Indeed, it immediately follows that

Theorem 4.2 ([47]). *Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and V is analytic, then the energy $E \in \mathbb{R}$ is*

(1) outside the spectrum *if* $L(E) > 0$ *and* $\hat{L}_1(E) > 0$*,*

(2) supercritical *if* $L(E) > 0$ *and* $\hat{L}_1(E) = 0$*,*

(3) critical *if* $L(E) = 0$ *and* $\hat{L}_1(E) = 0$*,*

(4) subcritical *if* $L(E) = 0$ *and* $\hat{L}_1(E) > 0$*.*

Thus, in the regime $L(E) = 0$, criticality is in the locus of vanishing of an additional continuous [12] function $\hat{L}_1(E)$, implying the prevalence of the acriticality claim. Theorem 4.2, of course, also contains the statement of Theorem 1.1, with $\hat{L} := \hat{L}_1$, as well as the fact that Schrödinger cocycle is subcritical if and only if its dual Lyapunov exponents are all positive. It also leads to a number of other powerful spectral corollaries, both for the general analytic case and several particular models [47]. It also has exciting physics applications [100].

5. PRECISE ANALYSIS OF SMALL DENOMINATORS

One of the most fascinating features of the spectral theory of one-frequency quasiperiodic operators in the supercritical regime is its delicate dependence on the arithmetics, that can be analyzed to a remarkable depth, and in some cases completely. There were many exciting recent developments where the arithmetics has played a crucial role (e.g., [9,15,89]) but here we focus only on the analysis of small denominators in the proofs of point spectrum and related study of the eigenfunctions.

The main difficulty in proving point spectrum (or the phenomenon of Anderson localization, that is, pure point spectrum with exponentially decaying eigenfunctions) and analyzing the corresponding eigenfunctions of ergodic operators is in the fact that the eigenvalues are dense in the spectrum. Formal perturbative expansions of eigenfunctions and eigenvalues include the $(V(T^n x) - V(T^m x))^{-1}$ terms that, of course, get arbitrarily large. More generally, when we have *resonances*, that is, restrictions to boxes that are not too far away from each other that have eigenvalues that are too close (something that is bound to happen for ergodic operators), small denominators are created. Thus localization for ergodic and, in particular, quasiperiodic operators can be viewed as a small denominator problem.

Indeed, it has been traditionally approached in a perturbative way: through KAM-type schemes for large couplings [39,44,109], which all required Diophantine conditions on the frequency α. Small denominators are not simply a nuisance, but lead to actual change in the spectral behavior, since in the opposite regime of very Liouville frequencies (*too* small denominators), there is no localization even with the positivity of the Lyapunov exponent; and delocalization (which in this case means singular continuous spectrum) can be proved by perturbation of nearby periodic operators [20,54]. At the same time, for exponentially approximated frequencies that are neither far from nor close enough to rationals, there is nothing left to perturb about or to remove. Tackling those cannot be approached perturbatively, but requires a precise analysis, giving the problem a strong number-theoretic flavor.

It should be noted that the topology of the one-dimensional line is such that even occasional barriers make it difficult to pass through, strongly favoring localization in the presence of even small irregularities. For example, in the one-dimensional random case, localization holds for all couplings λ, when considering a family of potentials λV, and the same is expected but is apparently difficult to prove even for the underlying dynamics (X, μ, T) with very weak chaotic properties, such as a skew shift. It has even been conjectured by Kotani and Last that absolutely continuous spectrum is impossible for one-dimensional operators that are not almost periodic, but it has been disproved [6,111], and with a particularly simple construction in [119]. Those examples notwithstanding, the presence of metal–insulator transitions (that roughly correspond to transitions between the spectral types) as couplings change remains a distinctive feature of quasiperiodic operators.

The transitions in coupling between absolutely continuous and singular spectrum are fully determined by the vanishing/nonvanishing of the Lyapunov exponent. In the supercritical regime, absolutely continuous spectrum is impossible, but whether the spectrum is point or singular continuous is resolved in the competition between the depth of the small denominators—the strength of the resonances—and the Lyapunov growth.

Two types of resonances have played a special role in the spectral theory of quasiperiodic operators. *Frequency* resonances, when $|V(x) - V(x + k\alpha)|$ is small simply because $\|(x + k\alpha) - x)\|_{\mathbb{R}/\mathbb{Z}} = \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}$ is small, where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \inf_{\ell\in\mathbb{Z}} |x - \ell|$, were first exploited in [21] based on [54] to prove the absence of eigenvalues (and therefore singular continuous spectrum in the hyperbolic regime) for quasiperiodic operators with Liouville frequencies. Their strength is measured by the arithmetic parameter

$$\beta(\alpha) = \limsup_{k\to\infty} -\frac{\ln \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{5.1}$$

that is equal to zero for Diophantine (thus a.e.) α. Frequency resonances are ubiquitous for all quasiperiodic potentials.

Another class of resonances, appearing for all *even* potentials, was discovered in [83], where it was shown that the arithmetic properties of the phase also play a role and may lead to singular continuous spectrum even for the Diophantine frequencies. Indeed, for even potentials, phases with almost symmetries, when $|V(x) - V(x + k\alpha|$ is small because $\|(x + k\alpha) - (-x)\|_{\mathbb{R}/\mathbb{Z}}$ is small, lead to resonances, regardless of the values of other parameters. The strength of *phase* resonances is measured by the arithmetic parameter

$$\delta(\alpha, \theta) = \limsup_{k\to\infty} -\frac{\ln \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|}. \tag{5.2}$$

Phase resonances are symmetry based and exist for all even functions V.

It was conjectured in [66] that for the almost Mathieu family no other resonances appear and the competition between the Lyapunov growth and combined exponential resonance strength resolves in a sharp way: there is a pure point spectrum for $L(E) >$

$\beta(\alpha) + \delta(\alpha, x)$ and a singular continuous spectrum in the regime $L(E) < \beta(\alpha) + \delta(\alpha, x)$. We note that for the special case of *α-rational* x, that is, such that $2x \in \mathbb{Z}\alpha + \mathbb{Z}$, we have $\delta(\alpha, x) = \beta(\alpha)$ so the resonances "double up" and the conjectured threshold becomes $2\beta(\alpha)$.

An early nonperturbative localization method was first developed in the 1990s for the almost Mathieu operator [84] and represented perhaps the first case of solving a traditionally KAM problem in a direct way, without an inductive procedure. It presented a (simple, but not sharp) technique to treat the *nonresonant* case, $\beta(\alpha) = \delta(\alpha) = 0$. Further breakthroughs came in [85] where the role of the Lyapunov exponents and corresponding deviations was first understood, allowing to achieve the nonresonant result up to the actual Lyapunov transition, and then in the work of Bourgain and collaborators [28,30] where robust nonperturbative methods were developed for general analytic potentials and more, leading to the proofs of localization for a.e. frequency throughout the supercritical regime. The ideas of [85] hold more generally, and have, in particular, led to very simple proofs of localization for the one-dimensional Anderson model [90]. Most importantly, however, their arithmetic nature has been crucial for further developments. For example, the fact that localization holds for α-rational x,[5] enabled Puig's proof [104] of the ten martini problem (that the spectrum is a Cantor set) for Diophantine α. The solution of the full ten martini problem [8,9] required, in particular, dealing with intermediate frequencies that are neither Diophantine nor Liouville, thus with the frequency resonances. A method to treat those has been devised in [9] leading to the proof of localization for $L(E) > \frac{16}{9}\beta$, but failing in the neighborhood of the actual transition. A sharp method to treat pure frequency resonances was developed in [77], and a sharp method to treat pure phase resonances in [78].

Therefore, the sharp arithmetic spectral transition conjecture of [66] has been established for single-type-resonances: for pure frequency resonances (that is, for the so-called α-Diophantine phases for which $\delta(\alpha, x) = 0$ so there are no exponential phase resonances) in [17,52,77],[6] and for pure phase resonances (that is, for Diophantine frequencies for which $\beta(\alpha) = 0$ so there are no exponential frequency resonances) in [78].

The methods to treat pure frequency and phase resonances in [77,78] are robust in a sense that weak exponential resonances of the other type can be added easily, but it is still an open problem to treat *combined* frequency and phase resonances in a sharp way. However, there were two very recent breakthroughs.

Namely, W. Liu has developed a way to sharply treat doubled resonances for the almost Mathieu operator, proving localization up to the conjectured threshold:

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5 This was, in fact, established in [72].

6 In [17] the pure frequency part of the conjecture of [66] has been proved by a completely different method, namely through quantitative reducibility [117] and duality, but in a measure-theoretic in x sense, i.e., losing the control over the arithmetics of x. A recent breakthrough by Ge–You [50] where an arithmetic version of quantitative reducibility was developed has lead to a way to obtain sharp arithmetic in phase results through duality as well, enabling, in particular, an arithmetic duality-based proof of the frequency part of the conjecture [52], that works also for all Aubry duals (2.1) of operators (0.1).

Theorem 5.1 ([99]). *Operator $H_{2\lambda \cos,\alpha,x}$ with α-rational x has Anderson localization whenever $L(E) > 2\beta(\alpha)$ (or equivalently, $\lambda > e^{2\beta(\alpha)}$).*

In Liu's earlier work, this was established for $L(E) > 3\beta(\alpha)$ [98], but a significant new understanding of treatment of doubled resonances was necessary to go sharp, and it was achieved in [99]. Also α-rational phases x hold special importance for various questions because eigenvalues for such x are located at gap edges [104]. Puig's proof of the ten martini problem for the Diophantine case [104] was based precisely on localization for α-rational x. The original plan to prove the full ten martini problem was to establish localization for α-rational x and $L(E) > \beta(\alpha)$ [8]. Not surprisingly, it failed, prompting the resonance doubling-up conjecture in [9] that is now solved [99]. It should be noted that the singular-continuous part of the conjecture, namely singular-continuous spectrum for α-rational x and $L(E) < 2\beta(\alpha)$, is still open.

In a different direction, R. Han, F. Yang, and I [58] developed a sharp method to treat the third type of resonances: high barriers (that effectively play the role of *antiresonances*), and, moreover, *combinations* of frequency resonances and high barriers, in another popular quasiperiodic family originating in physics, the Maryland model.

Maryland model is a family

$$(M_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda \tan\big(\pi(\theta + n\alpha)\big)u_n, \tag{5.3}$$

where $\lambda > 0$ is the coupling constant, irrational $\alpha \in \mathbb{T} = [0, 1]$ is the frequency, and $\theta \in \mathbb{T}$ is the phase with $\theta \notin \Theta = \{\frac{1}{2} + \alpha\mathbb{Z} + \mathbb{Z}\}$.

It was originally proposed by Grempel, Fishman, and Prange [56] as a linear version of the quantum kicked rotor and has attracted continuing interest from the physics community, see, e.g., [26, 42, 45], due to its exactly solvable nature. It has explicit expression for the Lyapunov exponent, integrated density of states, and even (a little less explicit) for the eigenvalues and eigenfunctions. In particular, the Lyapunov exponent $L_\lambda(E)$ is an explicit function of λ, E not dependent on α. However, the implicit expressions for the eigenfunctions do not allow for easy conclusions about their behavior, which is expected to be quite interesting, with transfer matrices satisfying certain exact renormalization [41].

Phase resonances do not exist for the Maryland model, and as a result, for Diophantine (i.e., nonresonant) frequencies it has localization for *all* phases [87, 107]. However, it does have barriers, when the trajectory of a given phase approaches the singularity too early. Barriers compensate for the resonances, and therefore serve as what we call in [58] the *antiresonances*, providing the reason why for the Maryland model there are phases with localization even for the most Liouville frequencies [76]. Thus Maryland model features a combination of frequency resonances and phase antiresonances.

Maryland model was the first one where the spectral decomposition has been resolved completely, for *all* values of the parameters [76].[7] Let p_n/q_n be the continued fraction approximants of α. We note that the frequency resonance index $\beta(\alpha)$ defined in (5.1)

7 It also remains the only one with spectral transitions where this could be claimed.

also satisfies $\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}$. A new index, $\delta^M(\alpha,\theta)$, was introduced in [76] as

$$\delta^M(\alpha,\theta) := \limsup_{n\to\infty} \frac{\ln q_{n+1} + \ln \|q_n(\theta - \frac{1}{2})\|_{\mathbb{T}}}{q_n}. \tag{5.4}$$

We have

Theorem 5.2 ([76]). *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum on $\{E : L_\lambda(E) < \delta^M(\alpha,\theta)\}$, and pure point spectrum on $\{E : L_\lambda(E) > \delta^M(\alpha,\theta)\}$.*[8]

This provides complete spectral analysis, for all α, θ, but was established implicitly: through the combination of Cayley and Fourier transforms and the study of a resulting explicit cohomological equation, making sharp the previous work in [56,107]. The extension of the analysis from a.e. θ in [107] to *all* θ in [76] required accounting for the effect of the barriers, and Cayley transform allowed to do it, albeit in a highly implicit way. In particular, this proof did not allow the analysis of the structure of eigenfunctions.

The method of [85] was adapted to the Maryland model in [87] where the nonresonant situation was treated and localization for Diophantine α was shown, developing the initial framework to study the eigenfunctions in the much more difficult resonant situation.

In [58] we show that $\delta(\alpha,\theta)$ can be interpreted as the exponential strength of frequency resonances, $\beta(\alpha)$, combined with the (negative) exponential strength of phase antiresonances, defined as the positions of exponential smallness of the $\cos(\pi(\theta + k\alpha))$,[9] and develop the approach to sharply treat the "resonance tamed by an antiresonance" situation. In particular, we give a constructive proof of the localization part of Theorem 5.2 and obtain

Theorem 5.3 ([58]). *For any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and any θ, the spectrum on $\{E : L_\lambda(E) \geq \delta^M(\alpha,\theta)\}$ is pure point and for any eigenvalue $E \in \{L_\lambda(E) > \delta^M(\alpha,\theta)\}$ and any $\epsilon > 0$, the corresponding eigenfunction ϕ_E satisfies $|\phi_E(k)| < e^{-(L_\lambda(E)-\delta^M(\alpha,\theta)-\epsilon)|k|}$ for sufficiently large $|k|$.*

Theorem 5.3 provides the sharp upper envelope, and develops the key tools to study the fine behavior of the eigenfunctions, see Section 6.2. In fact, such a study is the most exciting outcome of the proofs of localization based on sharp analysis of resonances.

There are several other models where sharp arithmetic spectral transitions have been conjectured and partially established, most notably the extended Harper's model, where for the complete analysis one would need to develop tools to study the simultaneous presence of three different types of resonances: frequency, phase, and singularity-induced antiresonances. However, for a.e. phase we expect the arithmetic frequency transition to be universal in the class of general analytic potentials. As for the arithmetic transitions in phase, we expect the same results to hold for general even analytic potentials for a.e. frequency. We note that the singular continuous part up to the conjectured transition is already established, even in a far greater generality, in [17,71,78].

8 It follows from the explicit formula for $L_\lambda(E)$ that the equality can only happen for two values of E.

9 So exponential largeness of the tan.

Finally, there is a question of arithmetic interfaces, e.g., what happens for the almost Mathieu operators with $L(E) = \beta(\alpha) + \delta(\alpha, \theta)$? It turns out that (in the pure resonance situations) both pure point and singular continuous spectra are possible depending on the finer arithmetic properties of parameters [13, 86, 88]. So far we do not even have a good conjecture on where the arithmetic thresholds within the transition lines lie. Making a significant progress on this problem would require a development of polynomial (as contrasted with current exponential) methods to tackle resonances, a very important problem in its own right, as it could lead to universal hierarchical structures (see Section 6) on polynomial scales.

6. EXACT ASYMPTOTICS AND UNIVERSAL HIERARCHICAL STRUCTURE OF EIGENFUNCTIONS

A very captivating question and a longstanding theoretical challenge is to explain the self-similar hierarchical structure visually obvious in the Hofstadter's butterfly, as well as the hierarchical structure of eigenfunctions, as related to the arithmetics of parameters. Such structure was first predicted for the almost Mathieu operator in the work of Azbel in 1964 [22], some 12 years before Hofstadter [64], and before numerical experimentation was possible. Such self-similar behavior is present for spectra and eigenfunctions of all quasiperiodic operators.

While this does not describe or explain the self-similarity, a step in the right direction is to prove that the spectrum is a Cantor set. Mark Kac offered ten martinis in 1982 for the proof of the Cantor set part of Azbel's 1964 conjecture. It was dubbed the Ten Martini problem by Barry Simon, who advertised it in his lists of 15 mathematical physics problems [106] and later, mathematical physics problems for the XXI century [108]. Most substantial partial solutions were made by Bellissard, Simon, Sinai, Helffer, Sjöstrand, Choi, Eliott, Yui, and Last [25, 36, 62, 96, 109], between 1983 and 1993. J. Puig [104] solved it for Diophantine α by noticing that localization at $\theta = 0$ [73, 85] leads to gaps at corresponding (dense) eigenvalues. The final solution was given in [9]. Cantor spectrum is also prevalent for general one-frequency operators with analytic potential: in the subcritical regime [10], and, by very different methods, in the supercritical regime [53] (and it is conjectured [11] also in the critical regime, which is nongeneric in itself [5]). Moreover, even all gaps predicted by the gap labeling are open in the noncritical almost Mathieu case [10, 16], the statement that is also expected to be true in the critical case, and recently claimed in the physics literature [27] to follow directly from [69].

As for the understanding the hierarchical behavior of the eigenfunctions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [116], it has remained an important open challenge even at the physics level, although some indications existed in the perturbative regime [33, 62, 109, 120].

Sharp analysis of resonances and small denominators has led to the discovery of universal self-similar structures of eigenfunctions defined by the type of resonance. The universal nature of these structures manifests in two ways: there is the same universal function that depends only on the type of the resonance, that governs the behavior around each exponential frequency or phase resonance (upon (possibly) reflection and renormalization), and it is the same structure for all the parameters involved: any (Diophantine) frequency α, (any α-Diophantine phase θ) with $\beta(\alpha) < L$ ($\delta(\alpha, \theta) < L$), and any eigenvalue E. It has been discovered and proved for the almost Mathieu operator [77,78] but is expected to be universal also throughout the class of analytic potentials, and more,[10] that is to hold in the regime of pure resonances. For example, the same universal structure for frequency resonances has already been proved for the Maryland model [59], for a.e. phase, namely, phases without the exponential antiresonances, see also a result on the hierarchical structure in the semiclassical regime [93]. However, for phases whose trajectories approach the barrier too fast, the hierarchical structure of the eigenfunctions is very different, and the complete analysis is extremely delicate.

Generally, one can identify four types of (anti)resonances that lead to different universal structures:

- frequency
- phase (only even potentials)
- barriers (antiresonance)
- singularity (antiresonance for Jacobi matrices)

We describe the universal structures for phase and frequency resonances [77,78] in the following subsections, and the one for the barrier antiresonances will appear in [59].

We expect that when different types of resonances are present, there will be further different self-similar structures, universal for all corresponding parameters and different resonance positions. Describing these structures for different combinations of resonances is very challenging but seems to be potentially within reach. In particular, in [58] we developed the tools to fully describe the universal structures for the Maryland model for all parameters, that is for combinations of frequency resonances and barrier antiresonances. We expect it to be done in [59]. We also expect the latter structures to be universal in the class of monotone potentials with a simple pole.

To give a glimpse into the universality results, we present two of them in more detail.

6.1. Frequency resonances

In [77] we find explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to α-Diophantine phase, see Theorem 6.1. This result holds for *all* frequency and coupling pairs in the frequency-

10 For example, C^2 cos-type potentials have been a popular object of study [43, 49, 51, 109, 115] and there are reasons to believe that they will feature the same structure, at least in the perturbative regime.

resonance localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure.

Since we are interested in exponential growth/decay, the behavior of f and g becomes most interesting in case of frequencies with exponential rate of approximation by the rationals.

These functions allow describing *precise* asymptotics of *arbitrary* solutions of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$ where E is an eigenvalue. The precise asymptotics of the norms of the transfer-matrices provides the first example of this sort for nonuniformly hyperbolic dynamics. Since those norms sometimes differ significantly from the reciprocals of the eigenfunctions, this leads to further interesting and unusual consequences, for example, exponential tangencies between contracted and expanded directions at the resonant sites.

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way. Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \le k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows:

Case 1. $q_{n+1}^{\frac{8}{9}} \ge \frac{q_n}{2}$ or $k \ge q_n$.
If $\ell q_n \le k < (\ell+1)q_n$ with $\ell \ge 1$, set

$$f(k) = e^{-|k-\ell q_n|\ln|\lambda|}\bar{r}_\ell^n + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\bar{r}_{\ell+1}^n, \tag{6.1}$$

and

$$g(k) = e^{-|k-\ell q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_{\ell+1}^n}, \tag{6.2}$$

where for $\ell \ge 1$,

$$\bar{r}_\ell^n = e^{-(\ln|\lambda| - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n})\ell q_n}.$$

Set also $\bar{r}_0^n = 1$ for convenience.
If $\frac{q_n}{2} \le k < q_n$, set

$$f(k) = e^{-k\ln|\lambda|} + e^{-|k-q_n|\ln|\lambda|}\bar{r}_1^n, \tag{6.3}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{6.4}$$

Case 2. $q_{n+1}^{\frac{8}{9}} < \frac{q_n}{2}$ and $\frac{q_n}{2} \le k \le \min\{q_n, \frac{q_{n+1}}{2}\}$.
Set

$$f(k) = e^{-k\ln|\lambda|}, \tag{6.5}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{6.6}$$

Notice that f, g only depend on α and λ, but not on θ or E; $f(k)$ decays and $g(k)$ grows exponentially, globally, at varying rates that depend on the position of k in the hierarchy defined by the continued fraction expansion of α, see Figures 3 and 4.

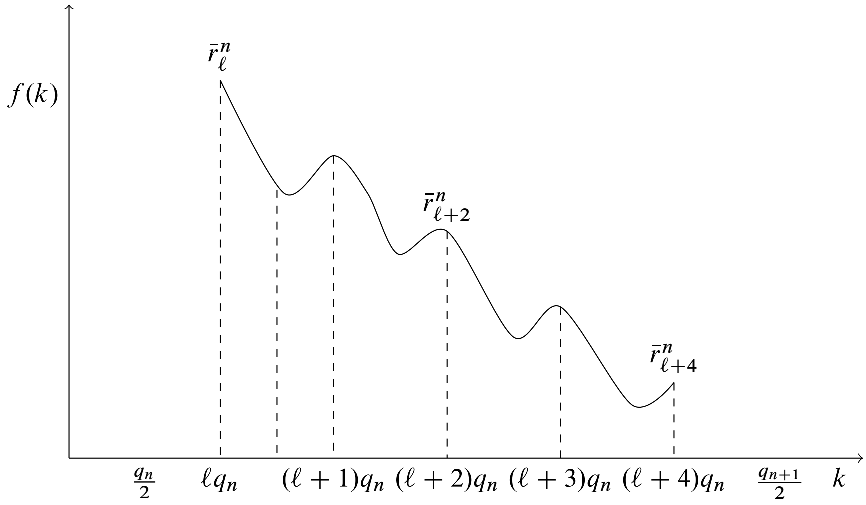

FIGURE 3
The universal behavior of eigenfunctions at scale n.

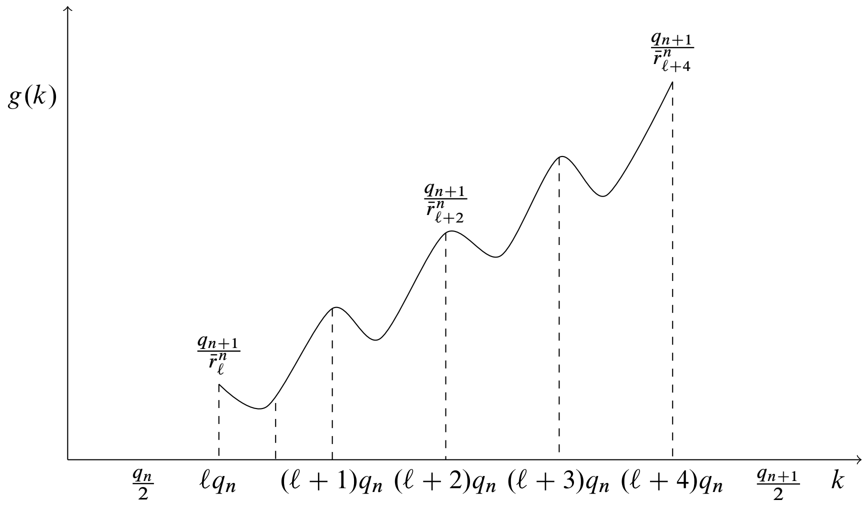

FIGURE 4
The universal behavior of transfer matrix norms at scale n.

It turns out that, in the entire regime $L(E) > \beta$, the exponential asymptotics of the eigenfunctions and norms of transfer matrices at the eigenvalues are completely determined by $f(k), g(k)$.

Theorem 6.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is an eigenvalue of $H_{\lambda,\alpha,\theta}$, and ϕ is the eigenfunction. Let $U(k) = \binom{\phi(k)}{\phi(k-1)}$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$) such that for any $|k| \geq K$, $U(k)$ and*

A_k[11] *satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leq \|U(k)\| \leq f(|k|)e^{\varepsilon|k|} \tag{6.7}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leq \|A_k\| \leq g(|k|)e^{\varepsilon|k|}. \tag{6.8}$$

In fact, the theorem is formulated in [77] for *generalized eigenfunctions*, thus can also be used to establish pure point spectrum throughout the indicated regime. Certainly, there is nothing special about $k = 0$, so the behavior described in Theorem 6.1 happens around an arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions: $U(k)$ behave as described at scale q_n but, when looked at in windows of size $q_k, q_k \leq q_{n-1}$, will demonstrate the same universal behavior around appropriate local maxima/minima.

To further illustrate the above, let ϕ be an eigenfunction and $U(k) = \binom{\phi(k)}{\phi(k-1)}$. An immediate corollary of Theorem 6.1 is the universality of behavior at all appropriately defined nonresonant local maxima. We will say k_0 is a local j-maximum of ϕ if $\|U(k_0)\| \geq \|U(k)\|$ for $|k - k_0| \sim q_j$. Then, with an appropriate notion of nonresonance (see [77]), we have

Theorem 6.2 ([77]). *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a nonresonant local j-maximum for $j > j(\epsilon)$, then*

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{\|U(k_0 + s)\|}{\|U(k_0)\|} \leq f(|s|)e^{\varepsilon|s|}, \tag{6.9}$$

for $|s - k_o| \sim q_j$.

In case $\beta(\alpha) > 0$, Theorem 6.1 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction.

Let k_0 be a global maximum. The self-similar hierarchical structure of local maxima can be described in the following way. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$. Then there is a *constant* scale $\hat{n}_0$, thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are all the local n_{j_0}-maxima in $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$.

The exponential behavior of the eigenfunction in the local neighborhood (of size of order $q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible.

Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denoting "depth 1" local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$, we then have a similar picture around $b_{a_{n_{j_0}}}$: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each $0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size of order $q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum, is given by f.

11 Products A_k are defined in (1.2).

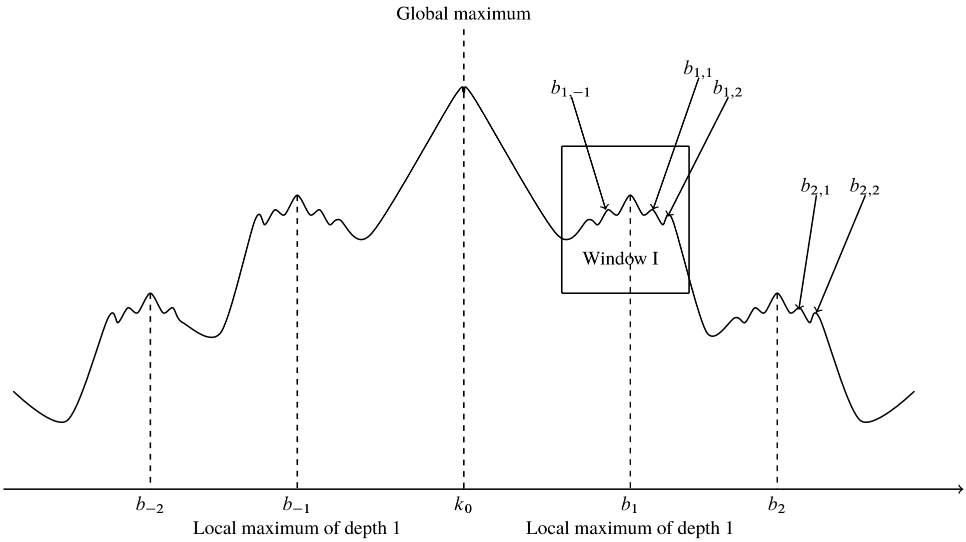

FIGURE 5
Universal self-similar structure of eigenfunctions

Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$ by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$, we then get the same picture taking the magnifying glass another level deeper, and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$, and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, yet the depth of the hierarchy that can be so achieved is at least $j/2 - C$, Figure 5 schematically illustrates the structure of local maxima of depth one and two, and Figure 6 illustrates that the neighborhood of a local maximum appropriately magnified looks like a picture of the global maximum. See **[77]** for the exact statement.

6.2. Phase resonances

In **[78]** we found another universal structure, this time for phase resonances. Once again, we found (different) functions f that determine universal asymptotics of the eigenfunctions, also locally around the resonances, which features a self-similar hierarchical structure. In particular, we have Theorem just like Theorem 6.1 but with new f and for $\beta(\alpha) = 0$ and $L > \delta(\alpha, \theta)$ **[78]**. The behavior described in this theorem happens around an arbitrary point. This, coupled with effective control of parameters at the local maxima, allows uncover-

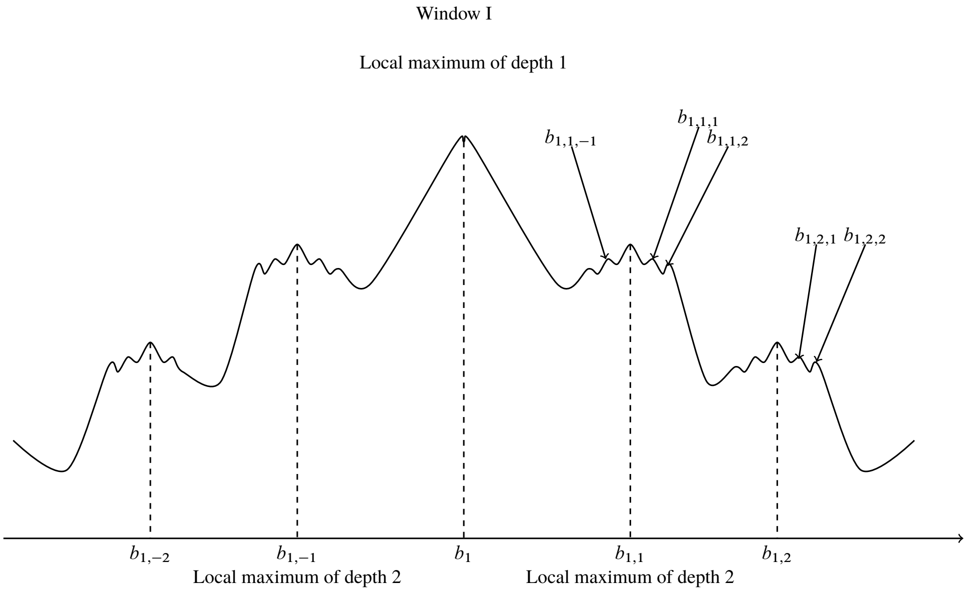

FIGURE 6
Universal self-similar structure of eigenfunctions, zoomed in

ing the self-similar nature of the eigenfunctions, but this time one needs not only the rescaling but also alternating reflections, leading to what we call the *reflective-hierarchical* structure.

Assume phase θ satisfies $0 < \delta(\alpha,\theta) < \ln\lambda$. Fix $0 < \varsigma < \delta(\alpha,\theta)$. Let k_0 be a global maximum of eigenfunction ϕ. Let K_i be the positions of exponential resonances of the phase $\theta' = \theta + k_0\alpha$ defined by

$$\|2\theta + (2k_0 + K_i)\alpha\|_{\mathbb{R}/\mathbb{Z}} \le e^{-\varsigma|K_i|}. \tag{6.10}$$

This means that $|v(\theta' + \ell\alpha) - v(\theta' + (K_i - \ell)\alpha)| \le Ce^{-\varsigma|K_i|}$, uniformly in ℓ, or, in other words, the potential $v_n = v(\theta + n\alpha)$ is $e^{-\varsigma|K_i|}$-almost symmetric with respect to $(k_0 + K_i)/2$.

Since α is Diophantine, we have

$$|K_i| \ge ce^{c|K_{i-1}|}, \tag{6.11}$$

where c depends on ς and α through the Diophantine constants κ, τ. On the other hand, K_i is necessarily an infinite sequence. Let ϕ be an eigenfunction, and $U(k) = \binom{\phi(k)}{\phi(k-1)}$. We say k is a local K-maximum if $\|U(k)\| \ge \|U(k+s)\|$ for all $s - k \in [-K, K]$.

The informal description of the *reflective-hierarchical* structure of local maxima is the following. There exists a *constant* $\hat{K}$ such that there is a local cK_j-maximum b_j within distance $\hat{K}$ of each resonance K_j. The exponential behavior of the eigenfunction in the local cK_j-neighborhood of each such local maximum, normalized by the value at the local maximum, is given by the *reflection* of f.

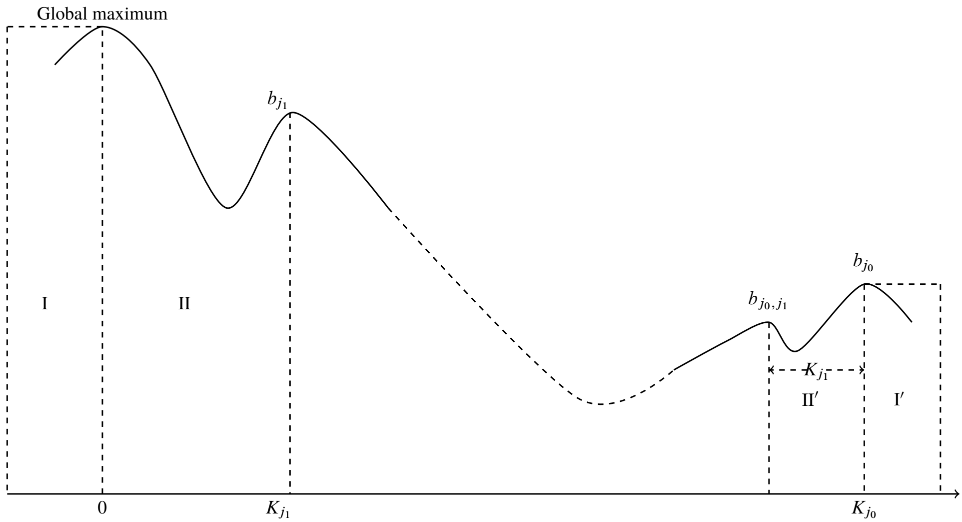

FIGURE 7
Reflective self-similarity of an eigenfunction.

Moreover, this describes the entire collection of local maxima of depth 1, that is, all K such that K is a cK-maximum. Then we have a similar picture in the vicinity of b_j: there are local cK_i-maxima $b_{j,i}$, $i < j$, within distance $\hat{K}^2$ of each $K_j - K_i$. The exponential (on the K_i scale) behavior of the eigenfunction in the local cK_i-neighborhood of each such local maximum, normalized by the value at the local maximum, is given by f.

Then we get the next level maxima $b_{j,i,s}$, $s < i$ in the $\hat{K}^3$-neighborhood of $K_j - K_i + K_s$ and reflected behavior around each, and so on, with reflections alternating with steps. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{j_0,j_1,\dots,j_s}$, with each "depth $s+1$" local maximum $b_{j_0,j_1,\dots,j_s}$ being in the corresponding vicinity of the "depth s" local maximum $b_{j_0,j_1,\dots,j_{s-1}} \approx k_0 + \sum_{i=0}^{s-1}(-1)^i K_{j_i}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, with $b_{j_0,j_1,\dots,j_{s-1}}$ determined with $\hat{K}^s$ precision, thus it presents an accurate picture as long as $K_{j_s} \gg \hat{K}^s$.

Thus the behavior of $\phi(x)$ is described by the same universal f in each $\sim K_{j_s}$ window around the corresponding local maximum $b_{j_0,j_1,\dots,j_s}$ after alternating reflections. The positions of the local maxima in the hierarchy are determined up to errors that at all but possibly the last step are superlogarithmically small in K_{j_s}. We call such a structure *reflective hierarchy*.

Figure 7 depicts reflective self-similarity of an eigenfunction with global maximum at 0. The self-similarity is seen as follows: I′ is obtained from I by scaling the x-axis proportional to the ratio of the heights of the maxima in I and I′; II′ is obtained from II by scaling the x-axis proportional to the ratio of the heights of the maxima in II and II′. The behavior in the regions I′, II′ mirrors the behavior in regions I, II upon reflection and corresponding dilation.

6.3. Universality and extensions

The hierarchical structures of Sections 6.1 and 6.2 are expected to hold universally for most in the appropriate sense (albeit not all, as for the almost Mathieu) local maxima for general analytic potentials. Establishing this fully would require certain new ideas since so far even an arithmetic version of localization for the Diophantine case has not been established for the general analytic family, the current state-of-the-art result by Bourgain–Goldstein [30] being measure-theoretic in α.

The universality of the hierarchical structures of Sections 6.1 and 6.2 is twofold: not only it is the same universal function that governs the behavior around each exponential frequency or phase resonance (upon reflection and renormalization), it is the same structure for all the parameters involved: any (Diophantine) frequency α (any α-Diophantine phase θ) with $\beta(\alpha) < L$ ($\delta(\alpha, \theta) < L$), and any eigenvalue E. The universal reflective-hierarchical structure in Section 6.2 requires the evenness of the function defining the potential and, moreover, resonances of other types may also be present in general. However, we conjectured in [78] that for general even analytic potentials for a.e. frequency only finitely many other exponentially strong resonances will appear, thus the structure described in Section 6.2 will hold for the corresponding class.

The key elements of the technique developed for the treatment of arithmetic resonances are robust and have made it possible to approach other questions and, in particular, study delicate properties of the singular continuous regime. Among other things, it has allowed obtaining upper bounds on fractal dimensions of the spectral measures and quantum dynamics for the singular continuous almost Mathieu operator [79], as well as potentials defined by general trigonometric analytic functions [75], and determining also the *exact* exponent of the exponential decay rate in expectation for the two-point function [74], the first result of this kind for any model. These methods are also expected to be applicable to many other models.

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REFERENCES

[1] S. Aubry and G. André, Analyticity breaking and Anderson localisation in incommensurate lattices. *Ann. Isr. Phys. Soc.* **3** (1980), 133–164.

[2] A. Avila, in preparation.

[3] A. Avila, The absolutely continuous spectrum of the almost Mathieu operator. 2008, arXiv:0810.2965.

[4] A. Avila, Almost reducibility and absolute continuity I. 2010, arXiv:1006.0704.

[5] A. Avila, Global theory of one-frequency Schrödinger operators. *Acta Math.* **215** (2015), no. 1, 1–54.

[6] A. Avila, On the Kotani–Last and Schrödinger conjectures. *J. Amer. Math. Soc.* **28** (2015), 579–616.

[7] A. Avila, B. Fayad, and R. Krikorian, A KAM Scheme for $SL(2,\mathbb{R})$ Cocycles with Liouvillean Frequencies. *Geom. Funct. Anal.* **21** (2011), no. 5, 1001–1019.

[8] A. Avila and S. Jitomirskaya, Solving the ten martini problem. *Lecture Notes in Phys.* **690** (2006), 5–16.

[9] A. Avila and S. Jitomirskaya, The Ten Martini Problem. *Ann. of Math. (2)* **170** (2009), no. 1, 303–342.

[10] A. Avila and S. Jitomirskaya, Almost localization and almost reducibility. *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 1, 93–131.

[11] A. Avila, S. Jitomirskaya, and C. Marx, Spectral theory of extended Harper's model and a question by Erdős and Szekeres. *Invent. Math.* **210** (2017), no. 1, 283–339.

[12] A. Avila, S. Jitomirskaya, and C. Sadel, Complex one-frequency cocycles. *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 9, 1915–1935.

[13] A. Avila, S. Jitomirskaya, and Q. Zhou, Second phase transition line. *Math. Ann.* **370** (2018), 271–285.

[14] A. Avila and R. Krikorian, Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles. *Ann. of Math.* **164** (2006), 911–940.

[15] A. Avila, Y. Last, M. Shamis, and Q. Zhou, On the abominable properties of the almost Mathieu operator with well approximated frequencies. 2021, arXiv:2110.07974.

[16] A. Avila, J. You, and Q. Zhou, in preparation.

[17] A. Avila, J. You, and Q. Zhou, Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.* **166** (2017), no. 14, 2697–2718.

[18] J. E. Avron, Topological quantum states and the 2016 Nobel prize. *Bull. IAMP* (2017), 3–22.

[19] J. Avron, D. Osadchy, and R. Seiler, A Topological Look at the Quantum Hall Effect. *Phys. Today* **56** (2003), 8–38.

[20] J. Avron and B. Simon, Singular continuous spectrum for a class of almost periodic Jacobi matrices. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 81–85.

[21] J. Avron and B. Simon, Almost periodic operators, II. The density of states. *Duke Math. J.* **50** (1983), 369–391.

[22] M. Y. Azbel, Energy spectrum of a conduction electron in a magnetic field. *Sov. Phys. JETP* **19** (1964), 634–645.

[23] S. Becker, R. Han, and S. Jitomirskaya, Cantor spectrum of graphene in magnetic fields. *Invent. Math.* **2** (2019), no. 18, 979–1041.

[24] J. Bellissard, R. Lima, and D. Testard, A metal–insulator transition for the almost Mathieu model. *Comm. Math. Phys.* **88** (1983), no. 2, 207–234.

[25] J. Bellissard, B. Simon, Cantor spectrum for the almost Mathieu equation. *J. Funct. Anal.* **48**, (1982), 408–419.

[26] M. Berry, Incommensurability in an exactly-soluble quantal and classical model for a kicked rotator. *Phys. D* **10** (1984), 369–378.

[27] D. S. Borgnia and R.-J. Slagerhttps, The Dry Ten Martini Problem at Criticality. 2021, arXiv:2112.06869.

[28] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*. Ann. of Math. Stud. 158, Princeton University Press, Princeton, NJ, 2005.

[29] J. Bourgain, Positivity and continuity of the Lyapunov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *J. Anal. Math.* **96** (2005), 313–355.

[30] J. Bourgain and M. Goldstein, On nonperturbative localization with quasiperiodic potential. *Ann. of Math.* **152** (2000), 835–879.

[31] J. Bourgain and S. Jitomirskaya, Absolutely continuous spectrum for 1D quasiperiodic operators. *Invent. Math.* **148** (2002), 453–463.

[32] J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Stat. Phys.* **108** (2002), 1203–1218. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[33] V. Buslaev and A. Fedotov, The monodromization and Harper equation. *Sémin. Équ. Dériv. Partielles* exp. no. **21** (1993–1994), 1–21.

[34] C. Cedzich, J. Fillman, and D. C. Ong, Almost everything about the unitary almost Mathieu operator. 2021, arXiv:2112.03216.

[35] H. Cheng, L. Ge, J. You, and Q. Zhou, Global rigidity for ultra-differentiable quasiperiodic cocycles and its spectral applications. 2021, arXiv:2101.11150.

[36] M. D. Choi, G. A. Elliott, and N. Yui, Gauss polynomials and the rotation algebra. *Invent. Math.* **99** (1990), 225–246.

[37] E. Dinaburg and Ya. Sinai, The one-dimensional Schrödinger equation with a quasi-periodic potential. *Funct. Anal. Appl.* **9** (1975), 279–289.

[38] P. Duarte and S. Klein, *Lyapunov exponents of linear cocycles: continuity via large deviations*. Atlantis Ser. Dyn. Syst. 3, Atlantis Press, 2016.

[39] L. H. Eliasson, Reducibility and point spectrum for linear quasi-periodic skew products. *Doc. Math.* Extra volume ICM 1998 II, 779–787.

[40] A. Fedotov and F. Klopp, Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case. *Comm. Math. Phys.* **227** (2002), 1–92.

[41] A. Fedotov and F. Sandomirskiy, An exact renormalization formula for the Maryland model. *Comm. Math. Phys.* **334** (2015), no. 2, 1083–1099.

[42] S. Fishman, Anderson localization and quantum chaos maps. *Scholarpedia* **5** (2010), no. 8, 9816.

[43] Y. Forman and T. VandenBoom, Localization and Cantor spectrum for quasiperiodic discrete Schrödinger operators with asymmetric, smooth, cosine-like sampling functions. 2021, arXiv:2107.05461.

[44] J. Fröhlich, T. Spencer, and P. Wittwer, Localization for a class of one dimensional quasi-periodic Schrödinger operators. *Comm. Math. Phys.* **132** (1990), 5–25.

[45] S. Ganeshan, K. Kechedzhi, and S. Das Sarma, Critical integer quantum hall topology and the integrable Maryland model as a topological quantum critical point. *Phys. Rev. B* **90** (2014), no. 4, 041405.

[46] L. Ge, in preparation.

[47] L. Ge, S. Jitomirskaya, J. You, and Q. Zhou, Multiplicative Jensen's formula and quantitative global theory of one-frequency Schrödinger operators, in preparation.

[48] L. Ge, Y. Wang, J. You, and X. Zhao, Transition space for the continuity of the Lyapunov exponent of quasiperiodic Schrödinger cocycles. 2021, arXiv:2102.05175.

[49] L. Ge, J. Xu, and J. Wang, The Hölder continuity of Lyapunov exponents for a class of cos-type quasiperiodic Schrödinger cocycles. 2020, arXiv:2006.03381.

[50] L. Ge and J. You, Arithmetic version of Anderson localization via reducibility. *Geom. Funct. Anal.* **30** (2020), no. 5, 1370–1401.

[51] L. Ge, J. You, and X. Zhao, Arithmetic version of Anderson localization for quasiperiodic Schrödinger operators with even cosine type potentials. 2021, arXiv:2107.08547.

[52] L. Ge, J. You, and X. Zhao, Arithmetic version of the frequency transition conjecture: new proof and generalization. *Peking J. Math.* (2021). DOI 10.1007/s42543-021-00040-y.

[53] M. Goldstein and W. Schlag, On resonances and the formation of gaps in the spectrum of quasi-periodic Schroedinger equations. *Ann. of Math.* **173** (2011), 337–475.

[54] A. Y. Gordon, The point spectrum of the one-dimensional Schrödinger operator. *Uspekhi Mat. Nauk* **31** (1976), 257–258.

[55] A. Y. Gordon, S. Jitomirskaya, Y. Last, and B. Simon, Duality and singular continuous spectrum in the almost Mathieu equation. *Acta Math.* **178** (1997), 169–183.

[56] D. Grempel, S. Fishman, and R. Prange, Localization in an incommensurate potential: an exactly solvable model. *Phys. Rev. Lett.* **49** (1982), no. 11, 833.

[57] R. Han and S. Jitomirskaya, Full measure reducibility and localization for quasiperiodic Jacobi operators: a topological criterion. *Adv. Math.* **319** (2017), 224–250.

[58] R. Han, S. Jitomirskaya, and F. Yang, Anti-resonances and sharp analysis of Maryland localization for all parameters, 2022, arXiv:2205.04021

[59] R. Han, S. Jitomirskaya, and F. Yang, Universal hierarchical structure of eigenfunctions in the Maryland model, in preparation.

[60] A. Haro and J. Puig, A Thouless formula and Aubry duality for long-range Schrödinger skew-products. *Nonlinearity* **26** (2013), 1163–1187.

[61] P. G. Harper, Single band motion of conducting electrons in a uniform magnetic field. *Proc. Phys. Soc. A* **68** (1955), 874–878.

[62] B. Helffer and J. Sjöstrand, Semi-classical analysis for Harper's equation. III. Cantor structure of the spectrum. *Mém. Soc. Math. Fr. (N.S.)* **39** (1989), 1–139.

[63] M. R. Herman, Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractere local d'un théoreme d'Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.* **58** (1983), 453–502.

[64] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. *Phys. Rev. B* **14** (1976), 22–39.

[65] X. Hou and J. You, Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Invent. Math.* **190** (2012), no. 1, 209–260.

[66] S. Jitomirskaya, Almost everything about the almost Mathieu operator. II. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pp. 373–382, International Press of Boston, 1994.

[67] S. Jitomirskaya, Ergodic Schrödinger operators (on one foot), Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, pp. 613–647, Proc. Sympos. Pure Math. 76, Part 2, Amer. Math. Soc., Providence, RI, 2007.

[68] S. Jitomirskaya, Critical phenomena, arithmetic phase transitions, and universality: some recent results on the almost Mathieu operator. *Curr. Dev. Math.* **2019** (2019), no. 1, 1–42.

[69] S. Jitomirskaya, On point spectrum at critical coupling. *Adv. Math.* **392** (2021), 6 pp.

[70] S. Jitomirskaya and I. Kachkovskiy, $L2$-reducibility and localization for quasiperiodic operators. *Math. Res. Lett.* **23** (2016), no. 2, 431–444.

[71] S. Jitomirskaya and S. Kocic, Spectral theory of Schrödinger operators over circle diffeomorphisms, *Int. Math. Res. Not.* **2** (2021). DOI 10.1093/imrn/rnaa362.

[72] S. Jitomirskaya, D. A. Koslover, and M. S. Schulteis, Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. *Ann. Henri Poincaré* **6** (2005), 103–124.

[73] S. Jitomirskaya, D. A. Koslover, and M. S. Schulteis, Continuity of the Lyapunov Exponent for analytic quasiperiodic cocycles. *Ergodic Theory Dynam. Systems* **29** (2009), 1881–1905.

[74] S. Jitomirskaya, H. Krüeger, and W. Liu, Exact dynamical decay rate for the almost Mathieu operator. *Math. Res. Lett.* **27** (2020), no. 3, 789–808.

[75] S. Jitomirskaya and W. Liu, in preparation.

[76] S. Jitomirskaya and W. Liu, Arithmetic spectral transitions for the Maryland model. *Comm. Pure Appl. Math.* **70** (2017), no. 6, 1025–1051.

[77] S. Jitomirskaya and W. Liu, Universal hierarchical structure of quasiperiodic eigenfunctions. *Ann. of Math. (2)* **187** (2018), 721–776.
[78] S. Jitomirskaya and W. Liu, Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. 2018, arXiv:1802.00781.
[79] S. Jitomirskaya, W. Liu, and S. Tcheremchantsev, Upper bounds on the fractal spectral dimensions and singular continuous spectrum near the arithmetic transition, in preparation.
[80] S. Jitomirskaya, W. Liu, and S. Zhang, Arithmetic spectral transitions: A competition between hyperbolicity and the arithmetics of small denominators IAS/Park City Math. Ser.: Harmon. Anal. Appl. **27**, pp. 35–72, 2020. DOI 10.1090/pcms/027/02.
[81] S. Jitomirskaya and C. Marx, Analytic quasi-periodic cocycles with singularities and the Lyapunov Exponent of Extended Harper's Model. *Comm. Math. Phys.* **316** (2012), 237–267.
[82] S. Jitomirskaya and C. Marx, Dynamics and spectral theory of quasi-periodic Schrödinger-type operators. *Ergodic Theory Dynam. Systems* **37** (2017), 2353–2393.
[83] S. Jitomirskaya and B. Simon, Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators. *Comm. Math. Phys.* **165** (1994), 201–205.
[84] S. Y. Jitomirskaya, Anderson localization for the almost Mathieu equation: a nonperturbative proof. *Comm. Math. Phys.* **165** (1994), 49–57.
[85] S. Y. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator. *Ann. of Math. (2)* **150** (1999), 1159–1175.
[86] S. Jitomirskaya and F. Yang, Phase transition in phase: the interface. 2021, preprint.
[87] S. Jitomirskaya and F. Yang, Pure point spectrum for the Maryland model: a constructive proof. *Ergodic Theory Dynam. Systems* **41** (2021), no. 1, 283–294.
[88] S. Jitomirskaya, F. Yang, and Q. Zhou, Second phase transition line: a constructive proof, in preparation.
[89] S. Jitomirskaya and S. Zhang, Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schödinger operators. *J. Eur. Math. Soc. (JEMS)* (2022). DOI 10.4171/jems/1139.
[90] S. Jitomirskaya and X. Zhu, Large deviations of the Lyapunov exponent and localization for the 1D Anderson model. *Comm. Math. Phys.* **370** (2019), 311–324.
[91] R. A. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients. *J. Differential Equations* **61** (1986), 54–78.
[92] S. Klein, Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. *J. Funct. Anal.* **218** (2005), 255–292.
[93] F. Klopp and A. A. Fedotov, On the hierarchical behavior of solutions of the Maryland equation in the semiclassical approximation. *Math. Notes* **108** (2020), no. 5, 906–910.

[94] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary one-dimensional Schrödinger operators. In *Stochastic Analysis*, pp. 225–248, N.-Holl. Math. Libr. 32, Elsevier, 1984.
[95] H. Kunz and B. Souillard, Sur le spectre des opérateurs aux différences finies aléatoires. *Comm. Math. Phys.* **78** (1980), 201–246.
[96] Last, Y.: Zero measure spectrum for the almost Mathieu operator. *Commun. Math Phys.* **164**(1994), 421–432 .
[97] Y. Last and B. Simon, Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators. *Invent. Math.* **135** (1999), 329–367.
[98] W. Liu, Almost Mathieu operators with completely resonant phases. *Ergodic Theory Dynam. Systems* **40** (2020), 1875–1893.
[99] W. Liu, Small denominators and large numerators of quasiperiodic Schrödinger operators. arXiv:2205.04648
[100] Y. Liu, Q. Zhou, and S. Chen, Localization transition, spectrum structure, and winding numbers for one-dimensional non-Hermitian quasicrystals. *Phys. Rev. B* **104** (2021), no. 2, 024201.
[101] V. A. Mandelshtam and S. Ya. Zhitomirskaya, 1D-quasiperiodic operators. Latent symmetries. *Comm. Math. Phys.* **139** (1991), 589–604.
[102] L. A. Pastur, Spectral properties of disordered systems in the one-body approximation. *Comm. Math. Phys.* **75** (1980), 179–196.
[103] R. Peierls, Zur Theorie des Diamagnetismus von Leitungselektronen. *Z. Phys. A: Hadrons Nucl.* **80** (1933), 763–791.
[104] J. Puig, Cantor spectrum for the almost Mathieu operator. *Comm. Math. Phys.* **244** (2004), 297–309.
[105] B. Simon, Almost periodic Schrödinger operators: A review. *Adv. in Appl. Math.* **3** (1982), 463–490.
[106] B. Simon, Fifteen problems in mathematical physics. *Oberwolfach Anniversary Volume* (1984), 423–454.
[107] B. Simon, Almost periodic Schrödinger operators. IV. The Maryland model. *Ann. Physics* **159** (1985), no. 1, 157–183.
[108] B. Simon, In *Schrödinger operators in the twenty-first century*, pp. 283–288, Math. Physics 2000, Imp. Coll. Press, London, 2000.
[109] Ya. Sinai, Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential. *J. Stat. Phys.* **46** (1987), 861–909.
[110] E. Sorets and T. Spencer, Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potential. *CMP* **142** (1991), 543–566.
[111] A. Volberg and P. Yuditskii, Kotani–Last problem and Hardy spaces on surfaces of Widom type. *Invent. Math.* **197** (2014), 683–740.
[112] Y. Wang, X. Xia, J. You, Z. Zheng, and Q. Zhou, Exact mobility edges for 1D quasiperiodic models. 2021, arXiv:2110.00962.

[113] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasi-periodic cocycles. *Duke Math. J.* **1** (2013), no. 62, 2363–2412.

[114] Y. Wang and J. You, Quasi-Periodic Schrödinger Cocycles with Positive Lyapunov Exponent are not Open in the Smooth Topology. *Comm. Math. Phys.* **362** (2018), no. 9, 801–826.

[115] Y. Wang and Z. Zhang, Cantor spectrum for a class of C^2 quasiperiodic Schrödinger operators. *Int. Math. Res. Not.* **2017** (2017), no. 8, 2300–2366.

[116] P. B. Wiegmann and A. V. Zabrodin, Quantum group and magnetic translations Bethe ansatz for the Azbel–Hofstadter problem. *Nucl. Phys. B* **422** (1994), 495–514.

[117] J. You, Quantitative almost reducibility and applications. In *Proceedings of ICM 2018*, pp. 2131–2154, World Scientific Publishing Co Pte Ltd, Singapore, 2019.

[118] J. You and Q. Zhou, Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications. *Comm. Math. Phys.* **323** (2013), no. 3, 975–1005.

[119] J. You and Q. Zhou, Simple Counter-Examples to Kotani–Last Conjecture Via Reducibility. *Int. Math. Res. Not.* **2015** (2015), no. 19, 9450–9455.

[120] S. Zhitomirskaya, Singular spectral properties of a one dimensional discrete Schrödinger operator with quasiperiodic potential. *Adv. Sov. Math.* **3** (1991), 215–254.

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