ABELIAN POLE SYSTEMS AND RIEMANN–SCHOTTKY-TYPE PROBLEMS

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ABSTRACT

In this survey of works on a characterization of Jacobians and Prym varieties among indecomposable principally polarized abelian varieties via the soliton theory, we focus on a certain circle of ideas and methods which show that the characterization of Jacobians as ppav whose Kummer variety admits a trisecant line and the Pryms as ppav whose Kummer variety admits a pair of symmetric quadrisecants can be seen as an abelian version of pole systems arising in the theory of elliptic solutions to the basic soliton hierarchies. We present also recent results in this direction on the characterization of Jacobians of curves with involution, which were motivated by the theory of two-dimensional integrable hierarchies with symmetries.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 14H42; Secondary 35Q51

KEYWORDS

Riemann–Schottky problem, soliton equations, Baker–Akhiezer functioon, Calogero–Moser system

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 2, pp. 1122–1153
DOI 10.4171/ICM2022/120

Published by EMS Press

1. INTRODUCTION

Novikov's conjecture on the Riemann–Schottky problem, namely that *the Jacobians of smooth algebraic curves are precisely those indecomposable principally polarized abelian varieties (ppavs) whose theta-functions provide solutions to the Kadomtsev–Petviashvili (KP) equation*, was the first evidence of nowadays well-established fact: connections between the algebraic geometry and the modern theory of integrable systems is beneficial for both sides. Novikov' conjecture was proved by T. Shiota in [48].

The first goal of this paper is to present the strongest known characterization of a Jacobian variety in this direction: *an indecomposable ppav X is the Jacobian of a curve if and only if its Kummer variety $K(X)$ has a trisecant line*, which was proved in [26,27]. This characterization is called *Welters' (trisecant) conjecture* after the work of Welters [54] which was motivated by Novikov's conjecture and Gunning's celebrated theorem [22]. The approach to its solution, proposed in [26], is general enough to be applicable to a variety of Riemann–Schottky-type problems. In [21,25] it was used for a characterization of principally polarized Prym varieties. The latter problem is almost as old and famous as the Riemann–Schottky problem, but is much harder.

Our second goal is to present recent results on characterization of Jacobians of curves with involution. The curves with involution naturally appear as a part of algebraic-geometrical data defining solutions to integrable system with symmetries. Numerous examples of such systems include the Kadomtsev–Petviashvili hierarchies of type B and C (BKP and CKP hierarhies, respectively) introduced in [11,12] and the Novikov–Veselov hierarchy introduced in [51,52].

The existence of an involution of a curve is central in proving that the constructed solutions have the necessary symmetry. The solutions corresponding to the same curve are usually parameterized by points of its Prym variety. In other words, the existence of involution plus some extra constraints on the divisor of the Baker–Akhiezer function are *sufficient* conditions ensuring required symmetry. The problem of proving that these conditions are *necessary* for *two-dimensional* integrable hierarchies is much harder and that is the problem solved in [28].

The third and to some extent our primary objective is to take this opportunity to elaborate on motivations underlining the proposed solution of the Riemann–Schottky-type problems and to introduce a certain collection of ideas and methods.

Maybe the most important among them is a mysterious *generating property* of two-dimensional linear differential, differential-functional, difference-functional equations. In fact, we will discuss two sources of generating properties. One of them is *local*, and it concerns equations with *meromorphic* coefficients in one of the variables that have *meromorphic solutions*. The other is *global* and concerns equations with elliptic coefficients that have solutions that are meromorphic sections of a line bundle over an elliptic curve [24].

The three main examples are:

(i) the differential equation

$$\big(\partial_t - \partial_x^2 + u(x,t)\big)\psi(x,t) = 0, \quad u = -2\partial_x^2\tau(x,t), \tag{1.1}$$

(ii) the differential-functional equation

$$\partial_t \psi(x,t) = \psi(x+1,t) + w(x,t)\psi(x,t), \quad w(x,t) = \partial_t \ln\left(\frac{\tau(x+1,t)}{\tau(x,t)}\right), \tag{1.2}$$

(iii) the difference-functional equation

$$\psi_{n+1}(x) = \psi_n(x+1) - v_n(x)\psi_n(x), \quad v_n(x) = \frac{\tau_n(x)\tau_{n+1}(x+1)}{\tau_n(x+1)\tau_{n+1}(x)} \tag{1.3}$$

with unknown functions $\psi_n(x)$, $n \in \mathbb{Z}$.

Each of these equations (after change of notations for independent variables) is one of two auxiliary linear problems for the three fundamental equations in the theory of integrable systems: the Kadomtsev–Petviashvili (KP) equation

$$3u_{yy} = (4u_t - 6uu_x + u_{xxx})_x, \tag{1.4}$$

the $2D$ Toda equation

$$\partial_\xi \partial_\eta \varphi_n = e^{\varphi_n - \varphi_n} - e^{\varphi_n - \varphi_{n+1}}, \quad \varphi_n = \varphi(x = n, \xi, \eta), \tag{1.5}$$

and the Bilinear Discrete Hirota equation (BDHE)

$$\begin{aligned} &\tau_n(l+1,m)\tau_n(l,m+1) - \tau_n(l,m)\tau_n(l+1,m+1) \\ &\quad + \tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1) = 0, \end{aligned} \tag{1.6}$$

respectively.

At the first glance, all three nonlinear equations, the KP equation, 2D Toda equation, and BDHE equation, look very different from each other. But in the theory of integrable systems, it is well known that these fundamental soliton equations share an intimate relation: the KP equation is as a continuous limit of the BDHE, and the 2D Toda equation can be obtained in an intermediate step.

Assume that in the first two cases $\tau(x,t)$ is an *entire* function of the variable x and a (local) smooth function of the variable t, and in the third case $\tau_n(x)$ is a sequence of entire functions of x. It turns out that under some generality assumption for each of the above linear equations, the answer to the question *when it has a meromorphic in x solution* is given in terms of equations describing the evolution of zeros of τ in the second variable.

To give an idea of these equations and why I called the very existence of them mysterious, as an instructive example, consider equation (1.1).

Let ψ be a meromorphic solution of (1.1) with $u = -2\partial_x^2 \ln \tau(x,t)$, where τ is an entire function of x and a smooth function of t in some neighborhood of $t = 0$. The generality assumption is that generic zeros of τ are simple. Consider the Laurent expansions of ψ and u in the neighborhood of a simple zero, $\tau(q(t),t) = 0$, $\partial_x \tau(q(t),t) \neq 0$:

$$\begin{aligned} u &= \frac{2}{(x-q)^2} + v + w(x-q) + \cdots; \\ \psi &= \frac{\alpha}{x-q} + \beta + \gamma(x-q) + \delta(x-q)^2 + \cdots. \end{aligned} \tag{1.7}$$

(The coefficients in these expansions $v, w, \ldots; \alpha, \beta, \ldots$ are smooth functions of the variable t). Substituting (1.7) into (1.1) gives an *infinite* system of equations. The first three of them are

$$\begin{aligned} &\alpha\dot{q} + 2\beta = 0; \\ &\dot{\alpha} + \alpha v + 2\gamma = 0; \\ &\dot{\beta} + v\beta - \gamma\dot{q} + \alpha w = 0. \end{aligned} \tag{1.8}$$

Taking the t-derivative of the first equation and using the other two, we get the equation

$$\ddot{q} = 2w, \tag{1.9}$$

derived first in **[6]**.

We would like to emphasize once again that there is no reason for the fact that the system (1.8) can be reduced to equations for the potential u only. Even more unexpected for the author was that, as we will see later, the existence of *one* meromorphic solution of equation (1.1) is sufficient for the existence of a *one-parameter* family of meromorphic solutions.

Formally, if we represent τ as an infinite product,

$$\tau(x,t) = c(t)\prod_i \big(x - q_i(t)\big), \tag{1.10}$$

then equation (1.9) can be written as the infinite system of equations

$$\ddot{q}_i = -4\sum_{j\neq i} \frac{1}{(q_i - q_j)^3}. \tag{1.11}$$

Equations (1.11) are purely formal because, even if τ has simple zeros at $t = 0$, in the general case there is no nontrivial interval of t where the zeros remain simple. One of the reasons to present (1.11) is that it shows that, when τ is a rational, trigonometric, or elliptic polynomial, equations (1.11) coincide with the equations of motion for the rational, trigonometrical, or elliptic Calogero–Moser (CM) system, respectively.

In a similar way, one can get that the existence of a meromorphic solution for equations (1.2) and (1.3) gives equations on zeros of τ which in the case when τ is an elliptic polynomial in x turned out to be the equations of motion of the elliptic Ruijsenaars–Schneider (RS) model and nested Bethe ansatz equations, respectively.

Recall that the elliptic CM system with k particles is a Hamiltonian system with coordinates $q = (q_1, \ldots, q_k)$, momentums $p = (p_1, \ldots, p_k)$, the canonical Poisson brackets $\{q_i, p_j\} = \delta_{ij}$, and the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{k} p_i^2 + \sum_{i\neq j} \wp(q_i - q_j). \tag{1.12}$$

The corresponding equations of motion admit the Lax representation $\dot{L} = [M, L]$ with

$$L_{ij} = p_i\delta_{ij} + 2(1-\delta_{ij})\Phi(q_i - q_j, z), \tag{1.13}$$

where

$$\Phi(x,z) = \frac{\sigma(z-x)}{\sigma(z)\sigma(z)} e^{x\zeta(z)} \tag{1.14}$$

and $\sigma, \zeta, \wp$ are classical Weierstrass functions.

The ellliptic RS system is a Hamiltonian system with coordinates $q = (q_1, \dots, q_k)$, momentums $p = (p_1, \dots, p_k)$, the canonical Poisson brackets $\{q_i, p_j\} = \delta_{ij}$, and the Hamiltonian

$$H = \sum_{i=1}^{k} f_i, \tag{1.15}$$

where

$$f_i := e^{p_i} \prod_{j \neq i} \left(\frac{\sigma(q_i - q_j - 1)\sigma(q_i - q_j + 1)}{\sigma(q_i - q_j)^2} \right)^{1/2}. \tag{1.16}$$

It is a completely integrable Hamiltonian system, whose equations of motion admit the Lax representation $\dot{L} = [M, L]$, where

$$L_{ij} = f_i \Phi(q_i - q_j - 1, z), \quad i, j = 1, \dots, k, \tag{1.17}$$

The elliptic nested Bethe ansatz equations are a system of algebraic equations

$$\prod_j \frac{\sigma(q_i^n - q_j^{n+1})\sigma(q_i^n - 1 - q_j^n)\sigma(q_i^n - q_j^{n-1} + 1)}{\sigma(q_i^n - q_j^{n-1})\sigma(q_i^{n+1} - q_j^n)\sigma(q_i^n - q_j^{n+1} - 1)} = -1 \tag{1.18}$$

for k unknown functions $q_i = \{q_i^n\}$, $i = 1, \dots, k$, of a discrete time variable $n \in \mathbb{Z}$.

The above systems are usually called elliptic *pole* systems, since they describe the dependence of the poles of the elliptic solutions of the KP, $2D$ Toda, and BDHE equations, respectively. A correspondence between finite-dimensional integrable systems and the pole systems of various soliton equations was considered in [7, 29, 32, 33]. In [2] it was generalized to the case of *field analogues* of CM type systems.

The most general form of the function τ, known to the author so far, for which the equations for its zeros are not formal, is the case of *abelian* functions, that is, when τ has the form

$$\tau = \tau(Ux + z, t), \tag{1.19}$$

where $x, t \in \mathbb{C}$ and $z \in \mathbb{C}^n$ are independent variables, $0 \neq U \in \mathbb{C}n$, and for all t the function $\tau(\cdot, t)$ is a holomorphic section of a line bundle $\mathcal{L} = \mathcal{L}(t)$ on an abelian variety $X = \mathbb{C}^n/\Lambda$, i.e., for all $\lambda \in \Lambda$ it satisfies the monodromy relations

$$\tau(z + \lambda, t) = e^{a_\lambda \cdot z + b_\lambda} \tau(z, t), \tag{1.20}$$

for some $a_\lambda \in \mathbb{C}^n$, $b_\lambda = b_\lambda(y, t) \in \mathbb{C}$.

It is tempting to call them *abelian* CM, RS, and nested Bethe ansatz equations. As we shall see below, they are central for the proof of three particular cases of the Welters conjecture.

2. RIEMANN–SCHOTTKY PROBLEM

Let $\mathbb{H}_g := \{B \in M_g(\mathbb{C}) \mid {}^tB = B, \mathrm{Im}(B) > 0\}$ be the Siegel upper half-space. For $B \in \mathbb{H}_g$ let $\Lambda := \Lambda_B := \mathbb{Z}^g + B\mathbb{Z}^g$ and $X := X_B := \mathbb{C}^g/\Lambda_B$. Riemann's theta function

$$\theta(z) := \theta(z, B) := \sum_{m \in \mathbb{Z}^g} e^{2\pi i(m,z)+\pi i(m,Bm)}, \quad (m, z) = m_1 z_1 + \cdots + m_g z_g, \tag{2.1}$$

is holomorphic and Λ-quasiperiodic in $z \in \mathbb{C}^g$.

The factor space $\mathbb{H}_g/Sp(2g, \mathbb{Z}) \simeq \mathcal{A}_g$ is the moduli space of g-dimensional ppavs. A ppav $(X, [\Theta]) \in \mathcal{A}_g$ is said to be *indecomposable* if the zero-divisor Θ of θ is irreducible.

Let $\mathcal{M}_g$ be the moduli space of nonsingular curves of genus g, and let $J : \mathcal{M}_g \to \mathcal{A}_g$ be the Jacobi map defined by the composition of maps $\mathcal{M}_g \to \mathbb{H}_g \to \mathcal{A}_g$. The first one requires a choice of a symplectic basis a_i, b_i $(i = 1, \ldots, g)$ of $H_1(\Gamma, \mathbb{Z})$ which defines a basis $\omega_1, \ldots, \omega_g$ of the space of holomorphic 1-forms on Γ such that $\int_{a_i} \omega_j = \delta_{ij}$, and then the *period matrix* and the *Jacobian variety* of Γ by

$$B := \left(\int_{b_i} \omega_j\right) \in \mathbb{H}_g \quad \text{and} \quad J(\Gamma) := \big(X_B, [\Theta_B]\big) \in \mathcal{A}_g,$$

respectively.

The above $J(\Gamma)$ is indecomposable and the Jacobi map J is injective (Torelli's theorem). The *(Riemann–)Schottky problem* is the problem of characterizing the Jacobi locus $\mathcal{J}_g := J(\mathcal{M}_g)$ or its closure $\overline{\mathcal{J}_g}$ in $\mathcal{A}_g$. For $g = 2, 3$, the dimensions of $\mathcal{M}_g$ and $\mathcal{A}_g$ coincide, and hence $\overline{\mathcal{J}_g} = \mathcal{A}_g$ by Torelli's theorem. Since $\mathcal{J}_4$ is of codimension 1 in $\mathcal{A}_4$, the case $g = 4$ is the first nontrivial case of the Riemann–Schottky problem.

A nontrivial relation for the Thetanullwerte of a curve of genus 4 was obtained by F. Schottky [45] in 1888, giving a modular form which vanishes on $\mathcal{J}_4$, and hence at least a *local* solution of the Riemann–Schottky problem in $g = 4$, i.e., $\overline{\mathcal{J}_4}$ is *an irreducible component of* the zero locus $\mathcal{S}_4$ of the Schottky relation. The irreducibility of $\mathcal{S}_4$ was proved by Igusa [23] in 1981, establishing $\overline{\mathcal{J}_4} = \mathcal{S}_4$, an effective answer to the Riemann–Schottky problem in genus 4.

A generalization of the Schottky relation to a curve of higher genus, the so-called Schottky–Jung relations, formulated as a conjecture by Schottky and Jung [46] in 1909, was proved by Farkas–Rauch [18] in 1970. Later, van Geemen [50] proved that the Schottky–Jung relations give a local solution of the Riemann–Schottky problem. They do not give a global solution when $g > 4$, since the variety they define has extra components already for $g = 5$ (Donagi [16]).

Over more than 120 year-long history of the Riemann–Schottky problem, quite a few geometric characterizations of the Jacobians have been obtained. None of them provides an explicit system of equations for the image of the Jacobian locus in the projective space under the level-two theta imbedding.

Following Mumford's review with a remark on Fay's trisecant formula [42], and the advent of algebraic geometrical integration scheme in the soliton theory [30, 31, 43] and Novikov's conjecture, significant progress was made in the 1980s in characterizing Jacobians and Pryms using Fay-like formulas and KP-like equations.

Let us first describe the trisecant identity in geometric terms. The Kummer variety $K(X)$ of $X \in \mathcal{A}_g$ is the image of the Kummer map

$$K = K_X : X \ni z \mapsto \big(: \Theta[\varepsilon, 0](z) :\big) \in \mathbb{CP}^{2^g - 1}, \tag{2.2}$$

where $\Theta[\varepsilon, 0](z) = \theta[\varepsilon, 0](2z, 2B)$ are the level-two theta-functions with half-integer characteristics $\varepsilon \in ((1/2)\mathbb{Z}/\mathbb{Z})^g$, i.e., they equal $\theta(2(z + B\varepsilon), 2B)$ up to some exponential factor so that we have

$$\theta(z + w)\theta(z - w) = \sum_{\varepsilon \in ((1/2)\mathbb{Z}/\mathbb{Z})^g} \Theta[\varepsilon, 0](z)\Theta[\varepsilon, 0](w). \tag{2.3}$$

We have $K(-z) = K(z)$ and $K(X) \simeq X/\{\pm 1\}$.

A *trisecant* of the Kummer variety is a projective line which meets $K(X)$ at three points. *Fay's trisecant formula* states that if $X = J(\Gamma)$, then $K(X)$ has a family of trisecants parameterized by 4 points A_i, $1 \leq i \leq 4$, on Γ. Gunning proved in [22] that, under certain nondegeneracy conditions, that the existence of a *one-parametric* family of trisecants characterizes the Jacobians.

Gunning's work was extended by Welters who proved that a Jacobian variety can be characterized by the existence of a formal one-parameter family of flexes of the Kummer variety [53]. A flex of the Kummer variety is a projective line which is tangent to $K(X)$ at some point up to order 2. It is a limiting case of trisecants when the three intersection points come together.

In [5] Arbarello and De Concini showed that the assumption in Welters' characterization is equivalent to an infinite sequence of partial differential equations known as the KP hierarchy, and proved that only a few first equations in the sequence are sufficient, by giving an explicit bound for the number of equations, $N = [(3/2)^g g!]$, based on the degree of $K(X)$.

An algebraic argument based on earlier results of Burchnall, Chaundy, and the author [10, 30, 31] characterizes the Jacobians using a commutative ring R of ordinary differential operators associated to a solution of the KP hierarchy. A simple counting argument then shows that only the first $2g + 1$ time evolutions in the hierarchy are needed to obtain R. The $2g + 1$ KP flows yield a finite number of differential equations for the Riemann theta function θ of X, to characterize a Jacobian. As for the number of equations, an easy estimate shows that $4g^2$ is enough, although a more careful argument should yield a better bound.

Novikov's conjecture, namely that just the first equation of the hierarchy ($N = 1$!) suffices to characterize the Jacobians, i.e.,

an indecomposable symmetric matrix B with positive definite imaginary part is the period matrix of a basis of normalized holomorphic differentials on a smooth algebraic curve Γ if and only if there are vectors $U \neq 0, V, W$, such that the function

$$u(x, y, t) = 2\partial_x^2 \ln \theta(Ux + Vy + Wt + Z|B), \tag{2.4}$$

satisfies the KP equation (1.4),

for quite some time seemed to be the strongest possible characterization within the reach of the soliton theory.

3. WELTER'S CONJECTURE

Novikov's conjecture is equivalent to the statement that the Jacobians are characterized by the existence of length 3 formal jet of flexes.

In [54] Welters formulated the question: *if the Kummer variety $K(X)$ has* one *trisecant, does it follow that X is a Jacobian?* In fact, there are three particular cases of the Welters conjecture, corresponding to three possible configurations of the intersection points (a, b, c) of $K(X)$ and the trisecant:

(i) all three points coincide ($a = b = c$);

(ii) two of them coincide ($a = b \neq c$);

(iii) all three intersection points are distinct ($a \neq b \neq c \neq a$).

Of course, the first two cases can be regarded as degenerations of the general case (iii). However, when the existence of only one trisecant is assumed, all three cases are independent and require their own approaches. The approaches used in [26,27] were based on the theories of three main soliton hierarchies (see details in [39]): the KP hierarchy for (i), the 2D Toda hierarchy for (ii) and the Bilinear Discrete Hirota Equations (BDHE) for (iii). Recently, pure algebraic proofs of the first two cases of the trisecant conjecture were obtained in [4].

Theorem 3.1. *An indecomposable principally polarized abelian variety (X, θ) is the Jacobian variety of a smooth algebraic curve of genus g if and only if there exist g-dimensional vectors $U \neq 0, V, A$, and constants p and E such that one of the following three equivalent conditions is satisfied:*

(A) *equality* (1.1) *with $\tau = \theta(Ux + Vt + Z)$ and*

$$\psi = \frac{\theta(A + Ux + Vt + Z)}{\theta(Ux + Vt + Z)} e^{px+Et} \tag{3.1}$$

holds, for an arbitrary vector Z;

(B) *for all theta characteristics $\varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$,*

$$\big(\partial_V - \partial_U^2 - 2p\partial_U + (E - p^2)\big)\Theta[\varepsilon, 0](A/2) = 0$$

(here and below ∂_U, ∂_V are the derivatives along the vectors U and V, respectively);

(C) *on the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$, the equation*

$$\begin{aligned} &\big[(\partial_V\theta)^2 - (\partial_U^2\theta)^2\big]\partial_U^2\theta + 2\big[\partial_U^2\theta\partial_U^3\theta - \partial_V\theta\partial_U\partial_V\theta\big]\partial_U\theta \\ &\quad + \big[\partial_V^2\theta - \partial_U^4\theta\big](\partial_U\theta)^2 = 0 \end{aligned} \tag{3.2}$$

holds.

The direct substitution of expression (3.1) into equation (1.1) and the use of the addition formula for the Riemann theta-functions shows the equivalence of conditions (A) and (B) in the theorem. Condition (B) means that the image of the point $A/2$ under the Kummer map is an inflection point (case (i) of Welters' conjecture).

Condition (C), which we call the abelian CM system, is the relation that is *really used* in the proof of the theorem. Formally, it is weaker than the other two conditions because its derivation does not use an explicit form of the solution ψ of equation (1.1), but requires only that ψ is a *meromorphic solution*. The latter, as we have seen, implies equation (1.9). Expanding the function θ in a neighborhood of a point $z \in \Theta := \{z \mid \theta(z) = 0\}$ such that $\partial_U \theta(z) \neq 0$, and noting that the latter condition holds on a dense subset of Θ since B is indecomposable, it is easy to see that equation (1.9) is equivalent to (3.2).

Equation (1.1) is one of the two auxiliary linear problems for the KP equation. For the author, the motivation to consider not the whole KP equation but just one of its auxiliary linear problems was his earlier work **[32]** on the elliptic Calogero–Moser (CM) system, where it was observed for the first time that equation (1.1) is all what one needs to construct the elliptic solutions of the KP equation.

The proof of Welters' conjecture was completed in **[27]**. First, here is the theorem which treats case (ii) of the conjecture:

Theorem 3.2. *An indecomposable, principally polarized abelian variety (X, θ) is the Jacobian of a smooth curve of genus g if and only if there exist nonzero g-dimensional vectors $U \neq A \pmod{\Lambda}$, V constants p, E, such that one of the following equivalent conditions holds:*

(A) *equation (1.2) with $\tau = \theta(Ux + Vt + Z)$ and ψ as in (3.1) holds for an arbitrary Z;*

(B) *the equations*

$$\partial_V \Theta[\varepsilon, 0]\big((A-U)/2\big) - e^p \Theta[\varepsilon, 0]\big((A+U)/2\big) + E\Theta[\varepsilon, 0]\big((A-U)/2\big) = 0,$$

are satisfied for all $\varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$. Here and below ∂_V is the constant vector field on $\mathbb{C}^g$ corresponding to the vector V;

(C) *the equation*

$$\partial_V\big[\theta(Z+U)\theta(Z-U)\big]\partial_V\theta(Z) = \big[\theta(Z+U)\theta(Z-U)\big]\partial^2_{VV}\theta(Z) \quad (3.3)$$

is valid on the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$.

Recall, that equation (1.2) is one of the two auxiliary linear problems for the $2D$ Toda lattice equation (1.5). The idea to use it for the characterization of the Jacobians was motivated by **[26]** and the author's earlier work with Zabrodin **[33]**, where a connection of the theory of elliptic solutions of the $2D$ Toda lattice equations and the theory of the elliptic Ruijsenaars–Schneider system was established.

Statement (B) is the second particular case of the trisecant conjecture: the line in $\mathbb{CP}^{2^g-1}$ passing through the points $K((A-U)/2)$ and $K((A+U)/2)$ of the Kummer variety is tangent to $K(X)$ at the point $K((A-U)/2)$.

Condition (C) is what we call the abelian RS equation.

The affirmative answer to the third particular case, (iii), of Welters' conjecture is given by the following statement.

Theorem 3.3. *An indecomposable, principally polarized abelian variety* (X,θ) *is the Jacobian of a smooth curve of genus* g *if and only if there exist nonzero* g*-dimensional vectors* $U \neq V \neq A \neq U \pmod{\Lambda}$ *such that one of the following equivalent conditions holds:*

(A) *equation* (1.3) *with* $\tau_n(x) = \theta(xU + nV + Z)$ *and*

$$\psi_n(x) = \frac{\theta(A + xU + nV + Z)}{\theta(xU + nV + Z)} e^{xp+nE}, \tag{3.4}$$

holds for an arbitrary Z*;*

(B) *the equations*

$$\Theta[\varepsilon,0]\left(\frac{A-U-V}{2}\right) + e^{p}\Theta[\varepsilon,0]\left(\frac{A+U-V}{2}\right) = e^{E}\,\Theta[\varepsilon,0]\left(\frac{A+V-U}{2}\right),$$

are satisfied for all $\varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{g}$*;*

(C) *the equation*

$$\frac{\theta(Z+U)\theta(Z-V)\theta(Z-U+V)}{\theta(Z-U)\theta(Z+V)\,\theta(Z+U-V)} = -1 \pmod{\theta} \tag{3.5}$$

is valid on the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$.

Under the assumption that the vector U *spans an elliptic curve* in X, Theorem 3.3 was proved in [29], where the connection of the elliptic solutions of BDHE and the so-called elliptic nested Bethe ansatz equations was established. Condition (C) is its abelian generalization.

4. THE PROBLEM OF CHARACTERIZATION OF PRYM VARIETIES

An involution $\sigma : \Gamma \to \Gamma$ on a smooth algebraic curve Γ naturally determines an involution $\sigma^* : J(\Gamma) \mapsto J(\Gamma)$ on its Jacobian. The odd subspace with respect to this involution is a sum of an Abelian subvariety of lower dimension, called the Prym variety, and a finite group. The restriction of the principal polarization of the Jacobian determines a polarization of the Prym variety which is principal if and only if the original involution of the curve has at most two fixed points. The problem of characterizing the locus $\mathcal{P}_g$ of Prym varieties of dimension g in the space $\mathcal{A}_g$ of all principally polarized Abelian varieties is well known and during its history has attracted considerable interest. This problem is much harder than the Riemann–Schottky problem and until relatively recently its solution in terms of a finite system of equations was completely open.

The problem of characterizing Prym varieties in the case of curves with an involution having two fixed points was solved in [25] in terms of the Schrödinger operators integrable with respect to one energy level. The theory of such operators was developed by Novikov and Veselov in [51, 52], where the authors also introduced the corresponding nonlinear equation, the so-called Novikov–Veselov equation. Curves with an involution having a pair of fixed points can be regarded as a limit of unramified covers. A characterization of the Prym varieties in the latter case in terms of the existence of quadrisecants was obtained the author and Grushevsky in [21].

The existence of families of quadrisecants for curves with an involution having at most two fixed points was proved in [9, 20]. An analogue of Gunning's theorem asserting that the existence of a family of secants characterizes Prym varieties was proved by Debarre [13]. We note that the existence of one quadrisecant does not characterize Prym varieties. A counterexample to the naive generalization of Welters' conjecture was constructed by Beauville and Debarre in [9].

It was proved in [21] that the existence of a symmetric pair of quadrisecants is a characteristic property for Prym varieties of unramified covers.

Theorem 4.1 (Geometric characterization of Prym varieties). *An indecomposable principally polarized Abelian variety $(X, \theta) \in \mathcal{A}_g$ is in the closure of the locus of Prym varieties of smooth unramified double covers if and only if there exist four distinct points $p_1, p_2, p_3, p_4 \in X$, none of them of order two, such that the images of the Kummer map of the eight points $p_1 \pm p_2 \pm p_3 \pm p_4$ lie on two quadrisecants (the corresponding quadruples of points are determined by the number of plus signs).*

We should note that the proof of this statement required constructing and developing the theory of a new integrable equation because before that, in contrast with all other cases, no nonlinear equations whose algebro-geometric solutions are associated to unramified double covers were known.

The auxiliary linear equation of the corresponding analogue of the Novikov–Veselov equation is a discrete analogue of the potential Schrödinger equation considered first in [15]. It has the form

$$\psi_{n+1,m+1} - u_{n,m}(\psi_{n+1,m} - \psi_{n,m+1}) - \psi_{n,m} = 0. \tag{4.1}$$

The analog of condition (C) in the previous theorem which can also be thought as the abelian generalization of some discrete time integrable system (which has not been studied so far) is as follows:

(C) *There are constants $c_i^{\pm}, i = 1, 2, 3$ such that two equations (one for the top choice of signs everywhere, and one for the bottom)*

$$\begin{aligned} & c_1^{\mp 2} c_3^2 \theta(Z + U - V)\theta(Z - U \pm W)\theta(Z + V \pm W) \\ & \quad + c_2^{\mp 2} c_3^2 \theta(Z - U + V)\theta(Z + U \pm W)\theta(Z - V \pm W) \\ & = c_1^{\mp 2} c_2^{\mp 2} \theta(Z - U - V)\theta(Z + U \pm W)\theta(Z + V \pm W) \\ & \quad + \theta(Z + U + V)\theta(Z - U \pm W)\theta(Z - V \pm W) \end{aligned} \tag{4.2}$$

are valid on the theta divisor $\{Z \in X : \theta(Z) = 0\}$.

5. ABELIAN SOLUTIONS OF THE SOLITON EQUATIONS

The general concept of *abelian solutions* of soliton equations was introduced by T. Shiota and the author in [37, 38]. It provides a unifying framework for the theory of the elliptic solutions of these equations and algebraic-geometrical solutions of rank 1 expressible in terms of Riemann (or Prym) theta-function. A solution $u(x, y, t)$ of the KP equation is called *abelian* if it is of the form

$$u = -2\partial_x^2 \ln \tau(Ux + z, y, t), \tag{5.1}$$

where $x, y, t \in \mathbb{C}$, and $z \in \mathbb{C}^n$ are independent variables, $0 \neq U \in \mathbb{C}^n$, and for all y, t the function $\tau(\cdot, y, t)$ is a holomorphic section of a line bundle $\mathcal{L} = \mathcal{L}(y, t)$ on an abelian variety $X = \mathbb{C}^n/\Lambda$, i.e., for all $\lambda \in \Lambda$ it satisfies the monodromy relations (1.19).

In the case of sections of the canonical line bundle on a principally polarized Abelian variety the corresponding theta-function is unique up to normalization. Hence the ansatz (5.1) takes the form $u = -2\partial_x^2 \ln \theta(Ux + Z(y, t) + z)$. Since flows commute with each other, the dependence of the vector $Z(y, t)$ must be linear,

$$u = -2\partial_x^2 \ln \theta(Ux + Vy + Wt + z). \tag{5.2}$$

Therefore, the problem of classification of such Abelian solutions is the same problem as posed by Novikov.

In the case of one-dimensional Abelian varieties, the problem of classification of Abelian solutions is the problem of classification of the elliptic solutions. The theory of elliptic solutions of the KP equation goes back to the remarkable work [1], where it was found that the dynamics of poles of the elliptic (rational or trigonometric) solutions of the Korteweg–de Vries equation can be described in terms of the elliptic (rational or trigonometric) Calogero–Moser (CM) system with certain constraints. It was observed in [32] that, when the constraints are removed, this restricted correspondence becomes an isomorphism when the elliptic solutions of the KP equation are considered. The elliptic solutions of the KP equation are distinguished amongst the general algebraic-geometric solutions by the condition that the corresponding vector U spans an elliptic curve embedded into the Jacobian of the curve. Note that, for any vector U, the closure of the group $\{Ux \mid x \in \mathbb{C}\}$, is an Abelian subvariety $X \subset J(\Gamma)$. So when this closure does not coincide with the whole Jacobian, we get nontrivial examples of Abelian solutions. Briefly, the main result on the classification of Abelian solutions of KP obtained in [37] can be formulated as the statement that all the Abelian solutions are obtained in this manner. To avoid some technical complications, we give the formulation of the corresponding theorem in the situation of general position.

Theorem 5.1. *Let $u(x, y, t)$ be an abelian solution of the KP such that the group $\mathbb{C}U \bmod \Lambda$ is dense in X. Then there exists a unique algebraic curve Γ with smooth marked point $P \in \Gamma$, holomorphic imbedding $j_0 : X \to J(\Gamma)$ and a torsion-free rank 1 sheaf $\mathcal{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ where $g = g(\Gamma)$ is the arithmetic genus of Γ, such that setting with the notation $j(z) = j_0(z) \otimes \mathcal{F}$*

$$\tau(Ux + z, y, t) = \rho(z, y, t)\widehat{\tau}\big(x, y, t, 0, \ldots \mid \Gamma, P, j(z)\big), \tag{5.3}$$

where $\widehat{\tau}(t_1, t_2, t_3, \ldots \mid \Gamma, P, \mathcal{F})$ *is the KP* τ*-function corresponding to the data* $(\Gamma, P, \mathcal{F})$*, and* $\rho(z, y, t) \not\equiv 0$ *satisfies the condition* $\partial_U \rho = 0$.

Note that if Γ is smooth then

$$\widehat{\tau}\big(x, t_2, t_3, \ldots \mid \Gamma, P, j(z)\big) = \theta\Big(Ux + \sum V_i t_i + j(z) \Bigm| B(\Gamma)\Big) e^{Q(x,t_2,t_3,\ldots)}, \tag{5.4}$$

where $V_i \in \mathbb{C}^n$, Q is a quadratic form, and $B(\Gamma)$ is the period matrix of Γ. A linearization on $J(\Gamma)$ of the nonlinear (y, t)-dynamics for $\tau(z, y, t)$ indicates the possibility of the existence of integrable systems on spaces of theta-functions of higher level. A CM system is an example of such a system for $n = 1$.

6. THE BAKER–AKHIEZER FUNCTIONS—GENERAL SCHEME

The “only if” part of all the theorems above is a corollary of the general algebraic-geometric construction of solutions of soliton equations based on a concept of the Baker–Akhiezer function.

Let Γ be a nonsingular algebraic curve of genus g with N marked points P_α and fixed local parameters $k_\alpha^{-1}(p)$ in neighborhoods of the marked points. The basic scalar *multipoint* and *multivariable* Baker–Akhiezer function $\psi(t, p)$ is a function of external parameters

$$t = (t_{\alpha,i}), \quad \alpha = 1, \ldots, N; \quad i = 0, \ldots; \quad \sum_\alpha t_{\alpha,0} = 0, \tag{6.1}$$

only finite number of which is nonzero, and a point $p \in \Gamma$. For each set of the external parameters t it is defined by its analytic properties on Γ.

Remark. For the simplicity we will begin with the assumption that the variables $t_{\alpha,0}$ are integers, i.e., $t_{\alpha,0} \in \mathbb{Z}$.

Lemma 6.1. *For any set of g points $\gamma_1, \ldots, \gamma_g$ in a general position there exists a unique (up to constant factor $c(t)$) function $\psi(t, p)$, such that:*

(i) *the function ψ (as a function of the variable $p \in \Gamma$) is meromorphic everywhere except for the points P_α and has at most simple poles at the points $\gamma_1, \ldots, \gamma_g$ (if all of them are distinct);*

(ii) *in a neighborhood of the point P_α the function ψ has the form*

$$\psi(t, p) = k_\alpha^{t_{\alpha,0}} \exp\left(\sum_{i=1}^\infty t_{\alpha,i} k_\alpha^i\right)\left(\sum_{s=0}^\infty \xi_{\alpha,s}(t) k_\alpha^{-s}\right), \quad k_\alpha = k_\alpha(p). \tag{6.2}$$

From the uniqueness of the Baker–Akhiezer function, we obtain

Theorem 6.1. *For each pair $(\alpha, n > 0)$, there exists a unique operator $L_{\alpha,n}$ of the form*

$$L_{\alpha,n} = \partial_{\alpha,1}^n + \sum_{j=0}^{n-1} u_j^{(\alpha,n)}(t) \partial_{\alpha,1}^j, \tag{6.3}$$

(where $\partial_{\alpha,n} = \partial/\partial t_{\alpha,n}$) such that

$$(\partial_{\alpha,n} - L_{\alpha,n})\psi(t,p) = 0. \tag{6.4}$$

The idea of the proof of the theorems of this type proposed in [30, 31] is universal.

For any formal series of the form (6.2), their exists a unique operator $L_{\alpha,n}$ of the form (6.3) such that

$$(\partial_{\alpha,n} - L_{\alpha,n})\psi(t,p) = O(k_\alpha^{-1}) \exp\left(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\right). \tag{6.5}$$

The coefficients of $L_{\alpha,n}$ are universal differential polynomials with respect to $\xi_{s,\alpha}$. They can be found after substitution of the series (6.2) into (6.5).

It turns out that if the series (6.2) is not formal, but is an expansion of the Baker–Akhiezer function in the neighborhood of P_α, the *congruence* (6.5) *becomes an equality*. Indeed, let us consider the function ψ_1,

$$\psi_1 = (\partial_{\alpha,n} - L_{\alpha,n})\psi(t,p). \tag{6.6}$$

It has the same analytic properties as ψ except for one. The expansion of this function in the neighborhood of P_α starts from $O(k_\alpha^{-1})$. From the uniqueness of the Baker–Akhiezer function it follows that $\psi_1 = 0$ and the equality (6.4) is proved.

Corollary 6.1. *The operators $L_{\alpha,n}$ satisfy the compatibility conditions*

$$[\partial_{\alpha,n} - L_{\alpha,n}, \partial_{\alpha,m} - L_{\alpha,m}] = 0. \tag{6.7}$$

Equations (6.7) are gauge invariant. For any function $c(t)$, operators

$$\tilde{L}_{\alpha,n} = c L_{\alpha,n} c^{-1} + (\partial_{\alpha,n} c) c^{-1} \tag{6.8}$$

have the same form (6.3) and satisfy the same operator equations (6.7). The gauge transformation (6.8) corresponds to the gauge transformation of the Baker–Akhiezer function

$$\widetilde{\psi}(t,p) = c(t)\psi(t,p). \tag{6.9}$$

In addition to differential equations (6.4), the Baker–Akhiezer function satisfies an infinite system of differential-difference equations. Recall that the discrete variables $t_{\alpha,0}$ are subject to the constraint $\sum_\alpha t_{\alpha,0} = 0$. Therefore, only the first $(N-1)$ of them are independent and $t_{N,0} = -\sum_{\alpha=1}^{N-1} t_{\alpha,0}$. Let us denote by T_α, $\alpha = 1, \dots, N-1$, the operator that shifts the arguments $t_{\alpha,0} \to t_{\alpha,0} + 1$ and $t_{N,0} \to t_{N,0} - 1$, respectively. For the sake of brevity, in the formulation of the next theorem we introduce the operator $T_N = T_1^{-1}$.

Theorem 6.2. *For each pair $(\alpha, n > 0)$, there exists a unique operator $\hat{L}_{\alpha,n}$ of the form*

$$\hat{L}_{\alpha,n} = T_\alpha^n + \sum_{j=0}^{n-1} v_j^{(\alpha,n)}(t) T_\alpha^j, \quad v_0^{(N,n)}(t) = 0 \tag{6.10}$$

such that

$$(\partial_{\alpha,n} - \hat{L}_{\alpha,n})\psi(t,p) = 0. \tag{6.11}$$

The proof is identical to that in the differential case.

Corollary 6.2. *The operators $\hat{L}_{\alpha,n}$ satisfy the compatibility conditions*

$$[\partial_{\alpha,n} - \hat{L}_{\alpha,n}, \partial_{\alpha,m} - \hat{L}_{\alpha,m}] = 0. \tag{6.12}$$

Theta-functional formulae. It should be emphasized that the algebro-geometric construction is not a sort of abstract "existence" and "uniqueness" theorems. It provides the explicit formulae for solutions in terms of the Riemann theta-functions. They are the corollary of the explicit formula for the Baker–Akhiezer function.

Let $a_i, b_i \in H_1(\Gamma, \mathbb{Z})$, $i = 1, \dots, g$, be a basis of cycles on Γ with the canonical intersection matrix, i.e., $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$, and let ω_i be the basis of holomorphic differentials on Γ normalized by the equations $\oint_{a_j} \omega_j = \delta_{ij}$. The matrix B of their b-periods $B_{ij} = \oint_{b_i} \omega_j$ is indecomposable symmetric matrix with positive definite imaginary part. By formula (2.1), it defines the Riemann theta-function $\theta(z) = \theta(z|B)$.

Theorem 6.3. *The Baker–Akhiezer function is given by the formula*

$$\psi(t, p) = c(t) \exp\Big(\sum t_{\alpha,i} \Omega_{\alpha,i}(p)\Big) \frac{\theta(A(p) + \sum U_{\alpha,i} t_{\alpha,i} + Z)}{\theta(A(p) + Z)}. \tag{6.13}$$

Here the sum is taken over all the indices $(\alpha, i > 0)$ and over the indices $(\alpha, 0)$ with $\alpha = 1, \dots, N-1$, and

(a) *$\Omega_{\alpha,i}(p)$ is the abelian integral, $\Omega_{\alpha,i}(p) = \int^p d\Omega_{\alpha,i}$, corresponding to the unique normalized, $\oint_{a_k} d\Omega_{\alpha,i} = 0$, meromorphic differential on Γ, which for $i > 0$ has the only pole of the form $d\Omega_{\alpha,i} = d(k_\alpha^i + O(1))$ at the marked point P_α and for $i = 0$ has simple poles at the marked point P_α and P_N with residues ± 1, respectively;*

(b) *$2\pi i U_{\alpha,j}$ is the vector of b-periods of the differential $d\Omega_{\alpha,j}$, i.e.,*

$$U_{\alpha,j}^k = \frac{1}{2\pi i} \oint_{b_k} d\Omega_{\alpha,j};$$

(c) *$A(p)$ is the Abel transform, i.e., a vector with the coordinates $A_i(p) = \int^p d\omega_i$;*

(d) *Z is an arbitrary vector (it corresponds to the divisor of poles of Baker–Akhiezer function).*

Notice that from the bilinear Riemann relations it follows that the expansion of the Abel transform near the marked point has the form

$$A(p) = A(P_\alpha) - \sum_{i=1}^{\infty} \frac{1}{i} U_{\alpha,i} k_\alpha^{-i}. \tag{6.14}$$

Example 1. One-point Baker–Akhiezer function. KP hierarchy. In the one-point case, the Baker–Akhiezer function has an exponential singularity at a single point P_1 and depends on a single set of variables $t_i = t_{1,i}$. Note that in this case there is no discrete variable,

$t_{1,0} \equiv 0$. Let us choose the normalization of the Baker–Akhiezer function with the help of the condition $\xi_{1,0} = 1$, i.e., an expansion of ψ in the neighborhood of P_1 equals

$$\psi(t_1, t_2, \ldots, p) = \exp\left(\sum_{i=1}^{\infty} t_i k^i\right)\left(1 + \sum_{s=1}^{\infty} \xi_s(t) k^{-s}\right). \tag{6.15}$$

Under this normalization (gauge), the corresponding operator L_n has the form

$$L_n = \partial_1^n + \sum_{i=0}^{n-2} u_i^{(n)} \partial_1^i. \tag{6.16}$$

For example, for $n = 2, 3$, after redefinition $x = t_1$ we have

$$L_2 = \partial_x^2 - u, \quad L_3 = \partial_x^3 - \frac{3}{2} u \partial_x - w, \tag{6.17}$$

with $u = 2\partial_x \xi_1$, $w = 3\partial_x \xi_2 + 3\partial_x^2 \xi_1 - \frac{3}{2} u \xi_1$.

If we define $y = t_2$, $t = t_3$, then from (6.7), with $n = 2$ and $m = 3$, it follows that $u(x, y, t, t_4, \ldots)$ satisfies the KP equation (1.4).

The normalization of the leading coefficient in (6.15) defines the function $c(t)$ in (6.13). This gives the following formula for the normalized one-point Baker–Akhiezer function:

$$\psi(t, p) = \exp\left(\sum t_i \Omega_i(p)\right) \frac{\theta(A(p) + \sum U_i t_i + Z)\theta(Z)}{\theta(\sum U_i t_i + Z)\theta(A(p) + Z)}, \tag{6.18}$$

(shifting Z if needed we may assumed that $A(P_1) = 0$). In order to get the explicit theta-functional form of the solution of the KP equation, it is enough to take the derivative of the first coefficient of the expansion at the marked point of the ratio of theta-functions in the formula (6.18).

Using (6.14), we get the final formula for the algebro-geometric solutions of the KP hierarchy [31], namely

$$u(t_1, t_2, \ldots) = -2\partial_1^2 \ln \theta\left(\sum_{i=1}^{\infty} U_i t_i + Z\right) + \text{const.} \tag{6.19}$$

Example 2. Two-point Baker–Akhiezer function. 2*D* Toda hierarchy. In the two-point case, the Baker–Akhiezer function has exponential singularities at two points $P_\alpha, \alpha = 1, 2$, and depends on two sets of continuous variables $t_{\alpha,i>0}$. In addition, it depends on one discrete variable $n = t_{1,0} = -t_{2,0}$. Let us choose the normalization of the Baker–Akhiezer function with the help of the condition $\xi_{1,0} = 1$.

According to Theorem 6.1, the function ψ satisfies two sets of differential equations. The compatibility conditions (6.7) within the each set can be regarded as two copies of the KP hierarchies. In addition the two-point Baker–Akhiezer function satisfies differential-difference equations (6.10). The first two of them have the form

$$(\partial_{1,1} - T + u)\psi = 0, \quad (\partial_{2,1} - wT^{-1})\psi = 0, \tag{6.20}$$

where

$$u = (T - 1)\xi_{1,1}(n, t), \quad w = e^{\varphi_n - \varphi_{n-1}}, \quad e^{\varphi_n(t)} = \xi_{2,0}(n, t). \tag{6.21}$$

The compatibility condition of these equations is equivalent to the $2D$ Toda equation with $\xi = t_{1,1}$ and $\eta = t_{2,1}$. The explicit formula for the solution $\varphi_n(t)$ is a direct corollary of the explicit formula for the Baker–Akhiezer function,

$$\varphi_n(t_{\alpha,i>0}) = \ln \frac{\theta((n+1)U + \sum U_{\alpha,i} t_{\alpha,i} + Z)}{\theta(nU + \sum U_{\alpha,i} t_{\alpha,i} + Z)}, \quad \alpha = 1, 2. \tag{6.22}$$

Example 3. Three-point Baker–Akhiezer function. Starting with three-point case, in which the number of discrete variables is 2, the Baker–Akhiezer function satisfies certain linear difference equations (in addition to the differential and the differential-difference equations (6.4), (6.11)). The origin of these equations is easy to explain. Indeed, if all the continuous variables vanish, $t_{\alpha,i>0} = 0$, then the Baker–Akhiezer function $\psi_{n,m} := \psi(n, m, p)$, where $n = -t_{1,0}$, $m = -t_{2,0}$, is a meromorphic function having a pole of order $n + m$ at P_3 and zeros of order n and m at P_1 and P_2, respectively, i.e.,

$$\psi_{n,m} \in H^0\big(D + n(P_3 - P_1) + m(P_3 - P_2)\big), \quad D = \gamma_1 + \cdots + \gamma_g. \tag{6.23}$$

The functions $\psi_{n+1,m}, \psi_{n,m+1}, \psi_{n,m}$ are all in the linear space $H^0(D + (n + m + 1)P_3 - nP_1 - mP_2)$. By Riemann–Roch theorem, for a generic D the latter space is 2-dimensional. Hence, these functions are linearly dependent, and they can be normalized such their linear dependence takes the form

$$\psi_{m,n+1} = \psi_{m+1,n} + u_{m,n}\psi_{m,n} \tag{6.24}$$

with

$$u_{n,m} = \frac{\tau_{m+1,n+1}\tau_{m,n}}{\tau_{m,n+1}\tau_{m+1,n}}, \quad \tau_{m,n} := \theta(mU + nV + Z). \tag{6.25}$$

At first glance, it seems that everything here is within the framework of classical algebraic-geometry. What might be new and brought to this subject by the soliton theory is understanding that *the discrete variables $t_{\alpha,0}$ can be replaced by continuous ones*. Of course, if in the formula (6.13) the variable $t_{\alpha,0}$ is not an integer, then ψ is not a single-valued function on Γ. Nevertheless, because the monodromy properties of ψ do not change if the shift of the argument by an integer, it satisfies the same type of linear equations with coefficients given by the same type of formulae. It is necessary to emphasize that in such a form the difference equation becomes a *functional* equation.

In the four-point case, there are three discrete variables n, m, and l. For each two of them the Baker–Akhiezer function satisfies a difference equation. The compatibility of these equations is expressed by the BDHE equation.

7. KEY IDEA AND STEPS OF THE PROOFS

As it was mentioned above, the proof of all the particular cases of Welters' trisecant conjecture uses different hierarchies: the KP, the 2D Toda, and the BDHE. In each case, there are some specific difficulties, but the main ideas and structures of the proof are the same. As an instructive example, we present in this section the idea and key steps of the proof of the first particular case of Welters' conjecture, namely, the proof of Theorem 3.1.

As it was mentioned above, the implication (A) $\to$ (C) is a direct corollary of (1.9). Now we are going to show that (1.9), which is satisfied when (1.1) has *one* meromorphic solution, is sufficient for the existence of *one-parametric* family of formal wave solutions below.

The wave solution of (1.1) is a solution of the form

$$\psi(x, y, k) = e^{kx+(k^2+b)t}\left(1 + \sum_{s=1}^{\infty} \xi_s(x, t)k^{-s}\right). \tag{7.1}$$

Lemma 7.1. *Suppose that equations* (1.9) *for the zeros of* $\tau(x, t)$ *hold. Then there exist meromorphic wave solutions of equation* (1.1) *that have simple poles at zeros* q *of* τ *and are holomorphic everywhere else.*

Proof. Substitution of (7.1) into (1.1) gives a recurrent system of equations

$$2\xi'_{s+1} = \partial_t \xi_s + u\xi_s - \xi''_s. \tag{7.2}$$

We are going to prove by induction that this system has meromorphic solutions with simple poles at all the zeros q of τ.

Let us expand ξ_s at q to get

$$\xi_s = \frac{r_s}{x-q} + r_{s0} + r_{s1}(x-q) + \cdots . \tag{7.3}$$

Suppose that ξ_s is defined and equation (7.2) has a meromorphic solution. Then the right-hand side of (7.2) has the zero residue at $x = q$, i.e.,

$$\mathrm{res}_q(\dot{\xi}_s + u\xi_s - \xi''_s) = \dot{r}_s + v_i r_s + 2r_{s1} = 0. \tag{7.4}$$

We need to show that the residue of the next equation vanishes also. From (7.2) it follows that the coefficients of the Laurent expansion for ξ_{s+1} are equal to

$$r_{s+1} = -\dot{q}r_s - 2r_{s0}, \quad 2r_{s+1,1} = \dot{r}_{s0} - r_{s1} + wr_s + vr_{s0}. \tag{7.5}$$

These equations imply

$$\dot{r}_{s+1} + vr_{s+1} + 2r_{s+1,1} = -r_s(\ddot{q} - 2w) - \dot{q}(\dot{r}_s + vr_s + 2r_{s1}) = 0, \tag{7.6}$$

and the lemma is proved. ■

λ-periodic wave solutions. Our next step in the proof is to fix a *translation-invariant* normalization of ξ_s which defines wave functions uniquely up to an x-independent factor. It is instructive to consider first the case of the periodic potentials $u(x + 1, t) = u(x, t)$ (see the details in [36]).

Equations (7.2) are solved recursively by the formulae

$$\xi_{s+1}(x, t) = c_{s+1}(t) + \xi^0_{s+1}(x, t), \tag{7.7}$$

$$\xi^0_{s+1}(x, t) = \frac{1}{2}\int_{x_0}^{x} (\dot{\xi}_s - \xi''_s + u\xi_s)dx = 0, \tag{7.8}$$

where $c_s(t)$ are *arbitrary* functions of the variable t. Let us show that the periodicity condition $\xi_s(x + 1, t) = \xi_s(x, t)$ defines the functions $c_s(t)$ uniquely up to an additive constant.

Assume that ξ_{s-1} is known and satisfies the condition that the corresponding function ξ_s^0 is periodic. The choice of the function $c_s(t)$ does not affect the periodicity property of ξ_s, but it does affect the periodicity in x of the function $\xi_{s+1}^0(x,t)$. In order to make $\xi_{s+1}^0(x,t)$ periodic, the function $c_s(t)$ should satisfy the linear differential equation

$$\partial_t c_s(t) + B(t)c_s(t) + \int_{x_0}^{x_0+1} \big(\dot{\xi}_s^0(x,t) + u(x,t)\xi_s^0(x,y)\big)\,dx, \tag{7.9}$$

where $B(t) = \int_{x_0}^{x_0+1} u\,dx$. This defines c_s uniquely up to a constant.

In the general case, when u is quasiperiodic, the normalization of the wave functions is defined along the same lines.

Let $Y_U = \langle \mathbb{C}U \rangle$ be the Zariski closure of the group $\mathbb{C}U = \{Ux \mid x \in \mathbb{C}\}$ in X. Shifting Y_U if needed, we may assume, without loss of generality, that Y_U is not in the singular locus Σ defined as the ∂_U-invariant subset of the theta-divisor Θ, i.e., $Y_U \not\subset \Sigma$. Then, for a sufficiently small t, we have $Y_U + Vt \notin \Sigma$ as well. Consider the restriction of the theta-function onto the affine subspace $\mathbb{C}^d + Vt$, where $\mathbb{C}^d :=$ (the identity component of $\pi^{-1}(Y_U)$), and $\pi : \mathbb{C}^g \to X = \mathbb{C}^g/\Lambda$ is the universal covering map of X:

$$\tau(z,t) = \theta(z + Vt), \quad z \in \mathbb{C}^d. \tag{7.10}$$

The function $u(z,t) = -2\partial_U^2 \ln \tau$ is periodic with respect to the lattice $\Lambda_U = \Lambda \cap \mathbb{C}^d$ and, for a fixed t, has a double pole along the divisor $\Theta^U(t) = (\Theta - Vt) \cap \mathbb{C}^d$.

Lemma 7.2. *Let equations (1.9) for the zeros of $\tau(Ux + z, t)$ hold. Then:*

(i) *equation (1.1) with the potential $u(Ux + z, t)$ has a wave solution of the form $\psi = e^{kx+k^2y}\phi(Ux + z, t, k)$ such that the coefficients $\xi_s(z,y)$ of the formal series*

$$\phi(z,t,k) = e^{bt}(1 + \sum_{s=1}^{\infty} \xi_s(z, tk^{-s}) \tag{7.11}$$

are meromorphic functions of the variable $z \in \mathbb{C}^d$ with a simple pole at the divisor $\Theta^U(t)$,

$$\xi_s(z + \lambda, t) = \xi_s(z,t) = \frac{\tau_s(z,t)}{\tau(z,t)}; \tag{7.12}$$

(ii) *$\phi(z,t,k)$ is quasiperiodic with respect to Λ_U, i.e., for $\lambda \in \Lambda_U$,*

$$\phi(z + \lambda, t, k; z_0) = \phi(z,t,k; z_0)\mu_\lambda(k); \tag{7.13}$$

(iii) *$\phi(z,t,k)$ is unique up to a ∂_U-invariant factor which is an exponent of the linear form,*

$$\phi_1(z,t,k) = \phi(z,t,k)e^{(\ell(k),z)}, \quad \big(\ell(k), U\big) = 0. \tag{7.14}$$

The spectral curve. The next goal is to show that λ-periodic wave solutions of equation (1.1) are common eigenfunctions of rings of commuting operators.

Note that a simple shift $z \to z + Z$, where $Z \notin \Sigma$, gives λ-periodic wave solutions with meromorphic coefficients along the affine subspaces $Z + \mathbb{C}^d$. These λ-periodic wave solutions are related to each other by a ∂_U-invariant factor. Therefore choosing, in a neighborhood of any $Z \notin \Sigma$, a hyperplane orthogonal to the vector U and fixing initial data on this hyperplane at $y = 0$, we define the corresponding series $\phi(z + Z, t, k)$ as a *local* meromorphic function of Z and the *global* meromorphic function of z.

Lemma 7.3. *Let the assumptions of Theorem 3.1 hold. Then there is a unique pseudo-differential operator*

$$\mathcal{L}(Z, \partial_x) = \partial_x + \sum_{s=1}^{\infty} w_s(Z) \partial_x^{-s} \tag{7.15}$$

such that

$$\mathcal{L}(Ux + Vy + Z, \partial_x)\psi = k\psi, \tag{7.16}$$

where $\psi = e^{kx+k^2y}\phi(Ux + Z, t, k)$ *is a* λ*-periodic solution of* (1.1)*. The coefficients* $w_s(Z)$ *of* $\mathcal{L}$ *are meromorphic functions on the abelian variety* X *with poles along the divisor* Θ*.*

Proof. Let ψ be a λ-periodic wave solution. The substitution of (7.11) into (7.16) gives a system of equations that recursively define $w_s(Z, t)$ as differential polynomials in $\xi_s(Z, t)$. The coefficients of ψ are local meromorphic functions of Z, but the coefficients of $\mathcal{L}$ are well-defined *global meromorphic functions* on $\mathbb{C}^g \setminus \Sigma$ because different λ-periodic wave solutions are related to each other by a ∂_U-invariant factor, which does not affect $\mathcal{L}$. The singular locus is of codimension ≥ 2. Then Hartogs' holomorphic extension theorem implies that $w_s(Z, t)$ can be extended to a global meromorphic function on $\mathbb{C}^g$.

The translational invariance of u implies the translational invariance of the λ-periodic wave solutions. Indeed, for any constant s, the series $\phi(Vs + Z, t - s, k)$ and $\phi(Z, t, k)$ correspond to λ-periodic solutions of the same equation. Therefore, they coincide up to a ∂_U-invariant factor. This factor does not affect $\mathcal{L}$. Hence, $w_s(Z, t) = w_s(Vt + Z)$.

The λ-periodic wave functions corresponding to Z and $Z + \lambda'$ for any $\lambda' \in \Lambda$ are also related to each other by a ∂_U-invariant factor. Hence, w_s are periodic with respect to Λ and therefore are meromorphic functions on the abelian variety X. The lemma is proved. ∎

Consider now the differential parts of the pseudodifferential operators $\mathcal{L}^m$. Let $\mathcal{L}_+^m$ be the differential operator such that $\mathcal{L}_-^m = \mathcal{L}^m - \mathcal{L}_+^m = F_m \partial^{-1} + O(\partial^{-2})$. The leading coefficient F_m of $\mathcal{L}_-^m$ is the residue of $\mathcal{L}^m$:

$$F_m = \operatorname{res}_\partial \mathcal{L}^m. \tag{7.17}$$

From the definition of $\mathcal{L}$, it follows that $[\partial_t - \partial_x^2 + u, \mathcal{L}^n] = 0$. Hence,

$$\left[\partial_t - \partial_x^2 + u, \mathcal{L}_+^m\right] = -\left[\partial_t - \partial_x^2 + u, \mathcal{L}_-^m\right] = 2\partial_x F_m. \tag{7.18}$$

The functions F_m are differential polynomials in the coefficients w_s of $\mathcal{L}$. Hence, $F_m(Z)$ are meromorphic functions on X. The next statement is crucial for the proof of the existence of commuting differential operators associated with u.

Lemma 7.4 ([26]). *The abelian functions F_m have at most second order poles on the divisor Θ.*

Let $\hat{F}$ be a linear space generated by $\{F_m, m = 0, 1, \ldots\}$, where we set $F_0 = 1$. It is a subspace of the 2^g-dimensional space of the abelian functions that have at most second order poles at Θ. Therefore, for all but $\hat{g} = \dim \hat{F}$ positive integers n, there exist constants $c_{i,n}$ such that

$$F_n(Z) + \sum_{i=0}^{n-1} c_{i,n} F_i(Z) = 0. \tag{7.19}$$

Let I denote the subset of integers n for which there are no such constants. We call this subset the gap sequence.

Lemma 7.5. *Let $\mathcal{L}$ be the pseudodifferential operator corresponding to a λ-periodic wave function ψ constructed above. Then, for the differential operators*

$$L_n = \mathcal{L}_+^n + \sum_{i=0}^{n-1} c_{i,n} \mathcal{L}_+^{n-i} = 0, \quad n \notin I, \tag{7.20}$$

the equations

$$L_n \psi = a_n(k)\psi, \quad a_n(k) = k^n + \sum_{s=1}^{\infty} a_{s,n} k^{n-s}, \tag{7.21}$$

where $a_{s,n}$ are constants, hold.

Proof. First note that from (7.18) it follows that

$$\left[\partial_t - \partial_x^2 + u, L_n\right] = 0. \tag{7.22}$$

Hence, if ψ is a λ-periodic wave solution of (1.1) corresponding to $Z \notin \Sigma$, then $L_n\psi$ is also a formal solution of the same equation. That implies the equation $L_n\psi = a_n(Z,k)\psi$, where a is ∂_U-invariant. The ambiguity in the definition of ψ does not affect a_n. Therefore, the coefficients of a_n are well-defined *global* meromorphic functions on $\mathbb{C}^g \setminus \Sigma$. The ∂_U-invariance of a_n implies that a_n, as a function of Z, is holomorphic outside of the locus. Hence it has an extension to a holomorphic function on $\mathbb{C}^g$. Equations (7.13) imply that a_n is periodic with respect to the lattice Λ. Hence a_n is Z-independent. Note that $a_{s,n} = c_{s,n}$, $s \leq n$. The lemma is proved. ■

The operator L_m can be regarded as a $Z \notin \Sigma$-parametric family of ordinary differential operators L_m^Z whose coefficients have the form

$$L_m^Z = \partial_x^n + \sum_{i=1}^{m} u_{i,m}(Ux + Z)\partial_x^{m-i}, \quad m \notin I. \tag{7.23}$$

Corollary 7.1. *The operators L_m^Z commute with each other,*

$$\left[L_n^Z, L_m^Z\right] = 0, \quad Z \notin \Sigma. \tag{7.24}$$

From (7.21) it follows that $[L_n^Z, L_m^Z]\psi = 0$. The commutator is an ordinary differential operator. Hence, the last equation implies (7.24).

Lemma 7.6. *Let $\mathcal{A}^Z$, $Z \notin \Sigma$, be a commutative ring of ordinary differential operators spanned by the operators L_n^Z. Then there is an irreducible algebraic curve Γ of arithmetic genus $\hat{g} = \dim \hat{F}$ such that $\mathcal{A}^Z$ is isomorphic to the ring $A(\Gamma, P_0)$ of the meromorphic functions on Γ with the only pole at a smooth point P_0. The correspondence $Z \to \mathcal{A}^Z$ defines a holomorphic imbedding of $X \setminus \Sigma$ into the space of torsion-free rank 1 sheaves $\mathcal{F}$ on Γ*

$$j : X \setminus \Sigma \mapsto \overline{\mathrm{Pic}}(\Gamma). \tag{7.25}$$

The statement of the lemma is a corollary of the following fundamental fact from the theory of commuting differential operators

Theorem 7.1 ([10, 30, 31, 43]). *There is a natural correspondence*

$$\mathcal{A} \leftrightarrow \left\{\Gamma, P_0, \left[k^{-1}\right]_1, \mathcal{F}\right\} \tag{7.26}$$

between regular *at $x = 0$ commutative rings $\mathcal{A}$ of ordinary linear differential operators containing a pair of monic operators of coprime orders, and sets of algebraic-geometrical data $\{\Gamma, P_0, [k^{-1}]_1, \mathcal{F}\}$, where Γ is an algebraic curve with a fixed first jet $[k^{-1}]_1$ of a local coordinate k^{-1} in the neighborhood of a smooth point $P_0 \in \Gamma$ and $\mathcal{F}$ is a torsion-free rank 1 sheaf on Γ such that*

$$H^0(\Gamma, \mathcal{F}) = H^1(\Gamma, \mathcal{F}) = 0. \tag{7.27}$$

The correspondence becomes one-to-one if the rings $\mathcal{A}$ are considered modulo conjugation $\mathcal{A}' = g(x)\mathcal{A}g^{-1}(x)$.

Note that in [10, 30, 31] the main attention was paid to the generic case of the commutative rings corresponding to smooth algebraic curves. The invariant formulation of the correspondence given above is due to Mumford [43].

The algebraic curve Γ is called the spectral curve of $\mathcal{A}$. The ring $\mathcal{A}$ is isomorphic to the ring $A(\Gamma, P_0)$ of meromorphic functions on Γ with the only pole at the point P_0. The isomorphism is defined by the equation

$$L_a\psi_0 = a\psi_0, \quad L_a \in \mathcal{A}, \quad a \in A(\Gamma, P_0). \tag{7.28}$$

Lemma 7.7 ([26]). *The linear space $\hat{F}$ generated by the abelian functions $\{F_0 = 1, F_m = \mathrm{res}_\partial \mathcal{L}^m\}$ is a subspace of the space H generated by F_0 and by the abelian functions $H_i = \partial_U \partial_{z_i} \ln \theta(Z)$.*

The construction of multivariate Baker–Akhiezer functions presented for smooth curves is a manifestation of a general statement valid for singular spectral curves: flows of the KP hierarchy define deformations of the commutative rings $\mathcal{A}$ of ordinary linear differential operators. The spectral curve is invariant under these flows. For a given spectral curve Γ, the orbits of the KP hierarchy are isomorphic to the generalized Jacobian $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$, which is the equivalence classes of zero degree divisors on the spectral curve (see the details in [30, 31, 47, 48]). Hence, for any $Z \notin \Sigma$, the orbit of the KP flows defines a holomorphic imbedding

$$i_Z : J(\Gamma) \mapsto X. \tag{7.29}$$

From (7.29) it follows that $J(\Gamma)$ is *compact*.

The generalized Jacobian of an algebraic curve is compact if and only if the curve is *smooth* [14]. On a smooth algebraic curve, a torsion-free rank 1 sheaf is a line bundle, i.e., $\overline{\mathrm{Pic}}(\Gamma) = J(\Gamma)$. Then (7.25) implies that i_Z is an isomorphism. Note that for the Jacobians of smooth algebraic curves, the bad locus Σ is empty [48], i.e., the imbedding j in (7.25) is defined everywhere on X and is inverse to i_Z. Theorem 3.1 is proved.

8. CHARACTERIZING JACOBIAN OF CURVES WITH INVOLUTION

As it was mentioned in the Introduction, the characterization problem of Jacobians of curves with involution addressed in [28] was motivated by the construction of solutions of two-dimensional integrable systems with symmetries. To the best of our knowledge, from a pure algebraic-geometrical perspective, the characterization problem of curves with involution in terms of their Jacobians has never been considered in its full generality. The only known to the author works in this direction are [8, 17, 44].

Two characterizations which distinguish such Jacobians were obtained in [28] within the framework of cases (i) and (ii) of Welter's conjecture. Both of them are limited to the case of involutions having at least one fixed point, i.e., to two-sheeted *ramified* covers.

In a certain sense, the setup we consider—the Jacobian and the Prym variety in it—resembles the setup arising in the famous Schottky–Yung relations, and it is tempting to find a way to get these relations by means of the soliton theory. Unfortunately, this challenging problem remains open.

The first characterization, related to the KP theory, is limited to the case of ramified cover for the obvious reason—a curve with one marked point is used in constructing its solutions.

Theorem 8.1. *An indecomposable principally polarized abelian variety (X, θ) is the Jacobian variety of a smooth algebraic curve Γ of genus g with involution $\sigma : \Gamma \to \Gamma$ having at least one point fixed if and only if there exist g-dimensional vectors $U \neq 0, V, A, \zeta$ and constants Ω_1, Ω_2, b_1 such that:*

Condition (A) *of Theorem* 3.1 *is satisfied and*

(B) *the intersection of the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$ with a shifted abelian subvariety $Y \subset X$ which is the Zariski closure of $\pi(Ux + \zeta) \subset X$ is reduced and the equation*

$$\partial_V \theta|_{\Theta \cap Y} = 0 \tag{8.1}$$

holds.

Moreover, the locus Π *of points* $\zeta \in X$ *for which equation* (8.1) *holds is the locus of points for which the equation* $\zeta + \sigma(\zeta) = 2P + K \in X$*, where* K *is the canonical class, holds.*

Condition (B) implies

(C) *there is a constant* b_2 *such that the equality*

$$\partial_U \partial_V \ln \theta|_{\hat{Y}} = b_2 \tag{8.2}$$

holds on Y.

From the addition theorem (2.3), it follows that (8.2) is equivalent to the condition that the vector $(\partial_U \partial_V K(0) - b_2 K(0))$ is *orthogonal to the image under the Kummer map* $K(\Pi)$ *of the shifted abelian subvariety* $\hat{Y}$:

$$\sum_{\varepsilon \in ((1/2)\mathbb{Z}/\mathbb{Z})^g} \big(\partial_U \partial_V \Theta[\varepsilon, 0](0) - b_2 \Theta[\varepsilon, 0](0)\big) \Theta[\varepsilon, 0](z) = 0, \quad z \in \hat{Y}, \tag{8.3}$$

whence follows the condition *of a kind of flatness* of the image under the Kummer map of the shifted Prym subvariety $\Pi \subset X$, that is, $K(\Pi)$ lies in a proper (projective) linear subspace.

The explicit meaning (B) is as follows. As shown in **[19, 48]**, the affine line $Ux + Z$ is not contained in Θ for any vector Z. Hence, the function $\tau(x, t) := \theta(Ux + Vt + z)$, $z \in Y$ is a *nontrivial* entire function of x. The statement that $\Theta \cap Y$ is reduced means that the zeros $q(t)$ of τ, considered as a function of x (depending on t), are generically simple, $\tau(q(t), t) = 0$, $\tau_x(q(t), t) \neq 0$. Then (8.2) is the equation

$$\partial_t q|_{t=0} = 0. \tag{8.4}$$

In the case when U spans an elliptic curve in the Jacobian, the statement that from (B) it follows that the corresponding curve Γ admits an involution is obvious. Indeed, in that case the curve is the normalization of the spectral curve of N-point elliptic CM systems. The latter is defined by the characteristic equation

$$\det\big(k \cdot \mathbb{I} - L(z)\big) = 0$$

of the matrix $L(z)$ defined in (1.13) with $q_i = q_i(0)$ and $p_i = \dot{q}_i(0)$, where $q_i(t)$ are roots of the equation $\theta(Ux + Vt + z) = 0$. If equation (8.4) holds, i.e., $p_i = 0$, then it is easy to see that the matrix $L(z)$ satisfies the equation $L^t(z) = -L(-z)$. The latter implies that the curve is invariant under the involution $(k, z) \to (-k, -z)$. That observation made in **[34]** was the main motivation behind **[28]**.

At the heart of the proof in the general case is the statement that if (B) is satisfied then there is a local coordinate k^{-1} such that if $\psi(x, t, k)$ is the wave solution of (1.1) as in Lemma 7.2 then $\psi(x, 0, -k) = \psi^*(x, 0, k)$ where ψ^* is a wave solution of the equation

$$(\partial_t + \partial_x^2 - u)\psi^*(x, t, k) = 0, \tag{8.5}$$

which is formally adjoint to (1.1).

The second characterization of the Jacobians of curves with involution is related to the 2D Toda theory. A priori, unlike in the KP case, there is no obvious reason why it is not applicable to all types of involution, including unramified covers. It turned out that there is an obstacle for the case of unramified covers, and our second theorem also gives a characterization of the Jacobians of curves with involution *with* fixed points.

Theorem 8.2. *An indecomposable, principally polarized abelian variety* (X,θ) *is the Jacobian of a smooth curve of genus g with involution having fixed points if and only if there exist nonzero g-dimensional vectors* $U \neq A \pmod{\Lambda}$, V, ζ, *constants* Ω_0, Ω_1, b_1 *such that:*

Condition (A) *of Theorem 3.2 is satisfied and*

(B) (i) *the intersection of the theta-divisor with the shifted Abelian variety* Y, *which is a closure of* $\pi(Ux+\zeta)$, *is reduced and is not invariant under the shift by* U, $\Theta \cap Y \neq (\Theta + U) \cap Y$, *and* (ii) *the equation*

$$\left(\left(\partial_V \theta(z)\right)^2 + \theta(z+U)\theta(z-U)\right)\Big|_{z\in\Theta\cap Y} = 0 \tag{8.6}$$

holds.

Moreover, the locus of the points $\zeta \in X$ *for which equation* (8.6) *holds is the locus of point for which the equation* $\zeta + \zeta^\sigma = K + P_1 + P_2 \in J(\Gamma)$, *where* (P_1, P_2) *are points of the curves permuted by* σ *and such that* $U = A(P_2) - A(P_1)$ *is satisfied.*

Remark 1. In the case when U spans an elliptic curve in the Jacobian, the statement of the theorem was proved first in [40].

The geometric form of the characterization is the condition that *the vector* $(2\partial_V^2 K(0) - b_2 K(U) - b_3 K(0))$ *is orthogonal to the image under the Kummer map of the abelian subvariety* Π:

$$\sum_{\varepsilon\in((1/2)\mathbb{Z}/\mathbb{Z})^g} \left(2\partial_V^2 \Theta[\varepsilon,0](0) - b_2\Theta[\varepsilon,0](U) - b_3\Theta[\varepsilon,0](0)\right)\Theta[\varepsilon,0](z) = 0, \tag{8.7}$$

where $z \in \Pi$ and b_3 is a constant.

9. NONLOCAL GENERATING PROBLEM

Until now our main focus was on equations that arise from the *local* generating properties of two-dimensional linear operators with meromorphic coefficients. The *nonlocal* generating properties of the same linear operators do not lead directly to equations of motion for zeros of the τ function. To begin with, they generate the Lax representation of these equations. That nonlocal perspective is known for the elliptic case. Its abelian generalization is an open and challenging problem.

Let $\mathcal{D}$ be a linear differential or difference operator in two variables (x,t) with coefficients which are scalar or matrix elliptic functions of the variable x (i.e., meromorphic double-periodic functions with the periods $2\omega_\alpha$, $\alpha = 1, 2$). We do not assume any special dependence of the coefficients with respect to the second variable. Then it is natural to introduce a notion of *double-Bloch* solutions of the equation

$$\mathcal{D}\Psi = 0. \tag{9.1}$$

We call a *meromorphic* vector-function $f(x)$ that satisfies the following monodromy properties:

$$f(x + 2\omega_\alpha) = B_\alpha f(x), \quad \alpha = 1, 2, \tag{9.2}$$

a *double-Bloch function*. The complex numbers B_α are called *Bloch multipliers*. (In other words, f is a meromorphic section of a vector bundle over the elliptic curve.)

In the most general form, a problem considered in the framework of elliptic pole systems is to *classify* and to *construct* all the operators L such that equation (9.1) has *sufficiently many* double-Bloch solutions.

It turns out that the existence of the double-Bloch solutions is so restrictive that only in exceptional cases such solutions do exist. A simple and general explanation of that is due to the Riemann–Roch theorem. Let D be a set of points q_i, $i = 1, \ldots, m$, on the elliptic curve Γ_0 with multiplicities d_i and let $V = V(D; B_1, B_2)$ be a linear space of the double-Bloch functions with the Bloch multipliers B_α that have poles at q_i of order less than or equal to d_i and holomorphic outside D. Then the dimension of D is equal to

$$\dim D = \deg D = \sum_i d_i.$$

Now let q_i depend on the variable t. Then for $f \in D(t)$, the function $\mathcal{D} f$ is a double-Bloch function with the same Bloch multipliers, but in general with higher orders of poles because taking derivatives and multiplication by the elliptic coefficients increase orders. Therefore, the operator $\mathcal{D}$ defines a linear operator

$$\mathcal{D}|_D : V\big(D(t); B_1, B_2\big) \mapsto V\big(D'(t); B_1, B_2\big), \quad N' = \deg D' > N = \deg D,$$

and (9.1) is *always* equivalent to an *overdetermined* linear system of N' equations in N unknown variables which are the coefficients $c_i = c_i(t)$ of an expansion of $\Psi \in V(t)$ with respect to a basis of functions $f_i(t) \in V(t)$. With some exaggeration, one may say that in the soliton theory the representation of a system in the form of the compatibility condition of an overdetermined system of the linear problems is considered as equivalent to integrability.

In all of known examples, $N' = 2N$ and the overdetermined system of equations has the form

$$LC = kC, \quad \partial_t C = MC, \tag{9.3}$$

where L and M are $N \times N$ matrix functions depending on a point z of the elliptic curve as a parameter. A compatibility condition of (9.3) has the standard Lax form $\partial_t L = [M, L]$, and is equivalent to a finite-dimensional integrable system.

The basis in the space of the double-Bloch functions can be written in terms of the fundamental function $\Phi(x, z)$ defined by the formula (1.14). Note that $\Phi(x, z)$ is a solution of the Lame equation

$$\left(\frac{d^2}{dx^2} - 2\wp(x)\right)\Phi(x, z) = \wp(z)\Phi(x, z). \tag{9.4}$$

From the monodromy properties, it follows that Φ, considered as a function of z, is doubly-periodic,

$$\Phi(x, z + 2\omega_\alpha) = \Phi(x, z),$$

though it is not elliptic in the classical sense due to an essential singularity at $z = 0$ for $x \neq 0$.

As a function of x, the function $\Phi(x, z)$ is a double-Bloch function, i.e.,

$$\Phi(x + 2\omega_\alpha, z) = T_\alpha(z)\Phi(x, z), \quad T_\alpha(z) = \exp\big(2\omega_\alpha\zeta(z) - 2\zeta(\omega_\alpha)z\big).$$

In the fundamental domain of the lattice defined by $2\omega_\alpha$, the function $\Phi(x, z)$ has a unique pole at the point $x = 0$,

$$\Phi(x, z) = x^{-1} + O(x). \tag{9.5}$$

The gauge transformation

$$f(x) \mapsto \tilde{f}(x) = f(x)e^{ax},$$

where a is an arbitrary constant, does not change the poles of any function and transforms a double Bloch-function into a double-Bloch function. If B_α are Bloch multipliers for f, then the Bloch multipliers for $\tilde{f}$ are equal to

$$\tilde{B}_1 = B_1 e^{2a\omega_1}, \quad \tilde{B}_2 = B_2 e^{2a\omega_2}. \tag{9.6}$$

The two pairs of Bloch multipliers that are connected with each other through the relation (9.6) for some a are called equivalent. Note that for all equivalent pairs of Bloch multipliers, the product $B_1^{\omega_2} B_2^{-\omega_1}$ is a constant depending on the equivalence class only.

From (9.5) it follows that a double-Bloch function $f(x)$ with simple poles q_i in the fundamental domain and with Bloch multipliers B_α (such that at least one of them is not equal to 1) may be represented in the form

$$f(x) = \sum_{i=1}^{N} c_i \Phi(x - q_i, z)e^{kx}, \tag{9.7}$$

where c_i is a residue of f at x_i and z, k are parameters related by

$$B_\alpha = T_\alpha(z)e^{2\omega_\alpha k}. \tag{9.8}$$

(Any pair of Bloch multipliers may be represented in the form (9.8) with an appropriate choice of the parameters z and k.)

To prove (9.7), it is enough to note that as a function of x the difference of the left- and right-hand sides is holomorphic in the fundamental domain. It is a double-Bloch function with the same Bloch multipliers as the function f. But a nontrivial double-Bloch function with at least one of the Bloch multipliers that is not equal to 1 has at least one pole in the fundamental domain.

Example: elliptic CM system. Let us consider equation (1.1) with an elliptic (in x) potential $u(x,t)$. Suppose that equation (1.1) has N linearly independent double-Bloch solutions with equivalent Bloch multipliers and N simple poles $q_i(t)$. The assumption that there exist N linearly independent double-Bloch solutions with equivalent Bloch multipliers implies that they can be written in the form

$$\Psi = \sum_{i=1}^{N} c_i(t,k,z)\Phi\big(x - q_i(t), z\big)e^{kx+k^2t}, \tag{9.9}$$

with the same z but different values of the parameter k.

Let us substitute (9.9) into (1.1). Then (1.1) is satisfied if and if we get a function holomorphic in the fundamental domain. First of all, we conclude that u has poles at q_i only. The vanishing of the triple poles $(x - q_i)^{-3}$ implies that $u(x,t)$ has the form

$$u(x,t) = 2\sum_{i=1}^{N} \wp\big(x - q_i(t)\big). \tag{9.10}$$

The vanishing of the double poles $(x - q_i)^{-2}$ gives the equalities that can be written as a matrix equation for the vector $C = (c_i)$,

$$\big(L(t,z) + k\mathbb{I}\big)C = 0, \tag{9.11}$$

where I is the unit matrix and the Lax matrix $L(t,z)$ is defined in (1.13). Finally, the vanishing of the simple poles gives the equations

$$\big(\partial_t - M(t,z)\big)C = 0, \tag{9.12}$$

where

$$M_{ij} = \Big(\wp(z) - 2\sum_{j\neq i}\wp(q_i - q_j)\Big)\delta_{ij} - 2(1-\delta_{ij})\Phi'(q_i - q_j, z). \tag{9.13}$$

The existence of N linearly independent solutions for (1.1) with equivalent Bloch multipliers implies that (9.11) and (9.12) have N independent solutions corresponding to different values of k. Hence, as a compatibility condition, we get the Lax equation $\dot{L} = [M, L]$ for the elliptic CM system.

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