

NONCOMMUTATIVE CREPANT RESOLUTIONS, AN OVERVIEW

MICHEL VAN DEN BERGH

ABSTRACT

Noncommutative crepant resolutions (NCCRs) are noncommutative analogues of the usual crepant resolutions that appear in algebraic geometry. In this paper we survey some results around NCCRs.

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1. INTRODUCTION

In this paper we will give an introduction to noncommutative crepant resolutions with some emphasis on our joint work with Špela Špenko about quotient singularities of reductive groups. Other surveys are [96, 115, 131].

1.1. Notation and conventions

We fix a few notations and definitions which are mostly self explanatory. For simplicity, we assume throughout that k is an algebraically closed field of characteristic zero, although this is often not necessary. In Section 5 we put $k = \mathbb{C}$ when invoking Hodge theory. For us an *algebraic variety* is a possibly singular integral separated scheme of finite type over k .

Modules over rings or sheaves of rings are left modules. Right modules are indicated by $(-)^{\circ}$. If Λ is a ring then we denote by $D(\Lambda)$ the unbounded derived category of complexes of Λ -modules and by $\text{Perf}(\Lambda)$ its full subcategory of perfect Λ -complexes. If Λ is noetherian then we write $\text{mod}(\Lambda)$ for the category of finitely generated Λ -modules. We also put $\mathcal{D}(\Lambda) = D^b(\text{mod}(\Lambda))$. We use similar notations in the geometric context. If \mathcal{X} is an Artin stack and Λ is a quasicoherent sheaf of rings on \mathcal{X} then $D_{\text{Qch}}(\Lambda)$ is the unbounded derived category of complexes of left Λ -modules with quasicoherent cohomology. The category of perfect Λ -complexes is denoted by $\text{Perf}(\Lambda)$, and we also put $\mathcal{D}(\Lambda) = D^b(\text{coh}(\Lambda))$ when Λ is noetherian. If $\Lambda = \mathcal{O}_{\mathcal{X}}$ then we replace Λ in the notations by \mathcal{X} .

A finitely generated R -module M over a normal noetherian domain is said to be *reflexive* if the canonical map $M \mapsto M^{\vee\vee}$ is an isomorphism where $M^{\vee} = \text{Hom}_R(M, R)$. This implies in particular that M is torsion free. If R a commutative noetherian domain then a *maximal Cohen–Macaulay* R -module is an R -module M such that M_m is maximal Cohen–Macaulay as R_m -module for every maximal ideal m . If R is has finite injective dimension then we say that R is *Gorenstein*. This implies that R is maximal Cohen–Macaulay.

1.2. Crepant resolutions and derived equivalences

Let X be a normal algebraic variety with Gorenstein singularities. A resolution of singularities $\pi : Y \rightarrow X$ is said to be *crepant* if $\pi^*\omega_X = \omega_Y$. In some sense, a crepant resolution is the tightest possible smooth approximation of an algebraic variety. Such crepant resolutions need not exist, however. For starters, their existence implies that X has rational singularities [89, COROLLARY 5.24] and this already strong restriction is far from sufficient. For example, the three-dimensional hypersurface singularities

$$x^2 + y^2 + z^2 + w^n = 0 \quad (n \geq 2) \tag{1.1}$$

have crepant resolutions if and only if n is even [111, COROLLARY 1.16]. When crepant resolutions do exist they are generally not unique. For example,

$$xy - zw = 0, \tag{1.2}$$

which corresponds to $n = 2$ in (1.1), has two distinct crepant resolutions given by blowing up (x, z) and (x, w) . This is the so-called “Atiyah flop.”

Nonetheless, experience has shown that such different crepant resolutions are strongly related. In particular, we have the following result:

Theorem 1.1 ([6, 98], see also Section 5 below). *Assume that X has canonical Gorenstein singularities. Then the Hodge numbers of Y for a crepant resolution $Y \rightarrow X$ are independent of Y .*

Kawamata and independently Bondal and Orlov in their lecture at ICM2002 conjectured an analogous categorical result, a variant¹ of which we state below.

Conjecture 1.2 ([22, CONJECTURE 4.4], [76, CONJECTURE 1.2]). *Assume X is a normal algebraic variety with Gorenstein singularities and $\pi_i : Y_i \rightarrow X$ for $i = 1, 2$ are two crepant resolutions (by schemes or DM-stacks). Then there is an equivalence of triangulated categories $F : \mathcal{D}(Y_1) \cong \mathcal{D}(Y_2)$, linear over X (cf. Remark 1.5 below).*

The conjecture is known (under some probably unnecessary projectivity hypotheses) in dimension ≤ 3 , by the work of Bridgeland [27] (see Section 1.4 below), and for toric varieties, by the work of Kawamata [79]. For symplectic singularities [10], it is true, up to an étale covering of X , by [72, THEOREM 1.6]. Furthermore, it is known for many specific crepant resolutions, e.g., those related by variation of GIT [5, 59, 60] (see also Section 4.2 below).

Remark 1.3. Conjecture 1.2 makes no statement about the nature of the equivalence $\mathcal{D}(Y_1) \cong \mathcal{D}(Y_2)$. In the case of the Atiyah flop, one possible equivalence is given by the Fourier–Mukai functor for the “fiber product kernel” $\mathcal{O}_{Y_1 \times_X Y_2}$ [21, THEOREM 3.6] (see also [18]) but this is far from the only possibility. Furthermore, $\mathcal{O}_{Y_1 \times_X Y_2}$ does not always work as Example 1.4 below shows.

It is now understood, thanks to intuition from mirror symmetry, that the equivalences in Conjecture 1.2 should be canonically associated to paths connecting two points in a topological space called the “stringy Kähler moduli space” (SKMS). In the case of the Atiyah flop, the SKMS is given by $\mathbb{P}^1 - \{0, 1, \infty\}$ [48, 59]. See also [63] and Section 4.3 below. The fact that the asserted equivalence in Conjecture 1.2 is expected to be noncanonical by itself might be the reason that the conjecture seems difficult to prove.

Below $\text{Gr}(d, n)$ is the Grassmannian of d -dimensional subspaces of the n -dimensional vector space k^n .

Example 1.4. The cotangent bundles $T^* \text{Gr}(d, n)$ and $T^* \text{Gr}(n - d, n)$, for complementary Grassmannians with $d \leq n/2$ are crepant resolutions of $\overline{B(d)} := \{X \in M_n(k) \mid X^2 = 0, \text{rk } X \leq d\}$ (e.g., [37, §6.1]). According to [37, §6], there is an equivalence $F : \mathcal{D}(T^* \text{Gr}(d, n)) \rightarrow \mathcal{D}(T^* \text{Gr}(n - d, n))$, but it is not given by the fiber product kernel (see [77, 103] for the case $(k, n) = (2, 4)$).

¹ We have omitted the projectivity hypotheses which appear in the original context and extended the conjecture to DM-stacks which is the natural context as will become clear below.

Remark 1.5. As said, one requires the derived equivalence F in Conjecture 1.2 to be linear over X . On the most basic level, this means the following: let $\text{Perf}(X)$ be the category of perfect complexes on X . Then $\mathcal{D}(Y_1), \mathcal{D}(Y_2)$ are $\text{Perf}(X)$ -modules, where $A \in \text{Perf}(X)$ acts as $L\pi_i^* A \otimes_{Y_i}^L -$, for $i = 1, 2$, and we want natural isomorphisms $F(L\pi_1^* A \otimes_{Y_1}^L -) \cong L\pi_2^* A \otimes_{Y_2}^L F(-)$ satisfying the appropriate compatibilities. *To simplify the exposition, we will implicitly assume in the rest of this paper that all constructions satisfy the appropriate linearity hypotheses.*

1.3. Noncommutative rings

Most of the results below will be based on the interplay between algebraic geometry and noncommutative rings. The relation between those subjects was first observed by Beilinson [11]. The connection is via tilting complexes.

Definition 1.6. Let Y be a noetherian scheme. A *partial tilting complex* \mathcal{T} on Y is a perfect complex such that $\text{Ext}_Y^i(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$. A *tilting complex* is a partial tilting complex that *generates* $D_{\text{Qch}}(Y)$ in the sense that its right orthogonal is zero, i.e., $\text{RHom}_Y(\mathcal{T}, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$. A *(partial) tilting bundle* is a (partial) tilting complex which is a vector bundle.

Below we will also use tilting complexes in slightly more general contexts (e.g., DM-stacks). Very general results concerning tilting complexes are [82, THEOREMS 1,2]. For simplicity, we state a slightly dumbed down version of them, although below we will sometimes silently rely on the stronger results in [82]. See also [19, 114].

Theorem 1.7 ([82, THEOREMS 1,2]). *If \mathcal{T} is a tilting complex on a noetherian scheme Y then $\text{RHom}_Y(\mathcal{T}, -)$ defines an equivalence of categories between $D_{\text{Qch}}(Y)$ and $D(\Lambda^\circ)$ for $\Lambda = \text{End}_Y(\mathcal{T})$. Moreover, if Y is regular then Λ has finite global dimension. If, furthermore, Λ is right noetherian then $\text{RHom}_Y(\mathcal{T}, -)$ restricts to an equivalence of categories $\mathcal{D}(Y) \cong \mathcal{D}(\Lambda^\circ)$.*

So a tilting complex reduces the homological algebra of Y to the usually noncommutative ring $\Lambda = \text{End}_Y(\mathcal{T})$. In the case of projective space \mathbb{P}^n , one can take $\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)$ [11].

1.4. Bridgeland's result

1.4.1. Flops

Let us return to Conjecture 1.2. In the absence of any specific conjectural construction of the asserted derived equivalence (see Remark 1.3), one may try to use the fact that if π_1, π_2 are projective then Y_1, Y_2 are connected by a sequence of “flops” [78, THEOREM 1], so that it is then sufficient to prove the conjecture for flops. Recall that crepant resolutions $\pi_1 : Y_1 \rightarrow X, \pi_2 : Y_2 \rightarrow X$ form a flop if X has terminal singularities [89, DEFINITION 2.12]

and there is a line bundle \mathcal{L} on Y_1 , relatively ample for π_1 , such that the corresponding² line bundle \mathcal{L}' on Y_2 is antiample.

In [27] Bridgeland proves that Conjecture 1.2 is true for three-dimensional flops (see also [38]). The key point is that the fibers of π_1, π_2 have dimension ≤ 1 . In the next section we explain a reinterpretation of Bridgeland’s proof, following [129].

1.4.2. Maps with fibers of dimension ≤ 1

Assume that $\pi : Y \rightarrow X$ is a projective map between noetherian schemes. We impose the following conditions:

- (1) $R\pi_*\mathcal{O}_Y = \mathcal{O}_X$.
- (2) The fibers of π have dimension ≤ 1 .

To simplify the discussion, we will restrict ourselves, furthermore, to the case that $X = \text{Spec } R$ is affine.³ It turns out that in this case $\text{coh}(Y)$ contains a tilting bundle which is of the form $\mathcal{T} := \mathcal{O}_Y \oplus \mathcal{T}_0$ where \mathcal{T}_0 is obtained as an extension

$$0 \rightarrow \mathcal{O}_Y^r \rightarrow \mathcal{T}_0 \rightarrow \mathcal{L} \rightarrow 0, \tag{1.3}$$

where \mathcal{L} is an ample line bundle on \mathcal{O}_Y generated by global sections and (1.3) is associated to an arbitrary finite set of generators of $H^1(Y, \mathcal{L}^{-1})$ as R -module (see [129, (3.1)]).

Remark 1.8. Note that by hypothesis (1), $\mathcal{O}_X, \mathcal{L}$ are partial tilting bundles on Y such that $\mathcal{O}_X \oplus \mathcal{L}$ generates $D_{\text{Qch}}(Y)$ [129, LEMMA 3.2.2]. Moreover, (2) and the fact that \mathcal{L} is generated by global sections imply $\text{Ext}_Y^{>0}(\mathcal{O}_Y, \mathcal{L}) = 0$. Likewise, (2) implies $\text{Ext}_Y^{>1}(\mathcal{L}, \mathcal{O}_Y) = 0$. The construction of the tilting bundle \mathcal{T} is based on the principle of “killing the remaining backward Ext^1 ” in the sequence $(\mathcal{O}_X, \mathcal{L})$ by a so-called “semiuniversal extension.” This principle extends to longer sequences. See, e.g., [61, LEMMA 2.4], [62, LEMMA 3.1]. See also Section 3.4 below for another application.

So if we put $\Lambda = \text{End}_Y(\mathcal{T})$, then we have $\text{End}_Y(\mathcal{T}^\vee) = \Lambda^\circ$, and from Theorem 1.7 we obtain equivalences⁴

$$\text{RHom}_Y(\mathcal{T}, -) : \mathcal{D}(Y) \cong \mathcal{D}(\Lambda^\circ), \quad \text{RHom}_Y(\mathcal{T}^\vee, -) : \mathcal{D}(Y) \cong \mathcal{D}(\Lambda). \tag{1.4}$$

To understand (1.4), we can ask what Λ looks like.

Example 1.9. Consider again the Atiyah flop (1.2). In this case $R = k[x, y, z, w]/(xy - zw)$. This is a toric singularity, and one can check that its class group is \mathbb{Z} with

2 This makes sense since Y_1 and Y_2 are isomorphic in codimension one.
 3 In [129] X is assumed to be quasiprojective.
 4 It is a fact that \mathcal{T} is tilting if and only if \mathcal{T}^\vee is tilting. The only nontrivial part is the generation property. To this end one may use that \mathcal{T} generates $D_{\text{Qch}}(Y)$ if and only if $\text{Perf}(Y)$ is the smallest épaisse subcategory of $D_{\text{Qch}}(Y)$ containing \mathcal{T} [105, LEMMA 2.2], together with the fact that $(-)^{\vee}$ is an autoequivalence of $\text{Perf}(Y)$.

generator $I = (x, z)$. The inverse of I is the fractional ideal $I^{-1} = x^{-1}(x, w)$. The ring Λ turns out to be the same (up to isomorphism) for both crepant resolutions of $\text{Spec } R$,

$$\Lambda = \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix}. \tag{1.5}$$

Interestingly, Λ is built up from the three indecomposable graded maximal Cohen–Macaulay R -modules: R , I , and I^{-1} . In particular, Λ is itself Cohen–Macaulay as R -module. This last fact turns out to be true more generally.

Theorem 1.10. *Assume that $X = \text{Spec } R$ is a normal Gorenstein variety. Assume that there exists a projective crepant resolution of singularities $\pi : Y \rightarrow X$ such that the dimensions of the fibers of π are ≤ 1 . Let \mathcal{T} be the tilting bundle defined above⁵ and put $T = \Gamma(Y, \mathcal{T})$. Then $\Lambda = \text{End}_Y(\mathcal{T}) = \text{End}_R(T)$. Furthermore, Λ and T are maximal Cohen–Macaulay R -modules.*

Proof. The fact that $\Lambda = \text{End}_Y(\mathcal{T})$ is maximal Cohen–Macaulay follows from [129, LEMMA 3.2.9] (see also [70, THEOREM 1.5], stated as Theorem 2.6 below). Now T is maximal Cohen–Macaulay because it is a direct summand of Λ as R -modules. Functoriality yields a map $i : \Lambda \rightarrow \text{End}_R(T)$ which is an isomorphism in codimension one (since the singular locus of X has codimension ≥ 2 , as X is normal). Since Λ is maximal Cohen–Macaulay, it is reflexive and hence i must be an isomorphism. ■

This result applies in particular if X has dimension 2 or if it is of dimension 3 with terminal singularities since then the condition on the dimension of the fibers is automatic.

Let us now assume that X in Conjecture 1.2 is 3-dimensional and π_1, π_2 form a flop (see Section 1.4.1). We will still be assuming that $X = \text{Spec } R$ is affine for simplicity. For $i = 1, 2$, we then have tilting bundles \mathcal{T}_i on Y_i defined via (1.3), using \mathcal{L} on Y_1 and $(\mathcal{L}')^{-1}$ on Y_2 (see Section 1.4.1 for $\mathcal{L}, \mathcal{L}'$). Let $(\Lambda_i)_{i=1,2}$ be the corresponding endomorphism rings. In this case Conjecture 1.2 follows from

$$\mathcal{D}(Y_1) \stackrel{(1.4)}{\cong} \mathcal{D}(\Lambda_1^\circ), \quad \mathcal{D}(Y_2) \stackrel{(1.4)}{\cong} \mathcal{D}(\Lambda_2), \quad \Lambda_1^\circ \stackrel{\text{Morita}}{\cong} \Lambda_2.$$

The asserted Morita equivalence is obtained in [129, §4.4] using the local structure of 3-dimensional terminal singularities (see [88, EXAMPLE 2.3]). Nowadays we may use [67, COROLLARY 8.8] (see also [68, THEOREM 1.5]) combined with [70, THEOREM 1.5] (stated as Theorem 2.6 below) to obtain that in any case $\Lambda_1, \Lambda_2^\circ$ are derived equivalent.

At the end of the day, we find that the two crepant resolutions Y_1, Y_2 of X are derived equivalent to the same noncommutative ring (either Λ_1° or Λ_2). It turns out to be fruitful to think of this intermediate noncommutative ring as a *third crepant resolution* of X , or of R , namely a *noncommutative crepant resolution*.

⁵ As we have stated in Section 1.2, the fact that X has a crepant resolution implies that it has rational singularities by [89, COROLLARY 5.24]. Thus in particular $R\pi_*\mathcal{O}_Y = \mathcal{O}_X$.

2. NONCOMMUTATIVE (CREPANT) RESOLUTIONS

2.1. Generalities

Below R is a normal noetherian domain with quotient field K . We denote by $\text{ref}(R)$ the category of reflexive R -modules and if Λ is a reflexive R -algebra then $\text{ref}(\Lambda)$ is the category of Λ -modules which are reflexive as R -modules. A *reflexive Azumaya algebra* [98] Λ is a reflexive R -algebra which is Azumaya in codimension one. A reflexive Azumaya algebra Λ is said to be trivial if it is of the form $\text{End}_R(M)$ for M a reflexive R -module. In that case $\text{ref}(R)$ and $\text{ref}(\Lambda)$ are equivalent. This is a particular case of “reflexive Morita equivalence” which is defined in the obvious way.

Definition 2.1. A *twisted noncommutative resolution* of R is a reflexive Azumaya algebra Λ over R such that $\text{gl dim } \Lambda < \infty$. If Λ is trivial then Λ is said to be a *noncommutative resolution (NCR)* of R .

Definition 2.2. Assume that R is Gorenstein. A *twisted noncommutative crepant resolution* Λ of R is a twisted NCR of R which is in addition a Cohen–Macaulay R -module. If Λ is an NCR then such a Λ is said to be a *noncommutative crepant resolution (NCCR)* of R .

The point of these definitions is that they provide reasonable noncommutative substitutes for “regularity,” “birationality,” and “crepancy.” This is explained in more detail in [129, §4].

Remark 2.3. We will sometimes use the concepts introduced in Definitions 2.1, 2.2 for schemes, possibly nonaffine. It is then understood that they reduce to the affine concepts, when restricting to open affine subschemes.

Remark 2.4. In the sequel we will be mostly concerned with NCCRs and thus the other definitions are mainly provided for context. Twisted NCCRs are natural generalizations of NCCRs, but the good properties of NCCRs (sometimes conjectural) are usually not shared by twisted NCCRs. See, e.g., Example 5.11 below. The definition of a (twisted) NCR is more tentative. In particular, the normality and reflexivity hypotheses do not seem very relevant. For example, there is a nice theory of noncommutative resolutions of nonnormal singularities in dimension one [96].

Example 2.5. It follows from Theorem 1.10 and Theorem 1.7 that if there exists a projective crepant resolution of singularities $\pi : Y \rightarrow X$ such that the dimensions of the fibers of π are ≤ 1 then R has an NCCR.

We mention the following theorem which gives another indication that the definition of an NCCR is the “correct one.”

Theorem 2.6 ([70, THEOREM 1.5]). *Let $f : Y \rightarrow \text{Spec } R$ be a projective birational morphism between Gorenstein varieties. Suppose that Y is derived equivalent to some ring Λ , then f is a crepant resolution if and only if Λ is an NCCR of R .*

The following conjecture is a natural extension of Conjecture 1.2.

Conjecture 2.7 ([128, CONJECTURE 4.6]). All crepant resolutions of X (commutative as well as noncommutative) are derived equivalent.

We have the following result which is proved in the same way as the 3-dimensional McKay correspondence [28].

Proposition 2.8 ([128, THEOREM 6.3.1, PROPOSITION 6.2.1]). *If X has three-dimensional Gorenstein singularities and it has an NCCR Λ , then it has a projective crepant resolution $Y \rightarrow X$ such that Λ and Y are derived equivalent.*

Proposition 2.9. *Conjecture 2.7 is true if X has dimension three, if we restrict to projective crepant resolutions.*

Proof. If X has an NCCR Λ then by Proposition 2.8 Λ is derived equivalent to a crepant resolution. Hence we are reduced to Bridgeland's result (see Section 1.4.1). Alternatively, to have a very nice direct argument that any two NCCRs are derived equivalent in dimension three, we may use [67, COROLLARY 8.8] (see also [68, THEOREM 1.5]). ■

Proposition 2.8 is false for arbitrary three-dimensional Gorenstein singularities as was shown by Dao [41].

Proposition 2.10 ([41, THEOREM 3.1, REMARK 3.2]). *Assume S is a regular local ring which is equicharacteristic or unramified, $0 \neq f \in S$ and $R = S/(f)$ is normal. If $\dim R = 3$ and R is factorial then R has no NCCR.*

Example 2.11. It turns out that there are 3-dimensional factorial hypersurface singularities that admit a crepant resolution. A concrete example is given by $R = k[[x_0, x_1, x_2, x_3]]/(x_0^4 + x_1^3 + x_2^3 + x_3^3)$ [100, THEOREM A,B]. In particular, a crepant resolution of such R does not admit a tilting complex by Theorem 2.6.

If X is a normal Gorenstein algebraic variety with a crepant resolution then it has rational singularities [89, COROLLARY 5.24]. A similar result is true for NCCRs.

Theorem 2.12 ([122, THEOREM 1.1]). *Let R be a normal finitely generated Gorenstein k -algebra. If R has a twisted NCCR then it has rational singularities.*

The actual result proved in [122] applies in a more general context and this has been further exploited in [64, 65] (see also [42, COROLLARY 1.7]).

Remark 2.13. In order to deal with singularities with a singular minimal model, Iyama and Wemyss generalize the definition of an NCCR [69, 70, 132] to certain rings, of possibly infinite global dimension, called *maximal modification algebras* (MMAs). Remarkably, many of the results about NCCRs extend to MMAs. However, in this overview we will restrict ourselves for simplicity to NCCRs.

2.2. Relation with crepant categorical resolutions

We conjecture that noncommutative crepant resolutions are examples of “strongly crepant categorical resolutions” as introduced by Kuznetsov in [93]. However, we can only prove this in special cases.

Let X be an algebraic variety. A *categorical resolution* [93] of $\mathcal{D}(X)$ is a “smooth” triangulated category $\tilde{\mathcal{D}}$ together with functors

$$\pi_* : \tilde{\mathcal{D}} \rightarrow \mathcal{D}(X), \quad \pi^* : \text{Perf}(X) \rightarrow \tilde{\mathcal{D}}$$

which are adjoint (i.e., $\text{Hom}_{\tilde{\mathcal{D}}}(\pi^* A, B) \cong \text{Hom}_{\mathcal{D}(X)}(A, \pi_* B)$ for $A \in \text{Perf}(X)$, $B \in \tilde{\mathcal{D}}$) such that the natural transformation $\text{id}_{\text{Perf}(X)} \rightarrow \pi_* \pi^*$, obtained by putting $B = \pi^* A$, is an isomorphism. This implies in particular that π^* is fully faithful. There is some variation possible in the definition of smoothness. For us it means that $\tilde{\mathcal{D}}$ is equivalent to the derived category of perfect modules over a smooth DG-algebra [75, DEFINITION 2.23].

Remark 2.14. If $\pi : Y \rightarrow X$ is a resolution of singularities of X then $(\mathcal{D}(Y), R\pi_*, L\pi^*)$ is a categorical resolution of $\mathcal{D}(X)$ if and only if X has rational singularities. Remarkably, however, it has been shown in [94] that $\mathcal{D}(Y)$ can be suitably enlarged to yield a categorical resolution. On the other hand, this result cannot be extended to more general dg-categories [54].

Following [93], we say that a categorical resolution $(\tilde{\mathcal{D}}, \pi_*, \pi^*)$ of $\mathcal{D}(X)$ is *weakly crepant* if π^* is both a left and a right adjoint to π_* .

There is also a notion of a *strongly crepant categorical resolution* for which we need the notion of a *relative Serre functor*. To define this, assume that X is Gorenstein and that $\tilde{\mathcal{E}}$ is a smooth triangulated category which is a $\text{Perf}(X)$ -module. We will denote the action of $A \in \text{Perf}(X)$ on $B \in \tilde{\mathcal{E}}$ as $A \otimes_X B$ and we assume that $- \otimes -$ is exact in both arguments. We also assume that the functor $\text{Perf}(X) \rightarrow \tilde{\mathcal{E}} : A \mapsto A \otimes_X B$ has a right adjoint $\tilde{\mathcal{E}} \rightarrow \mathcal{D}(X)$ which we denote by $\text{R}\mathcal{H}om_{\tilde{\mathcal{E}}/X}(B, -)$. That is, for $C \in \tilde{\mathcal{E}}$ we have functorial isomorphisms

$$\text{Hom}_{\tilde{\mathcal{E}}}(A \otimes_X B, C) \cong \text{Hom}_X(A, \text{R}\mathcal{H}om_{\tilde{\mathcal{E}}/X}(B, C)).$$

An autoequivalence $S_{\tilde{\mathcal{E}}/X} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is said to be a *relative Serre functor* for $\tilde{\mathcal{E}}/X$ if there are functorial isomorphisms

$$\text{R}\mathcal{H}om_X(\text{R}\mathcal{H}om_{\tilde{\mathcal{E}}/X}(B, C), \mathcal{O}_X) \cong \text{R}\mathcal{H}om_{\tilde{\mathcal{E}}/X}(C, S_{\tilde{\mathcal{E}}/X} B)$$

for $B, C \in \tilde{\mathcal{E}}$. We say that $\tilde{\mathcal{E}}/X$ is *strongly crepant* if the identity functor $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is a relative Serre functor.

A *strongly crepant categorical resolution* of X is a quadruple $(\tilde{\mathcal{D}}, \pi_*, \pi^*, \otimes_X)$ such that $(\tilde{\mathcal{D}}, \pi_*, \pi^*)$ is a categorical resolution of X , $- \otimes_X -$ is a $\text{Perf}(X)$ -module structure on $\tilde{\mathcal{D}}$ such that $\tilde{\mathcal{D}}/X$ is strongly crepant and π^* is \otimes_X -linear. The last condition means that for $A, B \in \text{Perf}(X)$ we have functorial isomorphisms $A \otimes_X \pi^* B \cong \pi^*(A \otimes_X B)$ satisfying the appropriate compatibilities.

It is shown in [93, §3] that a strongly crepant categorical resolution is weakly crepant, and, moreover, that if $\pi : Y \rightarrow X$ is a crepant resolution in the usual sense then $(\mathcal{D}(Y), R\pi_*, L\pi^*, L\pi^*(-) \otimes_Y -)$ is a strongly crepant categorical resolution of X .

The following easy lemma, which is an extension of [102, EXAMPLE 5.3], shows that, under suitable conditions, rings of the form $\text{End}_R(M)$ form crepant categorical resolutions. If M is an R -module then $\text{add}(M)$ is the category spanned by modules which are direct summands of some $M^{\oplus n}$.

Lemma 2.2.1. *Assume that $X = \text{Spec } R$ is an algebraic variety and let M be a finitely generated R -module such that $\Lambda = \text{End}_R(M)$ has finite global dimension. Then $\mathcal{D}(\Lambda)$ is smooth. Assume in addition that $R \in \text{add}(M)$. Then*

$$\text{Perf}(R) \rightarrow \mathcal{D}(\Lambda) : N \mapsto M \overset{L}{\otimes}_R N \tag{2.1}$$

yields a categorical resolution of singularities of X (since $\text{Perf}(R) \cong \text{Perf}(X)$). Moreover, assuming furthermore that R is normal Gorenstein:

- (1) *if M is maximal Cohen–Macaulay then this categorical resolution is weakly crepant;*
- (2) *if Λ is an NCCR then this categorical resolution is strongly crepant.*

Note that if (2) holds then M is maximal Cohen–Macaulay since we have assumed that $R \in \text{add}(M)$.

The hypotheses of Lemma 2.2.1 are actually too strong. For example, an NCCR is always a strongly crepant categorical resolution in dimension ≤ 3 . This follows from Proposition 2.15 below which can be proved using the methods of [67, 68].

Proposition 2.15. *Assume that $\Lambda = \text{End}_R(M)$ is an NCCR and $\dim R \leq 3$ then*

$$\text{Ext}_\Lambda^i(M, M) = 0 \quad \text{for } i > 0. \tag{2.2}$$

Proof. For the benefit of the reader, we give a proof. We may assume that R is local of dimension 3 (the case $\dim \leq 2$ is easy). By the Auslander–Buchsbaum formula [67, PROPOSITION 2.3] Λ has global dimension 3. Since M is reflexive, it has depth ≥ 2 , and hence, again by the Auslander–Buchsbaum formula, it has projective dimension ≤ 1 over Λ and, moreover, it is projective over Λ in codimension 2.

Hence we have a projective resolution of M as Λ -module

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Applying $\text{Hom}_\Lambda(-, M)$, we get a long exact sequence of R -modules

$$0 \rightarrow R \rightarrow \text{Hom}_\Lambda(P_0, M) \rightarrow \text{Hom}_\Lambda(P_1, M) \rightarrow \text{Ext}_\Lambda^1(M, M) \rightarrow 0. \tag{2.3}$$

Assume $\text{Ext}_\Lambda^1(M, M) \neq 0$. Since M is projective over Λ in codimension two, $\text{Ext}_\Lambda^1(M, M)$ is finite dimensional and hence it has depth 0 as R -module. On the other hand, since $\text{Hom}_\Lambda(P_i, M)$ is reflexive as R -module, it has depth ≥ 2 . Finally, R being maximal Cohen–Macaulay has depth 3. One may verify that these depth restrictions are incompatible with (2.3). ■

It seems too much to hope for that (2.2) would always be true, but the lack of time has prevented us from seriously looking for a counterexample. On the other hand, we are sufficiently optimistic to make the following conjecture.

Conjecture 2.16. If $X = \text{Spec } R$ is a normal algebraic variety with Gorenstein singularities then an NCCR of R always yields a strongly crepant categorical resolution of X .

To prove this conjecture, one would have to construct for an NCCR Λ of R a partial tilting complex P^\bullet of Λ -modules such that $\text{RHom}_\Lambda(P^\bullet, P^\bullet) = R$.

Remark 2.17. The strongly crepantness of \mathcal{E}/X as defined above is independent of the resolution property. One may check that if Λ/R is a twisted NCCR then $\mathcal{D}(\Lambda)$ is strongly crepant over $\text{Spec } R$. But one may also check that it is not a categorical resolution.

3. CONSTRUCTIONS OF NONCOMMUTATIVE CREPANT RESOLUTIONS

3.1. Quotient singularities

Here we will restrict ourselves to quotient singularities for finite groups. Quotient singularities for (infinite) reductive groups will be covered in Section 4.

If G is a finite group and W is a faithful finite-dimensional unimodular (i.e., $\det W = k$) representation of G then the skew group ring

$$\text{Sym}(W)\#G = \text{End}_{\text{Sym}(W)^G}(\text{Sym}(W))$$

is an NCCR for $R = \text{Sym}(W)^G$ (which is Gorenstein because of the unimodularity hypothesis).

In dimension ≤ 3 such quotient singularities always have a crepant resolution by the celebrated BKR-theorem [28]. In higher dimension this is not so. The simplest counterexample is given by \mathbb{Z}_2 acting with weights $(-1, -1, -1, -1)$ on $W = k^4$ because in that case R is \mathbb{Q} -factorial and terminal. See, e.g., [1].

3.2. Crepant resolutions with tilting complexes

In case R is a normal Gorenstein domain and $Y \rightarrow \text{Spec } R$ is a crepant resolution and \mathcal{T} is a tilting complex on Y then $\mathcal{E}nd_Y(\mathcal{T})$ is an NCCR of R by Theorem 2.6. Conversely, assuming a crepant resolution exists, any NCCR has to be of this form if we accept Conjecture 2.7 (\mathcal{T} is the dual of the image of Λ under the asserted derived equivalence $\mathcal{D}(\Lambda) \cong \mathcal{D}(Y)$).

This is a very general method for constructing NCCRs. Note, however, that even in dimension three there may be crepant resolutions without tilting complex. See Example 2.11. Furthermore, as indicated in Section 3.1, there are normal Gorenstein singularities that admit an NCCR but not a crepant resolution.

Example 3.1. A textbook example where this method works very well is the case of determinantal varieties [34, 36]. Let $n \geq 1$ and $0 \leq l < n$. Let $X_{l,n} = \text{Spec } R$ be the varieties of

matrices in $\text{Hom}_k(k^n, k^n) = M_{n \times n}(k)$ which have rank $\leq l$. It is a classical result that X is Gorenstein. It is also well known that X has a crepant Springer type resolution given by

$$Y = \{(\phi, V) \mid V \in \text{Gr}(l, n), \phi \in \text{Hom}_k(k^n, V)\},$$

where $\pi : Y \rightarrow X$ sends (ϕ, V) to the composition of ϕ with the inclusion $V \hookrightarrow k^n$. If \mathcal{R} denotes the universal subbundle on $\text{Gr}(l, n)$ then Y is the vector bundle $\text{Hom}(k^n, \mathcal{R})$ (i.e., $Y = \text{Spec Sym}((\mathcal{R}^{\oplus n})^\vee)$). Using Bott's theorem, one computes that the Kapranov tilting bundle on $\text{Gr}(l, n)$ [73] (see also [35][53] for the case of finite characteristic) pulls back to a tilting bundle on Y , which then gives an NCCR of R . For other approaches to this example, see [49] and Theorem 4.10 below.

Alas, things are often more complicated. For determinantal varieties associated to symmetric or skew-symmetric matrices, the Springer type resolutions are not crepant so a tilting bundle on them only gives an NCR (see [136]). NCCRs of such generalized determinantal varieties will be obtained in Section 4 using a different approach.

Example 3.2. Another beautiful and much deeper example [13, 16] is given by cotangent bundles of (partial) flag varieties $T^*(G/P)$. If P is a Borel subgroup of G then this is a crepant resolution of the nilpotent cone in $\text{Lie}(G)$. In general, they are crepant resolutions of closures of Richardson orbits [104]. It is shown in [13, 16] that $T^*(G/P)$ has a tilting bundle but it is not obtained as the pullback of a tilting bundle on G/P . In fact, the construction of the tilting bundle is highly nontrivial. To explain the construction, it is useful to exhibit a slightly different point of view on tilting bundles.

Let Y be a noetherian scheme. If \mathcal{A} is a quasicoherent sheaf of algebras on Y and $A = \Gamma(Y, \mathcal{A})$ then we say that \mathcal{A} is *derived affine* if $A = R\Gamma(Y, \mathcal{A})$ and the right orthogonal to \mathcal{A} in $D_{\text{Qch}}(\mathcal{A})$ is zero. In that case $R\Gamma(Y, -)$ defines an equivalence of categories between $D_{\text{Qch}}(\mathcal{A})$ and $D(A)$. It is not difficult to see that a vector bundle \mathcal{T} on Y which has everywhere nonzero rank is a tilting bundle, provided $\mathcal{E}nd_Y(\mathcal{T})$ is derived affine.

We say that Y is *derived \mathcal{D} -affine* if \mathcal{D}_Y is derived affine where \mathcal{D}_Y is the sheaf of differential operators on Y . In characteristic > 0 we mean by \mathcal{D}_Y the sheaf of *crystalline* differential operators, i.e., differential operators which may be expressed in terms of derivations, without using divided powers.

Now let $Z = G/P$. The Bernstein–Beilinson theorem [12], valid in characteristic zero, states that Z is even “ \mathcal{D} -affine” meaning that the equivalence $R\Gamma(Z, -)$ is also compatible with the natural \mathfrak{t} -structures. This is false in characteristic > 0 . However, Z is still derived \mathcal{D} -affine [16, THEOREM 3.2] whenever p is strictly bigger than the Coxeter number, which we will assume now.

We will give a rough sketch how this is used in [13, 16] to construct a tilting bundle on $Y = T^*Z$. Let us first assume that the characteristic of k is $p > 0$. To indicate this, we will adorn our notations with $(-)_p$. In that case \mathcal{D}_{Z_p} is coherent as a module over its center which is equal to $(\text{Sym}_{Z_p} \mathcal{E}_p)^{(1)}$ where $(-)^{(1)}$ denotes the Frobenius twist, and \mathcal{E}_p is the tangent bundle on Z_p . Hence we may view \mathcal{D}_{Z_p} as a sheaf of coherent algebras $\tilde{\mathcal{D}}$ on $\underline{\text{Spec}}(\text{Sym}_{Z_p} \mathcal{E}_p)^{(1)} = Y_p^{(1)}$ where $Y_p = T^*Z_p$. The sheaf $\tilde{\mathcal{D}}$ is still derived affine.

Now $\tilde{\mathcal{D}}$ is not of the form $\mathcal{E}nd_{Y_p^{(1)}}(\mathcal{T}_p^{(1)})$. However, if we let \hat{Y}_p be the formal completion of Y_p at the zero section then it turns out that the restriction $\hat{\mathcal{D}}$ of $\tilde{\mathcal{D}}$ to $\hat{Y}_p^{(1)}$ is of the form $\mathcal{E}nd_{\hat{Y}_p}(\hat{\mathcal{T}}_p)^{(1)}$ for a vector bundle $\hat{\mathcal{T}}_p$ on \hat{Y}_p . Moreover, $\hat{\mathcal{D}}$ is still derived affine and so $\hat{\mathcal{T}}_p$ is a tilting bundle on \hat{Y}_p . Then one uses deformation theory⁶ to lift $\hat{\mathcal{T}}_p$ to a tilting bundle $\hat{\mathcal{T}}$ in characteristic zero. Finally, one may use the fact that $Y = T^*Z$ (as a vector bundle) admits a nice G_m action to conclude by [72, THEOREM 1.8] that $\hat{\mathcal{T}}$ is actually the completion of a tilting bundle \mathcal{T} on Y .

Hidden behind this construction is the fact that \mathcal{D}_Z is, in some sense, a canonical noncommutative deformation of the symplectic variety T^*Z . If Y is a general symplectic variety then one may try to construct a noncommutative deformation using Fedosov quantization. This general idea has been used by Bezrukavnikov and Kaledin to prove an analogue of the BKR theorem [28] for crepant resolutions of symplectic quotient singularities [14] and by Kaledin to prove a suitable version of Conjecture 1.2 [72] for general symplectic singularities. To apply the method, one needs to be able to do Fedosov quantization in finite characteristic, a problem which has been solved to some extent in [15].

3.3. Resolutions with partial tilting complexes

Assume R is a normal Gorenstein domain with rational singularities and $Y \rightarrow \text{Spec } R$ is a resolution which is not crepant. A strengthening of Conjecture 2.7 inspired by [93] is that NCCRs are minimal in a categorical sense, i.e., their derived category embeds inside $\mathcal{D}(Y)$. This means that they are obtained as $\mathcal{E}nd_Y(\mathcal{T})$ for a partial tilting complex \mathcal{T} on Y . For a very general result in this direction, see [93, THEOREM 2]. We will restrict ourselves to a special case which will be useful in Section 5 and which can be easily proved directly.

Proposition 3.3 ([93]). *Let Z be a smooth projective variety with ample line bundle $\mathcal{O}_Z(1)$ and let $X = \text{Spec } R$ be the corresponding cone. Assume $\omega_Z = \mathcal{O}_Z(-n)$ for $n \geq 1$. Then R is Gorenstein. Moreover, a resolution of singularities $\pi : Y \rightarrow X$ of X is given by the line bundle over Z associated to $\mathcal{O}_Z(1)$. Assume $\mathcal{E} \in \mathcal{D}(Z)$ is such that:*

- (1) $\text{Ext}_Z^i(\mathcal{E}, \mathcal{E}(m)) = 0$ for $i > 0$ and $m \geq 0$;
- (2) $\text{Ext}_Z^i(\mathcal{E}, \mathcal{E}(m)) = 0$ for $i \geq 0$ and $m \in \{-1, \dots, -n + 1\}$;
- (3) $\mathcal{E} \oplus \mathcal{E}(1) \oplus \dots \oplus \mathcal{E}(n - 1)$ is a generator for $D_{\text{Qch}}(Z)$.

Let $\gamma : Y \rightarrow Z$ be the projection map and put $\mathcal{T} = \gamma^*\mathcal{E}$. Then $\text{End}_Y(\mathcal{T})$ is an NCCR of R .

Proof. We write $\mathcal{T}(m) = \gamma^*(\mathcal{E}(m))$. Then we have

$$\text{RHom}_Y(\mathcal{T}, \mathcal{T}(m)) = \bigoplus_{l \geq 0} \text{RHom}_Z(\mathcal{E}, \mathcal{E}(m + l)). \quad (3.1)$$

⁶ Tilting bundles have in particular vanishing $\text{Ext}^{1,2}$. Hence by classical deformation theory they are unobstructed and rigid.

Using (1) and (2), we deduce in particular that $\tilde{\mathcal{T}} := \mathcal{T} \oplus \cdots \oplus \mathcal{T}(-n + 1)$ is partial tilting (and hence this is also the case for \mathcal{T}). Furthermore, from (3) we obtain $\tilde{\mathcal{T}}^\perp = 0$. So $\tilde{\mathcal{T}}$ is in fact tilting. Put $\Lambda = \text{End}_Y(\mathcal{T})$, $\bar{\Lambda} = \text{End}_Y(\tilde{\mathcal{T}})$. By Theorem 1.7, $\bar{\Lambda}$ has finite global dimension.

Via the decomposition (3.1) Λ is an \mathbb{N} -graded ring. Put $\Lambda_{\geq u} = \bigoplus_{m \geq u} \Lambda_m$. Then (as ungraded rings) we have

$$\bar{\Lambda} = \begin{pmatrix} \Lambda & \Lambda_{\geq 1} & \cdots & \Lambda_{\geq n-1} \\ \Lambda & \Lambda & \cdots & \Lambda_{\geq n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda & \Lambda & \cdots & \Lambda_{\geq 1} \\ \Lambda & \Lambda & \cdots & \Lambda \end{pmatrix}.$$

If we put $\Gamma = M_n(\Lambda)$ then $\bar{\Lambda} \subset \Gamma$ and, moreover, Γ is (left and right) projective over $\bar{\Lambda}$ and in addition the multiplication map $\Gamma \otimes_{\bar{\Lambda}} \Gamma \rightarrow \Gamma$ is an isomorphism (it is a surjective map between projective Γ -modules of the same rank). We claim that Γ (and hence Λ) has finite global dimension. Indeed, if M is a right Γ -module and $P^\bullet \rightarrow M$ is a finite projective resolution of M as $\bar{\Lambda}$ -module (which exists since $\text{gl dim } \bar{\Lambda} < \infty$) then $\Gamma \otimes_{\bar{\Lambda}} P^\bullet$ is a finite Γ -projective resolution of $\Gamma \otimes_{\bar{\Lambda}} M = \Gamma \otimes_{\bar{\Lambda}} \Gamma \otimes_{\Gamma} M \cong M$.

Moreover, for $i > 0$,

$$\begin{aligned} \text{Ext}_R^i(\text{End}_Y(\mathcal{T}), \omega_R) &= \text{Ext}_X^i(\pi_* \mathcal{E}nd_Y(\mathcal{T}), \omega_X) \\ &= \text{Ext}_Y^i(\mathcal{E}nd_Y(\mathcal{T}), \omega_Y) \\ &= \text{Ext}_Y^i(\mathcal{T}, \mathcal{T}(-n + 1)) \\ &= 0, \end{aligned}$$

where in the second line we have used Grothendieck duality, in the third line the easily verified fact that $\omega_Y = \gamma^*(\omega_Z(1))$, and in the fourth line (3.1) and (1)–(2). It follows that $\text{End}_Y(\mathcal{T})$ is maximal Cohen–Macaulay over R . ■

3.4. Three-dimensional affine toric varieties

For simplicity, we define an affine toric variety as $X = \text{Spec } R$ where $R = k[\sigma^\vee \cap M]$ where M is a lattice and σ^\vee is a strongly convex full dimensional lattice cone in $M_{\mathbb{R}}$. Such an R is Gorenstein if there exists $m \in M$ such that σ (the dual cone of σ^\vee) is spanned by lattice vectors $x \in M^\vee$ satisfying $\langle x, m \rangle = 1$. The lattice polytope associated to R is defined as $P = \sigma \cap \langle m, - \rangle$.

In this case there is the following beautiful result by Broomhead [32, THEOREM 8.6].

Theorem 3.4. *The coordinate ring of a 3-dimensional Gorenstein affine toric variety admits a toric NCCR.*

By a toric NCCR we mean that the reflexive module defining the NCCR is isomorphic to a sum of ideals. Broomhead’s proof uses the theory of “dimer models” which is possible thanks to the combinatorics [57, 66]. A proof not using dimer models but using this combinatorics directly was given in [120].

A different method for constructing NCCRs for affine Gorenstein toric varieties was given in [119] and is based on a standard fact from toric geometry:

Lemma 3.5. *A subdivision of σ obtained by a regular triangulation of P with no extra vertices yields a projective crepant resolution of $\text{Spec } R$ by a toric Deligne–Mumford stack [24]. If $\dim X \leq 3$ then such a crepant resolution has fibers of dimension ≤ 1 .*

In dimension ≤ 3 one may then, starting from a sequence of generating line bundles, construct a tilting bundle using the principle of “killing backward Ext^1 ’s” (see Remark 1.8).

While this method yields an NCCR, it generally does not yield a toric one. On the other hand, it is also applicable to some higher dimensional toric singularities which do not have a toric NCCR.

Example 3.6 ([116, §9.1], [119, EXAMPLE 6.4]). Let $T = G_m^2$ be the two-dimensional torus and (after the identifying the character group $X(T)$ of T with \mathbb{Z}^2) consider the vector space W with weights $(3, 0)$, $(1, 1)$, $(0, 3)$, $(-1, 0)$, $(-3, -3)$, $(0, -1)$. Put $R := \text{Sym}(W)^T = k[x_1, x_2, x_3, x_4, x_5, x_6]^T = k[x_2x_4x_6, x_1x_3x_5, x_1x_4^3, x_3x_6^3, x_2^3x_5] \cong k[a, b, c, d, e]/(a^3b - cde)$. Clearly, R is the coordinate ring of a 4-dimensional affine toric variety, but it was shown in [116, §9.1] that R does not have a toric NCCR.

On the other hand, by [119, PROPOSITION 6.1], R does have a nontoric NCCR. In [119, EXAMPLE 6.4] an explicit NCCR is constructed which is given by a reflexive module which is the direct sum of 12 modules of rank 1 and 1 module of rank 2.

We conjecture:

Conjecture 3.7. An affine Gorenstein toric variety always has an NCCR.

Besides Theorem 3.4 this conjecture is also true for “quasisymmetric GIT quotients” for tori. See Corollary 4.7 below.

By [119, THEOREM A.1], the Grothendieck group of the DM-stack exhibited in Lemma 3.5 has rank $\text{Vol}(P)$. This suggests the following conjecture:

Conjecture 3.8. The number of indecomposable summands in the reflexive module defining an NCCR of R is equal to $\text{Vol}(P)$.

3.5. Mutations

It follows from the minimal model program that the number of crepant resolutions of an algebraic variety is finite.⁷ On the other hand, NCCRs can be modified by a process called “mutation” which is closely related to flopping of crepant resolutions. The difference is that the mutation process generally leads to an infinite number of different NCCRs (see, however, Example 3.12 below).

The following definitions and results are taken from [69]. Let R be a normal Gorenstein ring. Let M be a reflexive R -module such that $\Lambda = \text{End}_R(M)$ is an NCCR and let

⁷ I thank Shinnosuke Okawa for explaining to me how this follows from [17].

$0 \neq N \in \text{add}(M)$. Let K_0 be defined by the short exact sequence

$$0 \rightarrow K_0 \rightarrow N_0 \rightarrow M,$$

where $N_0 \in \text{add}(N)$ is a *right approximation* of M , i.e., any other map $N'_0 \rightarrow M$ with $N'_0 \in \text{add}(N)$ factors through N_0 . One defines the *right mutation* of M at N to be $\mu_N^+(M) := N \oplus K_0$. The *left mutation* of M at N is defined via duality as $\mu_N^-(M) = (\mu_{N^\vee}^+(M^\vee))^\vee$. We also put $\mu_N^\pm(\Lambda) = \text{End}_R(\mu_N^\pm(M))$. Note, however, that the passage from $\mu_N^\pm(M)$ to $\mu_N^\pm(\Lambda)$ loses some information.

Remark 3.9. Needless to say that $\mu_N^\pm(M)$ is only determined up to additive closure (i.e., up to taking $\text{add}(-)$). However, if R is complete local then we can make a minimal choice for $\mu_N^+(M)$ which we will do silently.

Theorem 3.10 ([69, THEOREMS 1.22, 1.23]). *Let M, N, Λ be as above.*

- (1) $\mu_N^\pm(M)$ define NCCRs.
- (2) $\Lambda, \mu_N^+(\Lambda)$ and $\mu_N^-(\Lambda)$ are all derived equivalent.
- (3) μ_N^+ and μ_N^- are mutually inverse operations (this statement makes sense since $N \in \text{add}(\mu_N^\pm(M))$).

If R is complete local of dimension 3, things simplify. Let us call a reflexive R -module *basic* if every indecomposable summand appears only once.

Theorem 3.11 ([69, THEOREMS 1.25]). *Assume that R is complete local of dimension 3. Let M be a basic reflexive R -module defining an NCCR, having at least two nonisomorphic indecomposable summands and let M_i be such an indecomposable summand. Then $\mu_{M/M_i}^+(M) \cong \mu_{M/M_i}^-(M)$.*

Example 3.12 ([63, 71, 133]). If R is complete local ring with a 3-dimensional terminal Gorenstein singularity then the basic reflexive modules yielding an NCCR correspond to the maximal cells in an affine hyperplane arrangement of dimension $\text{rk Cl}(R)$ with mutations at indecomposable summands corresponding to wall crossings [133, THEOREM 4.4]. The group $\text{Cl}(R)$ acts by translation on this hyperplane arrangement and the quotient consists of a finite number of cells which correspond to the NCCRs of R . The number of such NCCRs is generally higher than the number of crepant resolutions.

It is an interesting problem to understand this for other types of 3-dimensional singularities.

Remark 3.13. If $\Lambda = \text{End}_R(M)$ is an NCCR then because of the reflexive Morita equivalence $\text{ref}(\Lambda) = \text{ref}(R)$ the mutation procedure may also be defined on the level of reflexive Λ -modules (see [67, §5]). The resulting procedure also works for twisted NCCRs, where there is no reflexive Morita equivalence.

We now describe a different point of view on mutations, taken from [47]. For Q a quiver with n vertices let \widehat{kQ} be the completion of the path algebra of Q at path length.

A potential $w \in \widehat{kQ}$ is a convergent sum of cycles considered up to rotation (or, equivalently, $w \in \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$). If w is a potential then (∂w) denotes the (completed) two sided ideal generated by the cyclic derivatives $\partial_x w$ of w with respect to the arrows in Q , where for a cyclic path m we have $\partial_x m := \sum_{m=uxv} vu$ (note that this is invariant under path rotation). The *completed Jacobi algebra* associated to (Q, w) is defined as $\hat{J}(Q, w) := \widehat{kQ}/(\partial w)$. We say that w is *reduced* if it only contains cycles of length ≥ 3 . We can also consider the uncompleted version $J(Q, w) := kQ/(\partial w)$, in case w is a finite sum. We have the following result.

Theorem 3.14 ([130, THEOREMS A&B]). *If Λ is a basic (i.e., $\Lambda/\text{rad } \Lambda \cong k^{\oplus n}$) twisted NCCR of a 3-dimensional normal Gorenstein complete local ring then Λ is a completed Jacobi algebra $\hat{J}(Q, w)$ with w reduced.*

If Q does not have loops or 2-cycles then the mutations of $\Lambda := \hat{J}(Q, w)$ can be obtained by an alternative procedure described in [47]. The procedure to mutate at a vertex i of Q yields a new Jacobi algebra $\hat{J}(Q', w')$ defined as follows (see [83, §2.4]).

- (1) For each arrow β with target i and each arrow α with source i , add a new arrow $[\alpha\beta]$ from the source of β to the target of α .
- (2) Replace each arrow α with source or target i with an arrow α^* in the opposite direction.

The new potential w' is the sum of two potentials w'_1 and w'_2 . The potential w'_1 is obtained from w by replacing each composition $\alpha\beta$ (up to cyclic rotation) by $[\alpha\beta]$, where β is an arrow with target i . The potential w'_2 is given by

$$w'_2 = \sum_{\alpha, \beta} [\alpha\beta] \beta^* \alpha^*,$$

where the sum ranges over all pairs of arrows α and β such that β ends at i and α starts at i . It follows from [83, THEOREM 3.2] that this mutation coincides with the mutation defined in [69] and described above.

It may be that w' is not reduced, i.e., it contains 2-cycles. In that case the corresponding relations allow one to eliminate some arrows in Q' . By doing this, we find that the Jacobi algebra $\hat{J}(Q', w')$ can be more economically written as $J((Q')^{\text{red}}, (w')^{\text{red}})$ where $(w')^{\text{red}}$ is reduced.

If we are lucky that $(Q')^{\text{red}}$ does not contain any 2-cycles (it cannot contain loops) then we can repeat the mutation procedure at arbitrary vertices. If we can keep doing this forever then we call the original potential w *nondegenerate*.

Note that if $(Q')^{\text{red}}$ does not contain 2-cycles, it can be obtained from Q' by simply deleting all 2-cycles, so that the mutation procedure becomes to some extent combinatorial [81]. For a nondegenerate potential, this nice property persists under iterated mutations. The catch, however, is that in general it is not clear how to check that a potential is nondegenerate. A useful criterion, based on the theory of graded mutations [2], is given in [44].

Theorem 3.15 ([44, COROLLARY 1.3]). *Assume:*

- (1) Q is a \mathbb{Z} -graded quiver such that $(kQ)_{\leq 0}$ is finite dimensional.
- (2) Q has at least three vertices.
- (3) w is a homogeneous reduced potential of degree r (in particular, it is a finite sum).
- (4) $\Lambda = J(Q, w)$ is a twisted NCCR whose center is 3-dimensional with an isolated singularity.
- (5) $\Lambda/[\Lambda, \Lambda]$ does not contain any elements whose degree is in the interval $[1, r/2]$.

Then w is nondegenerate.

Note that (5) is automatic if $r = 1$. This gives an alternative proof why the potentials associated to “rolled up helix algebras” of Del Pezzo surfaces are nondegenerate (see [30, THEOREM 1.7], [44, THEOREM 4.2.1]). Theorem 3.15 also applies to many skew group rings $\Lambda = k[x, y, z]\#(\mathbb{Z}/n\mathbb{Z})$. For example, $n = 5$ and $\bar{1}$ acting with weights $(1/5, 2/5, 2/5)$ (see [67, §7]).

4. QUOTIENT SINGULARITIES FOR REDUCTIVE GROUPS

4.1. NCCRs via modules of covariants

In this section we discuss some results from [116]. Also G will always be a reductive group. Let S be the coordinate ring of a smooth affine G -variety X . Then S^G is the coordinate ring of the categorical quotient $X//G$. We will be interested in constructing (twisted) NCCRs for S^G . In the particular case when G is finite and X is a faithful unimodular G -representation, this was discussed in Section 3.1. An NCCR for S^G is given by the skew group ring $\Lambda = S\#G$. However, Λ can be described in a different way. For U a finite dimensional G -representation, put $M(U) := (U \otimes S)^G$. Then $M(U)$ is a reflexive S^G module (in fact, it is maximal Cohen–Macaulay). If every irreducible representation of G occurs at least once in U then Λ is Morita equivalent to $\text{End}_{S^G}(M(U))$. Hence $M(U)$ defines an NCCR of S^G .

The modules $M(U)$ we introduced are the so-called *modules of covariants* [31] and they make perfect sense for general reductive groups. A mild obstacle is that modules of covariants do not have to be reflexive in general [31]. This is not a serious problem, but if we want to avoid it anyway, we can restrict the pairs (G, X) we consider. We will say that G acts *generically* on a smooth affine variety if the locus of points with closed orbit and trivial stabilizer is nonempty and its complement has codimension ≥ 2 . If W is a G -representation then we will say that (G, W) is *generic* if G acts generically on $\text{Spec Sym } W \cong W^*$. We then have in particular

$$\text{End}_{S^G}(M(U)) = M(\text{End}(U)). \tag{4.1}$$

It is reasonable to search for NC(C)Rs of the form $\text{End}_{S^G}(M(U))$. However, if G is not finite there are nontrivial obstacles:

- (1) There are an infinite number irreducible representations so we cannot just take the sum of all of them. We need to make a careful selection.
- (2) Modules of covariants are usually not Cohen–Macaulay and so demanding that $\text{End}_{S^G}(M(U)) = M(\text{End}(U))$ (cf. (4.1)) is Cohen–Macaulay is a severe restriction on U .

The first issue is handled in [116, §10] where we construct certain nice complexes relating different modules of covariants (see also [135, CHAPTER 5]). The second issue is handled using results from [124] (see also [123, 125–127]).

Before we discuss NCCRs let us give a result on NCRs.

Proposition 4.1 ([116, COROLLARY 1.3.5]). *Assume that (G, W) is generic. Then there exists a finite dimensional G -representation U containing the trivial representation such that $\Lambda = \text{End}_{S^G}(M(U))$ is an NCR for S^G .*

Remark 4.2. The fact that U contains the trivial representation implies that Λ defines a categorical resolution by Lemma 2.2.1. It turns out that NCRs are easier to construct than NCCRs since it is sufficient to take U big enough, in a suitable sense.

To state our results about (twisted) NCCRs, we need to introduce some notation. Let G be a connected⁸ reductive group. Let $T \subset B \subset G$ be respectively a maximal torus and a Borel subgroup of G , with $\mathcal{W} = N(T)/T$ being the corresponding Weyl group. Put $X(T) = \text{Hom}(T, G_m)$ and let $\Phi \subset X(T)$ be the roots of G . By convention the roots of B are the negative roots Φ^- and $\Phi^+ = \Phi - \Phi^-$ is the set of positive roots. We write $\bar{\rho} \in X(T)_{\mathbb{R}}$ for half the sum of the positive roots. Let $X(T)_{\mathbb{R}}^+$ be the dominant cone in $X(T)_{\mathbb{R}}$ and let $X(T)^+ = X(T)_{\mathbb{R}}^+ \cap X(T)$ be the set of dominant weights. For $\chi \in X(T)^+$, we denote the simple G -representation with highest weight χ by $V(\chi)$.

Let W be a finite-dimensional G -representation of dimension d and put $S = \text{Sym}(W)$, $X = \text{Spec Sym}(W) = W^*$. Let $(\beta_i)_{i=1}^d \in X(T)$ be the T -weights of W .

Put

$$\Sigma = \left\{ \sum_i a_i \beta_i \mid a_i \in]-1, 0] \right\} \subset X(T)_{\mathbb{R}}.$$

The elements of the intersection $X(T)^+ \cap (-2\bar{\rho} + \Sigma)$ are called *strongly critical (dominant) weights* for G .

Theorem 4.3 ([116, THEOREM 3.4.3][124]). *Assume that X contains a point with closed orbit and finite stabilizer. Let $\chi \in X(T)^+$ be a strongly critical weight and $U = V(\chi)$. Then $M(U^*)$ is a Cohen–Macaulay S^G -module.*

⁸ In [116] we also consider the nonconnected case.

If we look at (4.1) and observe that the weights of $\text{End}(U)$ are very roughly speaking about twice those of U , then Theorem 4.3 suggests that to construct an NCCR we should restrict ourselves to representations whose highest weights are approximately contained in $-\bar{\rho} + (1/2)\Sigma$. This idea works for the class of “quasisymmetric” representations, which includes the class of self dual representations.

We say that W is *quasisymmetric* if for every line $\ell \subset X(T)_{\mathbb{R}}$ through the origin we have $\sum_{\beta_i \in \ell} \beta_i = 0$. This implies in particular that W is unimodular and hence S^G is Gorenstein if W is generic by a result of Knop [87].

From now on we assume that (G, W) is generic and W is quasisymmetric. Deviating slightly from [116], following [59], we introduce a certain affine hyperplane arrangement on $X(T)_{\mathbb{R}}^W$. Let $\bar{\mathcal{H}}$ be the collection of affine hyperplanes spanned by the facets of $-\bar{\rho} + (1/2)\bar{\Sigma}$. We consider the hyperplane arrangement in⁹ $X(T)_{\mathbb{R}}^W$ given by

$$\mathcal{H} = \bigcup_{H \in \bar{\mathcal{H}}} (-H + X(T)) \cap X(T)_{\mathbb{R}}^W. \quad (4.2)$$

Remark 4.4. The hyperplane arrangement (4.2) may be degenerate in the sense that $X(T)_{\mathbb{R}}^W \subset -H + \chi$ for some $\chi \in X(T)$.

Example 4.5. We give a simple example where degeneration occurs. Let $G = \text{SL}(2)$. If V is the standard representation and $W = V^n$ with n even, then $(1/2)\Sigma$ is the interval $]-n/2, n/2[$ (identifying $X(T) \cong \mathbb{Z}$). Moreover, $\bar{\rho} = 1$. Hence the “hyperplanes” $-H + X(T)$ are given by the integers. Furthermore, $X(T)_{\mathbb{R}}^W = 0$. Thus the induced hyperplane arrangement in $X(T)_{\mathbb{R}}^W$ is indeed degenerate. If n is odd, on the other hand, then it is nondegenerate. See [116, THEOREM 1.4.5] for a complete treatment of the case $G = \text{SL}(2)$.

This hyperplane arrangement is such that if δ is the complement of \mathcal{H} then

$$(-\bar{\rho} + \delta + 1/2\bar{\Sigma}) \cap X(T) = \emptyset.$$

The following result is a slight variation on [116, THEOREM 1.6.4].

Theorem 4.6. *Let (G, W) be generic and assume that W is quasisymmetric. Let δ be an element of the complement of \mathcal{H} . Put*

$$\mathcal{L}_{\delta} = X(T)^+ \cap (-\bar{\rho} + \delta + (1/2)\bar{\Sigma}), \quad (4.3)$$

$$U_{\delta} = \bigoplus_{\chi \in \mathcal{L}_{\delta}} V(\chi), \quad (4.4)$$

$$\Lambda_{\delta} = \text{End}_{S^G}(M(U_{\delta})). \quad (4.5)$$

If $\mathcal{L}_{\delta} \neq \emptyset$ then Λ is an NCCR for $\text{Sym}(W)^G$.

It is easy to see that \mathcal{L}_{δ} and hence U_{δ} depend only on the connected component of the complement of \mathcal{H} to which δ belongs.

⁹ Note that $X(T)_{\mathbb{R}}^W$ is just the character group $X(G)$ of G .

We obtain some evidence for Conjecture 3.7.¹⁰

Corollary 4.7 ([116, THEOREM 1.6.2]). *If $G = T$ is a torus and W is quasisymmetric then $\text{Sym}(W)^T$ has a (toric) NCCR.*

Remark 4.8. For reference we note that there is extension of Theorem 4.6 that may allow one to construct twisted NCCRs [116, THEOREM 1.6.4].

We now state some consequences of these results for determinantal varieties.

Theorem 4.9 ([116, THEOREM 1.4.1]). *For $l < n$, let $X_{l,n}$ be the variety of $n \times n$ -matrices of rank $\leq l$. Then $k[X_{l,n}]$ has an NCCR.*

The variety $X_{l,n}$ was already discussed in Example 3.1 and the NCCR obtained in [116] is the same as that we obtain in Theorem 4.9. To prove Theorem 4.9, we use the classical description of $k[X_{l,n}]$ as an invariant ring [134]. Put $G = \text{GL}(l)$ and let V be the standard representation of G . Put $W = V^n \oplus (V^*)^n$. Then $k[X_{l,n}] = \text{Sym}(W)^G$, and we show in [116] that Theorem 4.9 follows from Theorem 4.6. For the benefit of the reader, we describe the actual module of covariants that gives the NCCR. Let $B_{l,n-l}$ be the set of partitions that fit in a rectangle of size $l \times (n-l)$. In [116] it is shown that the following module of covariants defines an NCCR for R :

$$M = \bigoplus_{\lambda \in B_{l,n-l}} M(S^\lambda V), \quad (4.6)$$

where $S^\lambda V$ denotes the Schur functor indexed by λ applied to V .

Theorem 4.10. *For $2l < n$, let $X_{2l,n}^-$ be the variety of skew-symmetric $n \times n$ matrices of rank $\leq 2l$. If n is odd then $k[X_{2l,n}^-]$ has an NCCR.*

This time we put $G = \text{Sp}(2l)$ and $W = V^n$ where V is the standard representation of G .

Theorem 4.11 ([116, THEOREM 1.4.1]). *For $l < n$, let $X_{l,n}^+$ be the variety of symmetric $n \times n$ matrices of rank $\leq l$. If l and n have opposite parity then $k[X_{l,n}^+]$ has an NCCR. If l and n have the same parity then $k[X_{l,n}^+]$ has a twisted NCCR.*

Here we put $G = \text{O}(l)$ and again $W = V^n$ where V is the standard representation of G . A complication arises since $\text{O}(l)$ is not connected, so we cannot directly apply Theorem 4.6. So we have to perform a more refined analysis which is carried out in [116, §6]. Twisted NCCRs appear because $\text{SO}(l)$, the connected component of $\text{O}(l)$, is not simply connected.

The NCCRs given in Theorems 4.10, 4.11 have been crucial for establishing homological projective duality [91] for determinantal varieties of skew-symmetric matrices by Rennemo and Segal [113]. The corresponding results for symmetric matrices are work in progress by the same authors [112].

¹⁰ This is stated in [116] for W generic. However, one easily reduces to this case.

Even if W is not quasisymmetric then it is still possible that $\text{Sym}(W)^G$ has an NCCR given by a module of covariants but we are unaware of a general rule like Theorem 4.6 for constructing them. Three-dimensional affine toric varieties (see Section 3.4) are an example of this, since they can be written as $\text{Sym}(W)^G$ where W is generally not quasisymmetric. Another example is given by the recent work of Doyle:

Example 4.12 ([51, THEOREM 3.11]). Let $0 < l < n$ be integers such that $\gcd(l, n) = 1$. Let V be the standard representation of $G = \text{SL}(l)$ and put $W = V^n$.

Then $R := \text{Sym}(W)^G$ is the homogeneous coordinate ring of the Grassmannian $\text{Gr}(l, n)$ for the Plücker embedding. Let $P_{l, n-l}$ be the set of partitions whose young tableaux are above the diagonal in a rectangle of size $l \times (n - l)$. In [51] it is shown that the following module of covariants defines an NCCR for R :

$$M = \bigoplus_{\lambda \in P_{l, n-l}} M(S^\lambda V)$$

(compare with (4.6)).

We reiterate that even if an NCCR exists, there does not have to be one given by a module of covariants. See Example 3.6.

4.2. NCCRs via crepant resolutions obtained by GIT

Here we discuss some results from [59] that shows that in certain cases the NCCRs for the categorical quotients $X//G$ that we constructed in Section 4 can be obtained as the endomorphisms of a tilting bundle on a crepant resolution, i.e., the method of Section 3.2. This crepant resolution is constructed using a geometric invariant theory. It turns out that we have to allow crepant resolutions by Deligne–Mumford stacks. This occurred already before in Lemma 3.5 and Example 3.6.

Remark 4.13. To construct a resolution of $X//G$ using geometric invariant theory, one needs a linearized line bundle on X . Since here X is a representation, the only G -equivariant line bundles on X are those obtained from characters of G . If G is semisimple then there are no (nontrivial) characters so we cannot proceed. Thus we can, for example, not deal with determinantal varieties of symmetric and skew-symmetric matrices (see Theorems 4.10 and 4.11). In those cases the relevant groups were respectively $\text{Sp}(2l)$ and the connected component $\text{SO}(l)$ of $\text{O}(l)$, both of which are semisimple. On the other hand, ordinary determinantal varieties are fine since in that case $G = \text{GL}(l)$ which has a nontrivial character given by the determinant, which may be used to construct a crepant resolution.

Remark 4.14. Geometric invariant theory is still helpful for constructing a resolution of $X//G$ via a procedure invented by Kirwan [52, 85, 110]. However, these resolutions are usually not crepant. Consistent with expected minimality of NCCRs (see Section 3.3), we are able to show that some NCCRs embed inside them [118].

We retain the notations and assumptions of the previous section. We assume that W is a quasisymmetric representation of G and $X = \text{Spec Sym}(W) = W^*$. Recall that for a

character $\mu \in X(G) = X(T)^{\mathcal{W}}$ we may define a G -invariant open subset of X as

$$X^{ss,\mu} = \{x \in X \mid \exists k > 0 \text{ and } s \in \Gamma(\mathcal{O}_X \otimes \mu^k)^G \text{ such that } s(x) \neq 0\}.$$

The variety $X^{ss,\mu}$ admits a good quotient¹¹ $X^{ss,\mu} // G$ which is proper over $X // G$. For U a representation of G , we write $\mathcal{M}(U)$ for the vector bundle on $X^{ss,\mu} / G$ given by $U \otimes \mathcal{O}_{X^{ss,\mu} / G}$. The global sections of $\mathcal{M}(U)$ are equal to $M(U)$.

Below we let \mathcal{H}_0 be the *central* hyperplane arrangement on $X(G)_{\mathbb{R}} = X(T)_{\mathbb{R}}^{\mathcal{W}}$ corresponding to the affine hyperplane arrangement \mathcal{H} introduced in (4.2). Thus the hyperplanes in \mathcal{H}_0 are the hyperplanes which are induced from central hyperplanes in $X(T)_{\mathbb{R}}$ which are parallel to the facets of $\bar{\Sigma}$.

Proposition 4.15 ([59, PROPOSITION 2.1]). *Assume that the action of T on X has generically finite stabilizers and let $\mu \in X(G)$ be in the complement of \mathcal{H}_0 . Then $X^{ss,\mu} / G$ is a Deligne–Mumford stack.*

Lemma 4.16. *Assume that (G, W) is generic. Then the canonical map $X^{ss,\mu} / G \rightarrow X // G$ is crepant.*

Proof. This is proved in [119, LEMMA 4.5] in the case that G is a torus, but this assumption is not relevant for the proof. ■

The following is one of the main results of [59]. It is proved using similar combinatorics as in [116].

Theorem 4.17 ([59]). *Assume that the action of T on X has generically finite stabilizers and let $\mu \in X(G)$ be in the complement of \mathcal{H}_0 . Let U_{δ} be as in the statement of Theorem 4.6. Then $\mathcal{M}(U_{\delta})$ is a tilting bundle on $X^{ss,\mu} / G$ such that $\text{End}_{X^{ss,\mu} / G}(\mathcal{M}(U_{\delta})) = M(\text{End}(U_{\delta}))$.*

Proof. This follows from combining [59, THEOREM 1.2] with [59, LEMMA 2.9]. ■

In this way we obtain more evidence for Conjecture 1.2.

Corollary 4.18 ([59, COROLLARY 1.3]). *Under the hypotheses of Theorem 4.17, if $\mu, \mu' \in X(G)$ are in the complement of \mathcal{H}_0 and the complement of \mathcal{H} is nonempty (i.e., \mathcal{H} is nondegenerate) then $\mathcal{D}(X^{ss,\mu} / G) \cong \mathcal{D}(X^{ss,\mu'} / G)$.*

We also obtained the promised description of NCCRs via resolutions.

Corollary 4.19. *Assume that (G, W) is generic and let Λ be an NCCR constructed via Theorem 4.6. Let μ be in the complement of \mathcal{H}_0 . Then Λ is the endomorphism ring of a tilting bundle on the DM stack $X^{ss,\mu} / G$.*

Remark 4.20. Hidden behind what is discussed in Sections 4.1 and 4.2 is the idea of *windows*, pioneered in [49]. This is based on the fact that we have a restriction map

$$\text{Res} : \mathcal{D}(X/G) \rightarrow \mathcal{D}(X^{ss,\mu} / G).$$

¹¹ A G -equivariant map $Z \rightarrow Y$ is a good quotient if locally on Y it is of the form $U \rightarrow U // G$ for U affine.

It is then natural to try to find a full subcategory $\mathcal{D} \subset \mathcal{D}(X/G)$ such that the restriction of Res to \mathcal{D} yields an equivalence $\mathcal{D} \cong \mathcal{D}(X^{ss,\mu}/G)$. A very general result in this direction is [58, THEOREM 1.1], see also [5].

In concrete cases one may hope to define \mathcal{D} as the full subcategory of $\mathcal{D}(X/G)$ which is split generated by $U \otimes \mathcal{O}_{X/G}$ for a suitable G -representation U whose highest weights are restricted to a certain subset \mathcal{L} of $X(T)^+$ (a “window”). This is precisely what happens in Theorem 4.17, where we take $\mathcal{L} = \mathcal{L}_\delta$. The resulting category \mathcal{D} is a concrete realization of [58, THEOREM 1.1], see [59, LEMMA 3.5].

One does not actually need to have nontrivial $X^{ss,\mu}/G$ to apply the window principle. The proof of Theorem 4.6, is based on the fact that $\text{mod}(\Lambda_\delta)$ embeds in $\text{coh}(X/G)$ as the abelian category with a projective generator $U_\delta \otimes \mathcal{O}_{X/G}$.

4.3. Local systems, the SKMS, and schobers

In this slightly informal section we assume that the hypotheses of Theorem 4.17 hold. While Corollary 4.18 implies that two different $\mathcal{D}(X^{ss,\mu}/G)$, $\mathcal{D}(X^{ss,\mu'}/G)$ are derived equivalent, the actual derived equivalence depends on the choice of δ in the complement of \mathcal{H} . Moreover, by considering compositions $\mathcal{D}(X^{ss,\mu}/G) \xrightarrow{\delta} \mathcal{D}(X^{ss,\mu''}/G) \xrightarrow{\delta'} \mathcal{D}(X^{ss,\mu'}/G)$, we may produce more derived equivalences. This is consistent with the assertion in Remark 1.3 that there is no “god-given” derived equivalence between different crepant resolutions. A different way of saying this is that a crepant resolution may have a large group of derived autoequivalences.

If M is a Calabi–Yau variety then homological mirror symmetry predicts the existence of a space S (the “stringy Kähler moduli space,” or SKMS) such that $\pi_1(S)$ acts on $\mathcal{D}(M)$. More precisely, S is the moduli space of complex structures on the mirror dual M^\vee of M . In many cases there are good heuristic descriptions of M^\vee and S .

Even without access to the full mirror symmetric context, which may be technically challenging or even only heuristic, it turns out to be very illuminating to represent the derived autoequivalences of an algebraic variety (or stack) as elements of $\pi_1(S)$ for a suitable topological space S . Alternatively, we may think of such a representation as a *local system of triangulated categories on S* . Understanding this for $\mathcal{D}(X^{ss,\mu}/G)$ was, according to the authors, one of the main motivations for writing [59]. Indeed, when X is a quasisymmetric representation, under hypotheses of Theorem 4.17, one may take

$$S = (X(G)_{\mathbb{C}} - \mathcal{H}_{\mathbb{C}})/X(G),$$

where $\mathcal{H}_{\mathbb{C}}$ denotes the complexification of the real hyperplane arrangement \mathcal{H} [59, PROPOSITION 6.6].

In this case there is a nice way to understand that action of $\pi_1(S)$ on $\mathcal{D}(X^{ss}/G)$ [59, §6], [117]. Using Theorem 4.17 again, we may just as well describe the action of $\pi_1(S)$ on $\mathcal{D}(\Lambda_\delta)$ for δ contained in the complement of \mathcal{H} and $\Lambda_\delta = M(\text{End}(U_\delta))$.

For $\delta \in X(G)_{\mathbb{R}}$, define U_δ as in (4.4) and put $\mathcal{D}_\delta = \mathcal{D}(\Lambda_\delta)$. Now \mathcal{H} defines a cell decomposition of $X(G)_{\mathbb{R}}$ and it is easy to see that U_δ only depends on the cell to which δ belongs. Hence for a cell C let us write $\Lambda_C := \Lambda_\delta$, $\mathcal{D}_C := \mathcal{D}_\delta$ for $\delta \in C$. We will refer to

the cells of maximal dimension as chambers. These are also the connected components of the complement of \mathcal{H} .

If C' is a face of C then there is an idempotent $e_{C,C'} \in \Lambda_{C'}$ such that $\Lambda_C = e_{C,C'} \Lambda_{C'} e_{C,C'}$. If $C \neq C''$ are distinct adjacent chambers, sharing a codimension one face C' then the functor $e_{C'',C'} \Lambda_{C'} e_{C,C'} \overset{L}{\otimes}_{\Lambda_C} -$ defines an equivalence of categories $\phi_{C,C''} : \mathcal{D}_C \rightarrow \mathcal{D}_{C''}$.

Put $\tilde{S} = X(G)_C - \mathcal{H}_C$ and let $\Pi_1(\tilde{S})$ be the groupoid whose objects are the chambers and whose morphisms are given by the homotopy classes of paths in $X(G)_C - \mathcal{H}_C$ connecting the chambers. Then $\Pi_1(\tilde{S})$ is equivalent to the fundamental groupoid of $X(G)_C - \mathcal{H}_C$. If C, C'' are adjacent chambers separated by a hyperplane $H \in \mathcal{H}$ such that $H(C'') > 0$ then there is a canonical (up to homotopy) minimal path $\nu_{C,C''}$ in $X(G)_C - \mathcal{H}_C$ going from C to C'' and passing through $\{\text{Im } H_C > 0\}$. Sending C to \mathcal{D}_C and $\nu_{C,C''}$ to $\phi_{C,C''}$ defines a representation of the groupoid $\Pi_1(\tilde{S})$ in triangulated categories.

If $\chi \in X(G)$ then tensoring by χ defines an equivalence $\mathcal{D}_C \rightarrow \mathcal{D}_{C+\chi}$ and in this way the representation of $\Pi_1(\tilde{S})$ may be extended to a representation of $\Pi_1(\tilde{S}) \rtimes X(G)$ and the latter is equivalent to the fundamental groupoid $\Pi_1(S)$ of $S = \tilde{S}/X(G)$ [33, CHAPTER 11], [59, §6]. Hence, fixing a “base chamber” C , we get an action of $\pi_{1,C}(S)$ on \mathcal{D}_C .

Remark 4.21. It is shown in [117] that the family of triangulated categories $(\mathcal{D}_C)_C$ for all cells C is a so-called $X(G)$ -equivariant *perverse schober*. This is a categorification of a perverse sheaf on $X(G)_C/X(G)$ [74] (see also [20, 48]). Note that $X(G)_C/X(G)$ is a torus and S is the complement of a “toric hyperplane arrangement.” If G is itself a torus T then $X(T)_C/X(T)$ may be identified with the dual torus T^\vee .

Remark 4.22. The $X(G)$ -equivariant hyperplane arrangement constructed by Halpern–Leistner and Sam in [59] is very similar to the $\text{Cl}(R)$ -equivariant hyperplane arrangement associated to a 3-dimensional terminal complete Gorenstein ring R constructed by Iyama and Wemyss (see Example 4.12). One would expect there to be an associated equivariant schober also in this case. In the case of a single curve flop this is essentially contained in [50, §3].

Remark 4.23. As explained we have an action of $\pi_1(S)$ on \mathcal{D}_C for a chamber C and hence also an action of $\pi_1(S)$ on $K_0(\mathcal{D}_C)_C$. In other words, we have a local system L on S . It is then a natural question if this local system occurs as the solutions of a natural system of differential equations. In the case that G is a torus we show in [121] that a generic “equivariant” deformation of L is obtained as the solution of a well-known system of differential equations introduced by Gel’fand, Kapranov, and Zelevinsky [55]. This starts from a computation by Kite [86] which shows that the hyperplane arrangement constructed in [59] is up to translation defined by the so-called “principal A -determinant,” an important ingredient in the theory developed Gel’fand, Kapranov, and Zelevinsky. For more information, see [115].

Remark 4.24. The themes touched upon in this section occur in many different contexts. See, e.g., [3, 23, 29].

5. NCCRS AND STRINGY E -FUNCTIONS

In this section we discuss some ongoing work of Timothy De Deyn (see [43]). Let X be an algebraic variety over \mathbb{C} . The cohomology groups $H_c^i(X, \mathbb{C})$ carry a natural mixed Hodge structure. We denote by $h^{p,q}(H_c^i(X, \mathbb{C}))$ the dimension of the (p, q) -type component of $H_c^i(X, \mathbb{C})$. The *Hodge polynomial* of X is defined by

$$E(X, u, v) = \sum_{p,q,i} (-1)^i h^{p,q}(H_c^i(X, \mathbb{C})) u^p v^q.$$

The Hodge polynomial defines a ring homomorphism from the Grothendieck ring of algebraic varieties $K_0(\text{Var}/\mathbb{C})$ to $\mathbb{Z}[u, v]$.

We put $e(X) = E(X, 1, 1)$, i.e.,

$$e(X) = \sum_i (-1)^i \sum_{p,q} h^{p,q}(H_c^i(X, \mathbb{C})) = \sum_i (-1)^i \dim H_c^i(X, \mathbb{C}).$$

In other words, $e(X)$ is the *Euler characteristic* (with compact support¹²) of X . It defines a ring homomorphism from $K_0(\text{Var}/\mathbb{C})$ to \mathbb{Z} .

Definition 5.1 ([6, DEFINITION 3.1]). Assume that X is a normal \mathbb{Q} -Gorenstein algebraic variety/ \mathbb{C} with at most log-terminal singularities and let $\pi : Y \rightarrow X$ be a resolution of singularities whose exceptional locus is a normal crossing divisor. Let D_1, \dots, D_r be the irreducible components of the exceptional locus and put $I = \{1, \dots, r\}$. For any subset $J \subset I$ we set $D_J = \bigcap_{j \in J} D_j$, $D_J^\circ := D_J \setminus \bigcup_{j \in I \setminus J} D_j$. The *stringy E -function* of X is defined as

$$E_{st}(X, u, v) := \sum_{J \subset I} E(D_J^\circ, u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1}, \quad (5.1)$$

where the numbers $a_j \in \mathbb{Q} \cap]-1, \infty[$ are defined by

$$K_Y = \pi^* K_X + \sum_{j=1}^r a_j D_j.$$

Putting $e_{st}(X) = \lim_{u,v \rightarrow 1} E_{st}(X, u, v)$ defines the *stringy Euler characteristic* of X , with the formula

$$e_{st}(X) = \sum_{J \subset I} e(D_J^\circ) \prod_{j \in J} \frac{1}{a_j + 1}.$$

It follows from the theory of motivic integration (see, e.g., [8, 40, 45, 90, 95]) that $E_{st}(X, u, v)$ is independent of the chosen resolution Y [6, THEOREM 3.4]. Indeed, $E_{st}(X, u, v)$ may be obtained by integrating over the arc space associated to X [45]. In a similar vein, $E_{st}(X, u, v) = E_{st}(Y, u, v)$ holds for birational maps $\pi : Y \rightarrow X$ satisfying $\pi^* K_X = K_Y$ [6, THEOREM 3.12].

If X is smooth then the stringy E -function coincides with the Hodge polynomial. Hence one has

¹² If X is smooth then, by Poincaré duality, the Euler characteristic with compact support coincides with the usual Euler characteristic $\sum_i (-1)^i \dim H^i(X, \mathbb{C})$.

Theorem 5.2 ([6, THEOREM 3.12]). *If X has a crepant resolution Y then the stringy E -function of X coincides with the Hodge polynomial of Y . In particular, it is a polynomial. Similarly, the stringy Euler characteristic of X coincides with the usual Euler characteristic of Y and hence it is an integer.*

The following conjecture seems natural:

Conjecture 5.3 ([43]). *If X is a normal Gorenstein variety/ \mathbb{C} with an NCCR then its stringy E -function is a polynomial.*

We give some evidence for this conjecture below, but at this point it is probably safer to regard it as a question. We illustrate below in Remark 5.8 and Example 5.11 that reasonable extensions of this conjecture are false.

Example 5.4. Quotient varieties of the form $\mathbb{C}^n // G$ for $G \subset \mathrm{SL}(n)$ finite always have a stringy E -function which is a polynomial by [9], [46, THEOREM 3.6]. They also have an NCCR by Section 3.1. So in this case Conjecture 5.3 is true.

Example 5.5. Batyrev proves in [8, PROPOSITION 4.4] that the stringy E -function of any toric variety with Gorenstein singularities is a polynomial. Hence Conjecture 5.3 is compatible with Conjecture 3.7.

A good test for Conjecture 5.3 is given by cones over Fano varieties.

Proposition 5.6 ([43]). *Let Z be a smooth projective variety/ \mathbb{C} with ample line bundle $\mathcal{O}_Z(1)$ and let $X = \mathrm{Spec} R$ be the corresponding cone. Assume $\omega_Z = \mathcal{O}_Z(-n)$ for $n \geq 1$. Then R is Gorenstein and*

$$E_{st}(X, u, v) = E(Z, u, v) \frac{(q-1)q^n}{q^n - 1} \tag{5.2}$$

with $q = uv$. In particular,

$$e_{st}(X) = \frac{e(Z)}{n}. \tag{5.3}$$

Example 5.7. Consider the Grassmannian $Z := \mathrm{Gr}(d, n)$. Then (e.g., [25, PROPOSITION A.4])

$$E_{st}(Z, u, v) = \binom{n}{d}_q, \tag{5.4}$$

where

$$\binom{n}{d}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-d+1} - 1)}{(q - 1)(q^2 - 1) \cdots (q^d - 1)}.$$

Hence

$$e_{st}(Z) = \binom{n}{d}. \tag{5.5}$$

Let X be the cone over Z with respect to the Plucker embedding and let R be the coordinate ring of X . Using (5.2) and (5.4), it is shown in [43] that in this case $E_{st}(X, u, v)$ is a polynomial precisely when $\mathrm{gcd}(d, n) = 1$. On the other hand, by [51, THEOREM 3.11] (see

Example 4.12 above), R has an NCCR when $\gcd(d, n) = 1$. So Conjecture 5.3 is true in this case.

Remark 5.8. It is shown in [43, §4.2] that the cone over an arbitrary Grassmannian $\text{Gr}(d, n)$ always has a weakly crepant categorical resolution (see Section 2.2). Hence it follows that Conjecture 5.3 is false for weakly crepant categorical resolutions. On the other hand, it seems reasonable to extend Conjecture 5.3 to strongly crepant categorical resolutions.

Remark 5.9. In view of Theorem 5.2, one may naively ask if it is true that $E_{st}(X, u, v)$ being a polynomial implies that X has a crepant resolution. Not unexpectedly, this fails drastically. Finite group quotients and affine toric varieties have a polynomial stringy E -function, as we have seen above, but they need not have a crepant resolution. In fact, the example $\mathbb{C}^4 // \mathbb{Z}_2$ given in Section 3.1 of a Gorenstein singularity with an NCCR but without a crepant resolution, lives in both classes. On the other hand, these classes of counterexamples are not very convincing since they admit crepant resolutions by smooth Deligne–Mumford stacks, which is just as good. In the case of finite quotient singularities this is clear, and for toric varieties it follows from Lemma 3.5.

In contrast, one may show that for X as in Example 5.7 there is no crepant resolution by a smooth DM-stack (this is mainly because R is factorial). So in some sense it is a “better” counterexample (when $\gcd(d, n) \neq 1$).

GIT quotients form an important class of toric varieties (see [39, COROLLARY 14.2.16]). So in view of Example 5.5, as well as Example 5.4, the following question by Batyrev suggests itself:

Question 5.10 ([6, QUESTION 5.5]). Does a GIT quotient of \mathbb{C}^n for a linear action of $G \subset \text{SL}(n)$ always have a stringy E -function that is a polynomial?

Alas, the answer is negative. Indeed, Example 5.7 for $\gcd(d, n) \neq 1$ gives a simple counterexample since the cone over a Grassmannian is a GIT quotient for $\text{SL}(d)$ acting on n copies of its standard representation.

The first counterexample, however, was constructed much earlier in [84].

Example 5.11. Let W be given by three copies of the adjoint representation of $G = \text{SL}(2)$. Then by a quite involved computation it is shown in [84, COROLLARY 1.2] that the stringy E -function of $\text{Spec } R$ for $R = \text{Sym}(W)^G$ is not a polynomial. This example is interesting since, by [116, THEOREM 1.4.5], R has a twisted NCCR. In other words, Conjecture 5.3 is also false for twisted NCCRs.

As a side remark, we note that this twisted NCCR is a rather classical object. It is the trace ring generated by 3 generic traceless 2×2 matrices [4, 97, 99, 107–109].

In the setting of Theorem 5.2, the Euler characteristic of Y can be computed using periodic cyclic homology, thanks to the Hochschild–Kostant–Rosenberg theorem. Below we define the Euler characteristic $e(\Lambda)$ of an algebra Λ or a sheaf of algebras as

$\dim \text{HP}^{\text{even}}(\Lambda) - \dim \text{HP}^{\text{odd}}(\Lambda)$. If Λ is a quasicoherent sheaf of algebras then we use $\text{HP}^*(\Lambda) := \text{HP}^*(\text{Perf}_{\text{dg}} \Lambda)$ (where $\text{Perf}_{\text{dg}}(\Lambda)$ is a standard dg-enhancement of $\text{Perf}(\Lambda)$). The following conjecture appears plausible.

Conjecture 5.12. The stringy Euler characteristic of a normal Gorenstein variety can be computed as the Euler characteristic of an NCCR, computed via periodic cyclic homology.

Remark 5.13. One can again not expect this conjecture to hold for twisted NCCRs. An interesting example is given in [26]. It was shown by [22, 92] that for a generic complete intersection Y of n quadrics in \mathbb{P}^{2n-1} one has a derived equivalence between Y and $(\mathbb{P}^{n-1}, \mathcal{B}_0)$ where \mathcal{B}_0 is the even part of the universal Clifford algebra corresponding to the quadrics defining Y . Because of the derived equivalence, we then have $e(Y) = e(\mathcal{B}_0)$ [80]. One may show that \mathcal{B}_0 is a twisted NCCR of its center which is a double cover Z of \mathbb{P}^{n-1} [26, §1]. It is shown in [26] that in general $e(Y) \neq e_{st}(Z)$ and hence $e(\mathcal{B}_0) \neq e_{st}(Z)$. So Conjecture 5.12 does not extend to twisted NCCRs.

In suitable “local” contexts (e.g., [130, THEOREM 9.1]) Conjecture 5.12 leads to a more concrete conjecture:¹³

Conjecture 5.14 ([43]). Let R a normal Gorenstein ring which is either a complete local ring, or else connected \mathbb{N} -graded (i.e., $R_0 = \mathbb{C}$). Assume that R has an NCCR $\text{End}_R(M)$. Then the number of nonisomorphic indecomposable summands of M is equal to $e_{st}(X)$ for $X = \text{Spec } R$.

Example 5.15. If $X = \text{Spec } R$ is an affine toric variety as in Section 3.4, with Gorenstein singularities, then Batyrev [6, PROPOSITION 4.10] proves that $e_{st}(X)$ is equal to the volume of the associated polytope P (see Section 3.4). So Conjecture 5.14 is compatible with Conjecture 3.8.

Example 5.16. For varieties of the form $X = W//G = \text{Spec } \mathbb{C}[W]^G$ for $G \subset \text{SL}(n)$ finite and W a finite dimensional representation of G , it follows from [7, THEOREM 8.4] that $e_{st}(X)$ is equal to the number of conjugacy classes in G . This number is in turn equal to the number of irreducible representations of G and hence equal to the number of nonisomorphic indecomposable summands of the reflective $\mathbb{C}[W]^G$ -module $\mathbb{C}[W]$ which defines an NCCR for $\mathbb{C}[W]^G$ by Section 3.1. So Conjecture 5.14 is true in this specific example.

Example 5.17. Let X be the cone over $\text{Gr}(d, n)$ as in Example 5.7. Then by (5.3) and (5.5) we have

$$e_{st}(X) = \frac{1}{n} \binom{n}{d}.$$

¹³ To handle the complete case, one has to use a “completed” version of periodic cyclic homology. See [130].

We check that Conjecture 5.14 is compatible with the NCCR constructed by Doyle in [51, THEOREM 3.11] (see Example 4.12 above). Conjecture 5.14 amounts to

$$|P_{d,n-d}| = \frac{1}{n} \binom{n}{d},$$

which is indeed true by [101, §12.1].

Example 5.17 can be put in a more general context. Let us first state a lemma.

Lemma 5.18. *Let Z be a smooth projective variety with a tilting complex. Then one has $e(Z) = \text{rk } K_0(Z)$.*

Proof. Let \mathcal{T} be the tilting complex and put $A = \text{End}_Z(\mathcal{T})$. We have:

- (1) Euler characteristics may be computed with periodic cyclic homology.
- (2) Periodic cyclic homology is invariant under derived equivalence [89], and so are Grothendieck groups;
- (3) $\text{HP}^*(A) = \text{HP}^*(A/\text{rad } A)$ by Goodwillie’s theorem [56, THEOREM II.5.1], and the standard fact that $K_0(A) = K_0(A/\text{rad } A)$.

So we conclude

$$e(Z) \stackrel{(1,2)}{=} e(A) \stackrel{(3)}{=} \text{rk } K_0(A) \stackrel{(2)}{=} \text{rk } K_0(Z). \quad \blacksquare$$

Let us go back to the setting of Proposition 5.6 but assume now in addition that Z has a tilting complex. Then by (5.3) combined with Lemma 5.18, we get

$$e_{st}(X) = \frac{\text{rk } K_0(Z)}{n}.$$

Assuming that R has a graded NCCR Λ , Conjecture 5.14 implies

$$\text{rk } K_0(\Lambda) = \frac{\text{rk } K_0(Z)}{n}. \quad (5.6)$$

Example 5.19. This formula holds for the NCCRs constructed via Proposition 3.3. Indeed, $\text{rk } K_0(\Lambda)$ is given by the number u of nonisomorphic indecomposable summands of \mathcal{E} . On the other hand, $\mathcal{D}(Z)$ has a semiorthogonal decomposition consisting of n parts whose K_0 also has rank u . So (5.6) does indeed hold, and we obtain again some evidence for Conjecture 5.14.

Remark 5.20. One way to think of this example as the realization of the (conjectured) “motivic” identity (5.6) via semiorthogonal decompositions of derived categories. See [106] for another (deeper) instance of this principle.

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REFERENCES

- [1] Quotient singularities with no crepant resolution?, 2011, <https://mathoverflow.net/questions/66657/quotient-singularities-with-no-crepant-resolution/66702>.
- [2] C. Amiot and S. Oppermann, Cluster equivalence and graded derived equivalence. *Doc. Math.* **19** (2014), 1155–1206.
- [3] R. Anno, R. Bezrukavnikov, and I. Mirković, Stability conditions for Slodowy slices and real variations of stability. *Mosc. Math. J.* **15** (2015), no. 2, 187–203, 403.
- [4] M. Artin, On Azumaya algebras and finite dimensional representations of rings. *J. Algebra* **11** (1969), 532–563.
- [5] M. Ballard, D. Favero, and L. Katzarkov, Variation of geometric invariant theory quotients and derived categories. *J. Reine Angew. Math.* **746** (2019), 235–303.
- [6] V. V. Batyrev, Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pp. 1–32, World Sci. Publ., River Edge, NJ, 1998.
- [7] V. V. Batyrev, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *J. Eur. Math. Soc. (JEMS)* **1** (1999), no. 1, 5–33.
- [8] V. V. Batyrev, Stringy Hodge numbers and Virasoro algebra. *Math. Res. Lett.* **7** (2000), no. 2–3, 155–164.
- [9] V. V. Batyrev and D. I. Dais, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry. *Topology* **35** (1996), no. 4, 901–929.
- [10] A. Beauville, Symplectic singularities. *Invent. Math.* **139** (2000), no. 3, 541–549.
- [11] A. Beilinson, Coherent sheaves on \mathbb{P}^n and problems of linear algebra. *Funct. Anal. Appl.* **12** (1978), 214–216.
- [12] A. Beilinson and J. Bernstein, Localisation de \mathfrak{g} -modules. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 15–18.
- [13] R. Bezrukavnikov, Noncommutative counterparts of the Springer resolution. In *International Congress of Mathematicians. Vol. II*, pp. 1119–1144, Eur. Math. Soc., Zürich, 2006.
- [14] R. Bezrukavnikov and D. Kaledin, McKay equivalence for symplectic resolutions of quotient singularities. *Tr. Mat. Inst. Steklova* **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 20–42.

- [15] R. Bezrukavnikov and D. Kaledin, Fedosov quantization in positive characteristic. *J. Amer. Math. Soc.* **21** (2008), no. 2, 409–438.
- [16] R. Bezrukavnikov, I. Mirković, and D. Rumynin, Localization of modules for a semisimple Lie algebra in prime characteristic. *Ann. of Math. (2)* **167** (2008), no. 3, 945–991, With an appendix by Bezrukavnikov and Simon Riche.
- [17] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468.
- [18] A. Bondzenta and A. Bondal, Flops and spherical functors. 2015, arXiv:1511.00665v2.
- [19] A. Bondal, Representations of associative algebras and coherent sheaves. *Math. USSR, Izv.* **34** (1990), no. 1, 23–42.
- [20] A. Bondal, M. Kapranov, and V. Schechtman, Perverse sheaves and birational geometry. *Selecta Math. (N.S.)* **24** (2018), no. 1, 85–143.
- [21] A. Bondal and D. Orlov, Semiorthogonal decomposition for algebraic varieties. 1995, arXiv:alg-geom/9506012.
- [22] A. Bondal and D. Orlov, Derived categories of coherent sheaves. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pp. 47–56, Higher Ed. Press, Beijing, 2002.
- [23] L. A. Borisov and R. P. Horja, Mellin–Barnes integrals as Fourier–Mukai transforms. *Adv. Math.* **207** (2006), no. 2, 876–927.
- [24] L. Borisov, L. Chen, and G. Smith, The orbifold Chow ring of toric Deligne–Mumford stacks. *J. Amer. Math. Soc.* **18** (2005), no. 1, 193–215.
- [25] L. Borisov and A. Libgober, Stringy E -functions of Pfaffian–Grassmannian double mirrors. *Algebr. Geom.* **6** (2019), no. 4, 486–515.
- [26] L. Borisov and C. Wang, On stringy Euler characteristics of Clifford non-commutative varieties. *Adv. Math.* **349** (2019), 1117–1150.
- [27] T. Bridgeland, Flops and derived categories. *Invent. Math.* **147** (2002), no. 3, 613–632.
- [28] T. Bridgeland, A. King, and M. Reid, The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.* **14** (2001), no. 3, 535–554.
- [29] T. Bridgeland, Y. Qiu, and T. Sutherland, Stability conditions and the A2 quiver. *Adv. Math.* **365** (2020), 107049.
- [30] T. Bridgeland and D. Stern, Helices on del Pezzo surfaces and tilting Calabi–Yau algebras. *Adv. Math.* **224** (2010), no. 4, 1672–1716.
- [31] M. Brion, Sur les modules de covariants. *Ann. Sci. Éc. Norm. Supér. (4)* **26** (1993), 1–21.
- [32] N. Broomhead, Dimer models and Calabi–Yau algebras. *Mem. Amer. Math. Soc.* **215** (2012), no. 1011, viii+86 pp.
- [33] R. Brown, *Topology and groupoids*. BookSurge, LLC, Charleston, SC, 2006.

- [34] R.-O. Buchweitz, G. J. Leuschke, and M. Van den Bergh, Non-commutative desingularization of determinantal varieties I. *Invent. Math.* **182** (2010), no. 1, 47–115.
- [35] R.-O. Buchweitz, G. J. Leuschke, and M. Van den Bergh, On the derived category of Grassmannians in arbitrary characteristic. *Compos. Math.* **151** (2015), no. 7, 1242–1264.
- [36] R.-O. Buchweitz, G. J. Leuschke, and M. Van den Bergh, Non-commutative desingularization of determinantal varieties, II: arbitrary minors. *Int. Math. Res. Not. IMRN* **9** (2016), 2748–2812.
- [37] S. Cautis, J. Kamnitzer, and A. Licata, Derived equivalences for cotangent bundles of Grassmannians via categorical \mathfrak{sl}_2 actions. *J. Reine Angew. Math.* **675** (2013), 53–99.
- [38] J.-C. Chen, Flops and equivalences of derived categories for threefolds with only terminal Gorenstein singularities. *J. Differential Geom.* **61** (2002), no. 2, 227–261.
- [39] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*. Grad. Stud. Math. 124, American Mathematical Society, Providence, RI, 2011.
- [40] A. Craw, An introduction to motivic integration. In *Strings and geometry*, pp. 203–225, Clay Math. Proc. 3, Amer. Math. Soc., Providence, RI, 2004.
- [41] H. Dao, Remarks on non-commutative crepant resolutions of complete intersections. *Adv. Math.* **224** (2010), no. 3, 1021–1030.
- [42] H. Dao, O. Iyama, R. Takahashi, and M. Wemyss, Gorenstein modifications and \mathbb{Q} -Gorenstein rings. *J. Algebraic Geom.* **29** (2020), no. 4, 729–751.
- [43] T. De Deyn, A note on affine cones over Grassmannians and their stringy E -functions. 2022, arXiv:2203.06040.
- [44] L. de Thanhoffer de Völcsey and M. Van den Bergh, Some new examples of non-degenerate quiver potentials. *Int. Math. Res. Not. IMRN* **20** (2013), 4672–4686.
- [45] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.* **135** (1999), no. 1, 201–232.
- [46] J. Denef and F. Loeser, Motivic integration, quotient singularities and the McKay correspondence. *Compos. Math.* **131** (2002), no. 3, 267–290.
- [47] H. Derksen, J. Weyman, and A. Zelevinsky, Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)* **14** (2008), no. 1, 59–119.
- [48] W. Donovan, Perverse schobers on Riemann surfaces: constructions and examples. *Eur. J. Math.* **5** (2019), no. 3, 771–797.
- [49] W. Donovan and E. Segal, Window shifts, flop equivalences and Grassmannian twists. *Compos. Math.* **150** (2014), no. 6, 942–978.
- [50] W. Donovan and M. Wemyss, Stringy Kähler moduli, mutation and monodromy. 2019, arXiv:1907.10891.
- [51] B. Doyle, Homological Projective Duality for the Plücker embedding of the Grassmannian. 2021, arXiv:2110.10589.
- [52] D. Edidin and D. Rydh, Canonical reduction of stabilizers for Artin stacks with good moduli spaces. *Duke Math. J.* **170** (2021), no. 5, 827–880.

- [53] A. I. Efimov, Derived categories of Grassmannians over integers and modular representation theory. *Adv. Math.* **304** (2017), 179–226.
- [54] A. I. Efimov, Categorical smooth compactifications and generalized Hodge-to-de Rham degeneration. *Invent. Math.* **222** (2020), no. 2, 667–694.
- [55] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, Hypergeometric functions and toric varieties. *Funktsional. Anal. i Prilozhen.* **23** (1989), no. 2, 12–26.
- [56] T. G. Goodwillie, Cyclic homology, derivations, and the free loop space. *Topology* **24** (1985), no. 2, 187–215.
- [57] D. R. Gulotta, Properly ordered dimers, R -charges, and an efficient inverse algorithm. *J. High Energy Phys.* **10** (2008), 14.
- [58] D. Halpern-Leistner, The derived category of a GIT quotient. *J. Amer. Math. Soc.* **28** (2015), no. 3, 871–912.
- [59] D. Halpern-Leistner and S. Sam, Combinatorial constructions of derived equivalences. *J. Amer. Math. Soc.* **33** (2020), 735–773.
- [60] D. Halpern-Leistner and I. Shipman, Autoequivalences of derived categories via geometric invariant theory. *Adv. Math.* **303** (2016), 1264–1299.
- [61] W. Hara, On the Abuaf–Ueda flop via non-commutative crepant resolutions, SIGMA Symmetry Integrability Geom. *Methods Appl.* **17** (2021), Paper No. 044, 22 pp.
- [62] L. Hille and M. Perling, Tilting bundles on rational surfaces and quasi-hereditary algebras. *Ann. Inst. Fourier (Grenoble)* **64** (2014), no. 2, 625–644.
- [63] Y. Hirono and M. Wemyss, Stability conditions for 3-fold flops. 2019, arXiv:1907.09742.
- [64] C. Ingalls and T. Yasuda, Log centres of twisted noncommutative crepant resolutions are Kawamata log terminal: remarks on a paper of Stafford and Van den Bergh. Preprint, 2013.
- [65] C. Ingalls and T. Yasuda, Logarithmic centres of twisted noncommutative crepant resolutions are Kawamata log terminal, 2013, <https://people.math.carleton.ca/~cingalls/research/presentations/cmsWinter2013.pdf>.
- [66] A. Ishii and K. Ueda, Dimer models and the special McKay correspondence. *Geom. Topol.* **19** (2015), no. 6, 3405–3466.
- [67] O. Iyama and I. Reiten, Fomin–Zelevinsky mutation and tilting modules over Calabi–Yau algebras. *Amer. J. Math.* **130** (2008), no. 4, 1087–1149.
- [68] O. Iyama and M. Wemyss, On the noncommutative Bondal–Orlov conjecture. *J. Reine Angew. Math.* **683** (2013), 119–128.
- [69] O. Iyama and M. Wemyss, Maximal modifications and Auslander–Reiten duality for non-isolated singularities. *Invent. Math.* **197** (2014), no. 3, 521–586.
- [70] O. Iyama and M. Wemyss, Singular derived categories of \mathbb{Q} -factorial terminalizations and maximal modification algebras. *Adv. Math.* **261** (2014), 85–121.
- [71] O. Iyama and M. Wemyss, Tits cones intersections and applications, 2021, https://www.maths.gla.ac.uk/~mwemyss/MainFile_for_web.pdf.

- [72] D. Kaledin, Derived equivalences by quantization. *Geom. Funct. Anal.* **17** (2008), no. 6, 1968–2004.
- [73] M. M. Kapranov, On the derived categories of coherent sheaves on some homogeneous spaces. *Invent. Math.* **92** (1988), no. 3, 479–508.
- [74] M. Kapranov and V. Schechtman. Perverse sheaves 2015, arXiv:1411.2772.
- [75] L. Katzarkov, M. Kontsevich, and T. Pantev, Hodge theoretic aspects of mirror symmetry. In *From Hodge theory to integrability and TQFT tt*-geometry*, pp. 87–174, Proc. Sympos. Pure Math. 78, Amer. Math. Soc., Providence, RI, 2008.
- [76] Y. Kawamata, D -equivalence and K -equivalence. *J. Differential Geom.* **61** (2002), no. 1, 147–171.
- [77] Y. Kawamata, In *Derived equivalence for stratified Mukai flop on $G(2, 4)$, Mirror symmetry. V*, pp. 285–294, AMS/IP Stud. Adv. Math. 38, Amer. Math. Soc., Providence, RI, 2006.
- [78] Y. Kawamata, Flops connect minimal models. *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 419–423.
- [79] Y. Kawamata, Derived categories of toric varieties III. *Eur. J. Math.* **2** (2016), no. 1, 196–207.
- [80] B. Keller, On the cyclic homology of exact categories. *J. Pure Appl. Algebra* **136** (1999), no. 1, 1–56.
- [81] B. Keller, Quiver mutation in JavaScript and Java, <https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/>.
- [82] B. Keller and H. Krause, Tilting preserves finite global dimension. *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 5, 563–570.
- [83] B. Keller and D. Yang, Derived equivalences from mutations of quivers with potential. *Adv. Math.* **226** (2011), no. 3, 2118–2168.
- [84] Y.-H. Kiem, The stringy E -function of the moduli space of rank 2 bundles over a Riemann surface of genus 3. *Trans. Amer. Math. Soc.* **355** (2003), no. 5, 1843–1856.
- [85] F. C. Kirwan, Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. *Ann. of Math. (2)* **122** (1985), no. 1, 41–85.
- [86] A. Kite, Discriminants and quasi-symmetry. 2017, arXiv:1711.08940.
- [87] F. Knop, Über die Glattheit von Quotientenabbildungen. *Manuscripta Math.* **56** (1986), no. 4, 419–427.
- [88] J. Kollár, Flops. *Nagoya Math. J.* **113** (1989), 15–36.
- [89] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*. Cambridge Tracts in Math. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original
- [90] M. Kontsevich, *Motivic integration*. Lecture at Orsay, December 1995.
- [91] A. Kuznetsov, Homological projective duality. *Publ. Math. Inst. Hautes Études Sci.* **105** (2007), 157–220.

- [92] A. Kuznetsov, Derived categories of quadric fibrations and intersections of quadrics. *Adv. Math.* **218** (2008), no. 5, 1340–1369.
- [93] A. Kuznetsov, Lefschetz decompositions and categorical resolutions of singularities. *Selecta Math. (N.S.)* **13** (2008), no. 4, 661–696.
- [94] A. Kuznetsov and V. A. Lunts, Categorical resolutions of irrational singularities. *Int. Math. Res. Not.* **13** (2015), 4536–4625.
- [95] E. León-Cardenal, J. Martín-Morales, W. Veys, and J. Viu-Sos, Motivic zeta functions on \mathbb{Q} -Gorenstein varieties. *Adv. Math.* **370** (2020), 107192.
- [96] G. J. Leuschke, Non-commutative crepant resolutions: scenes from categorical geometry. In *Progress in commutative algebra 1*, pp. 293–361, de Gruyter, Berlin, 2012.
- [97] L. Le Bruyn, Trace rings of generic 2 by 2 matrices. *Mem. Amer. Math. Soc.* **66** (1987), no. 363, vi+100 pp.
- [98] L. Le Bruyn, Quiver concomitants are often reflexive Azumaya. *Proc. Amer. Math. Soc.* **105** (1989), no. 1, 10–16.
- [99] L. Le Bruyn and M. Van den Bergh, Regularity of trace rings of generic matrices. *J. Algebra* **117** (1988), no. 1, 19–29.
- [100] H.-W. Lin, On crepant resolution of some hypersurface singularities and a criterion for UFD. *Trans. Amer. Math. Soc.* **354** (2002), no. 5, 1861–1868.
- [101] N. A. Loehr, In *Bijjective combinatorics*, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2011.
- [102] V. A. Lunts, Categorical resolution of singularities. *J. Algebra* **323** (2010), no. 10, 2977–3003.
- [103] Y. Namikawa, Mukai flops and derived categories. II. In *Algebraic structures and moduli spaces*, pp. 149–175, CRM Proc. Lecture Notes 38, Amer. Math. Soc., Providence, RI, 2004.
- [104] Y. Namikawa, Induced nilpotent orbits and birational geometry. *Adv. Math.* **222** (2009), no. 2, 547–564.
- [105] A. Neeman, The connection between the K -theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. *Ann. Sci. Éc. Norm. Supér. (4)* **25** (1992), no. 5, 547–566.
- [106] A. Polishchuk and M. Van den Bergh, Semiorthogonal decompositions of the categories of equivariant coherent sheaves for some reflection groups. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 9, 2653–2749.
- [107] C. Procesi, The invariant theory of $n \times n$ matrices. *Adv. Math.* **19** (1976), no. 3, 306–381.
- [108] C. Procesi, A formal inverse to the Cayley–Hamilton theorem. *J. Algebra* **107** (1987), no. 1, 63–74.
- [109] Ju. P. Razmyslov, Identities with trace in full matrix algebras over a field of characteristic zero. *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 723–756.
- [110] Z. Reichstein, Stability and equivariant maps. *Invent. Math.* **96** (1989), no. 2, 349–383.

- [111] M. Reid, Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, pp. 131–180, Adv. Stud. Pure Math. 1, North-Holland, Amsterdam, 1983.
- [112] J. V. Rennemo, *Homological projective duality for symmetric rank loci*, lecture at the conference “Workshop on Derived Categories, Moduli Spaces, and Deformation Theory”, June 2019.
- [113] J. V. Rennemo and E. Segal, Hori-mological projective duality. *Duke Math. J.* **168** (2019), no. 11, 2127–2205.
- [114] J. Rickard, Morita theory for derived categories. *J. Lond. Math. Soc. (2)* **39** (1989), 436–456.
- [115] Š. Špenko, HMS symmetries and hypergeometric systems. In *Proceeding of the 8th European Congress of Mathematics, 2021*, to appear.
- [116] Š. Špenko and M. Van den Bergh, Non-commutative resolutions of quotient singularities for reductive groups. *Invent. Math.* **210** (2017), no. 1, 3–67.
- [117] Š. Špenko and M. Van den Bergh, A class of perverse schobers in geometric invariant theory. 2019, arXiv:1908.04213.
- [118] Š. Špenko and M. Van den Bergh, Comparing the Kirwan and noncommutative resolutions of quotient varieties. 2019, arXiv:1912.01689.
- [119] Š. Špenko and M. Van den Bergh, Non-commutative crepant resolutions for some toric singularities I. *Int. Math. Res. Not. IMRN* **21** (2020), 8120–8138.
- [120] Š. Špenko and M. Van den Bergh, Non-commutative crepant resolutions for some toric singularities. II. *J. Noncommut. Geom.* **14** (2020), no. 1, 73–103.
- [121] Š. Špenko and M. Van den Bergh, Perverse schobers and GKZ systems, arXiv:2007.04924. To appear in *Adv. Math.*, 2020.
- [122] J. T. Stafford and M. Van den Bergh, Noncommutative resolutions and rational singularities. *Michigan Math. J.* **57** (2008), 659–674.
- [123] M. Van den Bergh, Modules of covariants. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pp. 352–362, Birkhäuser, 1995.
- [124] M. Van den Bergh, Cohen–Macaulayness of modules of covariants. *Invent. Math.* **106** (1991), 389–409.
- [125] M. Van den Bergh, Cohen–Macaulayness of semi-invariants for tori. *Trans. Amer. Math. Soc.* **336** (1993), no. 2, 557–580.
- [126] M. Van den Bergh, A converse to Stanley’s conjecture for Sl_2 . *Proc. Amer. Math. Soc.* **121** (1994), no. 1, 47–51.
- [127] M. Van den Bergh, Local cohomology of modules of covariants. *Adv. Math.* **144** (1999), no. 2, 161–220.
- [128] M. Van den Bergh, Non-commutative crepant resolutions. In *The legacy of Niels Henrik Abel*, pp. 749–770, Springer, Berlin, 2004.
- [129] M. Van den Bergh, Three-dimensional flops and noncommutative rings. *Duke Math. J.* **122** (2004), no. 3, 423–455.

- [130] M. Van den Bergh, Calabi—Yau algebras and superpotentials. *Selecta Math. (N.S.)* **21** (2015), no. 2, 555–603.
- [131] M. Wemyss, Noncommutative resolutions. In *Noncommutative algebraic geometry*, pp. 239–306, Math. Sci. Res. Inst. Publ. 64, Cambridge Univ. Press, New York, 2016.
- [132] M. Wemyss, Flops and clusters in the homological minimal model programme. *Invent. Math.* **211** (2018), no. 2, 435–521.
- [133] M. Wemyss, A lockdown survey on cDV singularities. 2021, arXiv:[2103.16990](https://arxiv.org/abs/2103.16990).
- [134] H. Weyl, *The classical groups*. Princeton University Press, 1946.
- [135] J. Weyman, *Cohomology of vector bundles and syzygies* 149, Cambridge University Press, 2003.
- [136] J. Weyman and G. Zhao, Noncommutative desingularization of orbit closures for some representations of GL_n . 2012, arXiv:[1204.0488](https://arxiv.org/abs/1204.0488).

MICHEL VAN DEN BERGH

Free University of Brussels, Pleinlaan 2, 1050 Brussel, Belgium, and
 Hasselt University, Martelarenlaan 42, 3500 Hasselt, Belgium, and
 Research Foundation – Flanders, Egmontstraat 5, 1000 Brussel, Belgium,
michel.vandenbergh@uhasselt.be, michel.van.den.bergh@vub.be