RAMSEY THEORY OF HOMOGENEOUS STRUCTURES: CURRENT TRENDS AND OPEN PROBLEMS

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This paper is dedicated to Norbert Sauer for his seminal works on the partition theory of homogeneous structures, and for his mathematical and personal generosity.

ABSTRACT

This article highlights historical achievements in the partition theory of countable homogeneous relational structures, and presents recent work, current trends, and open problems. Exciting recent developments include new methods involving logic, topological Ramsey spaces, and category theory. The paper concentrates on big Ramsey degrees, presenting their essential structure where known and outlining areas for further development. Cognate areas, including infinite dimensional Ramsey theory of homogeneous structures and partition theory of uncountable structures, are also discussed.

1. INTRODUCTION

Ramsey theory is a beautiful subject which interrelates with a multitude of mathematical fields. In particular, since its inception, developments in Ramsey theory have often been motivated by problems in logic; in turn, Ramsey theory has instigated some seminal developments in logic. The intent of this article is to provide the general mathematician with an introduction to the intriguing subject of Ramsey theory on homogeneous structures while being detailed enough to describe the state-of-the-art and the main ideas at play. We present historical highlights and discuss why solutions to problems on homogeneous structures require more than just straightforward applications of finite structural Ramsey theory. In the following sections, we map out collections of recent results and methods which were developed to overcome obstacles associated with forbidden substructures. These new methods involve applications from logic (especially forcing but also ideas from model theory), topological Ramsey spaces, and category theory.

The subject of Ramsey theory on infinite structures begins with this lovely theorem.

Theorem 1.1 (Ramsey, [58]). *Given positive integers k and r and a coloring of the k-element subsets of the natural numbers $\mathbb{N}$ into r colors, there is an infinite set of natural numbers $N \subseteq \mathbb{N}$ such that all k-element subsets of N have the same color.*

There are two natural interpretations of Ramsey's theorem in terms of infinite structures. First, letting $<$ denote the standard linear order on $\mathbb{N}$, Ramsey's theorem shows that given any finite coloring of all linearly ordered substructures of $(\mathbb{N}, <)$ of size k, there is an isomorphic substructure $(N, <)$ of $(\mathbb{N}, <)$ such that all linearly ordered substructures of $(N, <)$ of size k have the same color. Second, one may think of the k-element subsets of $\mathbb{N}$ as k-hyperedges. Then Ramsey's theorem yields that, given any finite coloring of the k-hyperedges of the complete k-regular hypergraph on infinitely many vertices, there is an isomorphic subgraph in which all k-hyperedges have the same color.

Given this, one might naturally wonder about other structures.

Question 1.2. Which infinite structures carry an analogue of Ramsey's theorem?

The rational numbers $(\mathbb{Q}, <)$ as a dense linearly ordered structure (without endpoints) was the earliest test case. It is a fun exercise to show that given any coloring of the rational numbers into finitely many colors, there is one color-class which contains a dense linear order, that is, an isomorphic subcopy of the rationals in one color. Thus, the rationals satisfy a structural pigeonhole principle known as *indivisibility*.

The direct analogy with Ramsey's theorem ends, however, when we consider pairs of rationals. It follows from the work of Sierpiński in [65] that there is a coloring of the pairs of rationals into two colors so that both colors persist in every isomorphic subcopy of the rationals. Sierpiński's coloring provides a clear understanding of one of the fundamental issues arising in partition theory of infinite structures not occurring in finite structural Ramsey theory. Let $\{q_i : i \in \mathbb{N}\}$ be a listing of the rational numbers, without repetition, and for $i < j$ define $c(\{q_i, q_j\}) = $ blue if $q_i < q_j$, and $c(\{q_i, q_j\}) = $ red if $q_j < q_i$. Then in

each subset $Q \subseteq \mathbb{Q}$ forming a dense linear order, both color classes persist; that is, there are pairs of rationals in Q colored red and also pairs of rationals in Q colored blue. Since it is impossible to find an isomorphic subcopy of the rationals in which all pairsets have the same color, a direct analogue of Ramsey's theorem does not hold for the rationals.

The failure of the straightforward analogue of Ramsey's theorem is not the end, but rather just the beginning of the story. Galvin (unpublished) showed a few decades later that there is a bound on the number of unavoidable colors: Given any coloring of the pairs of rationals into finitely many colors, there is a subcopy of the rationals in which all pairs belong to the union of two color classes. Now one sees that Question 1.2 ought to be refined.

Question 1.3. For which infinite structures **S** is there a Ramsey-analogue in the following sense: Let **A** be a finite substructure of **S**. Is there a positive integer T such that for any coloring of the copies of **A** into finitely many colors, there is a subcopy **S'** of **S** in which there are no more than T many colors for the copies of **A**?

The least such integer T, when it exists, is denoted $T(\mathbf{A})$ and called the *big Ramsey degree* of **A** in **S**, a term coined in Kechris–Pestov–Todorcevic (2005). The "big" refers to the fact that we require an isomorphic subcopy of an *infinite* structure in which the number of colors is as small as possible (in contrast to the concept of *small Ramsey degree* in finite structural Ramsey theory).

Notice how Sierpiński played the enumeration $\{q_i : i \in \mathbb{N}\}$ of the rationals against the dense linear order to construct a coloring of pairsets of rationals into two colors, each of which persists in every subcopy of the rationals. This simple, but deep idea sheds light on a fundamental difference between finite and infinite structural Ramsey theory. The interplay between the enumeration and the relations on an infinite structure has bearing on the number of colors that must persist in any subcopy of that structure. We will see examples of this at work throughout this article and explain the general principles which have been found for certain classes of structures with relations of arity at most two, even as the subject aims towards a future overarching theory of big Ramsey degrees.

2. THE QUESTIONS

Given a finite relational language $\mathcal{L} = \{R_i : i < k\}$ with each relation symbol R_i of some finite arity, say, n_i, an $\mathcal{L}$-*structure* is a tuple $\mathbf{A} = \langle A, R_0^{\mathbf{A}}, \ldots, R_{k-1}^{\mathbf{A}} \rangle$, where $A \neq \emptyset$ is the *universe* of **A** and for each $i < k$, $R_i^{\mathbf{A}} \subseteq A^{n_i}$. For $\mathcal{L}$-structures **A** and **B**, an *embedding* from **A** into **B** is an injection $e : A \to B$ such that for all $i < k$, $R_i^{\mathbf{A}}(a_1, \ldots, a_{n_i}) \leftrightarrow R_i^{\mathbf{B}}(e(a_1), \ldots, e(a_{n_i}))$. The e-image of **A** is a *copy* of **A** in **B**. If e is the identity map, then **A** is a *substructure* of **B**. An *isomorphism* is an embedding which is onto its image. We write $\mathbf{A} \leq \mathbf{B}$ exactly when there is an embedding of **A** into **B**, and $\mathbf{A} \cong \mathbf{B}$ exactly when **A** and **B** are isomorphic.

A class $\mathcal{K}$ of finite structures for a relational language $\mathcal{L}$ is called a *Fraïssé class* if it is hereditary, satisfies the joint embedding and amalgamation properties, contains (up to isomorphism) only countably many structures, and contains structures of arbitrarily large

finite cardinality. Class $\mathcal{K}$ is *hereditary* if whenever $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \leq \mathbf{B}$, then also $\mathbf{A} \in \mathcal{K}$; $\mathcal{K}$ satisfies the *joint embedding property* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \leq \mathbf{C}$; $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and there are embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$. A Fraïssé class $\mathcal{K}$ satisfies the *strong amalgamation property* (SAP) if given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $e : \mathbf{A} \to \mathbf{B}$ and $f : \mathbf{A} \to \mathbf{C}$, there is some $\mathbf{D} \in \mathcal{K}$ and embeddings $e' : \mathbf{B} \to \mathbf{D}$ and $f' : \mathbf{C} \to \mathbf{D}$ such that $e' \circ e = f' \circ f$, and $e'[B] \cap f'[C] = e' \circ e[A] = f' \circ f[A]$. We say that $\mathcal{K}$ satisfies the *free amalgamation property* (FAP) if it satisfies the SAP and, moreover, $\mathbf{D}$ can be chosen so that $\mathbf{D}$ has no additional relations other than those inherited from $\mathbf{B}$ and $\mathbf{C}$.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be $\mathcal{L}$-structures such that $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$. We use $\binom{\mathbf{B}}{\mathbf{A}}$ to denote the set of all copies of $\mathbf{A}$ in $\mathbf{B}$. The Erdős–Rado arrow notation $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$ means that for each coloring of $\binom{\mathbf{C}}{\mathbf{A}}$ into k colors, there is a $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}}$ is *monochromatic*, meaning every member of $\binom{\mathbf{B}'}{\mathbf{A}}$ has the same color.

Definition 2.1. A Fraïssé class $\mathcal{K}$ has the *Ramsey property* if for any two structures $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$ and any $k \geq 2$, there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{B} \leq \mathbf{C}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$.

Many Fraïssé classes, such as the class of finite graphs, do not have the Ramsey property. However, by allowing a finite expansion of the language, often by just a linear order, the Ramsey property becomes more feasible. Letting $<$ be a binary relation symbol not in the language $\mathcal{L}$ of $\mathcal{K}$, an $\mathcal{L} \cup \{<\}$-structure is in $\mathcal{K}^<$ if and only if its universe is linearly ordered by $<$ and its $\mathcal{L}$-reduct is a member of $\mathcal{K}$. A highlight is the work of Nešetřil and Rödl in [51] and [52], proving that for any Fraïssé class $\mathcal{K}$ with FAP, its ordered version $\mathcal{K}^<$ has the Ramsey property. The recent paper [40] by Hubička and Nešetřil presents the state-of-the-art in finite structural Ramsey theory. Examples of Fraïssé classes with the Ramsey property include the class of finite linear orders, and the classes of finite ordered versions of graphs, digraphs, tournaments, triangle-free graphs, posets, metric spaces, hypergraphs, hypergraphs omitting some irreducible substructures, and many more.

A structure $\mathbf{K}$ is called *universal* for a class of structures $\mathcal{K}$ if each member of $\mathcal{K}$ embeds into $\mathbf{K}$. A structure $\mathbf{K}$ is *homogeneous* if each isomorphism between finite substructures of $\mathbf{K}$ extends to an automorphism of $\mathbf{K}$. Unless otherwise specified, we will write *homogeneous* to mean *countably infinite homogeneous*, such structures being the focus of this paper. The *age* of an infinite structure $\mathbf{K}$, denoted Age($\mathbf{K}$), is the collection of all finite structures which embed into $\mathbf{K}$. A fundamental theorem of Fraïssé from [31] shows that each Fraïssé class gives rise to a homogeneous structure via a construction called the *Fraïssé limit*. Conversely, given any countable homogeneous structure $\mathbf{K}$, Age($\mathbf{K}$) is a Fraïssé class and, moreover, the Fraïssé limit of Age($\mathbf{K}$) is isomorphic to $\mathbf{K}$. The Kechris–Pestov–Todorcevic correspondence between the Ramsey property of a Fraïssé class and extreme amenability of the automorphism group of its Fraïssé limit in [41] propelled a burst of discoveries of more Fraïssé classes with the Ramsey property.

First we state an esoteric but driving question in the area.

Question 2.2. What *is* a big Ramsey degree?

What is the essential nature of a big Ramsey degree? Why is it that given a Fraïssé class $\mathcal{K}$ satisfying the Ramsey property, its Fraïssé limit usually fails to carry the full analogue of Ramsey's Theorem 1.1 (i.e., all big Ramsey degrees being one)? A theorem of Hjorth in [37] showed that for any homogeneous structure **K** with $|\text{Aut}(\mathbf{K})| > 1$, there is a structure in Age(**K**) with big Ramsey degree at least two. While much remains open, we now have an answer to Question 2.2 for FAP and some SAP homogeneous structures with finitely many relations of arity at most two, and these results will be discussed in the following sections.

We say that **S** has *finite big Ramsey degrees* if $T(\mathbf{A})$ exists for each finite substructure **A** of **S**. We say that *exact big Ramsey degrees* are known if there is either a computation of the degrees or a characterization from which they can be computed. *Indivisibility* holds if $T(\mathbf{A}) = 1$ for each one-element substructure **A** of **S**. The following questions progress in order of strength: A positive answer to (3) implies a positive answer to (2), which in turn implies a positive answer to (1).

Question 2.3. Given a homogeneous structure **K**,

(1) Does **K** have finite big Ramsey degrees? That is, can one find upper bounds ensuring that big Ramsey degrees exist?

(2) If **K** has finite big Ramsey degrees, is there a characterization of the exact big Ramsey degrees via canonical partitions? If yes, calculate or find an algorithm to calculate them.

(3) Does **K** carry a big Ramsey structure?

Part (2) of this question involves finding *canonical partitions*.

Definition 2.4 (Canonical Partition, [44]). Given a Fraïssé class $\mathcal{K}$ with Fraïssé limit **K**, and given $\mathbf{A} \in \mathcal{K}$, a partition $\{P_i : i < n\}$ of $\binom{\mathbf{K}}{\mathbf{A}}$ is *canonical* if the following hold: For each finite coloring of $\binom{\mathbf{K}}{\mathbf{A}}$, there is a subcopy $\mathbf{K}'$ of **K** such that for each $i < n$, all members of $P_i \cap \binom{\mathbf{K}'}{\mathbf{A}}$ have the same color; and *persistence*: For every subcopy $\mathbf{K}'$ of **K** and each $i < n$, $P_i \cap \binom{\mathbf{K}'}{\mathbf{A}}$ is nonempty.

Canonical partitions recover an exact analogue of Ramsey's theorem for each piece of the partition. In practice such partitions are characterized by adding extra structure to **K**, including the enumeration of the universe of **K** and a tree-like structure capturing the relations of **K** against the enumeration.

Part (3) of Question 2.3 has to do with a connection between big Ramsey degrees and topological dynamics, in the spirit of the Kechris–Pestov–Todorcevic correspondence, proved by Zucker in [70]. A *big Ramsey structure* is essentially a finite expansion $\mathbf{K}^*$ of **K** so that each finite substructure of $\mathbf{K}^*$ has big Ramsey degree one, and, moreover, the unavoidable colorings cohere in that for $\mathbf{A}, \mathbf{B} \in \text{Age}(\mathbf{K})$ with **A** embedding into **B**, the canonical partition for copies of **B** when restricted to copies of **A** recovers the canonical partition for

copies of **A**. Big Ramsey structures imply canonical partitions. The reverse is not known in general, but certain types of canonical partitions are known to imply big Ramsey structures (Theorem 6.10 in [8]), and it seems reasonable to the author to expect that (1)–(3) are equivalent.

Canonical partitions and big Ramsey structures are really getting at the question of whether we can find an optimal finite expansion $\mathbf{K}^*$ of a given homogeneous structure $\mathbf{K}$ so that $\mathbf{K}^*$ carries an exact analogue of Ramsey's theorem. In this sense, big Ramsey degrees are not quite so mysterious, but are rather saying that an exact analogue of Ramsey's theorem holds for an appropriately expanded structure. The question then becomes: What is the appropriate expansion?

3. CASE STUDY: THE RATIONALS

The big Ramsey degrees for the rationals were determined by 1979. Laver in 1969 (unpublished, see [10]) utilized a Ramsey theorem for trees due to Milliken [50] (Theorem 3.2) to find upper bounds. Devlin completed the picture in his PhD thesis [10], calculating the big Ramsey degrees of the rationals. These surprisingly turn out to be related to the odd coefficients in the Taylor series of the tangent function: The big Ramsey degree for n-element subsets of the rationals is $T(n) = (2n-1)!c_{2n-1}$, where c_k is the kth coefficient in the Taylor series for the tangent function, $\tan(x) = \sum_{k=0}^{\infty} c_k x^k$. As Todorcevic states, the big Ramsey degrees for the rationals "characterize the Ramsey theoretic properties of the countable dense linear ordering $(\mathbb{Q}, <)$ in a very precise sense. The numbers $T(n)$ are some sort of Ramsey degrees that measure the complexity of an arbitrary finite coloring of the n-element subsets of $\mathbb{Q}$ modulo, of course, restricting to the n-element subsets of X for some appropriately chosen dense linear subordering X of $\mathbb{Q}$." (page 143, [66], notation modified)

We present Devlin's characterization of the big Ramsey degrees of the rationals and the four main steps in his proof. (A detailed proof appears in Section 6.3 of [66].) Then we will present a method from [8] using coding trees of 1-types which bypasses nonessential constructs, providing what we see as a satisfactory answer to Question 2.2 for the rationals.

We use some standard mathematical logic notation, providing definitions as needed for the general mathematician. The set of all natural numbers $\{0, 1, 2, \dots\}$ is denoted by ω. Each natural number $k \in \omega$ is equated with the set $\{0, \dots, k-1\}$ and its natural linear ordering. For us $k \in \omega$ and $k < \omega$ are synonymous. For $k \in \omega$, $k^{<\omega}$ denotes the tree of all finite sequences with entries in $\{0, \dots, k-1\}$, and $\omega^{<\omega}$ denotes the tree of all finite sequences of natural numbers. Finite sequences with any sort of entries are thought of as functions with domain some natural number. Thus, for a finite sequence t the *length* of t, denoted $|t|$, is the domain of the function t, and for $i \in \text{dom}(t)$, $t(i)$ denotes the ith entry of the sequence t. For $\ell \in \omega$, we write $t \upharpoonright \ell$ to denote the initial segment of t of length ℓ if $\ell \leq |t|$, and t otherwise. For two finite sequences s and t, we write $s \sqsubseteq t$ when s is an initial segment of t, and we write $s \sqsubset t$ when s is a proper initial segment of t, meaning that $s \sqsubseteq t$ and $s \neq t$. We write $s \wedge t$ to denote the *meet* of s and t, that is, the longest sequence which is an initial segment of both s and t. Given a subset S of a tree of finite sequences, the *meet*

closure of S, denoted cl(S), is the set of all nodes in S along with the set of all meets $s \wedge t$, for $s, t \in S$.

A Ramsey theorem for trees, due to Milliken, played a central role in Devlin's work and has informed subsequent approaches to finding upper bounds for big Ramsey degrees. In this area, a subset $T \subseteq \omega^{<\omega}$ is called a *tree* if there is a subset $L_T \subseteq \omega$ such that $T = \{t \upharpoonright \ell : t \in T, \ell \in L_T\}$. Thus, a tree is closed under initial segments of lengths in L_T, but not necessarily closed under all initial segments in $\omega^{<\omega}$. The *height* of a node t in T, denoted $\mathrm{ht}_T(t)$, is the order-type of the set $\{s \in T : s \sqsubset t\}$, linearly ordered by $\sqsubset$. We write $T(n)$ to denote $\{t \in T : \mathrm{ht}_T(t) = n\}$. For $t \in T$, let $\mathrm{Succ}_T(t) = \{s \upharpoonright (|t| + 1) : s \in T \text{ and } t \sqsubset s\}$, noting that $\mathrm{Succ}_T(t) \subseteq T$ only if $|t| + 1 \in L_T$.

A subtree $S \subseteq T$ is a *strong subtree* of T if $L_S \subseteq L_T$ and each node s in S branches as widely as T will allow, meaning that for $s \in S$, for each $t \in \mathrm{Succ}_T(s)$ there is an extension $s' \in S$ such that $t \sqsubseteq s'$. For the next theorem, define $\prod_{i<d} T_i(n)$ to be the set of sequences $(t_0, \ldots, t_{d-1})$ where $t_i \in T_i(n)$, the product of the nth levels of the trees T_i. Then let

$$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n). \tag{3.1}$$

The following is the strong tree version of the Halpern–Läuchli theorem.

Theorem 3.1 (Halpern–Läuchli, [34]). *Let d be a positive integer, $T_i \subseteq \omega^{<\omega}$ ($i < d$) be finitely branching trees with no terminal nodes, and $r \geq 2$. Given a coloring $c : \bigotimes_{i<d} T_i \to r$, there is an increasing sequence $\langle m_n : n < \omega \rangle$ and strong subtrees $S_i \leq T_i$ such that for all $i < d$ and $n < \omega$, $S_i(n) \subseteq T_i(m_n)$, and c is constant on $\bigotimes_{i<d} S_i$.*

The Halpern–Läuchli theorem has a particularly strong connection with logic. It was isolated by Halpern and Lévy as a key juncture in their work to prove that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over the Zermelo–Fraenkel Axioms of set theory. Once proved by Halpern and Läuchli, Halpern and Lévy completed their proof in [35].

Harrington (unpublished) devised an innovative proof of the Halpern–Läuchli theorem which used Cohen forcing. The forcing helps find good nodes in the trees T_i from which to start building the subtrees S_i. From then on, the forcing is used ω many times, each time running an unbounded search for finite sets $S_i(n)$ which satisfy that level of the Halpern–Läuchli theorem. Being finite, each $S_i(n)$ is in the ground model. The proof entails neither passing to a generic extension nor any use of Shoenfield's Absoluteness Theorem.

A *k-strong subtree* is a strong subtree with k many levels. The following theorem is proved inductively using Theorem 3.1.

Theorem 3.2 (Milliken, [50]). *Let $T \subseteq \omega^{<\omega}$ be a finitely branching tree with no terminal nodes, $k \geq 1$, and $r \geq 2$. Given a coloring of all k-strong subtrees of T into r colors, there is an infinite strong subtree $S \subseteq T$ such that all k-strong subtrees of S have the same color.*

For more on the Halpern–Läuchli and Milliken theorems, see [21, 46, 66]. Now we look at Devlin's proof of the exact big Ramsey degrees of the rationals, as it has bearing on many current approaches to big Ramsey degrees.

The rationals can be represented by the tree $2^{<\omega}$ of binary sequences with the lexicographic order $\lhd$ defined as follows: Given $s, t \in 2^{<\omega}$ with $s \neq t$, and letting u denote $s \wedge t$, define $s \lhd t$ to hold if and only if $(|u| < |s|$ and $s(|u|) = 0)$ or $(|u| < |t|$ and $t(|u|) = 1)$. Then $(2^{<\omega}, \lhd)$ is a dense linear order. The following is Definition 6.11 in [66], using the terminology of [62]. For $|s| < |t|$, the number $t(|s|)$ is called the *passing number* of t at s.

Definition 3.3. For $A, B \subseteq \omega^{<\omega}$, we say that A and B are *similar* if there is a bijection $f : \mathrm{cl}(A) \to \mathrm{cl}(B)$ such that for all $s, t \in \mathrm{cl}(A)$,

    (a) (preserves end-extension) $s \sqsubseteq t \Leftrightarrow f(s) \sqsubseteq f(t)$,

    (b) (preserves relative lengths) $|s| < |t| \Leftrightarrow |f(s)| < |f(t)|$,

    (c) $s \in A \Leftrightarrow f(s) \in B$,

    (d) (preserves passing numbers) $t(|s|) = f(t)(|f(s)|)$ whenever $|s| < |t|$.

Similarity is an equivalence relation; a similarity equivalence class is called a *similarity type*. We now outline the four main steps to Devlin's characterization of big Ramsey degrees in the rationals. Fix $n \geq 1$.

I. (Envelopes) Given a subset $A \subseteq 2^{<\omega}$ of size n, let k be the number of levels in $\mathrm{cl}(A)$. An *envelope* of A is a k-strong subtree $E(A)$ of $2^{<\omega}$ such that $A \subseteq E(A)$. Given any k-strong subtree S of $2^{<\omega}$, there is exactly one subset $B \subseteq S$ which is similar to A. This makes it possible to transfer a coloring of the similarity copies of A in $2^{<\omega}$ to the k-strong subtrees of $2^{<\omega}$ in a well-defined manner.

II. (Finite Big Ramsey Degrees) Apply Milliken's theorem to obtain an infinite strong subtree $T \subseteq 2^{<\omega}$ such that every similarity copy of A in T has the same color. As there are only finitely many similarity types of sets of size n, finitely many applications of Milliken's theorem results in an infinite strong subtree $S \subseteq 2^{<\omega}$ such that the coloring is monochromatic on each similarity type of size n. This achieves finite big Ramsey degrees.

III. (Diagonal Antichain for Better Upper Bounds) To obtain the exact big Ramsey degrees, Devlin constructed a particular antichain of nodes $D \subseteq 2^{<\omega}$ such that $(D, \lhd)$ is a dense linear order and no two nodes in the meet closure of D have the same length, a property called *diagonal*. He also required ($*$): All passing numbers at the level of a terminal node or a meet node in $\mathrm{cl}(D)$ are 0, except of course the rightmost extension of the meet node. Diagonal antichains turn out to be essential to characterizing big Ramsey degrees, whereas the additional requirement ($*$) is now seen to be nonessential when viewed through the lens of coding trees of 1-types.

IV. (Exact Big Ramsey Degrees) To characterize the big Ramsey degrees, Devlin proved that the similarity type of each subset of D of size n persists in every subset $D' \subseteq D$ such that $(D', \lhd)$ is a dense linear order. The similarity types of antichains in D thus form a canonical partition for linear orders of size n. By calculating the number of different similarity types of subsets of D of size n, Devlin found the big Ramsey degrees for the rationals.
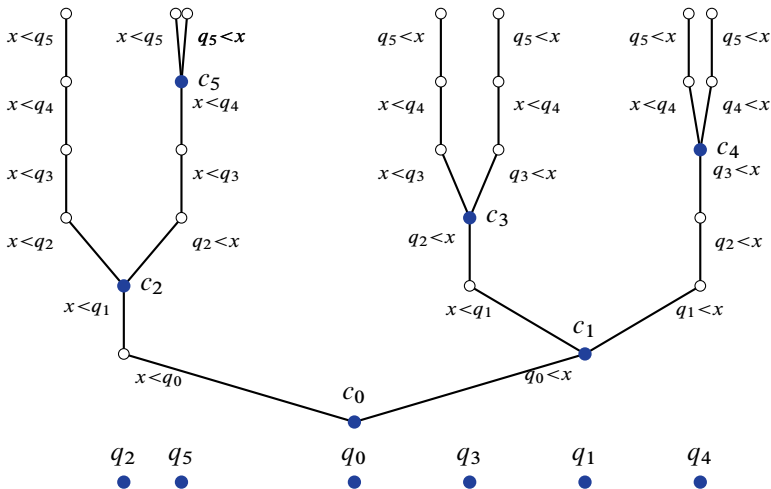
FIGURE 1

Coding tree $\mathbb{S}(\mathbb{Q})$ of 1-types for $(\mathbb{Q}, <)$ and the linear order represented by its coding nodes.

Now we present the characterization of the big Ramsey degrees for the rationals using coding trees of 1-types. Coding trees on $2^{<\omega}$ were first developed in [13] to solve the problem of whether or not the triangle-free homogeneous graph has finite big Ramsey degrees. The presentation given here is from [8], where the notion of coding trees was honed using model-theoretic ideas. We hope that presenting this view here will set the stage for a concrete understanding of big Ramsey degree characterizations discussed in Section 5.

Fix an enumeration $\{q_0, q_1, \dots\}$ of $\mathbb{Q}$. For $n < \omega$, we let $\mathbb{Q} \upharpoonright n$ denote the substructure $(\{q_i : i \in n\}, <)$ of $(\mathbb{Q}, <)$, which we refer to as an *initial substructure*. One can think of $\mathbb{Q} \upharpoonright n$ as a finite approximation in a construction of the rationals. The definition of a coding tree of 1-types in [8] uses complete realizable quantifier-free 1-types over initial substructures. Here, we shall retain the terminology of [8] but (with apologies to model-theorists) will use sets of literals instead, since this will convey the important aspects of the constructions while being more accessible to a general readership. For now, we call a set of formulas $s \subseteq \{(q_i < x) : i \in n\} \cup \{(x < q_i) : i \in n\}$ a 1-*type* over $\mathbb{Q} \upharpoonright n$ if (a) for each $i < n$ exactly one of the formulas $(q_i < x)$ or $(x < q_i)$ is in s, and (b) there is some (and hence infinitely many) $j \geq n$ such that q_j *satisfies* s, meaning that replacing the variable x by the rational number q_j in each formula in s results in a true statement. In other words, s is a 1-type if s prescribes a legitimate way to extend $\mathbb{Q} \upharpoonright n$ to a linear order of size $n + 1$.

Definition 3.4 (Coding Tree of 1-Types for $\mathbb{Q}$, [8]). For a fixed enumeration $\{q_0, q_1, \dots\}$ of the rationals, the *coding tree of* 1-*types* $\mathbb{S}(\mathbb{Q})$ is the set of all 1-types over initial substructures along with a function $c : \omega \to \mathbb{S}(\mathbb{Q})$ such that $c(n)$ is the 1-type of q_n over $\mathbb{Q} \upharpoonright n$. The tree-ordering is simply inclusion.

Given $s \in \mathbb{S}(\mathbb{Q})$ let $|s| = j + 1$ where j is maximal such that one of $(x < q_j)$ or $(q_j < x)$ is in s. For each $i < |s|$, we let $s(i)$ denote the formula from among $(x < q_i)$ or $(q_i < x)$ which is in s. The *coding nodes* $c(n)$, in practice usually denoted by c_n, are special distinguished nodes representing the rational numbers; c_n represents the rational q_n, because c_n is the 1-type with parameters from among $\{q_i : i \in n\}$ that q_n satisfies. Notice that this tree $\mathbb{S}(\mathbb{Q})$ has at most one splitting node per level. The effect is that any antichain of coding nodes in $\mathbb{S}(\mathbb{Q})$ will automatically be diagonal. (See Figure 1, reproduced from [8].)

Fix an ordering $<_{\text{lex}}$ on the literals: For $i < j$, define $(x < q_i) <_{\text{lex}} (q_i < x) <_{\text{lex}} (x < q_j)$. Extend $<_{\text{lex}}$ to $\mathbb{S}(\mathbb{Q})$ by declaring for $s, t \in \mathbb{S}(\mathbb{Q})$, $s <_{\text{lex}} t$ if and only if s and t are incomparable and for $i = |s \wedge t|$, $s(i) <_{\text{lex}} t(i)$.

Definition 3.5. For A, B sets of coding nodes in $\mathbb{S}(\mathbb{Q})$, we say that A and B are *similar* if there is a bijection $f : \text{cl}(A) \to \text{cl}(B)$ such that for all $s, t \in \text{cl}(A)$, f satisfies (a)–(c) of Definition 3.3 and (d') $s <_{\text{lex}} t \iff f(s) <_{\text{lex}} f(t)$,

When B is similar to A, we call B a *similarity copy* of A. Condition (d) in Definition 3.3 implies that the lexicographic order on $2^{<\omega}$ is preserved, and, moreover, that passing numbers at meet nodes and at terminal nodes are preserved. In (d') we only need to preserve lexicographic order.

Extending Harrington's method, forcing is utilized to obtain a pigeonhole principle for coding trees of 1-types in the vein of the Halpern–Läuchli Theorem 3.1, but for colorings of finite sets of coding nodes, rather than antichains. Via an inductive argument using this pigeonhole principle, we obtain the following Ramsey theorem on coding trees.

Theorem 3.6 ([8]). *Let $\mathbb{S}(\mathbb{Q})$ be a coding tree of 1-types for the rationals. Given a finite set A of coding nodes in $\mathbb{S}(\mathbb{Q})$ and a finite coloring of all similarity copies of A in $\mathbb{S}(\mathbb{Q})$, there is a coding subtree S of $\mathbb{S}(\mathbb{Q})$ similar to $\mathbb{S}(\mathbb{Q})$ such that all similarity copies of A in S have the same color.*

Fix $n \geq 1$. By applying Theorem 3.6 once for each similarity type of coding nodes of size n, we prove finite big Ramsey degrees, accomplishing step II while bypassing step I in Devlin's proof. Upon taking *any* antichain D of coding nodes in $\mathbb{S}(\mathbb{Q})$ representing a dense linear order, we obtain better upper bounds which are then proved to be exact, accomplishing steps III and IV.

Big Ramsey degrees of the rationals. In [8], we show that given $n \geq 1$, the big Ramsey degree $T(n)$ for linear orders of size n in the rationals is the number of similarity types of antichains of coding nodes in $\mathbb{S}(\mathbb{Q})$.

What then *is* the big Ramsey degree $T(n)$ in the rationals? It is the number of different ways to order the indexes of an increasing sequence of rationals $\{q_{i_0} < q_{i_1} < \cdots < q_{i_{n-1}}\}$ with incomparable 1-types along with the number of ways to order the first differences of their 1-types over initial substructures of $\mathbb{Q}$. The first difference between the 1-types of the rationals q_i and q_j occurs at the least k such that $q_i < q_k$ and $q_k < q_j$, or vice versa. This means that q_i and q_j are in the same interval of $\mathbb{Q} \upharpoonright k$ but in different intervals of

$\mathbb{Q} \upharpoonright (k+1)$. Concretely, $T(n)$ is the number of $<$-isomorphism classes of $(2n-1)$-tuples of integers $(i_0, \ldots, i_{n-1}, k_0, \ldots, k_{n-2})$ with the following properties: $\{q_{i_0} < q_{i_1} < \cdots < q_{i_{n-1}}\}$ is a set of rationals in increasing order, and for each $j < n-1$, $q_{i_j} < q_{k_j} < q_{i_{j+1}}$ where $k_j < \min(i_j, i_{j+1})$ and is the least integer satisfying this relation.

4. HISTORICAL HIGHLIGHTS, RECENT RESULTS, AND METHODS

We now highlight some historical achievements, and present recent results and the main ideas of their methods. For an overview of results up to the year 2000, see the appendix by Sauer in Fraïssé's book [32]; for an overview up to the year 2013, see Nguyen Van Thé's habilitation thesis [54]. Those interested in open problems intended for undergraduate research may enjoy [18].

The Rado graph is the second example of a homogeneous structure with nontrivial big Ramsey degrees which has been fully understood in terms of its partition theory. The Rado graph **R** is up to isomorphism the homogeneous graph on countably many vertices which is universal for all countable graphs. It was known to Erdős and other Hungarian mathematicians in the 1960s, though possibly earlier, that the Rado graph is indivisible. In their 1975 paper [30], Erdős, Hajnal, and Pósa constructed a coloring of the edges in **R** into two colors such that both colors persist in each subcopy of **R**. Pouzet and Sauer later showed in [57] that the big Ramsey degree for edge colorings in the Rado graph is exactly two. The complete characterization of the big Ramsey degrees of the Rado graph was achieved in a pair of papers by Sauer [62] and by Laflamme, Sauer, and Vuksanovic [44], both appearing in 2006, and the degrees were calculated by Larson in [45]. The two papers [62] and [44] in fact characterized exact big Ramsey degrees for all unrestricted homogeneous structures with finitely many binary relations, including the homogeneous digraph, homogeneous tournament, and random graph with finitely many edges of different colors. Milliken's theorem was used to prove existence of upper bounds, alluding to a deep connection between big Ramsey degrees and Ramsey theorems for trees. These results are discussed in Section 5.1.

In [43], for each $n \geq 2$, Laflamme, Nguyen Van Thé, and Sauer calculated the big Ramsey degrees of $\mathbb{Q}_n$, the rationals with an equivalence relation with n many equivalence classes each of which is dense in $\mathbb{Q}$. This hinged on proving a "colored version" of Milliken's theorem, where the levels of the trees are colored, to achieve upper bounds. Applying their result for $\mathbb{Q}_2$, they calculated the big Ramsey degrees of the dense local order, denoted **S**(2). In his PhD thesis [38], Howe proved finite big Ramsey degrees for the generic bipartite graph and the Fraïssé limit of the class of finite linear orders with a convex equivalence relation.

A robust and streamlined approach applicable to a large class of homogeneous structures, and recovering the previously mentioned examples (except for **S**(2)), was developed by Coulson, Patel, and the author in [8], building on ideas in [13] and [12]. In [8], it was shown that homogeneous structures with relations of arity at most two satisfying a strengthening of SAP, called SDAP$^+$, have big Ramsey structures which are characterized in a simple manner, and therefore their big Ramsey degrees are easy to compute. The proof proceeds via a Ramsey theorem for colorings of finite antichains of coding nodes on *diagonal coding*

trees of 1-*types*. This approach bypasses any need for envelopes, the theorem producing of its own accord exact upper bounds. Moreover, the Halpern–Läuchli-style theorem, which is proved via forcing arguments to achieve a ZFC result and used as the pigeonhole principle in the Ramsey theorem, immediately yields indivisibility for all homogeneous structures satisfying SDAP$^+$, with relations of any arity. These results and their methods are discussed in Section 5.1.

The k-clique-free homogeneous graphs, denoted $\mathbf{G}_k$, $k \geq 3$, were constructed by Henson in his 1971 paper [36], where he proved these graphs to be weakly indivisible. In their 1986 paper [42], Komjáth and Rödl proved that $\mathbf{G}_3$ is indivisible, answering a question of Hajnal. A few years later, El-Zahar and Sauer gave a systematic approach in [24], proving that for each $k \geq 3$, the k-clique-free homogeneous graph $\mathbf{G}_k$ is indivisible. In 1998, Sauer proved in [60] that the big Ramsey degree for edges in $\mathbf{G}_3$ is two. Further progress on big Ramsey degrees of $\mathbf{G}_3$, however, needed a new approach. This was achieved by the author in [13], where the method of coding trees was first developed. In [12], the author extended this work, proving that $\mathbf{G}_k$ has finite big Ramsey degrees, for each $k \geq 3$. In [13] and [12], the author proved a Ramsey theorem for colorings of finite antichains of coding nodes in diagonal coding trees. These diagonal coding trees were designed to achieve very good upper bounds and directly recover the indivisibility results in [42] and [24], discovering much of the essential structure involved in characterizing their exact big Ramsey degrees. (Milliken-style theorems on nondiagonal coding trees which fully branch at each level do not directly prove indivisibility results, and produce looser upper bounds.) In particular, after a minor modification, the trees in [13] produced exact big Ramsey degrees for $\mathbf{G}_3$, as shown in [14]. Around the same time, exact big Ramsey degrees for $\mathbf{G}_3$ were independently proved by Balko, Chodounský, Hubička, Konečný, Vena, and Zucker, instigating the collaboration of this group with the author.

Given a finite relational language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathbf{A}$ is called *irreducible* if each pair of its vertices are in some relation of $\mathbf{A}$. Given a set $\mathcal{F}$ of finite irreducible $\mathcal{L}$-structures, Forb($\mathcal{F}$) denotes the class of all finite $\mathcal{L}$-structures into which no member of $\mathcal{F}$ embeds. Fraïssé classes of the form Forb($\mathcal{F}$) are exactly those with free amalgamation. Zucker in [71] proved that for any Fraïssé class of the form Forb($\mathcal{F}$), where $\mathcal{F}$ is a finite set of irreducible substructures and all relations have arity at most two, its Fraïssé limit has finite big Ramsey degrees. His proof used coding trees which branch at each level and a forcing argument to obtain a Halpern–Läuchli-style theorem which formed the pigeonhole principle for a Milliken-esque theorem for these coding trees. An important advance in this paper is Zucker's abstract, top-down approach, providing simplified and relatively short proof of finite big Ramsey degrees for this large class of homogeneous structures. On the other hand, his Milliken-style theorem does not directly recover indivisibility (more work is needed afterwards to show this), and the upper bounds in [71] did not recover those in [13] or [12] for the homogeneous k-clique-free graphs. However, by further work done in [6], by Balko, Chodounský, Hubička, Konečný, Vena, Zucker, and the author, indivisibility results are proved and exact big Ramsey degrees are characterized. Thus, the picture for FAP classes

with finitely many relations of arity at most two is now clear. These results will be discussed in Section 5.2.

Next, we look at homogeneous structures with relations of arity at most two which do not satisfy SDAP$^+$ and whose ages have strong (but not free) amalgamation. Nguyen Van Thé made a significant contribution in his 2008 paper [53], in which he proved that the ultrametric Urysohn space $\mathbf{Q}_S$ has finite big Ramsey degrees if and only if S is a finite distance set. In the case that S is finite, he calculated the big Ramsey degrees. Moreover, he showed that for an infinite countable distance set S, $\mathbf{Q}_S$ is indivisible if and only if S with the reverse order as a subset of the reals is well ordered. His proof used infinitely wide trees of finite height and his pigeonhole principle was actually Ramsey's theorem. All countable Urysohn metric spaces with finite distance set were proved to be indivisible by Sauer in [63], completing the work that was initiated in [55] in relation to the celebrated distortion problem from Banach space theory and its solution by Odell and Schlumprecht in [56].

Mašulović instigated the use of category theory to prove transport principles showing that finite big Ramsey degrees can be inferred from one category to another. After proving a general transport principle in [47], he applied it to prove finite big Ramsey degrees for many universal structures and also for homogenous metric spaces with finite distance sets with a certain property which he calls *compact with one nontrivial block*. Mašulović proved in [48] that in categories satisfying certain mild conditions, small Ramsey degrees are minima of big Ramsey degrees. In the paper [49] with Šobot (not using category theory), finite big Ramsey degrees for finite chains in countable ordinals were shown to exist if and only if the ordinal is smaller than ω^ω. Dasilva Barbosa in [9] proved that categorical precompact expansions grant upper bounds for big and small Ramsey degrees. As an application, he calculated the big Ramsey degrees of the circular directed graphs $\mathbf{S}(n)$ for all $n \geq 2$, extending the work in [43] for $\mathbf{S}(2)$.

Hubička recently developed a new method to handle forbidden substructures utilizing topological Ramsey spaces of parameter words due to Carlson and Simpson [7]. In [39], he applied his method to prove that the homogeneous partial order and Urysohn S-metric spaces (where S is a set of nonnegative reals with $0 \in S$ satisfying the 4-values condition) have finite big Ramsey degrees. He also showed that this method is quite broad and can be applied to yield a short proof of finite big Ramsey degrees in $\mathbf{G}_3$. Beginning with the upper bounds in [39], the exact big Ramsey degrees of the generic partial order have been characterized in [5] by Balko, Chodounský, Hubička, Konečný, Vena, Zucker, and the author. Also utilizing techniques from [39], Balko, Chodounský, Hubička, Konečný, Nešetřil, and Vena in [2] have found a condition which guarantees finite big Ramsey degrees for binary relational homogeneous structures with strong amalgamation. Examples of structures satisfying this condition include the S-Urysohn space for finite distance sets S, Λ-ultrametric spaces for a finite distributive lattice, and metric spaces associated to metrically homogeneous graphs of a finite diameter from Cherlin's list with no Henson constraints.

For homogeneous structures with free amalgamation, a recent breakthrough of Sauer proving indivisibility in [64] culminates a long line of work in [25–28, 61]. Complementary work appeared in [8], where it was proved that for finitely many relations of any

arity, SDAP$^+$ implies indivisibility. On the other hand, big Ramsey degrees of structures with relations of arity greater than two has only recently seen progress, beginning with [3] and [4], where Balko, Chodounský, Hubička, Konečný, and Vena found upper bounds for the big Ramsey degrees of the generic 3-hypergraph. Work in this area is ongoing and promising.

5. EXACT BIG RAMSEY DEGREES

This section presents characterizations of exact big Ramsey degrees known at the time of writing. These hold for homogeneous structures with finitely many relations of arity at most two. Two general classes have been completely understood: Structures satisfying a certain strengthening of strong amalgamation called SDAP$^+$ (Section 5.1) and structures whose ages have free amalgamation (Section 5.2). Lying outside of these two classes, the generic partial order has been completely understood in terms of exact big Ramsey degrees and will be briefly discussed at the end of Section 5.2. These characterizations all involve the notion of a diagonal antichain, in various trees or spaces of parameter words, representing a copy of an enumerated homogeneous structure. Here, we present these notions in terms of structures, as they are independent of the representation.

Let $\mathbf{K}$ be an enumerated homogeneous structure with universe $\{v_n : n < \omega\}$. Let $\mathbf{A} \leq \mathbf{K}$ be a finite substructure of $\mathbf{K}$, and suppose that the universe of $\mathbf{A}$ is $\{v_i : i \in I\}$ for some finite set $I \subseteq \omega$. We say that $\mathbf{A}$ is an *antichain* if for each pair $i < j$ in I there is a $k(i, j) < i$ such that the set $\{k(i, j) : i, j \in I \text{ and } i < j\}$ is disjoint from I, and

$$\mathbf{K} \upharpoonright (\{v_\ell : \ell < k(i, j)\} \cup \{v_i\}) \cong \mathbf{K} \upharpoonright (\{v_\ell : \ell < k(i, j)\} \cup \{v_j\}), \tag{5.1}$$

$$\mathbf{K} \upharpoonright (\{v_\ell : \ell \leq k(i, j)\} \cup \{v_i\}) \ncong \mathbf{K} \upharpoonright (\{v_\ell : \ell \leq k(i, j)\} \cup \{v_j\}). \tag{5.2}$$

An antichain $\mathbf{A}$ is called *diagonal* if $\{k(i, j) : i < j \leq m\}$ has cardinality m. We call $k(i, j)$ the *meet level* of the pair v_i, v_j.

The notion of diagonal antichain is central to all characterizations of big Ramsey degrees obtained so far. It seems likely that antichains will be essential to all characterizations of big Ramsey degrees. However, preliminary work shows that some homogeneous binary relational structures, such as two or more independent linear orders, will have characterizations in their trees of 1-types involving antichains which are not diagonal, but could still be characterized via products of finitely many diagonal antichains.

The indexing of the relation symbols $\{R_\ell : \ell < L\}$ in the language $\mathcal{L}$ of $\mathbf{K}$ induces a lexicographic ordering on trees representing relational structures. Here, we present this idea directly on the structures. For $m \neq n$, we declare $v_m <_{\text{lex}} v_n$ if and only if $\{v_m, v_n\}$ is an antichain and, letting k be the meet level of the pair v_m, v_n, and letting ℓ denote the least index in L such that v_m and v_n disagree on their R_ℓ-relationship with v_k, either $R_\ell(v_k, v_n)$ holds while $R_\ell(v_k, v_m)$ does not, or else $R_\ell(v_n, v_k)$ holds while $R_\ell(v_m, v_k)$ does not.

Two diagonal antichains $\mathbf{A}$ and $\mathbf{B}$ in an enumerated homogeneous structure $\mathbf{K}$ are *similar* if they have the same number of vertices, and the increasing bijection from the universe $A = \{v_{m_i} : i \leq p\}$ of $\mathbf{A}$ to the universe $B = \{v_{n_i} : i \leq p\}$ of $\mathbf{B}$ induces an isomorphism

from **A** to **B** which preserves $<_{\text{lex}}$ and induces a map on the meet levels which, for each $i < j \leq p$, sends $k(m_i, m_j)$ to $k(n_i, n_j)$. This implies that the map sending the coding node c_{m_i} to c_{n_i} $(i \leq p)$ in the coding tree of 1-types $\mathbb{S}(\mathbf{K})$ (see Definition 3.4) induces a map on the meet-closures of $\{c_{m_i} : i \leq p\}$ and $\{c_{n_i} : i \leq p\}$ satisfying Definition 3.5.

Similarity is an equivalence relation, and an equivalence class is called a *similarity type*. We say that **K** has *simply characterized big Ramsey degrees* if for $\mathbf{A} \in \text{Age}(\mathbf{K})$, the big Ramsey degree of **A** is exactly the number of similarity types of diagonal antichains representing **A**. In the next subsection, we will see many homogeneous structures with simply characterized big Ramsey degrees.

5.1. Exact big Ramsey degrees with a simple characterization

The decades-long investigation of the big Ramsey degrees of the Rado graph culminated in the two papers [62] and [44]. These two papers moreover characterized the big Ramsey degrees for all *unrestricted* binary relational homogeneous structures. Unrestricted binary relational structures are determined by a finite language $\mathcal{L} = \{R_0, \ldots, R_{l-1}\}$ of binary relation symbols and a nonempty constraint set $\mathcal{C}$ of $\mathcal{L}$-structures with universe $\{0, 1\}$ with the following property: If **A** and **B** are two isomorphic $\mathcal{L}$-structures with universe $\{0, 1\}$, then either both are in $\mathcal{C}$ or neither is in $\mathcal{C}$. We let $\mathbf{H}_{\mathcal{C}}$ denote the homogeneous structure such that each of its substructures with universe of size two is isomorphic to one of the structures in $\mathcal{C}$. Examples of unrestricted binary relational homogeneous structures include the Rado graph, the generic directed graph, the generic tournament, and random graphs with more than one edge relation.

Given a universal constraint set $\mathcal{C}$, letting $k = |\mathcal{C}|$, Sauer showed in [62] how to form a structure, call it $\mathbf{U}_{\mathcal{C}}$, with nodes in the tree $k^{<\omega}$ as vertices, such that $\mathbf{H}_{\mathcal{C}}$ embeds into $\mathbf{U}_{\mathcal{C}}$. Fix a bijection $\lambda : \mathcal{C} \to k$. Given two nodes $s, t \in k^{<\omega}$ with $|s| < |t|$, declare that $t(|s|) = j$ if and only if the induced substructure of $\mathbf{U}_{\mathcal{C}}$ on universe $\{s, t\}$ is isomorphic to the structure $\lambda(j)$ in $\mathcal{C}$, where the isomorphism sends s to 0 and t to 1. For two nodes $s, t \in k^{<\omega}$ of the same length, declare that for s lexicographically less than t, the induced substructure of $\mathbf{U}_{\mathcal{C}}$ on universe $\{s, t\}$ is isomorphic to the structure $\lambda(0)$ in $\mathcal{C}$, where the isomorphism sends s to 0 and t to 1. As a special case, a universal graph is constructed as follows: Let each node in $2^{<\omega}$ be a vertex. Define an edge relation E between vertices by declaring that, for $s \neq t$ in $2^{<\omega}$, $s \, E \, t$ if and only if $|s| \neq |t|$ and $(|s| < |t| \implies t(|s|) = 1)$. Then $(2^{<\omega}, E)$ is universal for all countable graphs. In particular, the Rado graph embeds into the graph $(2^{<\omega}, E)$, and vice versa.

In trees of the form $k^{<\omega}$, the notion of similarity is exactly that of Definition 3.3, and steps I–IV discussed in Section 3 outline the proof of exact big Ramsey degrees contained in the pair of papers [62] and [44]. Milliken's theorem was used to prove existence of upper bounds via strong tree envelopes. For step III, Sauer constructed in [62] a diagonal antichain $D \subseteq k^{<\omega}$ such that the substructure of $\mathbf{U}_{\mathcal{C}}$ restricted to universe D is isomorphic to $\mathbf{H}_{\mathcal{C}}$, achieving upper bounds shown to be exact in [44], finishing step IV. The big Ramsey degree of a finite substructure **A** of $\mathbf{H}_{\mathcal{C}}$ is exactly the number of distinct similarity types of subsets of D whose induced substructure in $\mathbf{U}_{\mathcal{C}}$ is isomorphic to **A**.

The work in [62] and [44] greatly influenced the author's development of *coding trees* and their Ramsey theorems in [13] and [12] (discussed in Section 5.2). Those papers along with a suggestion of Sauer to the author during the Banff 2018 Workshop on *Unifying Themes in Ramsey Theory*, to try moving the forcing arguments in those papers from coding trees to structures, informed the approach taken in the paper [8], which is now discussed.

Let $\mathbf{K}$ be an enumerated Fraïssé structure with vertices $\{v_n : n < \omega\}$. For $n < \omega$, we let $\mathbf{K}_n$ denote $\mathbf{K} \restriction \{v_i : i < n\}$, the induced substructure of $\mathbf{K}$ on its first n vertices, and call $\mathbf{K}_n$ an *initial substructure* of $\mathbf{K}$. We write 1-*type* to mean complete realizable quantifier-free 1-type over $\mathbf{K}_n$ for some n.

Definition 5.1 (Coding Tree of 1-Types, [8]). The *coding tree of* 1-*types* $\mathbb{S}(\mathbf{K})$ for an enumerated Fraïssé structure $\mathbf{K}$ is the set of all 1-types over initial substructures of $\mathbf{K}$ along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ such that $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is simply inclusion.

A substructure $\mathbf{A}$ of $\mathbf{K}$ with universe $A = \{v_{n_0}, \ldots, v_{n_m}\}$ is represented by the set of coding nodes $\{c(n_0), \ldots, c(n_m)\}$ as follows: For each $i \leq m$, since $c(n_i)$ is the quantifier-free 1-type of v_{n_i} over $\mathbf{K}_{n_i}$, substituting v_{n_i} for the variable x into each formula in $c(n_i)$ which has only parameters from $\{v_{n_j} : j < i\}$ uniquely determines the relations in $\mathbf{A}$ on the vertices $\{v_{n_j} : j \leq i\}$. In [8], we formulated the following strengthening of SAP in order to extract a general property ensuring that big Ramsey degrees have simple characterizations.

Definition 5.2 (SDAP). A Fraïssé class $\mathcal{K}$ has the *Substructure Disjoint Amalgamation Property (SDAP)* if $\mathcal{K}$ has strong amalgamation, and the following holds: Given $\mathbf{A}, \mathbf{C} \in \mathcal{K}$, suppose that $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say v and w. Then there exist $\mathbf{A}', \mathbf{C}' \in \mathcal{K}$, where $\mathbf{A}$ is a substructure of $\mathbf{A}'$ and $\mathbf{C}'$ is a disjoint amalgamation of $\mathbf{A}'$ and $\mathbf{C}$ over $\mathbf{A}$, such that letting v', w' denote the two vertices in $C' \setminus A'$ and assuming (1) and (2), the conclusion holds:

(1) Suppose $\mathbf{B} \in \mathcal{K}$ is any structure containing $\mathbf{A}'$ as a substructure, and let σ and τ be 1-types over $\mathbf{B}$ satisfying $\sigma \restriction \mathbf{A}' = \mathrm{tp}(v'/\mathbf{A}')$ and $\tau \restriction \mathbf{A}' = \mathrm{tp}(w'/\mathbf{A}')$,

(2) Suppose $\mathbf{D} \in \mathcal{K}$ extends $\mathbf{B}$ by one vertex, say v'', such that $\mathrm{tp}(v''/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say, w'', such that $\mathrm{tp}(w''/\mathbf{B}) = \tau$ and $\mathbf{E} \restriction (A \cup \{v'', w''\}) \cong \mathbf{C}$.

This amalgamation property can, of course, be presented in terms of embeddings, but the form here is indicative of how it is utilized. A free amalgamation version called *SFAP* is obtained from SDAP by restricting to FAP classes and requiring $\mathbf{A}' = \mathbf{A}$ and $\mathbf{C}' = \mathbf{C}$. Both of these amalgamation properties are preserved under free superposition. A *diagonal subtree* of $\mathbb{S}(\mathbf{K})$ is a subtree such that at any level, at most one node branches, the branching degree is two, and branching and coding nodes never occur on the same level. Diagonal coding trees are subtrees of $\mathbb{S}(\mathbf{K})$ which are diagonal and represent a subcopy of $\mathbf{K}$. The property SDAP$^+$ holds for a homogeneous structure $\mathbf{K}$ if (a) its age satisfies SDAP, (b) there is a

diagonal coding subtree of $\mathbb{S}(\mathbf{K})$, and (c) a technicality called the Extension Property which in most cases is trivially satisfied. Classes of the form Forb($\mathscr{F}$) where $\mathscr{F}$ is a finite set of 3-irreducible structures, meaning each triple of vertices is in some relation, satisfy SFAP; their ordered versions satisfy SDAP$^+$.

A version of the Halpern–Läuchli theorem for diagonal coding trees was proved in [8] using the method of forcing to obtain a ZFC result, with the following theorem as an immediate consequence.

Theorem 5.3 ([8]). *Let* $\mathbf{K}$ *be a homogeneous structure satisfying* SDAP$^+$*, with finitely many relations of any arity. Then* $\mathbf{K}$ *is indivisible.*

For relations of arity at most two, an induction proof then yields a Ramsey theorem for finite colorings of finite antichains of coding nodes in diagonal coding trees. This accomplishes steps I–III simultaneously and directly, without any need for envelopes, providing upper bounds which are then proved to be exact, finishing step IV.

Theorem 5.4 ([8]). *Let* $\mathbf{K}$ *be a homogeneous structure satisfying* SADP$^+$*, with finitely many relations of arity at most two. Then* $\mathbf{K}$ *admits a big Ramsey structure and, moreover, has simply characterized big Ramsey degrees.*

Theorem 5.4 provides new classes of examples of big Ramsey structures while recovering results in [10,38,43,44] and special cases of the results in [71]. Theorem 5.3 provides new classes of examples of indivisible Fraïssé structures, in particular for ordered structures such as the ordered Rado graph, while recovering results in [24,27,42] and certain cases of Sauer's results in [64] for FAP classes, while providing new SAP examples with indivisibility.

5.2. Big Ramsey degrees for free amalgamation classes

An obstacle to progress in partition theory of homogeneous structures had been the fact that Milliken's theorem is not able to handle forbidden substructures, for instance, triangle-free graphs. Most results up to 2010 had either utilized Milliken's theorem or a variation (as in [43,62]) or else used difficult direct methods (as in [60]) which did not lend naturally to generalizations. The idea of coding trees came to the author during the her stay at the Isaac Newton Institute in 2015 for the programme, *Mathematical, Foundational and Computational Aspects of the Higher Infinite*, culminating in the work [13]. The ideas behind coding trees included the following: Knowing that at the end of the process one will want a diagonal antichain representing a copy of $\mathbf{G}_3$, starting with a tree where vertices in $\mathbf{G}_3$ are represented by special nodes on different levels should not hurt the results. Further, by using special nodes to code the vertices of $\mathbf{G}_3$ into the trees, one might have a chance to prove Milliken-style theorems on a collection of trees, each of which codes a subcopy of $\mathbf{G}_3$.

The author had made a previous attempt at this problem starting early in 2012. Upon stating her interest this problem, Todorcevic (2012, at the Fields Institute Thematic Program on Forcing and Its Applications) and Sauer (2013, at the Erdős Centenary Meeting) each told the author that a new kind of Milliken theorem would need to be developed in order to handle triangle-free graphs, which intrigued her even more. Though unknown to her at the

time, a key piece to this puzzle would be Harrington's forcing proof of the Halpern–Läuchli theorem, which Laver was kind enough to outline to her in 2011. (At that time, the author was unaware of the proof in [67].) While at the INI in 2015, Bartošová reminded the author of her interest in big Ramsey degrees of $\mathbf{G}_3$. Having had time by then to fill out and digest Laver's outline, it occurred to the author to try approaching the problem first with the strongest tool available, namely forcing.

Forcing is a set-theoretic method which is normally used to extend a given universe satisfying a given set of axioms (often ZFC) to a larger universe in which the same set of axioms hold while some other statement or property is different than in the original universe. The beautiful thing about Harrington's proof is that, while it does involve the method of forcing, the forcing is only used as a search engine for an object which already exists in the universe in which one lives. In the context of the Fraïssé limit $\mathbf{K}$ of a class $\mathrm{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a finite set of finite irreducible structures, by carefully designing forcings on coding trees with partial orders ensuring that new levels obtained by the search engine are capable of extending a given fixed finite coding tree to a subcoding tree representing a copy of $\mathbf{K}$, one is able to prove Halpern–Läuchli-style theorems for coding trees. These form the pigeonhole principles of various Milliken-style theorems for coding trees.

As the results and main ideas of the methods in [12,13,71] have been discussed in the previous section, we now present the characterization of big Ramsey degrees in [6].

Theorem 5.5 ([6]). *Let $\mathbf{K}$ be a homogeneous structure with finitely many relations of arity at most two such that $\mathrm{Age}(\mathbf{K}) = \mathrm{Forb}(\mathcal{F})$ for some finite set $\mathcal{F}$ of finite irreducible structures. Then $\mathbf{K}$ admits a big Ramsey structure.*

Given a Fraïssé class $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$ with relations of arity at most two, where $\mathcal{F}$ is a finite set of finite irreducible structures, let $\mathbf{K}$ denote an enumerated Fraïssé limit of $\mathcal{K}$. Coding trees for $\mathbf{K}$ appearing in various papers are all essentially coding trees of 1-types. The proof of Theorem 5.5 uses the upper bounds of Zucker in [71] as the starting point. It then proceeds by constructing a diagonal antichain of coding nodes which represent the structure $\mathbf{K}$, with additional requirements if there are any forbidden irreducible substructures of size three or more. While the exact characterization in its full generality is not short to state, the simpler version for the structures $\mathbf{G}_k$ include the following: All coding nodes $c_n \in \mathbf{A}$ code an edge with v_m for some $m < n$ and have the following property: If $\mathbf{B}$ is any finite graph which has the same relations over $\mathbf{G}_k \upharpoonright |c_n|$ as c_n does, then $\mathbf{B}$ has no edges. Furthermore, changes in the sets of structures which are allowed to extend a given truncation of $\mathbf{A}$ (as a level set in the coding tree) happen as gradually as possible. From the characterization in [6], one can make an algorithm to compute the big Ramsey degrees.

As a concrete example, we present the exact characterization for triangle-free graphs. In Figure 2, on the left is the beginning of $\mathbf{G}_3$ with some fixed enumeration of the vertices as $\{v_n : n < \omega\}$. The nth *coding node* in the tree $\mathbb{S} = \mathbb{S}(\mathbf{G}_3) \subseteq 2^{<\omega}$ represents the nth vertex v_n in $\mathbf{G}_3$, where passing number 0 represents a nonedge and passing number 1 represents an edge. Equivalently, $\mathbb{S}$ is the coding tree of 1-types for $\mathbf{G}_3$, as the left branch at the level of c_n represents the literal $(x \not{E} v_n)$ and the right branch represents $(x E v_n)$.
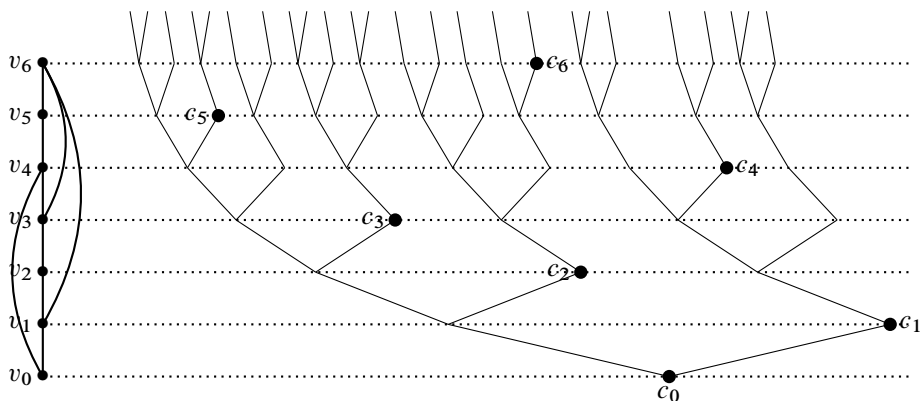
FIGURE 2

Coding tree $\mathbb{S}(\mathbf{G}_3)$ and the triangle-free graph represented by its coding nodes.

Given an antichain $\mathbf{A} \subseteq \mathbf{K}$, we say that $\mathbf{A}$ is a *diagonal substructure* if, letting I be the set of indices of vertices in $\mathbf{A}$, the following hold: (a) For each $i \in I$, v_i has an edge with v_m for some $m < i$; let m_i denote the least such m. (b) If $i < j$ are in I with $v_i \not\mathrel{E} v_j$ and $m_j < i$, then there is some $n \in i$ such that $v_i \mathrel{E} v_n$ and $v_j \mathrel{E} v_n$, and the least such n, denoted $n(i, j)$ is not in I. (c) For each $i, j, k, \ell \in I$ (not necessarily distinct) with $i < j$, $k < \ell$, $(i, j) \neq (k, \ell)$, $n_j < i$, and $n_\ell < k$, we have $n(i, j) \neq n(k, \ell)$. Given a finite triangle-free graph $\mathbf{A}$, the big Ramsey degree $T(\mathbf{A})$ in $\mathbf{G}_3$ is the number of different diagonal substructures representing a copy of $\mathbf{A}$.

We conclude this section by mentioning the exact big Ramsey degrees in the generic partial order in [5]. This result begins with the upper bounds proved by Hubička in [39] and then proceeds by taking a diagonal antichain D representing the generic partial order with additional structure of interesting levels built into D. A level ℓ of D is *interesting* if there are exactly two nodes, say s, t, in that level so that ($*$) for exactly one relation $\rho \in \{<, >, \perp\}$, given any $s', t' \in D$ extending s, t, respectively, $s' \rho t'$, while there is no such relation for the pair $s \restriction (\ell - 1), t \restriction (\ell - 1)$. Since an interesting level for a pair of nodes s, t predetermines the relations between any pair s', t' extending s, t, respectively, passing numbers are unnecessary to the characterization. The big Ramsey degree of a given finite partial order $\mathbf{P}$ is then the number of different diagonal antichains $A \subseteq D$ representing $\mathbf{P}$ along with the order in which the interesting levels are interspersed between the splitting levels and the nodes in A.

6. OPEN PROBLEMS AND RELATED DIRECTIONS

Section 2 laid out the guiding questions for big Ramsey degrees. Here we discuss some of the major open problems in big Ramsey degrees and ongoing research in cognate areas.

Problem 6.1. For which SAP Fraïssé classes does the Fraïssé limit have finite big Ramsey degrees?

Subquestions are the following: Given an SAP Fraïssé class with finitely many relations and a finite set of forbidden substructures, does its Fraïssé limit have finite big Ramsey degrees? Results in [40] give evidence for a positive answer to this question. For such classes with relations of arity at most two, do big Ramsey degrees always exist? We would like a general condition on SAP classes characterizing those with finite big Ramsey degrees. We point out that Problem 6.1 in its full generality is still open for small Ramsey degrees

Problem 6.2. For results whose proofs use the method of forcing, find new proofs which are purely combinatorial.

This has been done for the triangle-free graph by Hubička in [39], but new methods will be needed for k-clique-free homogeneous graphs for $k \geq 4$ and other such FAP classes.

The next problem has to do with topological dynamics of automorphism groups of homogeneous structures. The work of Zucker in [70] has established a connection but not a complete correspondence yet.

Problem 6.3. Does every homogeneous structure with finite big Ramsey degrees also carry a big Ramsey structure? Is there an exact correspondence, in the vein of the KPT-correspondence, between big Ramsey structures and topological dynamics?

The hope in Problem 6.3 is to obtain as complete a dynamical understanding of big Ramsey degrees as we have for small Ramsey degrees, where a result of [69] shows that given a Fraïssé class $\mathcal{K}$ with Fraïssé limit $\mathbf{K}$, then $\mathcal{K}$ has finite small Ramsey degrees if and only if the universal minimal flow of $\mathrm{Aut}(\mathbf{K})$ is metrizable.

Finally, we mention several areas of ongoing study related to the main focus of this paper. Computability-theoretic and reverse mathematical aspects have been investigated by Anglès d'Auriac, Cholak, Dzhafarov, Monin, and Patey. In their treatise [1], they show that the Halpern–Läuchli theorem is computably true and find reverse-mathematical strengths for various instances of the product Milliken theorem and the big Ramsey structures of the rationals and the Rado graph. As these structures both have simply characterized big Ramsey degrees, it will be interesting to see if different reverse mathematical strengths emerge for structures such as the triangle-free homogeneous graph or the generic partial order.

Extending Harrington's forcing proof to the uncountable realm, Shelah in [59] showed that it is consistent, assuming certain large cardinals, that the Halpern–Läuchli theorem holds for trees $2^{<\kappa}$, where κ is a measurable cardinal. Džamonja, Larson, and Mitchell applied a slight modification of his theorem to characterize the big Ramsey degrees for the κ-rationals and the κ-Rado graph in [22] and [23]. Their characterizations have as their basis the characterizations of Devlin and Laflamme–Sauer–Vuksanovic for the rationals and Rado graph, respectively, but also involve well-orderings of each level of the tree $2^{<\kappa}$, necessitated by κ being uncountable. The field of big Ramsey degrees for uncountable

homogeneous structures is still quite open, but the fleshing out of the Ramsey theorems on trees of uncountable height has seen some recent work in [19, 20, 68].

The next problem comes from a general question in [41].

Problem 6.4. Develop infinite-dimensional Ramsey theory on spaces of copies of a homogeneous structure.

For a set $N \subseteq \omega$, let $[N]^\omega$ denote the set of all infinite subsets of N, and note that $[\omega]^\omega$ represents the Baire space. The infinite-dimensional Ramsey theorem of Galvin and Prikry [33] says that given any Borel subset $\mathcal{X}$ of the Baire space, there is an infinite set N such that $[N]^\omega$ is either contained in $\mathcal{X}$ or is disjoint from $\mathcal{X}$. Ellentuck's theorem in [29] found optimality in terms of sets with the property of Baire with respect to a finer topology. The question in [41] asks for extensions of these theorems to subspaces of $[\omega]^\omega$, where each infinite set represents a copy of some fixed homogeneous structure. A Galvin–Prikry-style theorem for spaces of copies of the Rado graph has been proved by the author in [17]. By a comment of Todorcevic in Luminy in 2019, the infinite-dimensional Ramsey theorem should ideally also recover exact big Ramsey degrees. Such a theorem is being written down by the author for structures satisfying SDAP$^+$ with relations of arity at most two. This is one instance where coding trees are necessitated to be diagonal in order for the infinite dimensional Ramsey theorem to directly recover exact big Ramsey degrees.

We close by mentioning that structural Ramsey theory has been central in investigations of ultrafilters which are relaxings of Ramsey ultrafilters in the same way that big Ramsey degrees are relaxings of Ramsey's theorem. An exposition of recent work appearing in [16] will give the reader yet another view of the power of Ramsey theory.

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