MEASURABLE GRAPH COMBINATORICS

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ABSTRACT

We survey some recent results in the theory of measurable graph combinatorics. We also discuss applications to the study of hyperfiniteness and measurable equidecompositions.

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1. INTRODUCTION

Measurable graph combinatorics focuses on finding measurable solutions to combinatorial problems on infinite graphs. This study involves ideas and techniques from combinatorics, ergodic theory, probability theory, descriptive set theory, and theoretical computer science. We survey some recent progress in this area, focusing on the study of *locally finite* graphs: graphs where each vertex has finitely many neighbors. We also discuss applications to the study of hyperfiniteness of Borel actions of groups, and measurable equidecompositions.

Without any constraints such as measurability conditions, combinatorial problems on locally finite graphs often simplify to studying their restriction to finite subgraphs. This is the case with the problem of graph coloring. Recall that if $G = (V, E)$ is a graph, a *(proper) Y-coloring* of G is a map $c: V \to Y$ so that for every two adjacent vertices $\{x, y\} \in E$, the colors assigned to these two vertices are distinct, $c(x) \neq c(y)$. The *chromatic number* $\chi(G)$ of G is the smallest cardinality of a set Y so there is a Y-coloring of G. A classical theorem of De Bruijn and Erdős states that for a locally finite graph G, the chromatic number of G is equal to the supremum of the chromatic number of all finite subgraphs of G. That is, $\chi(G) = \sup_{\text{finite } H \subseteq G} \chi(H)$. The proof of this theorem is a straightforward compactness argument using the Axiom of Choice.

In contrast, many phenomena can influence measurable chromatic numbers beyond just the constraints imposed by finite subgraphs. We illustrate this change in behavior with a simple example. Let S^1 be the circle, let $T: S^1 \to S^1$ be an irrational rotation, and let μ be Lebesgue measure on S^1. Consider the graph G_T with vertex set S^1 and where x, y are adjacent if $T(x) = y$ or $T(y) = x$. Every vertex in G_T has degree 2 and every connected component of G_T is infinite. Hence, by alternating between two colors, it is easy to see that the classical chromatic number of G_T is 2. However, there can be no Lebesgue measurable 2-coloring of G_T. Suppose $c: S^1 \to \{0, 1\}$ was a Lebesgue measurable coloring of G_T, and $A_0 = \{x : c(x) = 0\}$ and $A_1 = \{x : c(x) = 1\}$ were the two color sets. Then since the coloring must alternate between the two colors, we must have $T(A_0) = A_1$, and since T is measure preserving and A_0 and A_1 are disjoint and cover S^1, we therefore have $\lambda(A_0) = \lambda(A_1) = \frac{1}{2}$. However, the transformation T^2 is also an irrational rotation and hence T^2 is *ergodic*, meaning any set invariant under T^2 must be null or conull. Since $T^2(A_0) = A_0$, A_0 must be null or conull. Contradiction!

In this paper we focus on the study of combinatorial problems on *Borel graphs*: graphs where the set V of vertices is a standard Borel space and where the edge relation E is Borel as a subset of $V \times V$. In the setting where each vertex has at most countably many neighbors, this is equivalent to saying that there are countably many Borel functions $f_0, f_1, \ldots : V \to V$ that generate G in the sense that $x \mathrel{E} y$ if and only if $f_i(x) = y$ for some i. The equivalence follows from the Lusin–Novikov theorem [28, 18.15]. An important example of a Borel graph is the following type of *Schreier graph*. If a is a Borel action of a countable group Γ on a standard Borel space X and S is a symmetric set of generators for Γ, then let $G(a, S)$ be the graph on the vertex set $V = X$ where $x, y \in V$ are adjacent if there is a $\gamma \in S$ such that $\gamma \cdot x = y$. For example, the graph associated to the irrational rotation described above is a graph of this form.

For more comprehensive surveys of this area, the reader should consult the papers [30, 44]. A notable recent development we will not discuss is the connections that have been found between measurable combinatorics and the study of distributed algorithms in theoretical computer science, particularly the LOCAL model. This model of computing takes place on a large graph where each vertex represents a computer which is assigned a unique identifier, and each edge is a communication link. These processors execute the same algorithm in parallel, communicating with their neighbors in rounds to construct a global solution to some combinatorial problem. Recent work [2, 3, 6, 17] has established some precise connections between measurable combinatorics and LOCAL algorithms which have already led to new theorems in both areas (see, e.g., [2, 4]).

2. MEASURABLE COLORINGS

If G is a Borel graph, we define the Borel chromatic number $\chi_B(G)$ of G to be the smallest cardinality of a standard Borel space Y so that there is a Borel measurable Y-coloring of G. We clearly have that $\chi(G) \leq \chi_B(G)$ where $\chi(G)$ is the classical chromatic number of G. Borel chromatic numbers were first studied in a foundational paper of Kechris, Solecki, and Todorcevic [32].

Let $G = (V, E)$ be a graph. If $x \in V$ is a vertex, we let $N(x) = \{y : \{x, y\} \in E\}$ denote the set of *neighbors* of x. The *degree* of x is the cardinality of $N(x)$. We say that a graph is Δ-*regular* if every vertex has degree Δ. A basic result about graph coloring is that, given any finite graph G of finite maximum degree Δ, there is a $(\Delta + 1)$-coloring of G. This is easy to see by coloring the vertices of G one by one. If we have a partial coloring of G, then any uncolored vertex x has at most Δ neighbors so there must be a color from the set of $\Delta + 1$ colors we can use to extend this partial coloring to x. The analogous fact remains true about Borel colorings:

Theorem 2.1 (Kechris, Solecki, Todorcevic [32, PROPOSITION 4.6]). *If G is a Borel graph of finite maximum degree Δ, then G has a Borel $(\Delta + 1)$-coloring.*

One method of proving this theorem is to adapt the greedy algorithm described above. Recall that a set of vertices is *independent* if it does not contain two adjacent vertices. First, we may find a countable sequence of Borel sets A_n such that each A_n is independent, and their union is all vertices $\bigcup_n A_n = V(G)$. Then we can iteratively construct a coloring of G in countably many steps where at step n we color all the elements of A_n the least color not already used by one of its neighbors. In general, the connection between algorithms for solving combinatorial problems and measurable combinatorics is deep. Many techniques for constructing measurable colorings are based on algorithmic ideas, since algorithms for solving combinatorial problems will often yield an explicitly definable solutions to them.

The upper bound given by Theorem 2.1 is tight; a complete graph on $\Delta + 1$ vertices has maximum degree Δ and chromatic number $\Delta + 1$. Surprisingly, the upper bound of

Theorem 2.1 is also optimal even in the case of acyclic Borel graphs. Hence, for bounded degree Borel graphs, the Borel chromatic number and classical chromatic number may be very far apart since any acyclic graph has classical chromatic number at most 2.

Theorem 2.2 (Marks [38]). *For every finite Δ, there is an acyclic Borel graph of degree Δ with no Borel Δ-coloring.*

The graphs used to establish Theorem 2.2 are quite natural, and arise from Schreier graphs of actions of free products of Δ many copies of $\mathbb{Z}/2\mathbb{Z}$. Theorem 2.2 is proved using Martin's theorem of Borel determinacy [41] which states that in any infinite two-player game of perfect information with a Borel payoff set, one of the two players has a winning strategy. The direct use of Borel determinacy to prove this theorem leads to an interesting question of reverse mathematics since Borel determinacy requires a great deal of set-theoretic power to prove: the use of uncountably many iterates of the powerset of $\mathbb{R}$ [19]. We currently do not know of any simpler proof of Theorem 2.2 that avoids the use of Borel determinacy or can be proved in second-order arithmetic (which suffices for most theorems of descriptive set theory).

Problem 2.3. Is Theorem 2.2 provable in the theory Z_2 of full second-order arithmetic?

Recently, Brandt, Chang, Grebík, Grunau, Rozhoň, and Vidnyánszky [6] have shown that characterizing the set of Borel graphs of maximum degree $\Delta \geq 3$ that have no Borel $(\Delta + 1)$-coloring is as hard as possible in a precise sense: the set of such graphs is $\mathbf{\Sigma}^1_2$ complete. Here $\mathbf{\Sigma}^1_2$ completeness is a logical measurement of the complexity of this problem. The proof of their theorem combines the techniques of [39] with earlier work of Todorcevic and Vidnyánszky [48] proving $\mathbf{\Sigma}^1_2$ completeness for the set of locally finite Borel graphs generated by a single function that have finite Borel chromatic number. In contrast to the work of [6] for $\Delta \geq 3$, in the case $\Delta = 2$, a dichotomy theorem of Carroy, Miller, Schrittesser, and Vidnyánszky [8] characterizes the 2-colorable Borel graphs in a simple way as those for which there is no Borel homomorphism from a canonical non-Borel-2-colorable graph known as $\mathbb{L}_0$.

This type of theorem—proving it is hard to characterize the set of graphs with some combinatorial property—is familiar in finite graph theory via computational complexity theory. For example, it is a well-known theorem that the set of finite graphs that are k-colorable for $k \geq 3$ is NP-complete. Indeed, there are some surprising newly found connections between computational complexity theory and complexity in measurable combinatorics. Thornton [47] has used techniques adapted from the celebrated CSP (constraint satisfaction problem) dichotomy theorem [7, 51] in theoretical computer science to bootstrap the results of [6] to show many other natural combinatorial problems on locally finite Borel graphs are either $\mathbf{\Sigma}^1_2$ complete or a $\mathbf{\Pi}^1_1$. The CSP dichotomy theorem concerns a certain class of natural problems in NP: general constraint satisfaction problems like graph coloring with k colors, k-SAT, or, more generally, computing the set of finite structures X that have a homomorphism to a given fixed finite structure D. The CSP dichotomy states that all such constraint satisfaction problems are either in P (like 2-coloring or 2-SAT), or they are NP-complete (like 3-coloring or 3-SAT).

The results in [6] rule out any simple theory for understanding Borel chromatic number for locally finite Borel graphs in general. In contrast, if we weaken our measurability condition to study μ-measurable colorings with respect to some Borel probability measure μ instead of Borel colorings, the theory of μ-measurable colorings appears to have a much closer connection to finite graph theory. If μ is a Borel measure on the vertex set of a Borel graph G, let $\chi_\mu(G)$ be the least size of a set Y so there is a μ-measurable coloring of G. So $\chi(G) \leq \chi_\mu(G) \leq \chi_B(G)$, since every Borel function is μ-measurable.

For finite graphs of maximum degree Δ, a theorem of Brooks characterizes those connected graphs which have chromatic number of $\Delta + 1$. They are precisely the complete graphs on $\Delta + 1$ vertices, and odd cycles in the case $\Delta = 2$. Analogously, we have the following generalization of Brooks's theorem for μ-measurable colorings:

Theorem 2.4 (Conley, Marks, Tucker-Drob [13]). *Suppose that G is a Borel graph with degree bounded by a finite $\Delta \geq 3$. Suppose further that G contains no complete graph on $\Delta + 1$ vertices. If μ is any Borel probability measure on $V(G)$, then G admits a μ-measurable Δ-coloring.*

Several important open problems in descriptive set theory concern whether there is a difference between being able to find a Borel solution to a problem versus being able to find a μ-measurable solution with respect to every Borel probability measure μ (e.g., the hyperfiniteness vs measure hyperfiniteness problem [29, PROBLEM 8.29]). Theorems 2.2 and 2.4 are encouraging in this context because they point the way towards tools that may be able to resolve these types of questions.

The proof of Theorem 2.4 is based on a technique for finding *one-ended spanning subforests* in Borel graphs: acyclic subgraphs on the same vertex set where each connected component has exactly one end. More recently, these techniques for finding one-ended spanning subforests were applied to prove new results in the theory of cost: a real valued invariant of countable groups arising from their ergodic actions [9].

Bernshteyn has substantially strengthened Theorem 2.4 by showing for k within a factor of $\sqrt{\Delta}$ of Δ, there is a μ-measurable k-coloring of G if and only if there is any k-coloring of G.

Theorem 2.5 (Bernshteyn [2]). *There is a Δ_0 so that if G is a Borel graph with finite maximum degree $\Delta \geq \Delta_0$ and μ is a Borel probability measure on $V(G)$, then if c satisfies $c \leq \sqrt{D} - 5/2$, then G has a $(\Delta - c)$-coloring if and only if G has a μ-measurable $(\Delta - c)$-coloring.*

The above results give cases where the μ-measurable chromatic number behaves similarly to the classical chromatic number. These two quantities may still differ by a large amount, however. Let $\mathbb{F}_n$ be the free group on n generators and let $S_n \subseteq \mathbb{F}_n$ be a free symmetric generating set. Let a_n be the action of $\mathbb{F}_n$ on the space $[0,1]^{\mathbb{F}_n}$ via the *Bernoulli shift*: $(\gamma \cdot x)(\delta) = x(\gamma^{-1}\delta)$ restricted to its free part. Let $G_n = G(a_n, S_n)$ be the Schreier graph of this action, and let $\mu_n = \lambda^{\mathbb{F}_n}$ be the product of Lebesgue measure λ on [0, 1]. Since G_n is acyclic, the classical chromatic number is $\chi(G_n) = 2$. However, $\chi_{\mu_n}(G_n) \geq \frac{n}{\log 2n}$ which can be shown using results about the size of independent sets in random $(2n)$-regular graphs and an ultraproduct argument. This argument was first suggested by [36]; see [30] for a detailed proof. Bernshteyn has recently proven an upper bound on $\chi_{\mu_n}(G_n)$ which is within a factor of two of this lower bound [1]. However, it remains an open problem to compute the precise rate of growth of $\chi_{\mu_n}(G_n)$.

Bernshteyn's Theorem 2.5 and the above upper bound on $\chi_{\mu_n}(G_n)$ are based on an adaptation of the powerful Lovász Local Lemma (LLL) to the setting of measurable combinatorics. The LLL is a tool of probabilistic combinatorics which can show the existence of objects which are described by constraints that are local in the sense that each constraint is independent of all but a small number of other constraints, and each constraint has a high probability of being satisfied. Precisely, the *symmetric LLL* states that if $A_1, \ldots, A_n$ are events in a probability space which each occur with probability at most p, each event A_i is independent of all but at most d of the other events, and $ep(d+1) \leq 1$, then there is a positive probability none of these events occur.

The LLL is a pure existence result, and since the desired object typically exists with exponentially small probability, it was a major open problem to find an algorithmic way to quickly find satisfying assignments where none of the events $A_1, \ldots, A_n$ happen. In particular, a naive attempt to randomly sample from the probability distribution until a solution is found would take at least exponential time. In a breakthrough result in 2009, Moser and Tardos [42] gave an efficient randomized algorithm that can quickly compute satisfying assignments for the LLL.

Adaptations of the Moser–Tardos algorithm and the LLL to the setting of measurable combinatorics began with work of Kun [33], who used a version of the Moser–Tardos algorithm to find spanning subforests to prove a strengthening of the Gaboriau–Lyons [20] theorem in ergodic theory. More recently, Csoka, Grabowski, Mathe, Pikhurko, and Tyros [14] have proved a Borel version of the symmetric LLL for Borel graphs of subexponential growth, and Bernshteyn has proved μ-measurable versions for Bernoulli shifts of groups, and probability measure preserving Borel graphs [1, 2]. These results, combined with the large literature in combinatorics using the LLL to construct colorings of graphs, are the main tool in the proof of Theorem 2.5.

It is known that there cannot be a Borel version of the symmetric LLL for bounded degree Borel graphs in general [12]. Indeed, the existence of such a theorem combined with standard coloring techniques using the LLL would contradict Theorem 2.2. However, an interesting special case remains open: a Borel version of the symmetric LLL for Borel Schreier graphs generated by Borel actions of *amenable* groups, which are defined in the next section. Such a version of the local lemma could be a useful tool for making progress on the open problems discussed in the next section.

The theorems we have described above are a small selection of what is now known about measurable chromatic numbers. We hope they give the reader some sense of the variety of results and tools of the subject.

3. CONNECTIONS WITH HYPERFINITENESS

A major research program in modern descriptive set theory has been to understand the relative complexity of equivalence relations under Borel reducibility. Precisely, if E and F are equivalence relations on standard Borel spaces X and Y, say that E is *Borel reducible* to F if there is a Borel function $f: X \to Y$ such that for all $x, y \in X$, we have $x \, E \, y \iff f(x) \, F \, f(y)$. Such a function induces a definable injection from X/E to Y/F. If we think of E and F as classification problems, this means E is simpler than F in the sense that any invariants that can be used to classify F can also be used to classify E. In the study of Borel reducibility of equivalence relations, there has been success both in understanding the abstract structure of all Borel equivalence relations under Borel reducibility, and also in proving particular nonclassification results of interest to working mathematicians. For example, Hjorth's theory of turbulence [26] gives a precise dichotomy for when an equivalence relation generated by a Polish group action can be classified by invariants that are countable structures, and turbulence has been applied to prove nonclassifiability results in C^* algebras [18].

A Borel equivalence relation E is said to be *countable* if every E-class is countable. The countable Borel equivalence relations are an important and well-studied subclass of Borel equivalence relations with rich connections with operator algebras and ergodic theory. One reason for this is the Feldman–Moore theorem [31, THEOREM 1.3], which states that every countable Borel equivalence relation is induced by a Borel action of a countable group. Results proved about the dynamics of measure preserving actions of countable groups have played a played an important role in our understanding of the theory of countable Borel equivalence relations.

Understanding how the descriptive-set-theoretic complexity of countable Borel equivalence relations is related to the dynamics of the group actions that generate them is a deep problem. An important simplicity notion for Borel reducibility is hyperfiniteness: a Borel equivalence relation is *hyperfinite* if it can be written as an increasing union of Borel equivalence relations whose classes are all finite. The hyperfinite equivalence relations are the simplest nontrivial class of Borel equivalence relations as made precise by the Glimm–Effros dichotomy of Harrington, Kechris, and Louveau [25]. Weiss has asked if the group-theoretic notion of amenability precisely corresponds to hyperfiniteness:

Problem 3.1 (Weiss, [50]). Suppose E is a Borel equivalence relation generated by a Borel action of a countable amenable group. Is E hyperfinite?

Amenability was defined by von Neumann in reaction to the Banach–Tarski paradox. It is a group-theoretic notion of dynamical tameness. Precisely, a group Γ is *amenable* if and only if for every $\varepsilon > 0$ and every finite $S \subseteq \Gamma$ there exists some nonempty finite $F \subseteq \Gamma$ such that $|SF \triangle F|/|F| < \varepsilon$. Such an F is called an (ε, S)-Følner set. Examples of amenable groups include finite, abelian, and solvable groups, while the free group on two generators is nonamenable. If Weiss's question has a positive answer, then amenability precisely characterizes hyperfiniteness since every nonamenable group has a nonhyperfinite Borel action.

Evidence that Weiss's question has a positive answer is given by a theorem in ergodic theory of Ornstein and Weiss [43] that every measure preserving action of an amenable group on a standard probability space is hyperfinite modulo a nullset.

Progress on Weiss's question has grown out of progress on the problem of finding Borel tilings of group actions by Følner sets. Precisely, if $a\colon \Gamma \curvearrowright X$ is an action of a finitely generated group Γ, and $F_1, \ldots, F_n \subseteq \Gamma$ are finite subsets of Γ, a *tiling* of a by the shapes $F_1, \ldots, F_n$ is a collection of subsets $A_1, \ldots, A_n \subseteq X$ so that the sets $F_1 \cdot A_1, \ldots, F_n \cdot A_n$ are pairwise disjoint and form a partition of X. Finding tilings of a group action can be thought of as a generalized coloring problem or constraint satisfaction problem of the type often studied in measurable combinatorics, and can be approached using many of the same tools. For example, Jackson, Kechris, and Louveau [27] have shown that Weiss's question has a positive answer for groups of polynomial volume growth. Their argument uses Voronoi regions around Borel maximal independent sets to make Borel tilings with desirable properties. Gao and Jackson [21] have shown that Weiss's question has a positive answer for abelian groups. Their argument centers around a more refined inductive argument to find tilings of $\mathbb{Z}^n$ by hyperrectangles. These tilings are found by iteratively adjusting the location of the boundaries of hyperrectangular tiles that cover the space until their parallel boundaries are far apart. Schneider and Seward have extended Gao and Jackson's machinery to all locally nilpotent groups [45]. All these tilings are the building blocks out of which witnesses to hyperfiniteness are constructed.

A positive answer to the following open problem would be progress towards a positive solution to Weiss's question:

Problem 3.2. Let Γ be an amenable group with finite symmetric generating set S and $a\colon \Gamma \curvearrowright X$ be a free Borel action of a on a standard Borel space X. For every $\varepsilon > 0$, do there exist (ε, S)-Følner sets $F_1, \ldots, F_n \subseteq \Gamma$ such that the action a has a Borel tiling with shapes $F_1, \ldots, F_n$?

The existence of such tilings without any measurability conditions was only recently established by Downarowicz, Huczek, and Zhang [15]. A key step in their proof is to use Hall's matching theorem to match untiled points in a Ornstein–Weiss style *quasitiling* [43] to construct an exact tiling. Recall that if $G = (V, E)$ is a graph, a *perfect matching* of G is a subset $M \subseteq E$ of edges so that each vertex $x \in V$ is incident to exactly one edge in M. Hall's matching theorem states that a bipartite graph with bipartition $A, B \subseteq V$ has a perfect matching if and only if for every finite set $F \subseteq A$ or $F \subseteq B$,

$$\left|N(F)\right| \geq |F|.$$

Recently, Problem 3.2 has been shown to have a positive answer modulo a nullset [10]. A key part of the proof is a measurable matching result proved using an idea of Lyons and Nazarov [36] that was originally used to find factor of i.i.d. perfect matchings of regular trees. Lyons and Nazarov's argument uses the *Hungarian matching algorithm* (repeatedly flipping augmenting paths) to show that if a bipartite Borel graph G satisfies a certain measure-theoretic expansion condition strengthening Hall's condition, then it has a measurable perfect matching.

Conley, Jackson, Marks, Seward, and Tucker-Drob have proven the following:

Theorem 3.3 (Conley, Jackson, Marks, Seward, Tucker-Drob [11]). *Let Γ be a countable group admitting a normal series where each quotient of consecutive terms is a finite group or a torsion-free abelian group with finite $\mathbb{Q}$-rank, except that the top quotient can be any group of uniform local polynomial volume-growth or the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. Then every free Borel action of Γ is hyperfinite.*

By combining this with prior work of Seward and Schneider [45, COR. 8.2] they obtain the following corollary:

Corollary 3.4. *Weiss's question has a positive answer for polycyclic groups.*

This is the best partial result on Weiss's question that is currently known. Significantly, Corollary 3.4 applies to groups of exponential volume growth such as certain semidirect products of $\mathbb{Z}^n$. All the previous work on Weiss's question applied only to groups locally of polynomial volume growth, and this seemed an inherent limitation to previous methods.

The central idea of [11] is to adapt the machinery of Gromov's theory of asymptotic dimension of groups to the setting of descriptive set theory, making a theory of Borel asymptotic dimension. These ideas were implicitly hidden in all previous work on Weiss's question, but were first made explicit in [11]. Asymptotic dimension was introduced by Gromov as a quasiisometry invariant of metric spaces, used to study geometric group theory. The asymptotic dimension of a metric space (X, ρ) is the least d such that for every $r > 0$ there is a uniformly bounded cover U of X so that every closed r-ball intersects at most $d + 1$ sets in U. Essentially, asymptotic dimension is a "large-scale" analogue of Lebesgue covering dimension. There are actually several different ways to define asymptotic dimension whose equivalences are nontrivial to prove. Proving that these different definitions still define the same notion in the Borel context is one of the keys to the work in [11]. Alternate definitions allow the conversion between Voronoi cell-type tilings patterned on the work of Jackson, Kechris, and Louveau, and covers with far apart facial boundaries pioneered by Gao and Jackson.

Resolving Weiss's question for all amenable groups appears to be a difficult problem. In general, we have a poor understanding of the geometry and structure of Følner sets in arbitrary amenable groups. Problem 3.1 for arbitrary amenable groups seems to either require significant advances in our geometric understanding of amenable groups, or completely different descriptive-set theoretic tools for attacking the hyperfiniteness problem. One question which gets at the heart of this difficulty is the following:

Problem 3.5. Suppose G is a bounded degree Borel graph having uniformly bounded polynomial growth. Is the connectedness relation of G hyperfinite?

The obstacle in resolving Problem 3.5 is that while polynomial growth groups have tight both upper and lower bound on their growth, Problem 3.5 only posits an upper bound on the growth of G, which may consequently have much less uniformity in its growth than the Schreier graph associated to an action of a polynomial growth group. This lack of a lower bound on growth means that the techniques of Jackson, Kechris, and Louveau for proving hyperfiniteness of groups of polynomial growth cannot resolve Problem 3.5 Finding techniques for resolving Problem 3.5 where there is far less regular geometric structure would be one way of making progress towards resolving Weiss's question in general since we know little about any regular geometric structure in arbitrary amenable groups.

4. MEASURABLE EQUIDECOMPOSITIONS

If $a\colon \Gamma \curvearrowright X$ is an action of a group Γ on a space X, then we say sets $A, B \subseteq X$ are *a-equidecomposable* if there are a finite partition $\{A_0, \dots, A_n\}$ of A and group elements $\gamma_0, \dots, \gamma_n \in \Gamma$ so that $\gamma_0 A_0, \dots, \gamma_n A_n$ is a partition of B. For example, in this language, the Banach–Tarski paradox says that one unit ball is equidecomposable with two unit balls under the group action of isometries of $\mathbb{R}^3$. In the last few years several new results proved about these types of geometrical paradoxes with the unifying theme that the "paradoxical" sets in many classical geometrical paradoxes can surprisingly be much nicer than one would naively expect.

A classical generalization of the Banach–Tarski paradox states that any two bounded sets $A, B \subseteq \mathbb{R}^3$ with nonempty interior are equidecomposable. Of course, the pieces used in these equidecompositions must be nonmeasurable in general, since A and B may have different measure. However, a remarkable theorem of Grabowski, Máthé, and Pikhurko states that there is always an equidecomposition using measurable sets when A and B have the same Lebesgue measure.

Theorem 4.1 (Grabowski, Máthé, Pikhurko [24]). *If $A, B \subseteq \mathbb{R}^3$ are bounded sets with nonempty interior and if additionally A and B are assumed to have the same Lebesgue measure, then A and B can be equidecomposed using Lebesgue measurable pieces.*

It is an open problem whether Theorem 4.1 can be strengthened to yield a Borel equidecomposition, assuming A and B are Borel.

Key to Theorem 4.1 and other advances in measurable equidecompositions has been progress made on measurable matching problems. The connection comes from the following graph-theoretic reformulation of equidecompositions as perfect matchings. Let $a\colon \Gamma \curvearrowright X$ be a Borel action of a group Γ on a space X, let $A, B, \subseteq X$ be subsets of X, and let $S \subseteq \Gamma$ be finite. Let $G(A, B, S)$ be the graph whose set of vertices is the disjoint union $A \sqcup B$ and where $x \in A$ and $Y \in B$ are adjacent if there is a $\gamma \in S$ so that $\gamma \cdot x = y$. Then it is easy to see that A, B are equidecomposable using group elements from S if and only if there is a perfect matching of the graph $G(A, B, S)$.

Theorem 4.1 and other new results about measurable equidecompositions rely on combining process made on measurable matching problems with modern results about the dynamics of the group actions being studied. For example, Theorem 4.1 uses the local spectral gap of Boutonnet, Ioana, and Salehi Golsefidy [5] for certain lattices in the group $\mathsf{SO}_3(\mathbb{R})$ of rotations in $\mathbb{R}^3$. This result is used to check that the graph $G(A, B, S)$ satisfies the expansion condition of Lyons and Nazarov [36] which ensures the existence of a measurable matching.

Some other recent theorems about measurable equidecompositions concern Tarski's famous circle squaring problem from 1925: the question of whether a disk and square of the same area in $\mathbb{R}^2$ are equidecomposable by isometries. Tarski's circle squaring problem arose from the fact that the analogue of the Banach–Tarski paradox is false in $\mathbb{R}^2$. This is because there are so-called *Banach measures* in $\mathbb{R}^2$: finitely additive isometry-invariant measures that extend Lebesgue measure. Their existence is proved using the amenability of the isometry group of $\mathbb{R}^2$. Hence, if Lebesgue measurable sets $A, B \subseteq \mathbb{R}^2$ are equidecomposable, they must have the same Lebesgue measure. The real thrust of Tarski's circle squaring problem is the converse of this: the general problem of to what extent there is an equivalence between equidecomposability and having the same measure.

In 1990, Laczkovich [34] (see also [35]) gave a positive answer to Tarski's circle squaring problem using the Axiom of Choice. His proof involved sophisticated tools from Diophantine approximation and discrepancy theory to prove strong quantitative bounds on the ergodic theorem for translation actions of the torus, as well as the graph-theoretic approach to equidecomposition described above.

Marks and Unger have shown that there is a Borel solution to Tarski's circle squaring problem, building on an earlier result of Grabowski, Máthé, and Pikhurko, [23] that the circle can be squared using Lebesgue measurable pieces.

Theorem 4.2 (Marks, Unger [40]). *Tarski's circle squaring problem has a positive solution using Borel pieces. More generally, for all $n \geq 1$, if $A, B \subseteq \mathbb{R}^n$ are bounded Borel sets with the same positive Lebesgue measure whose boundaries have upper Minkowski dimension less than n, then A and B are equidecomposable using Borel pieces.*

Hence, for Borel sets whose boundaries are not wildly fractal, having the same measure is actually equivalent to having an explicitly definable Borel equidecomposition.

Theorem 4.2 uses new techniques for constructing Borel perfect matchings in Borel graphs based on first finding a real-valued Borel *flow* as an intermediate step. Precisely, if $f: V \to \mathbb{R}$ is a function on the vertices of a graph G, then an f-flow on G is a real-valued function ϕ on the edges of G such that $\phi(x, y) = -\phi(y, x)$ for every directed edge (x, y) of G, and such that for every $x \in V$ the flow ϕ satisfies Kirchoff's law,

$$f(x) = \sum_{y \in N(x)} \phi(x, y).$$

Given a circle and square $A, B \subseteq [0, 1)^2$ of the same area, the first step in the proof of Theorem 4.2 is finding an explicit $(1_A - 1_B)$-flow of an appropriate Borel graph whose vertices are all the elements of $[0, 1)^2$ and whose edges are generated by finitely many translations.

The advantage of working with the generality of flows is twofold. First, a flow can be constructed in countably many steps, making the error in Kirchoff's law above continuously approach 0 whereas the error in a partial matching that makes it imperfect is discrete. Second, the average of f-flows is an f-flow and so it is possible to integrate families of definable flows to get another definable flow. Finally, there are well known combinatorial equivalences between flows and matchings which are used in the last step of the proof of Theorem 4.2 to "round" a real-valued flow into an integer valued flow and then use it to construct a matching.

Another key ingredient in the proof of Theorem 4.2 is the hyperfiniteness of Borel actions of abelian groups. In particular, the proof of Theorem 4.2 uses a recent refinement due to Gao, Jackson, Krohne, and Seward [22] of Gao and Jackson's [21] theorem that Borel actions of abelian groups are hyperfinite. These witnesses to hyperfiniteness are used to ensure that the Ford–Fulkerson algorithm converges when it is used to round the Borel real-valued flow described above into a Borel integer-valued flow.

This flow approach to equidecomposition problems may be useful for attacking other open questions such as the Borel–Ruziewicz problem:

Problem 4.3 (Wagon [49]). Suppose $n \geq 2$. Is Lebesgue measure the unique finitely additive rotation invariant probability measure defined on the Borel subsets of the n-sphere S^n?

This question is inspired by a theorem of Margulis [37] and Sullivan [46] ($n \geq 4$), and Drinfeld [16] ($n = 2, 3$), who proved that Lebesgue measure is the unique finitely additive rotation invariant measure on the Lebesgue measurable subsets of S^n. Wagon's proposed strengthening would be a more natural result since the Borel sets are the canonical σ-algebra to measure.

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