THE PARIS–HARRINGTON PRINCIPLE AND SECOND-ORDER ARITHMETIC—BRIDGING THE FINITE AND INFINITE RAMSEY THEOREM

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ABSTRACT

The *Paris–Harrington principle* (PH) is known as one of the earliest examples of "mathematical" statements independent from the standard axiomatization of natural numbers called *Peano Arithmetic (*PA*)*. In this article, we discuss various variations of PH and examine the relations between finite and infinite Ramsey's theorem and systems of arithmetic.

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1. INTRODUCTION

To prove a statement about natural numbers, we usually rely explicitly or implicitly on reasoning by mathematical induction. In the setting of mathematical logic, the axiomatic system for natural numbers consists of the axioms for discrete ordered semirings and the scheme of mathematical induction, which is known as *Peano Arithmetic* (PA). Within PA, one can prove many theorems in number theory or finite combinatorics, such as the existence of infinitely many prime numbers or the following finite Ramsey theorem (FRT):

(FRT) For any $n, k, m, a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $f : [[a, b)_{\mathbb{N}}]^n \to k$ there exist $H \subseteq [a, b)_{\mathbb{N}}$ and $c < k$ such that $[H]^n \subseteq f^{-1}(c)$ and $|H| = m$.

(Here, $[a, b)_{\mathbb{N}} = \{x \in \mathbb{N} : a \leq x < b\}$ and $[X]^n = \{F \subseteq X : |F| = n\}$ where $|F|$ denotes the cardinality of F. We write k for the set $[0, k)_{\mathbb{N}}$.) Thus, the question might arise: can we prove all true numerical statements within PA?

The answer is known to be negative. The famous *incompleteness theorem* by Kurt Gödel says that there is a numerical statement which is independent from PA (i.e., cannot be proved or disproved from PA). Such an independent statement is provided by diagonalization or self-reference as the liar paradox, and in particular, the numerical statement which intends to say "PA is consistent" is independent from PA. This leads to another question whether there is a "mathematical" statement which is independent from PA. The *Paris–Harrington principle* (PH) [33] is one of the earliest and most important such examples. It is a variant of the finite Ramsey theorem which states the following:

(PH) For any $n, k, a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $f : [[a, b)_{\mathbb{N}}]^n \to k$ there exist $H \subseteq [a, b)_{\mathbb{N}}$ and $c < k$ such that $[H]^n \subseteq f^{-1}(c)$ and $|H| > \min H$.

Here, a set H is said to be *relatively large* if $|H| > \min H$, so PH says "for any $a \in \mathbb{N}$, there exists a large enough finite set X above a such that any coloring on X for the Ramsey theorem has a solution which is relatively large." By some standard coding of finite sets of natural numbers as single natural numbers (e.g., by binary expansion), PH can be considered as a purely numerical statement. By easy combinatorics, one can prove PH from the infinite Ramsey theorem (RT), thus PH is a *true* statement about natural numbers.

So how can we know that PH is not provable from PA? The reason is again provided by the Gödel incompleteness, namely, $\mathsf{PA} + \mathrm{PH}$ implies the consistency of PA and thus it is not provable from PA. Indeed, Paris and Harrington showed that PH is equivalent over PA to the correctness of PA with respect to $\forall\exists$-sentences (the statement "any $\forall\exists$-sentence provable from PA is true"), which is a strengthening of the consistency of PA.

On the other hand, many variants of the infinite Ramsey theorem are widely studied in the setting of second-order arithmetic. This is one of the central topics in the project named *reverse mathematics* whose ultimate goal is to determine the logical strength of mathematical theorems in various fields and classify them from viewpoints of several fields in logic. Typically, the strength of variants of the infinite Ramsey theorem is precisely calibrated from the viewpoints of computability and proof theory. Particularly, precise analyses for variants of the Paris–Harrington principle are important approaches to identify the consistency strength of variants of the infinite Ramsey theorem.

In this article, we will overview the relations between the Paris–Harrington principle, the infinite Ramsey theorem and correctness statements (also known as reflection principles) mainly in the setting of second-order arithmetic. For this purpose, we will work with nonstandard models of arithmetic and relate the finite and infinite Ramsey theorem in them. A brief idea here is that if a nonstandard model satisfies some variant of finite Ramsey theorem with a solution of nonstandard size, then it should include a model for infinite Ramsey theorem. This can be realized by the theory of *indicators* introduced by Kirby and Paris [23]. We reformulate their argument and connect variants of PH with the correctness of the infinite Ramsey theorem.

The structure of this article is the following. In Section 2, we set up basic definitions and review the studies on the Ramsey theorem in arithmetic. We give several formulations of the Paris–Harrington principle and their equivalents within second-order arithmetic in Sections 3 and 4. In Section 5, we see how the Paris–Harrington principle is related to the infinite Ramsey theorem by means of indicators. Some proofs in Section 5 require basic knowledge of nonstandard models of arithmetic.

2. FIRST- AND SECOND-ORDER ARITHMETIC AND THE RAMSEY THEOREM

In this section, we introduce fragments of first- and second-order arithmetic and set up basic definitions. For precise definitions, basic properties and other information, see, e.g., [16,21] for first-order arithmetic and [17,39] for second-order arithmetic.

We write $\mathcal{L}_1$ for the language of first-order arithmetic, which consists of constants 0, 1, function symbols $+$, $\times$, and binary relation symbols $=$, $\leq$, and write $\mathcal{L}_2$ for the language of second-order arithmetic which consists of $\mathcal{L}_1$ plus another binary relation $\in$. We use $x, y, z, \dots$ for first-order (number) variables and $X, Y, Z, \dots$ for second-order (set) variables. An $\mathcal{L}_2$-formula φ is said to be *bounded* or Σ^0_0 if it does not contain any second-order quantifiers and all first-order quantifiers are of the form $\forall x \leq t$ or $\exists x \leq t$, and it is said to be Σ^0_n (resp. Π^0_n) if it is of the form $\exists x_1 \forall x_2 \dots Q x_n \theta$ (resp. $\forall x_1 \exists x_2 \dots Q x_n \theta$) where θ is Σ^0_0. An $\mathcal{L}_2$-formula φ is said to be *arithmetical* or Σ^1_0 if it does not contain any second-order quantifiers, and it is said to be Σ^1_n (resp. Π^1_n) if it is of the form $\exists X_1 \forall X_2 \dots Q X_n \theta$ (resp. $\forall X_1 \exists X_2 \dots Q X_n \theta$) where θ is Σ^1_0. If a Σ^0_n-formula (resp. Π^0_n-formula) φ does not contain any set variables (i.e., φ is an $\mathcal{L}_1$-formula), it is said to be Σ_n (resp. Π_n). We can extend $\mathcal{L}_1$ with unary relation symbols $\vec{U} = U_1, \dots, U_k$. Here, we identify U_i's as second-order (set) constants and consider $\mathcal{L}_1 \cup \vec{U}$-formulas as Σ^1_0-formulas (with extra constants). Then, an $\mathcal{L}_1 \cup \vec{U}$-formula is said to be $\Sigma^{\vec{U}}_n$ (resp. $\Pi^{\vec{U}}_n$) if it is Σ^0_n (resp. Π^0_n).

For our discussions, we need to distinguish the actual ("standard") natural numbers from natural numbers formalized in axiomatic systems. Here, we use $\mathfrak{N}$ for the set of standard natural numbers, and $\mathbb{N}$ for natural numbers formalized in the system. When we write

"$n = 2, 3, 4, \ldots$," it is intended that n ranges over $\mathfrak{N}$ and $n \geq 2$, while "$n \geq 2$" means that n ranges over $\mathbb{N}$ and $n \geq 2$.

2.1. The Paris–Harrington principle in first-order arithmetic

We adopt the *elementary function arithmetic* (EFA) for our base system of first-order arithmetic. It consists of the axioms of discrete ordered semirings, the totality of exponentiation[1] and the *induction axiom (*IND*)* of the form

$$\text{(IND)}\quad \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$$

for each Σ_0-formula $\varphi(x)$. Then, the system $\mathsf{I}\Sigma_n$ is defined as EFA plus the induction axioms for Σ_n-formulas, and the *Peano arithmetic* (PA) is defined as $\mathsf{PA} = \bigcup_{n \in \mathfrak{N}} \mathsf{I}\Sigma_n$. We may also expand EFA with unary predicates. If $\vec{U} = U_1, \ldots, U_k$ are unary predicates, $\mathsf{EFA}^{\vec{U}}$ consists of EFA plus the induction axioms for $\Sigma_0^{\vec{U}}$-formulas.

Within EFA, finite sets of natural numbers, finite sequences of natural numbers, functions on finite sets, or other finite objects on $\mathbb{N}$ are coded by numbers. We write $[\mathbb{N}]^{<\mathbb{N}}$ for the set of all (codes) of finite subsets of $\mathbb{N}$. For each $F \in [\mathbb{N}]^{<\mathbb{N}}$, we can define $|F|$ as the (unique) smallest $m \in \mathbb{N}$ such that there is a bijection between F and $m = [0, m)_{\mathbb{N}}$. In the context of the Ramsey theorem, a function of the form $c : [X]^n \to k$ is often called a *coloring*. (Recall that $[X]^n = \{F \in [\mathbb{N}]^{<\mathbb{N}} : |F| = n \wedge F \subseteq \mathbb{N}\}$.) Then, a set $H \subseteq X$ is said to be c*-homogeneous* if there exists $i < k$ such that $[H]^n \subseteq c^{-1}(i)$.

We first define the key notion introduced by Paris **[32]**. The following definition can be made within EFA.

Definition 2.1 (Density). Let $n \geq 1$ or $n = \infty$ and $k \geq 2$ or $k = \infty$. For given $m \in \mathbb{N}$, we define m*-density for* (n, k) as follows:

- a finite set F is said to be 0-dense(n, k) if $|F| > \min F$ (F is relatively large),
- a finite set F is said to be $(m+1)$-dense(n, k) if for any $c : [F]^{n'} \to k'$ where $n' \leq \min\{n, \min F\}$ and $k' \leq \min\{k, \min F\}$, there exists a c-homogeneous set $H \subseteq F$ such that H is m-dense(n, k). (Here, we set $\min\{\infty, a\} = a$ for $a \in \mathbb{N}$.)

Although the notion is defined inductively, the statement that F is m-dense(n, k) is Σ_0, in other words, there exists a Σ_0-formula $\psi(n, k, F, m)$ such that $\psi(n, k, F, m)$ holds if and only if F is m-dense(n, k).

Definition 2.2 (The Paris–Harrington principle). Let $n \geq 1$ or $n = \infty$, $k \geq 2$ or $k = \infty$ and $m \in \mathbb{N}$. Then, the *Paris–Harrington principle,* $m\mathrm{PH}^n_k$ and ItPH^n_k, is defined as follows:

- $m\mathrm{PH}^n_k$: $\forall a \exists b \geq a([a, b)_{\mathbb{N}}$ is m-dense$(n, k))$.
- $\mathrm{ItPH}^n_k :\equiv \forall m\, m\mathrm{PH}^n_k$.

1 Technically, it is not easy (but possible) to define the exponential function in this setting, see **[16]**. Alternatively, one may safely add an extra function symbol $\exp(x) = 2^x$ and its recursive definition.

We simply write PH^n_k for $1\mathrm{PH}^n_k$. Additionally, we usually omit ∞ and write PH^n for PH^n_∞, PH for $\mathrm{PH}^\infty_\infty$, and so on.

It is known that $\mathsf{I}\Sigma_1$ proves $\mathrm{PH}^{n+1}_2 \to \mathrm{PH}^n$. Thus there is a hierarchy of implications

$$\mathrm{PH}^1 \le \mathrm{PH}^2_2 \le \mathrm{PH}^2_3 \le \cdots \le \mathrm{PH}^2 \le \mathrm{PH}^3_2 \le \mathrm{PH}^3_3 \le \cdots \le \mathrm{PH}^3 \le \mathrm{PH}^4_2 \le \cdots.$$

It is known that this hierarchy is strict above PH^2 over $\mathsf{I}\Sigma_1$, whereas $\mathsf{I}\Sigma_n$ proves PH^{n+1}_k for $k = 2, 3, \ldots$ On the other hand, calibrating the strength of $m\mathrm{PH}^n_k$ for $m \ge 2$ is much harder, except for the implication $m\mathrm{PH}^n_2 \to \mathrm{PH}^n_{m+1}$ which directly follows from the definition.

We next formalize the correctness of theories of arithmetic. Within EFA, basic notions of first-order logic such as (well-formed) formulas, formal proofs (by the Hilbert-style proof system or other formal systems) are formalizable by means of Gödel numbering. Typically, we can encode the provability for first- and second-order arithmetic within EFA, namely, there exists a Σ_1-formula $\mathrm{Prov}(T, x)$ which means that a formula (encoded by) x is provable from a theory (i.e., a finite or recursive set of sentences) T.[2] On the other hand, we can also formalize the truth on $\mathbb{N}$, but only partially. By formalizing Tarski's truth definition, for each tuples of variables $\vec{Z}$ and $\vec{z}$, there exists a Π^0_1-formula $\pi(\vec{Z}, \vec{z}, x)$ such that for any unary predicates $\vec{U}$ and a $\Sigma^{\vec{U}}_0$-formula $\varphi(\vec{z})$, $\mathsf{EFA}^{\vec{U}}$ proves $\forall \vec{z}(\pi(\vec{U}, \vec{z}, \ulcorner\varphi\urcorner) \leftrightarrow \varphi(\vec{z}))$ where $\ulcorner\varphi\urcorner$ is the Gödel number encoding φ. Then, for $n = 1, 2, \ldots,$ there exists a Π^0_n-formula $\mathrm{Tr}_n(\vec{Z}, \vec{z}, x)$ such that for any unary predicates $\vec{U}$ and a $\Pi^{\vec{U}}_n$-formula $\varphi(\vec{z})$, $\mathsf{EFA}^{\vec{U}}$ proves $\forall \vec{z}(\mathrm{Tr}_n(\vec{U}, \vec{z}, \ulcorner\varphi\urcorner) \leftrightarrow \varphi(\vec{z}))$. This formula is called the Π_n*-truth predicate*. The formalized correctness statements (also known as reflection principles) are defined as follows. (Formally, π and Tr_n depend on the number of variables, but we may assume that $\vec{Z}$ and $\vec{z}$ contains all variables which will appear in the entire discussion. We may ignore variables not appearing in the formula encoded by x by substituting 0 into them.)

Definition 2.3 (Correctness). Let $n = 1, 2, \ldots,$ and let T be an $\mathcal{L}_1$- or $\mathcal{L}_2$-theory. Then the Π_n-correctness of T (Π_n-corr(T)) is the following statement:

$$\forall x(\text{“}x \text{ is (a Gödel number of) a } \Pi_n\text{-sentence”} \land \mathrm{Prov}(T, x) \to \mathrm{Tr}_n(x)).$$

Note that Π_n-corr(T) is a Π_n-statement, and it implies the consistency of T since it implies $\neg(0 = 1) \to \neg\mathrm{Prov}(T, \ulcorner 0 = 1\urcorner)$.

Now we are ready to state the theorem by Paris and Harrington.

Theorem 2.1 (Paris and Harrington **[32, 33]**). *The following are equivalent over* $\mathsf{I}\Sigma_1$[3]*:*

1. PH.
2. ItPH^n_k *($n = 3, 4, \ldots,$ $k = 2, 3, \ldots$ or $k = \infty$).*
3. Π_2-corr(PA).

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2 We encode T, e.g., by its recursive index.

3 In **[32]**, Paris showed that ItPH^3_2 is independent of PA, while his argument implies the equivalence of statements 2 and 3. See Section 5.2.

Here, ItPH^3_2 is the original statement independent of PA introduced by Paris [32]. The equivalence of ItPH^3_2 and PH can be proved in a combinatorial way, while we see that both are equivalent to Π_2-corr(PA) in Section 5. Moreover, the Π_2-correctness of fragments of PA can be characterized by PH as well.

Theorem 2.2 (Paris, see [16]). *Let* $n = 1, 2, \ldots$. *Then* Π_2-corr$(\mathsf{I}\Sigma_n)$ *is equivalent to* PH^{n+1} *over* $\mathsf{I}\Sigma_1$.

There are many other combinatorial or other numerical principles known to be independent of PA such as the Kanamori–McAloon theorem (KM) [20] and the termination of the Goodstein sequence [15]. Many of them are equivalent to the Π_2-correctness of PA, while some others are strictly stronger. A typical such example is a finite variant of Kruskal's tree theorem introduced by Friedman. See [13, 38].

2.2. Second-order arithmetic and the infinite Ramsey theorem

The system of second-order induction $\mathsf{I}\Sigma^i_n$ consists of EFA plus the induction axioms for Σ^i_n-formulas. It is not difficult to see that $\mathsf{I}\Sigma^0_n$ is a conservative extension of $\mathsf{I}\Sigma_n$, in other words, they prove the same $\mathcal{L}_1$-sentences. Our base system for second-order arithmetic is RCA_0, which consists of $\mathsf{I}\Sigma^0_1$ plus the following *recursive comprehension axiom (*RCA*)*: for each pair of Σ^0_1-formulas $\varphi(x)$, $\psi(x)$,

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)).$$

The next system is WKL_0, which consists of RCA_0 plus *weak Kőnig's lemma (*WKL*)*. Here, we define WKL in a slightly stronger form (but still equivalent to the original definition over RCA_0, see [39, Lemma IV.1.4]). A *tree* T is a family of functions of the form $p : [0, m)_{\mathbb{N}} \to \mathbb{N}$ ($m \in \mathbb{N}$) such that for any $p \in T$ and $\ell \in \mathbb{N}$ with $[[0, \ell)_{\mathbb{N}}]^n \subseteq \mathrm{dom}(p)$, $p \restriction [[0, \ell)_{\mathbb{N}}]^n$ is also a member of T. A tree T is said to be *bounded* if there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that $p(i) \le h(i)$ for any $p \in T$ and $i \in \mathrm{dom}(p)$. Then WKL asserts the following:

> for any infinite bounded tree T, there exists a function (a *path of* T) f such that $f \restriction [0, m)_{\mathbb{N}} \in T$ for any $m \in \mathbb{N}$.

Finally, the system ACA_0 consists of RCA_0 plus the *arithmetical comprehension axiom (*ACA*)*: for each Σ^1_0-formula $\varphi(x)$,

$$\exists X \forall x(x \in X \leftrightarrow \varphi(x)).$$

The strength of these three systems is precisely known and WKL_0 is strictly in-between RCA_0 and ACA_0. On the other hand, the $\mathcal{L}_1$-consequences of RCA_0 and WKL_0 are the same and they coincide with those of $\mathsf{I}\Sigma_1$, while the $\mathcal{L}_1$-consequences of ACA_0 coincide with those of PA.

Over RCA_0, the infinite Ramsey theorem is directly formalizable as follows.

Definition 2.4 (The infinite Ramsey theorem). The *infinite Ramsey theorem* RT^n_k is defined as follows:

- RT^n_k: for any $c : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is c-homogeneous ($n \geq 1$ and $k \geq 2$).
- $\mathrm{RT}^n_\infty :\equiv \forall k \mathrm{RT}^n_k$, $\mathrm{RT}^\infty_\infty :\equiv \forall n \mathrm{RT}^n_\infty$.

We usually omit ∞ and write RT^n for RT^n_∞, RT for $\mathrm{RT}^\infty_\infty$.

Within RCA_0, it is known that RT^n_k implies RT^n_{k+1} and RT^{n+1}_2 implies RT^n. Be aware that the former does not imply $\mathrm{RT}^n_2 \to \mathrm{RT}^n$ because of the lack of induction. So, we have the hierarchy

$$\mathrm{RT}^1_2 \leq \mathrm{RT}^1 \leq \mathrm{RT}^2_2 \leq \mathrm{RT}^2 \leq \mathrm{RT}^3_2 \leq \cdots .$$

However, this hierarchy collapses at the level of $n = 3$.

Theorem 2.3 (Jockusch [19], reformulated by Simpson [39]). *Let $n = 3, 4, \ldots,$ and let $k = 2, 3, \ldots$ or $k = \infty$. Then, over RCA_0, RT^n_k is equivalent to ACA_0.*

On the other hand, the full infinite Ramsey theorem RT is strictly stronger than ACA_0. This is unavoidable since RT implies PH over RCA_0, and thus it implies the consistency of PA. To prove RT, we need the system ACA_0' which consists of ACA_0 plus the assertion that for any $n \in \mathbb{N}$ and any set X, the nth Turing jump of X exists.

Theorem 2.4 (McAloon [29], see also [17]). *Over RCA_0, RT is equivalent to ACA_0'.*

The situations of RT^2_2 and RT^2 are complicated. There are many important results on the reverse mathematical and computability theoretic strength of RT^2_2 or RT^2 such as [7,8,30,37]. Typically, RT^2_2 and RT^2 are strictly in between RCA_0 and ACA_0, but still different from WKL_0 even with full induction.

Theorem 2.5 (Jockusch [19], Liu [28]). *RT^2_2 and RT^2 are incomparable with WKL_0 over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_\infty$ (where $\mathsf{I}\Sigma^i_\infty = \{\mathsf{I}\Sigma^i_n : n \in \mathfrak{N}\}$).*

The Π^1_1-consequences (or equivalently, $\mathcal{L}_1$-consequences with second-order constants) of RT^2_2 and RT^2 are also studied precisely. A Π^1_n-formula $\forall X_1 \ldots QX_n\theta$ is said to be *restricted* Π^1_n *(*$\mathrm{r}\Pi^1_n$*)* if θ is Σ^0_2 and n is odd or θ is Π^0_2 and n is even, and $\mathrm{r}\Sigma^1_n$-formulas are defined in the dual way.

Theorem 2.6.

1. *$\mathsf{RCA}_0 + \mathrm{RT}^2_2$ proves $\mathsf{B}\Sigma^0_2$ and it is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ (i.e., any Π^1_1-sentences which are provable from $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ are provable from $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$). (Hirst [18] and Cholak/Jockusch/Slaman [7])*[4]
2. *$\mathsf{RCA}_0 + \mathrm{RT}^2_2$ is $\mathrm{r}\Pi^1_1$-conservative over RCA_0. (Patey/Yokoyama [34], see also Kołodziejczyk/Yokoyama [25])*
3. *$\mathsf{RCA}_0 + \mathrm{RT}^2$ proves $\mathsf{B}\Sigma^0_3$ and it is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_3$. (Hirst [18] and Slaman/Yokoyama [40])*

4 $\mathsf{B}\Sigma^0_n$ is called a *bounding principle*, see [16] for the definition.

The above theorem decides the consistency strength (or proof-theoretic strength) of RT^2_2 and RT^2, and more precise studies have been carried out for RT^2_2 with respect to the size of proofs [24,25]. However, the exact $\mathcal{L}_1$-consequences of $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ are still not identified. Meanwhile, several hybrid approaches of computability and proof/model-theory are currently being developed such as [9,10] which may help to calibrate the $\mathcal{L}_1$-consequences of various combinatorial principles.

3. THE PARIS–HARRINGTON PRINCIPLE IN SECOND-ORDER ARITHMETIC

In this section, we consider the Paris–Harrington principle in the setting of second-order arithmetic. The main difference is that we can now consider the Paris–Harrington principle within an infinite set. Then, Theorems 2.1 and 2.2 are reformulated as Theorems 3.2–3.6.

3.1. Second-order formulations of PH

Recall that PH^n_k asserts that there exists an arbitrary large finite set which is 1-dense(n,k). Indeed, a 1-dense(n,k) set should exist within any infinite subset of $\mathbb{N}$ by the infinite Ramsey theorem (see the proof of Proposition 3.1 below). We reformulate PH^n_k based on this idea in second-order arithmetic.

Definition 3.1 (The Paris–Harrington principle, second-order form). Let $n \geq 1$ or $n = \infty$, $k \geq 2$ or $k = \infty$ and $m \in \mathbb{N}$. Then, the *Paris–Harrington principle, $m\overline{\mathrm{PH}}^n_k$* and $\mathrm{It}\overline{\mathrm{PH}}^n_k$, is defined as follows:

- $m\overline{\mathrm{PH}}^n_k$: for any infinite set X_0, there exists a finite set $F \subseteq X_0$ such that F is m-dense(n,k).
- $\mathrm{It}\overline{\mathrm{PH}}^n_k :\equiv \forall m\, m\overline{\mathrm{PH}}^n_k$.

Just like for PH, we write $\overline{\mathrm{PH}}^n_k$ for $1\overline{\mathrm{PH}}^n_k$, $\overline{\mathrm{PH}}^n$ for $\overline{\mathrm{PH}}^n_\infty$, and so on.

We first see that any of these variants of the Paris–Harrington theorem are true since they are consequences of the infinite Ramsey theorem by the following "compactness" argument.

For given $n \geq 1$ and $k \geq 2$, an (n,k)*-coloring tree* T on a set X is a family of functions of the form $p : [m \cap X]^n \to k$ $(m \in \mathbb{N})$ such that for any $p \in T$ and $\ell \in \mathbb{N}$ with $[\ell \cap X]^n \subseteq \mathrm{dom}(p)$, $p\restriction[\ell \cap X]^n$ is also a member of T. Then, WKL_0 proves that any infinite (n,k)-coloring tree T on an infinite set X has a path $f : [X]^n \to k$ in the sense that $f\restriction[m \cap X]^n \in T$ for any $m \in \mathbb{N}$.

Proposition 3.1. *Let $n \geq 1$ or $n = \infty$, $k \geq 2$ or $k = \infty$ and $m \in \mathbb{N}$. $\mathsf{WKL}_0 + \mathrm{RT}^n_k$ proves $m\overline{\mathrm{PH}}^n_k \to m + 1\overline{\mathrm{PH}}^n_k$. In particular, $\mathsf{WKL}_0 + \mathrm{RT}^n_k$ proves $\overline{\mathrm{PH}}^n_k$, and $\mathsf{WKL}_0 + \mathrm{RT}^n_k + \mathsf{I}\Sigma^1_1$ proves $\mathrm{It}\overline{\mathrm{PH}}^n_k$.*

Proof. We prove for the case $n \geq 1$ and $k \geq 2$. Assume that $m + 1\overline{\mathrm{PH}}^n_k$ fails on some infinite set X. Let T be an (n, k)-coloring tree on X such that $p \in T$ if and only if there is no p-homogeneous set which is m-dense(n, k). Then, T is infinite since any finite subset of X is not $m + 1$-dense(n, k), and thus it has a path $f : [X]^n \to k$. By RT^n_k, there is an infinite set $H \subseteq X$ which is f-homogeneous. Then $m\mathrm{PH}^n_k$ fails on H by the definition of f. ■

Proving $\overline{\mathrm{PH}}^n_k$ just from the induction is much harder, but if $n = 1, 2, \ldots$, $\mathsf{I}\Sigma^0_n$ still proves $\overline{\mathrm{PH}}^{n+1}_k$ for $k = 2, 3, \ldots$ On the other hand, stronger induction does not help with the absence of the infinite Ramsey theorem. Indeed, $\mathsf{RCA}_0 + \mathsf{I}\Sigma^1_\infty$ does not prove $\overline{\mathrm{PH}}$ or even PH.[5]

Within RCA_0, the statement of $\mathrm{r}\Pi^1_n$-correctness of a theory T ($\mathrm{r}\Pi^1_n$-corr(T)) can be defined like in Definition 2.3, and $\mathrm{r}\Pi^1_n$-corr(T) is an $\mathrm{r}\Pi^1_n$-statement. Second-order versions of the Paris–Harrington principle are closely related to $\mathrm{r}\Pi^1_1$-correctness of the infinite Ramsey theorem and other systems, and also related to well-orderedness of ordinals, which is naturally formalizable within RCA_0. Here we summarize the relations between the Paris–Harrington principles, $\mathrm{r}\Pi^1_1$-correctness and well-foundedness of ordinals.

Theorem 3.2. *The following are equivalent over* RCA_0*:*

1. $\overline{\mathrm{PH}}^2$.
2. $\mathrm{It}\overline{\mathrm{PH}}^2_2$.
3. $\mathrm{r}\Pi^1_1$-corr$(\mathsf{I}\Sigma^0_1)$.
4. $\mathrm{r}\Pi^1_1$-corr$(\mathsf{WKL}_0 + \mathrm{RT}^2_2)$.
5. *Well-foundedness of* ω^ω.

Theorem 3.3. *The following are equivalent over* RCA_0*:*

1. $\overline{\mathrm{PH}}^3$.
2. $\mathrm{It}\overline{\mathrm{PH}}^2$.
3. $\mathrm{r}\Pi^1_1$-corr$(\mathsf{I}\Sigma^0_2)$.
4. $\mathrm{r}\Pi^1_1$-corr$(\mathsf{WKL}_0 + \mathrm{RT}^2)$.
5. *Well-foundedness of* ω^{ω^ω}.

Theorem 3.4. *The following are equivalent over* RCA_0 *(for* $n = 1, 2, \ldots$*):*

1. $\overline{\mathrm{PH}}^{n+1}$.
2. $\mathrm{r}\Pi^1_1$-corr$(\mathsf{I}\Sigma^0_n)$.
3. *Well-foundedness of* ω_{n+1}.

5 Indeed, $\mathsf{WKL}_0 + \mathsf{I}\Sigma^1_\infty$ is a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_\infty$.

Theorem 3.5. *The following are equivalent over* RCA_0*:*

1. $\overline{\mathrm{PH}}$.
2. $\mathrm{It}\overline{\mathrm{PH}}^n_k$ *($n = 3, 4, \ldots, k = 2, 3, \ldots, \infty$).*
3. $\mathrm{r}\Pi^1_1$-corr(ACA_0).
4. *Well-foundedness of* ε_0.

Theorem 3.6. *Over* RCA_0, $\mathrm{It}\overline{\mathrm{PH}}$ *is equivalent to* $\mathrm{r}\Pi^1_1$-corr(ACA_0').

Over ACA_0, any Π^1_1-formula is equivalent to a $\mathrm{r}\Pi^1_1$-formula. Thus, $\mathsf{ACA}_0 + \overline{\mathrm{PH}}$ implies Π_n-corr(PA) for any $n \in \mathfrak{N}$, in other words, the $\mathcal{L}_1$-correctness schema of PA.

Many of the equivalences in the above theorems have been known to experts in one formulation or another for a long time, although at least some of them are hard to find in the literature. On the other hand, 3 $\leftrightarrow$ 4 of Theorems 3.2 and 3.3 are more recent, and not easy since they correspond to the study of the first-order strength of the infinite Ramsey theorem for pairs, which we have seen in Theorem 2.6. The equivalences between variants of PH and the well-orderedness of ordinals are obtained by measuring the largeness of finite sets using ordinals, as presented in the next subsection. In Section 5, we explain how to prove the equivalences between variants of PH and the correctness statements by the method of indicators.

3.2. PH and the notion of α-largeness

The Paris–Harrington principle is closely related to a notion of largeness for finite sets defined using ordinals. In [22], Ketonen and Solovay introduced the notion of α-largeness for ordinal $\alpha < \varepsilon_0$ and calibrated how large set is needed for PH.

Definition 3.2 (α-largeness, within RCA_0[6]). For $\alpha < \varepsilon_0$ and $m \in \mathbb{N}$, define $\alpha[m] = 0$ if $\alpha = 0$, $\alpha[m] = \beta$ if $\alpha = \beta + 1$, $\alpha[m] = \beta + \omega^\gamma \cdot m$ if $\alpha = \beta + \omega^{\gamma+1}$, and $\alpha[m] = \beta + \omega^{\gamma[m]}$ if $\alpha = \beta + \omega^\gamma$ and γ is a limit ordinal. Then a finite set $X = \{x_0 < \cdots < x_{\ell-1}\} \subseteq \mathbb{N}$ ($\{x_i\}_i$ is the increasing enumeration of X) is called *α-large* if $\alpha[x_0]\ldots[x_{\ell-1}] = 0$.

The well-foundedness of ordinals and the notion of α-largeness is closely related. Indeed, if α is well-founded and $X = \{x_0 < x_1 < \cdots\}$ is infinite, then $\alpha[x_0][x_1]\ldots$ should terminate at 0 within finitely many steps, which means that X contains an α-large set. It is not difficult to see the converse, and we have the following.

Proposition 3.7. *Let* $\alpha < \varepsilon_0$. *The following assertions are equivalent over* RCA_0*:*

1. *Any infinite set contains an α-large finite subset.*
2. *α is well-founded.*

6 Indeed, this definition still works within EFA with primitive recursive descriptions of ordinals.

The relations between PH and α-largeness are well-studied and have been the topic of ordinal analysis; see, e.g., [3–5, 22, 25, 27, 41]. Here we list several (digested) results from those papers. Let $\omega_0^\alpha = \alpha$ and $\omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}$, and let $\omega_n = \omega_n^1$.

Theorem 3.8. *The following are provable within* RCA_0. *Let $F \subseteq \mathbb{N}$ be a finite set with* $\min F \geq 3$, *and let $n, k \geq 1$ and $m \geq 0$.*

1. *If F is ω^{k+4}-large, then F is* 1-dense$(2, k)$. *(Ketonen/Solovay [22])*
2. *If F is* 1-dense$(2, k+1)$, *then F is ω^k-large. (folklore)*
3. *If F is $\omega_n^{\omega \cdot k+1}$-large, then F is* 1-dense$(n+1, k)$. *(essentially [22])*
4. *If F is* 1-dense$(n+1, 3^n)$, *then F is ω_n-large. (Kotlarski/Piekart/Weiermann [27])*
5. *If F is ω^{300^m}-large, then F is m-*dense$(2, 2)$. *(Kołodziejczyk/Yokoyama [25])*
6. *If F is ω_{3m+2}-large, then F is m-*dense$(3, 2)$. *(Bigorajska/Kotlarski [4])*

Many implications of Theorems 3.2–3.5 follow from the above theorem. Indeed, $1 \leftrightarrow 2 \leftrightarrow 5$ of Theorem 3.2 follows from statements 1, 2 and 5 of the above, and $5 \to 1$ of Theorem 3.3, $3 \to 1$ of Theorem 3.4, and $1 \leftrightarrow 4 \to 2$ of Theorem 3.5 follow from statements 3, 4, and 6. We see other implications in Section 5.

Well-foundedness of ordinals is also heavily related with correctness statements and their relations are widely studied. For the recent developments, see, e.g., [1, 31].

4. GENERALIZATIONS OF PH

In this section, we see several generalizations of the Paris–Harrington principle by modifying the relative largeness condition "$|H| > \min H$." They are still natural strengthenings of the finite Ramsey theorem and quickly follow from the infinite Ramsey theorem and a compactness argument of the kind presented in Proposition 3.1. Nonetheless, a strong enough form of the Paris–Harrington principle recovers the infinite Ramsey theorem (Theorem 4.5) and its iterations provide the rΠ_2^1-correctness of the infinite Ramsey theorem (Theorems 4.6–4.8).

4.1. Phase transition

A natural generalization of PH_k^n would be provided by changing the relative largeness condition $|H| > \min H$ to $|H| > f(\min H)$ for some function f. We write $\mathrm{PH}_{k,f}^n$ or $\overline{\mathrm{PH}}_{k,f}^n$ for the statement defined as PH_k^n or $\overline{\mathrm{PH}}_k^n$ but with $|H| > \min H$ replaced by $|H| > f(\min H)$. Unfortunately, this does not make PH stronger in most cases. Indeed, one can easily prove the following.

Proposition 4.1. 1. *Let $n = 2, 3, \ldots$ or $n = \infty$, and let f be a primitive recursive function. Then* $\mathsf{I}\Sigma_1 + \mathrm{PH}^n$ *proves* PH_f^n.

2. *Let f be a provably recursive function of* PA. *Then* $\mathsf{I}\Sigma_1 + \mathrm{PH}$ *proves* PH_f.

3. *Let $n = 1, 2, \dots$ or $n = \infty$ and let $k = 2, 3, \dots$ or $k = \infty$. Then* $\mathsf{RCA}_0 + \overline{\mathrm{PH}}^n_k$ *proves that for any function f, $\overline{\mathrm{PH}}^n_{k,f}$ holds.*

On the other hand, PH_f can be weaker if f is slower growing than the identity function. Indeed, if f is a constant function, then PH_f is just the finite Ramsey theorem, and thus it is provable within PA. Weiermann [44] revealed the border of the provability and unprovability in this context as part of his research program called *phase transition*.

Theorem 4.2 (Weiermann [44]). *Let $\log_n$ be the inverse function of the nth iterated exponential function $\exp^n(x)$ where $\exp(x) = 2^x$, and let $\log_*$ be the inverse function of the superexponential (tower) function 2_x.*

1. $\mathrm{PH}_{\log_n}$ *is not provable from* PA *for any $n \geq 1$.*

2. $\mathrm{PH}_{\log_*}$ *is provable from* PA.

A sharper border is revealed in [44], and similar analyses have been done for KM and other principles as well [35].

4.2. PH with generalized largeness

To obtain further generalization of PH, we want to consider some condition of the form $|H| > f(H)$ where f assigns some "required size" for each finite set. Inspired by Terrence Tao's blog [43], Gaspar and Kohlenbach [14] introduced several "finitary" versions of the infinite pigeonhole principle (RT^1 in our terminology) which are formulated based on this idea. Then, Pelupessy generalizes it to the infinite Ramsey theorem as follows.

Definition 4.1 (Gaspar/Kohlenbach [14], Pelupessy [36]). A function $f : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ is said to be *asymptotically stable* if for any increasing sequence of finite sets $F_0 \subseteq F_1 \subseteq \cdots$ $\{f(F_i)\}_{i\in\mathbb{N}}$ converges. Then, the finitary infinite Ramsey theorem FIRT^n_k states the following:

- FIRT^n_k: for any asymptotically stable function $f : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$, there exists $r \in \mathbb{N}$ such that for any $c : [[0, r)_\mathbb{N}]^n \to k$, there exists a homogeneous set $H \subseteq [0, r)_\mathbb{N}$ such that $|H| > f(H)$.
- $\mathrm{FIRT}^n_\infty \equiv \forall k \mathrm{FIRT}^n_k$, $\mathrm{FIRT}^\infty_\infty \equiv \forall n \mathrm{FIRT}^n_k$.

The finitary infinite pigeonhole principle FIPP_2 in [14] is the same as FIRT^1_∞.

Gaspar/Kohlenbach and Pelupessy showed that FIRT^n_k is equivalent to RT^n_k over WKL_0 (we will see this in detail later). Thus, FIRT^n_k could be considered as a "finitary" rephrasing of infinite combinatorics.

Remark 4.3. In [14], another form of the finitary infinite pigeonhole principle FIPP_3 is also studied, and the question is raised which is more appropriate as the finitary version of infinite pigeonhole principle. However, FIPP_3 is equivalent to ACA_0 [45], and it does not fit with the general form of the Ramsey theorem.

Then, can we consider more general statements? Remember that the original idea of the finite Ramsey theorem or the Paris–Harrington principle is that if a large enough set is given, one must find a homogeneous set which is still "large" in some sense. Here, we consider a general concept of largeness for finite sets as follows.

Definition 4.2 (Largeness notion). A family of finite sets $\mathbb{L} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is said to be a *prelargeness notion* if it is upward closed, in other words, $F_0 \in \mathbb{L}$ and $F_0 \subseteq F_1$ implies $F_1 \in \mathbb{L}$. A prelargeness notion $\mathbb{L}$ is said to be a *largeness notion* if for any infinite set $X \subseteq \mathbb{N}$, there exists a finite set $F \subseteq X$ such that $F \in \mathbb{L}$.

The idea of the above definition is that an infinite set is always large enough and thus it should contain a "large finite set" in the sense of $\mathbb{L}$. For example, $\mathbb{L}_\omega = \{F \in [\mathbb{N}]^{<\mathbb{N}} : |F| > \min F\}$ is a largeness notion. Note that "$\mathbb{L}$ is a prelargeness notion" is just a $\Pi^{\mathbb{L}}_1$-statement and thus it is available within $\mathsf{EFA}^{\mathbb{L}}$. On the other hand, "$\mathbb{L}$ is a largeness notion" is an $\mathrm{r}\Pi^1_1$-statement, so it strictly requires the second-order language. Next, we generalize the density notion. The following definition can be made within $\mathsf{EFA}^{\mathbb{L}}$.

Definition 4.3 (Density with respect to $\mathbb{L}$). Let $n \geq 1$ or $n = \infty$ and $k \geq 2$ or $k = \infty$. Let $\mathbb{L}$ be a prelargeness notion. We define the density for $(n, k, \mathbb{L})$ as follows:

- a finite set F is said to be 0-dense$(n, k, \mathbb{L})$ if $F \in \mathbb{L}$,
- a finite set F is said to be $m+1$-dense$(n, k, \mathbb{L})$ if for any $c : [F]^{n'} \to k'$ where $n' \leq \min\{n, \min F\}$ and $k' \leq \min\{k, \min F\}$, there exists a c-homogeneous set $H \subseteq F$ such that H is m-dense$(n, k, \mathbb{L})$.

The statement that F is m-dense$(n, k, \mathbb{L})$ is $\Sigma^{\mathbb{L}}_0$.

Now we define the generalized Paris–Harrington principle. The following definition can be made within RCA_0.

Definition 4.4 (Generalized PH). Let $n \geq 1$ or $n = \infty$, $k \geq 2$ or $k = \infty$ and $m \in \mathbb{N}$. Then, the generalized Paris–Harrington principle, $m\mathrm{GPH}^n_k$ and ItGPH^n_k, is defined as follows:

- $m\mathrm{GPH}^n_k$: for any largeness notion $\mathbb{L}$ and for any infinite set X_0, there exists a finite set $F \subseteq X_0$ such that F is m-dense$(n, k, \mathbb{L})$.
- $\mathrm{ItGPH}^n_k :\equiv \forall m\, m\mathrm{GPH}^n_k$.

Just like for PH, we write GPH^n_k for $1\mathrm{GPH}^n_k$, GPH^n for GPH^n_∞ and so on.

Unlike $\overline{\mathrm{PH}}^n_k$, GPH^n_k is "iterable." Indeed, GPH^n_k states that if $\mathbb{L}$ is a largeness notion, then the family of all 1-dense$(n, k, \mathbb{L})$ sets is also a largeness notion, and thus GPH^n_k can be applied to it again. Furthermore, any infinite subset $X \subseteq \mathbb{N}$ is "isomorphic to $\mathbb{N}$" in the following sense; if $h : \mathbb{N} \to X$ is a monotone increasing bijection and $\mathbb{L}$ is a largeness notion, then $h^{-1}(\mathbb{L})$ is a largeness notion and for any $F \subseteq_{\mathrm{fin}} \mathbb{N}$, F is 1-dense$(n, k, h^{-1}(\mathbb{L}))$ if and only if $h(F)$ is 1-dense$(n, k, \mathbb{L})$. Using these ideas, we can get the following.

Proposition 4.4. *Let $n = 1, 2, 3, \ldots$. The following are equivalent over* RCA_0*:*

1. $m\mathrm{GPH}^n_k$ *($k = 2, 3, 4, \ldots$, $m = 1, 2, 3, \ldots$).*
2. GPH^n_2 *on $\mathbb{N}$: for any largeness notion $\mathbb{L}$, there exists a finite set $F \subseteq \mathbb{N}$ such that F is* 1-dense$(n, 2, \mathbb{L})$.

To give a characterization of GPH, we consider the following variants of the infinite Ramsey theorem which was originally introduced by Flood [11].

Definition 4.5 (Ramsey-type weak Kőnig's lemma). An *infinite homogeneous function* for an infinite (n, k)-coloring tree T on X_0 is a function $h : [X]^n \to k$ such that $X \subseteq X_0$ is infinite and for any $m \in \mathbb{N}$, there exists $p \in T$ such that $h \restriction [X \cap m]^n = p \restriction [X \cap m]^n$.

We define two forms of the *Ramsey-type weak Kőnig's lemma*, RWKL^n_k and RWKL^{n-}_k, as follows:

- RWKL^{n-}_k: for any infinite (n, k)-coloring tree T on $\mathbb{N}$, there exists an infinite homogeneous function for T ($n \geq 1$ and $k \geq 2$),
- $\mathrm{RWKL}^{n-}_\infty \equiv \forall k \mathrm{RWKL}^{n-}_k$, $\mathrm{RWKL}^{\infty-}_\infty \equiv \forall n \mathrm{RWKL}^{n-}_\infty$,
- RWKL^n_k: for any infinite (n, k)-coloring tree T on $\mathbb{N}$, there exists a constant infinite homogeneous function for T ($n \geq 1$ and $k \geq 2$),
- $\mathrm{RWKL}^n_\infty \equiv \forall k \mathrm{RWKL}^n_k$, $\mathrm{RWKL}^\infty_\infty \equiv \forall n \mathrm{RWKL}^n_\infty$.

Note that the original definition of Ramsey-type weak Kőnig's lemma by Flood is our RWKL^1_2.[7] Over RCA_0, it is strictly in-between WKL and DNR (see [11, 12]). Variants of Ramsey-type weak Kőnig's lemma with homogeneous functions are introduced and studied by Bienvenu, Patey, and Shafer in [2] and the definition of RWKL^{n-}_k is inspired by them.

Theorem 4.5. *Let $n \geq 1$ or $n = \infty$ and $k \geq 2$ or $k = \infty$. The following are equivalent over* RCA_0*:*

1. GPH^n_k.
2. FIRT^n_k.
3. RWKL^n_k.
4. $\mathrm{RT}^n_k + \mathrm{RWKL}^{n-}_k$.

Proof. It is enough to show the equivalence for the case $n \geq 1$ and $k \geq 2$. Equivalence $3 \leftrightarrow 4$ is easy from the definition. If $f : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ is asymptotically stable, then $\mathbb{L} = \{F : \exists G \subseteq$

7 The original name in [11] was "Ramsey-type Kőnig's lemma", but "Ramsey-type weak Kőnig's lemma" turned to be the standard name in the later works.

$F|G| > f(G)\}$ is a largeness notion, which implies $1 \to 2$. Conversely, if $\mathbb{L}$ is a largeness notion, then a function f defined as $f(F) = \min\{|G| - 1 : G \subseteq F \wedge G \in \mathbb{L}\} \cup \{|F|\}$ is asymptotically stable and $F \in \mathbb{L} \leftrightarrow |F| > f(F)$. This implies $2 \to 1$. Implication $3 \to 1$ is a standard compactness argument which we have seen in Proposition 3.1. To show $1 \to 3$, let T be an infinite (n,k)-coloring tree on $\mathbb{N}$ with no infinite constant homogeneous function. Define $\mathbb{L}$ as $F \in \mathbb{L}$ if there is no $p \in T$ such that p is constant on $[F]^n$. Then, one can check that $\mathbb{L}$ is a largeness notion, and hence by 1, there exists a finite set $F_0 \subseteq \mathbb{N}$ which is 1-dense$(n,k,\mathbb{L})$. Take some $p \in T$ so that $\mathrm{dom}(p) \supseteq [F_0]^n$, then there must exist $H \subseteq F_0$ such that $H \in \mathbb{L}$ and p is constant on $[H]^n$, which is a contradiction. ■

In case $n = 3, 4, 5, \ldots$, any of the statements in the above theorem is just equivalent to ACA_0, so we mostly interested in the case $n = 1$ and 2. On the other hand, unlike RT^1_2 or PH^1_2, the principle GPH^1_2 is still not trivial since RWKL^1_2 (which is equivalent to RWKL^{1-}_2) is not provable within RCA_0. This may be interpreted as saying that the generalized version of the Paris–Harrington principle cannot be proved without using some compactness argument. In general, RWKL^{n-}_k is easily implied by WKL_0, but we do not know whether it is strictly weaker than WKL over RCA_0 or not in case $n \geq 2$.

4.3. Iterations of generalized PH and correctness statements

The iterated version of GPH can be related to stronger correctness statements.

Theorem 4.6. *Let $k = 2$ or $k = \infty$. Then* ItGPH^2_k *is equivalent to* $\mathrm{r}\Pi^1_2\text{-corr}(\mathsf{WKL}_0 + \mathrm{RT}^2_k)$ *over* WKL_0.

Over ACA_0', any Π^1_2-formula (of possibly nonstandard length) is equivalent to a $\mathrm{r}\Pi^1_2$-formula, and thus $\mathrm{r}\Pi^1_2$-truth predicate is actually the truth predicate for all Π^1_2-formulas. Furthermore, It is known that $\mathrm{r}\Pi^1_2\text{-corr}(\mathsf{ACA}_0)$ is equivalent to ACA_0'.[8] So we simply write $\Pi^1_2\text{-corr}(T)$ for $\mathrm{r}\Pi^1_2\text{-corr}(T)$ if $T \supseteq \mathsf{ACA}_0$.

Theorem 4.7. *The following are equivalent over* RCA_0*:*

1. RT.
2. GPH.
3. ItGPH^n_k *($n = 3, 4, \ldots, k = 2, 3, \ldots, \infty$).*
4. $\Pi^1_2\text{-corr}(\mathsf{ACA}_0)$.

Theorem 4.8. *Over* RCA_0*,* ItGPH *is equivalent to* $\Pi^1_2\text{-corr}(\mathsf{ACA}_0')$.

We will see the proofs of these theorems using indicators in the next section.

The strength of ItGPH^2_2 or ItGPH^2 is rather unclear. It is not difficult to check that $\mathsf{RCA}_0 + \mathrm{ItGPH}^2_2$ implies RT^2 and $\mathsf{WKL}_0 + \mathrm{RT}^2_2 + \mathsf{I}\Sigma^1_1$ implies ItGPH^2 as in the proof of Proposition 3.1. (Note that even ItGPH does not imply $\mathsf{I}\Sigma^1_1$ since $\mathsf{I}\Sigma^1_1$ is never implied from

8 This follows from the proof of [39, THEOREM IX.4.5].

any true Π^1_2-statement.) In particular, they are true in any ω-models of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$. Meanwhile, the following questions are still open.

Question 4.6. 1. Is ItGPH^2_2 equivalent to RT^2 over WKL_0?

2. Does ACA_0 imply ItGPH^2 or ItGPH^2_2?

5. INDICATORS AND CORRECTNESS STATEMENTS

The notion of indicators is introduced by Kirby and Paris [23, 32] to show several independence results from PA, and its theory is organized systematically by Kaye [21]. The argument of indicators can connect first-order objects with second-order objects by means of nonstandard models. Recently, indicators have been used to calibrate the proof-theoretic strength of the infinite Ramsey theorem in the context of reverse mathematics [6, 24, 34, 46].

5.1. Models of first- and second-order arithmetic

To introduce the argument of indicators, we first set up basic model theory of first- and second-order arithmetic. For the details, see [16, 21, 26, 39]. A structure for $\mathcal{L}_1$ is a 6-tuple $M = (M; 0^M, 1^M, +^M, \times^M, \leq^M)$. (We often omit the superscript M if it is clear from the context.) An $\mathcal{L}_1$-structure $\mathfrak{N} = (\mathfrak{N}; 0, 1, +, \times, \leq)$ where $0, 1, +, \times, \leq$ are usual is called the *standard model*, and an $\mathcal{L}_1$-structure is said to be *nonstandard* if it is not isomorphic to $\mathfrak{N}$. When we consider an expanded language $\mathcal{L}_1 \cup \vec{U}$ where $\vec{U} = U_1, \ldots, U_k$ are second-order constants, an $\mathcal{L}_1 \cup \vec{U}$-structure is a pair $(M, \vec{U}^M)$ where M is an $\mathcal{L}_1$-structure and $U_i \subseteq M$. We may consider $\mathbb{N}$ as a special second-order constant which satisfies $\forall x x \in \mathbb{N}$, in other words, $\mathbb{N}^M = M$ for any M. For second-order arithmetic, we use Henkin semantics. A structure for $\mathcal{L}_2$ is a pair (M, S) where M is an $\mathcal{L}_1$-structure and $S \subseteq \mathcal{P}(M)$. Thus, any $\mathcal{L}_1 \cup \vec{U}$-structure can be considered as an $\mathcal{L}_2$-structure.

Let M be a nonstandard model of $\mathsf{EFA}^{\vec{U}}$. We write $[M]^{<M}$ for the set of all "finite sets in M" (also called M-finite sets), in other words, $[M]^{<M} = ([\mathbb{N}]^{<\mathbb{N}})^M$. A nonempty proper subset $I \subsetneq M$ is said to be a *cut* if $a < b \wedge b \in I$ implies $a \in I$ for any $a, b \in M$ (denoted by $I \subseteq_e M$) and $a + 1 \in I$ for any $a \in I$. If I is a cut and $\varphi(x)$ is a $\Sigma^{\vec{U}}_0$-formula such that $M \models \varphi(a)$ for any $a \in I$ (resp. $a \in M \setminus I$), then there exists $a \in M \setminus I$ (resp. $a \in I$) such that $M \models \varphi(a)$. This principle is called *overspill* (resp. *underspill*). A cut $I \subseteq_e M$ is said to be *semiregular* if for any $F \in [M]^{<M}$ with $|F| \leq \min F$, $F \cap I$ is bounded in I.

In our study, models of WKL_0 play central roles. Here are two important theorems.

Theorem 5.1 (Harrington, see Section IX.2 of [39]).

1. *For any countable model* $(M, S) \models \mathsf{RCA}_0$, *there exists* $\bar{S} \supseteq S$ *such that* $(M, \bar{S}) \models \mathsf{WKL}_0$.[9]

2. WKL_0 *is* Π^1_1-*conservative over* RCA_0.

9 Model (M, S) is said to be countable if both of M and S are countable.

Theorem 5.2 (see, e.g., Theorems 7.1.5 and 7.1.7 of [26]). *Let M be a model of* EFA *and $I \subsetneq_e M$ be a cut. Then, I is semiregular if and only if $(I, \mathrm{Cod}(M/I)) \models \mathsf{WKL}_0$, where* $\mathrm{Cod}(M/I) = \{F \cap I : F \in [M]^{<M}\}$.

5.2. Indicators

Now we give the definition of indicators. Here, we slightly arrange the definition in [21] so as to fit better with second-order arithmetic.

Definition 5.1 (Indicators). Let $\vec{U} = U_1, \dots, U_k$ be second-order constants, and let $T \supseteq \mathsf{EFA}$ be an $\mathcal{L}_2$-theory.

1. Let M be a countable nonstandard model of $\mathsf{EFA}^{\vec{U}}$. A $\Sigma_0^{\vec{U}}$-definable function $Y : [M]^{<M} \to M$ is said to be an *indicator for T on M* if for each $F, F' \in [M]^{<M}$, $Y(F) \leq \max F$, $Y(F) \leq Y(F')$ if $F \subseteq F'$, and

   (cut) $Y(F) > m$ for any $m \in \mathfrak{N}$ if and only if there exists a cut $I \subsetneq_e M$ and $S \subseteq \mathrm{Cod}(M/I)$ such that $(I, S) \models T$, $U_i^M \cap I \in S$ for each $U_i \in \vec{U}$ and $F \cap I$ is unbounded in I.

2. A $\Sigma_0^{\vec{U}}$-formula $Y(F, m)$ is said to be an *indicator for T* if for any countable nonstandard model $M \models \mathsf{EFA}^{\vec{V}}$ with $\vec{U} \subseteq \vec{V}$ ($\vec{U}$ is a subtuple of $\vec{V}$), Y defines an indicator for T on M.

For a given indicator Y, we define two statements "$Y \geq m$" and "$Y^{\mathrm{int}} \geq m$" as follows:

$$Y \geq m \equiv \forall X_0 (X_0 \text{ is infinite} \to \exists F \subseteq_{\mathrm{fin}} X_0 \; Y(F) \geq m),$$

$$Y^{\mathrm{int}} \geq m \equiv \forall a \exists b Y([a, b)_{\mathbb{N}}) \geq m.$$

Note that $Y \geq m$ is a $\mathrm{r}\Pi^1_1$-statement while $Y^{\mathrm{int}} \geq m$ is a Π_2-statement.

Theorem 5.3. *Define Σ_0-formulas $Y_{\mathrm{PH}^n}(F, m)$, $Y_{\mathrm{PH}}(F, m)$, and $Y_{\mathrm{ItPH}^n_k}(F, m)$ as follows:*

- $Y_{\mathrm{PH}^n}(F, m) \leftrightarrow m = \max\{k' \leq \max F : F \textit{ is } 1\text{-dense}(n, k')\} \cup \{0\}$ *($n = 2, 3, \dots$),*
- $Y_{\mathrm{PH}}(F, m) \leftrightarrow m = \max\{n' \leq \max F : F \textit{ is } 1\text{-dense}(n', 2)\} \cup \{0\}$,
- $Y_{\mathrm{ItPH}^n_k}(F, m) \leftrightarrow m = \max\{m' \leq \max F : F \textit{ is } m'\text{-dense}(n, k)\} \cup \{0\}$ *($n = 2, 3, \dots$ or $n = \infty$ and $k = 2, 3, 4, \dots$ or $k = \infty$).*

Then, we have the following:

1. Y_{PH^n} *is an indicator for* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_{n-1}$.
2. Y_{PH} *is an indicator for* ACA_0.
3. $Y_{\mathrm{ItPH}^n_k}$ *is an indicator for* $\mathsf{WKL}_0 + \mathrm{RT}^n_k$.

In addition, these facts are provable within WKL_0.

Proof. For statements 1 and 2, one can reformulate the discussions of **[21, SECTION 14.3]**. Statement 3 is essentially due to Paris **[32, EXAMPLE 2]** (see also **[6, THEOREM 1]** and **[34, LEMMA 3.2]**). We sketch the proof for statement 3 for the case $n = 2, 3, \dots$ and $k = 2, 3, \dots$

It is enough to check the condition (cut) for $Y_{\mathrm{ItPH}^n_k}$. The right-to-left direction follows from Proposition 3.1 and overspill. For the left-to-right direction, let M be a countable nonstandard model of $\mathsf{EFA}^{\vec{U}}$ and let $F \in [M]^{<M}$ be m-dense(n,k) for any $m \in \mathfrak{N}$. By overspill, take $d \in M \setminus \mathfrak{N}$ such that F is d-dense(n,k). We will construct a countable decreasing sequence of M-finite sets $\{F_i\}_{i\in\mathfrak{N}}$ such that F_i is $(d-i)$-dense(n,k) and

(i) if $E \in [M]^{<M}$ and $|E| \le E$, then $E \cap [\min F_i, \max F_i)_{\mathbb{N}} = \emptyset$ for some $i \in \mathfrak{N}$,

(ii) if $p \in [M]^{<M}$ and $p : [F]^n \to k$, then for some $i \in \mathfrak{N}$, F_i is p-homogeneous.

Once such a sequence is constructed, put $I = \{a \in M : \exists i \in \mathfrak{N}(a < \min F_i)\}$. Then, $F_i \cap I$ is unbounded in I and $U_i^M \cap I \in \mathrm{Cod}(M/I)$. By Theorem 5.2, $(I, \mathrm{Cod}(M/I)) \models \mathsf{WKL}_0$ since I is a semiregular cut by (i), and (ii) implies $(I, \mathrm{Cod}(M/I)) \models \mathrm{RT}^n_k$.

Finally, we construct $\{F_i\}_{i\in\mathfrak{N}}$. Since $[M]^{<M}$ is countable, it is enough to show:

(i)′ if $E \in [M]^{<M}$, $|E| \le \min E$ and F is $\ell+1$-dense(n,k) then there exists $F' \subseteq F$ which is ℓ-dense(n,k) such that $E \cap [\min F_i, \max F_i)_{\mathbb{N}} = \emptyset$,

(ii)′ if $p \in [M]^{<M}$, $p : [F]^n \to k$ and F is $\ell+1$-dense(n,k) with $\ell \ge 1$, then there exists $F' \subseteq F$ which is ℓ-dense(n,k) such that F is p-homogeneous.

Indeed, (ii)′ is trivial from the definition of density. For (i)′, define $c : [F]^2 \to 2$ as $c(\{x,y\}) = 0 \leftrightarrow [x,y)_{\mathbb{N}} \cap E = \emptyset$, and take a c-homogeneous set $F' \subseteq F$ such that F' is ℓ-dense(n,k). If $[F']^2 \subseteq c^{-1}(1)$, then put $F'' = F' \setminus \{\min F'\}$ and we have $|F''| \le |E| \le \min E < \min F''$, but F'' must be relatively large since it is at least 0-dense(n,k). Hence $[F']^2 \subseteq c^{-1}(0)$, which we are done. ■

For the next theorem, we want to formalize model-theoretic arguments within second-order arithmetic. Within WKL_0, one can set up basic (countable) model theory for first-order logic, and then prove Gödel's completeness theorem **[39, SECTIONS II.8 AND IV.3]**. Standard techniques for countable nonstandard models of arithmetic such as the compactness theorem, over/underspill, back and forth, recursive saturation and forcing are naturally formalizable once a countable model with a full evaluation function (truth definition) is provided. On the other hand, it is not possible in general to consider $\mathbb{N}$ itself as a model of first-order arithmetic since its truth definition is too complicated,[10] hence it is not easy to guarantee that a family of true sentences are consistent. Still, we can deal with the consistency of Π_2-sentences as follows.

10 Some strong enough system such as ACA_0^+ can do this, but WKL_0 is not enough.

Lemma 5.4. RCA_0 *proves the following. Let $\vec{A} = A_1, \dots, A_k$ be sets, and let Γ be a set of true $\Pi_2^{\vec{A}}$-sentences. Then, Γ is consistent (with considering $\vec{A}$ as second-order constants).*[11]

Proof. We work within RCA_0 and show that $\mathbb{N}$ (together with $\vec{A}$) is a weak model of Γ in the sense of **[39, Definition II.8.9]**. It is enough to construct a function $f : S^\Gamma \to 2$ which satisfies Tarski's truth definition, where S^Γ is the set of all substitution instances of subformulas of Γ. Let S_0^Γ be the set of all substitution instances of $\Sigma_0^{\vec{A}}$-subformulas of Γ. Since there is a $\Pi_1^{\vec{A}}$-formula which defines the truth of all $\Sigma_0^{\vec{A}}$-formulas, one can take a function $f : S_0^\Gamma \to 2$ which satisfies the truth definition. Then f can be expanded to S^Γ by putting the truth value 1 for all sentences in $S^\Gamma \setminus S_0^\Gamma$. (They are $\Sigma_1^{\vec{A}}$ or $\Pi_2^{\vec{A}}$ and always true.) ■

Theorem 5.5. *Let $T \supseteq \mathsf{RCA}_0$ be an $\mathcal{L}_2$-theory, and let Y be an indicator for T.*

1. *For any $\mathrm{r}\Pi_1^1$-sentence φ, $T \vdash \varphi$ if and only if $\mathsf{RCA}_0 + \{Y \geq m : m \in \mathfrak{N}\} \vdash \varphi$.*
2. *For any Π_2-sentence φ, $T \vdash \varphi$ if and only if $\mathsf{I}\Sigma_1 + \{Y^{\mathrm{int}} \geq m : m \in \mathfrak{N}\} \vdash \varphi$.*[12]

If Y is an indicator for T provably in WKL_0, we also have the following:

3. *Over RCA_0, $\mathrm{r}\Pi_1^1$-corr(T) is equivalent to $\forall m Y \geq m$.*
4. *Over $\mathsf{I}\Sigma_1$, Π_2-corr(T) is equivalent to $\forall m Y^{\mathrm{int}} \geq m$.*

Proof. We show statements 1 and 3. (Statements 2 and 4 can be shown similarly.)

The right-to-left direction of statement 1 follows from Theorem 5.1.1 and Tanaka's self-embedding theorem **[42]**. Indeed, if (M, S) is a countable nonstandard model of T, then there exists a model $(\bar{M}, \bar{S})$ which is isomorphic to (M, S) such that $M \subsetneq_e \bar{M}$ and $S \subseteq \mathrm{Cod}(\bar{M}/M)$. If $X \in S$ is infinite in (M, S) and $m \in \mathfrak{N}$, then there exists $\bar{F} \in [\bar{M}]^{<\bar{M}}$ such that $X = \bar{F} \cap M$. By the condition (cut), $Y(\bar{F}) \geq m$, hence there exists a set $F \in [M]^{<M}$ such that $F \subseteq X$ and $Y(F) \geq m$ by underspill.

For the left-to-right direction of statement 1, it is enough to show that if

$$\{\forall x \exists y \theta(U, x, y)\} \cup \mathsf{RCA}_0 \cup \{Y \geq m : m \in \mathfrak{N}\}$$

is consistent with a second-order constant U and a Σ_0^U-formula θ, then

$$\{\forall x \exists y \theta(U, x, y)\} \cup T$$

is consistent. Let (M, S) be a countable nonstandard model of

$$\{\forall x \exists y \theta(U, x, y)\} \cup \mathsf{RCA}_0 \cup \{Y \geq m : m \in \mathfrak{N}\}.$$

Then there exists an infinite set A in (M, S) such that for any $a, b \in A$ with $a < b$, $\forall x < a \exists y < b \theta(U, x, y)$. By overspill, there exists an M-finite set $F \subseteq A$ with $Y(F) \geq m$

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11 This lemma also follows from (the relativization of) the fact that $\mathsf{I}\Sigma_1$ is equivalent to Π_3-corr(EFA). See **[1]**.

12 For statements 1 and 2, the base theories RCA_0 and $\mathsf{I}\Sigma_1$ can be weakened to RCA_0^* and $\mathsf{EFA} + \mathsf{B}\Sigma_1$ (the proof still works using recursively saturated models).

for any $m \in \mathfrak{N}$. By (cut), take $I \subsetneq_e M$ and $S' \subseteq \mathrm{Cod}(M/I)$ such that $(I, S') \models T$ and $F \cap I$ is unbounded in I. The latter implies $(I, S') \models \forall x \exists y \theta(U, x, y)$.

For the left-to-right direction of statement 3, we first formalize the right-to-left direction of statement 1 within WKL_0. In other words, "for each $m \in \mathbb{N}$, $Y \geq m$ is provable in T" is provable within WKL_0. Thus it is provable within RCA_0 by Theorem 5.1.2 since it is a Π^0_2-statement, and hence $\mathrm{r}\Pi^1_1$-corr(T) implies $\forall m Y \geq m$.

For the right-to-left direction, again we first work within WKL_0. It is enough to show that if $\forall x \exists y \theta(U, x, y)$ holds for some set U and a Σ^U_0-formula θ, then $\{\forall x \exists y \theta(U, x, y)\} \cup T$ is consistent. Take an infinite set A such that A is Δ^U_1-definable and for any $a, b \in A$ with $a < b$, $\forall x < a \exists y < b \theta(U, x, y)$. Then, by the assumption, for any $m \in \mathbb{N}$, there exists a finite set $F \subseteq A$ such that $Y(F) \geq m$. Thus, by Lemma 5.4, a set of Π^U_2-sentences $\Gamma = \mathsf{EFA}^U \cup \{\forall a \in F \forall b \in F(a < b \to \forall x < a \exists y < b \theta(U, x, y))\} \cup \{Y(F) \geq m : m \in \mathbb{N}\}$ is consistent (consider F as a new number constant). Take a countable nonstandard model of Γ and formalize the argument for the left-to-right direction of statement 1, then we see that $\{\forall x \exists y \theta(U, x, y)\} \cup T$ is consistent.

The above argument actually showed that for any set U, "$\forall m Y \geq m$ with respect to any infinite set $A \leq_T U$" implies Σ^U_2-corr(T). This is a Π^1_1-statement provable in WKL_0, so it is also provable within RCA_0 by Theorem 5.1.2. Thus RCA_0 proves that $\forall m Y \geq m$ implies $\mathrm{r}\Pi^1_1$-corr(T). ■

Theorems 5.3 and 5.5 directly connect PH and the correctness statements, and Theorems 2.1 and 2.2 are direct consequences of them. They also imply conservation theorems. Indeed, Theorem 2.6.2 is a direct consequence of Theorems 5.3 and 5.5 plus Theorem 3.8.5 (see **[25]**).

Proofs of Theorems 3.2–3.6. By definitions, $\overline{\mathrm{PH}}^n$, $\overline{\mathrm{PH}}$, and $\mathrm{It}\overline{\mathrm{PH}}^n_k$ are equivalent to $\forall m Y_{\mathrm{PH}^n} \geq m$, $\forall m Y_{\mathrm{PH}} \geq m$ and $\forall m Y_{\mathrm{ItPH}^n_k} \geq m$, respectively. Then, equivalences between variants of PH and corresponding $\mathrm{r}\Pi^1_1$-correctness statements ($1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ of Theorems 3.2 and 3.3, $1 \leftrightarrow 2$ of Theorem 3.4, $1 \leftrightarrow 2 \leftrightarrow 3$ of Theorem 3.5 and Theorem 3.6) follow from Theorems 5.3 and 5.5. Implications between variants of PH and well-foundedness statements follow from Theorem 3.8 (see the paragraph below Theorem 3.8). Other implications can be shown as follows: $3 \to 5$ of Theorem 3.3 and $2 \to 3$ of Theorem 3.4 are implied from the formalization of the fact that $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ proves well-foundedness of ω^k_n for each $k \in \mathfrak{N}$, and $3 \leftrightarrow 4$ of Theorem 3.3 is implied from the formalization of the conservation result for $\mathsf{WKL}_0 + \mathrm{RT}^2$ in **[40]**. ■

5.3. Indicators corresponding to largeness notions

To obtain a characterization of $\mathrm{r}\Pi^1_2$-correctness, we modify Theorem 5.5 using indicators which can preserve largeness notions.

Given two finite sets $F_0 = \{x_0 < \cdots < x_{\ell-1}\}$ and $F_1 = \{x'_0 < \cdots < x'_{\ell'-1}\}$, define $F_0 \trianglelefteq F_1$ as $\ell \leq \ell'$ and $x_i \geq x'_i$ for any $i < \ell$. A prelargeness notion $\mathbb{L}$ is said to be *normal* if $F_0 \in \mathbb{L}$ and $F_0 \trianglelefteq F_1$ implies $F_1 \in \mathbb{L}$. It is not difficult to check that $\mathbb{L}_\omega$ is a normal largeness notion. For a given prelargeness notion $\mathbb{L}$, put $\mathbb{L}^+ = \{F \in \mathbb{L} : \forall G \subseteq [0, \max F]_{\mathbb{N}} (G \trianglerighteq F \to G \in \mathbb{L})\}$. Then $\mathbb{L}^+$ is a normal prelargeness notion.

Lemma 5.6. *The following is provable within* WKL_0. *For any largeness notion* $\mathbb{L}$, $\mathbb{L}^+$ *is a largeness notion.*

Proof. Assume that $\mathbb{L}$ is a prelargeness notion and there exists an infinite set $X = \{x_0 < x_1 < \cdots\}$ such that no finite subset of X is a member of $\mathbb{L}^+$. Define a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ as $\sigma \in T$ if and only if σ is strictly increasing, $\{\sigma(i) : i < |\sigma|\} \trianglerighteq \{x_i : i < |\sigma|\}$ and $\{\sigma(i) : i < |\sigma|\} \notin \mathbb{L}$. Then, T is a bounded tree and T is infinite. Take a path $h \in [T]$, then $Y = \{h(i) : i \in \mathbb{N}\}$ is an infinite set and any finite subset of Y is not a member of $\mathbb{L}$. ∎

Now we generalize the notion of semiregularity with a normal (pre)largeness notion and consider a variant of Theorem 5.5.

Definition 5.2 ($\mathbb{L}$-semiregularity). Let M be a nonstandard model of $\mathsf{EFA}^{\mathbb{L}}$, and let $\mathbb{L}$ be a normal prelargeness notion in M. Then, a cut $I \subseteq_e M$ is said to be $\mathbb{L}$-*semiregular* if for any finite set $F \notin \mathbb{L}$, $F \cap I$ is bounded in I, or equivalently, $\mathbb{L} \cap I$ is a normal largeness notion in $(I, \mathrm{Cod}(M/I))$.

A $\Sigma_0^{\vec{U}}$-formula $Y^{\mathbb{L}} \equiv Y(\mathbb{L}, F, m)$ (where $\mathbb{L} \in \vec{U}$) is said to be an $\mathbb{L}$-*semiregular indicator* for an $\mathcal{L}_2$-theory T if for any countable nonstandard model $M \models \mathsf{EFA}^{\vec{V}}$ with $\vec{U} \subseteq \vec{V}$ such that $\mathbb{L}$ is a normal prelargeness notion in M, $Y^{\mathbb{L}}$ defines an indicator for T on M but the condition (cut) replaced by

($\mathbb{L}$-cut) $Y(F) > m$ for any $m \in \mathfrak{N}$ if and only if there exists an $\mathbb{L}$-semiregular cut $I \subsetneq_e M$ and $S \subseteq \mathrm{Cod}(M/I)$ such that $(I, S) \models T$, $U_i^M \cap I \in S$ for each $U_i \in \vec{U}$ and $F \cap I$ is unbounded in I.

Theorem 5.7. *Let* $T \supseteq \mathsf{WKL}_0$ *be an* $\mathcal{L}_2$*-theory, and let* $Y^{\mathbb{L}}$ *be an* $\mathbb{L}$*-semiregular indicator for* T *provably in* WKL_0. *Then the following assertions are equivalent over* WKL_0*:*

1. $\mathrm{r}\Pi^1_2$-corr(T).
2. *For any* $\mathbb{L}$, *if* $\mathbb{L}$ *is a normal largeness notion, then* $\forall m Y^{\mathbb{L}} \geq m$.

Proof. Implication 1 → 2 follows from the same discussion as the proof for Theorem 5.5. To show 2 → 1, we reason within WKL_0 and show that, assuming statement 2 is true, if $\theta(U)$ holds for some set U and an $\mathrm{r}\Pi^1_1$-formula $\theta(U)$, then $\{\theta(U)\} \cup T$ is consistent. By **[34, PROPOSITION 2.5]**, take a Σ^0_0-formula $\eta(G, F)$ such that WKL_0 proves

$$\forall V(\theta(V) \leftrightarrow \forall Z(Z \text{ is infinite} \to \exists F \subseteq_{\mathrm{fin}} Z \eta(V \cap [0, \max F]_{\mathbb{N}}, F))).$$

Define $\mathbb{L}_0 \subseteq [\mathbb{N}]^{<\mathbb{N}}$ as $G \in \mathbb{L}_0 \leftrightarrow \exists F \subseteq G \eta(U \cap [0, \max F]_{\mathbb{N}}, F)$, and let $\mathbb{L} = \mathbb{L}_0^+$. Since $\theta(U)$ holds, $\mathbb{L}$ is a normal largeness notion. By assumption, we have $Y^{\mathbb{L}} \geq m$ for any $m \in \mathbb{N}$. Thus, by Lemma 5.4, a set of $\Pi_2^{U,\mathbb{L}}$-sentences $\Gamma = \mathsf{EFA}^{U,\mathbb{L}} \cup \{\mathbb{L}$ is a normal prelargeness notion$\} \cup \{\forall G(G \in \mathbb{L} \to \exists G' \eta(U \cap [0, \max G']_{\mathbb{N}}, G'))\} \cup \{Y^{\mathbb{L}}(F) \geq m : m \in \mathbb{N}\}$ is consistent (consider F as a new number constant).

Take a countable nonstandard model $M \models \Gamma$. Then, $M \models Y^{\mathbb{L}}(F^M) \geq m$ for any $m \in \mathbb{N}$ and thus there exists an $\mathbb{L}$-semiregular cut $I \subsetneq_e M$ and $S \subseteq \mathrm{Cod}(M/I)$ such that $U^I = U^M \cap I \in S$, $\mathbb{L}^I = \mathbb{L}^M \cap I \in S$ and $(I, S) \models T$. Since I is $\mathbb{L}^M$-semiregular, $\mathbb{L}^I$ is a largeness notion in (I, S). Since $M \models \forall G(G \in \mathbb{L} \to \exists G' \eta(U \cap [0, \max G']_{\mathbb{N}}, G'))$, we have $(I, S) \models \theta(U^I)$. ■

Theorem 5.8. *Let $n = 2, 3, 4 \ldots$ or $n = \infty$ and $k = 2, 3, 4, \ldots$ or $k = \infty$. Define $\Sigma_0^{\mathbb{L}}$-formula $Y^{\mathbb{L}}_{\mathrm{ItGPH}^n_k}$ as follows:*

$$Y_{\mathrm{ItGPH}^n_k}(\mathbb{L}, F, m) \leftrightarrow m = \max\{m' \leq \max F : F \text{ is } m'\text{-dense}(n, k, \mathbb{L} \cap \mathbb{L}_\omega)\} \cup \{0\}.$$

Then, $Y^{\mathbb{L}}_{\mathrm{ItGPH}^n_k}$ is an $\mathbb{L}$-semiregular indicator for $\mathsf{WKL}_0 + \mathrm{RT}^n_k$. *Moreover, this fact is provable within* WKL_0.

Proof. Essentially the same as the proof for Theorem 5.3.3. We additionally need to show the following (which is an analogous of (i)′):

> If $\mathbb{L}$ is a normal prelargeness notion, F is $\ell + 1$-dense$(n, k, \mathbb{L} \cap \mathbb{L}_\omega)$ with $\ell \geq 1$ and G is a finite set such that $G \notin \mathbb{L}$, then there exists $F' \subseteq F$ such that F' is ℓ-dense$(n, k, \mathbb{L} \cap \mathbb{L}_\omega)$ and $[\min F', \max F')_{\mathbb{N}} \cap G = \emptyset$.

Given $\ell, \mathbb{L}$, F and G as above, define $c : [F]^2 \to 2$ as $c(\{x, y\}) = 1 \leftrightarrow [x, y)_{\mathbb{N}} \cap G \neq \emptyset$. Take a c-homogeneous set $F' \subseteq F$ such that F' is ℓ-dense$(n, k, \mathbb{L} \cap \mathbb{L}_\omega)$. If $[F']^2 \subseteq c^{-1}(0)$, we are done, so assume $[F']^2 \subseteq c^{-1}(1)$. Put $G' = G \cap [\min F', \max F')_{\mathbb{N}}$ and $F'' = F' \setminus \{\min F'\}$. Then F'' is at least 0-dense$(n, k, \mathbb{L} \cap \mathbb{L}_\omega)$ and thus $F'' \in \mathbb{L}$. On the other hand, $G' \trianglerighteq F''$ by the definition of c, and thus $G' \in \mathbb{L}$. This is a contradiction since $G' \subseteq G$ and $G \notin \mathbb{L}$. ■

Proofs of Theorems 4.6, 4.7, and 4.8. By Lemma 5.6, ItGPH^n_k is equivalent to the statement that if $\mathbb{L}$ is a normal largeness notion, then $\forall m Y^{\mathbb{L}}_{\mathrm{ItGPH}^n_k} \geq m$. Then, implications between ItGPH^n_k and $\mathrm{r}\Pi^1_2$-corr$(\mathsf{WKL}_0 + \mathrm{RT}^n_k)$ follow from Theorems 5.7 and 5.8. ■

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