THE ZARISKI CANCELLATION PROBLEM AND RELATED PROBLEMS IN AFFINE ALGEBRAIC GEOMETRY

NEENA GUPTA

ABSTRACT

In this article, we shall discuss the solution to the Zarsiki Cancellation Problem in positive characteristic, various approaches taken so far towards the possible solution in characteristic zero, and several other questions related to this problem.

1. INTRODUCTION

"Polynomials and power series
May they forever rule the world."   —*Shreeram S. Abhyankar, 1970* [4]

Right from the beginning of the 19th century, mathematicians have been involved in studying polynomial rings (over $\mathbb{C}$ and over $\mathbb{R}$). Some of the early breakthroughs on polynomial rings have led to the foundation of Commutative Algebra. One such result is the Hilbert Basis Theorem, a landmark result on the finite generation of ideals, which solved a central problem on invariant theory. This was followed by the Hilbert Nullstellensatz which connects affine varieties (zero locus of a set of polynomials) with rings of regular functions on varieties and thus enables one to make use of the algebraic machinery of commutative algebra to study geometric properties of varieties.

Affine Algebraic Geometry deals with the study of affine spaces (and certain closed subspaces), equivalently, polynomial rings (and certain quotients). There are many fundamental problems on polynomial rings which can be formulated in an elementary mathematical language but whose solutions remain elusive. Any significant progress requires the development of new and powerful methods and their ingenious applications.

One of the most challenging problems in Affine Algebraic Geometry is the Zariski Cancelation Problem (ZCP) on polynomial rings (Question $1'$ below). In this article, we shall discuss the solution to the ZCP in positive characteristic, various approaches taken so far towards the possible solution in characteristic zero, and several other questions related to this problem. For a survey on problems in Affine Algebraic Geometry, one may look at [42, 62, 69].

Throughout the article, all rings will be assumed to be commutative with unity and k will denote a field. For a ring R, R^* will denote the group of units of R. We shall use the notation $R^{[n]}$ for a polynomial ring in n variables over a commutative ring R. Thus, $E = R^{[n]}$ will mean that $E = R[t_1, \ldots, t_n]$ for some elements $t_1, \ldots, t_n$ in E which are algebraically independent over R. Unless otherwise stated, capital letters like $X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_m, X, Y, Z, T$ will be used as variables of polynomial rings.

2. CANCELLATION PROBLEM

Let A be an affine (finitely generated) algebra over a field k. The k-algebra A is said to be cancellative (over k) if, for any k-algebra B, $A[X] \cong_k B[X]$ implies that $A \cong_k B$. A natural question in this regard is: which affine domains are cancellative? More precisely:

Question 1. Let A be an affine algebra over a field k. Suppose that B is a k-algebra such that the polynomial rings $A[X]$ and $B[X]$ are isomorphic as k-algebras. Does it follow that $A \cong_k B$? In other words, is the k-algebra A cancellative?

A special case of Question 1, famously known as the Zariski Cancellation Problem, asks whether affine spaces are cancellative, i.e., whether any polynomial ring in n variables over a field k is cancellative. More precisely:

Question 1'. Suppose that B is an affine k algebra satisfying $B[X] \cong_k k[X_1, \dots, X_{n+1}]$ for some positive integer n. Does it follow that $B \cong_k k[X_1, \dots, X_n]$? In other words, is the polynomial ring $k[X_1, \dots, X_n]$ cancellative?

Abhyankar, Eakin, and Heinzer have shown that any domain A of transcendence degree one over any field k is cancellative [3]. In fact, they showed that, for any UFD R, the polynomial ring $R[X]$ is cancellative over R. This was further generalized by Hamann to a ring R which either contains $\mathbb{Q}$ or is a seminormal domain [52].

In 1972, Hochster demonstrated the first counterexample to Question 1 [53]. His example, a four-dimensional ring over the field of real numbers $\mathbb{R}$, is based on the fact that the projective module defined by the tangent bundle over the real sphere with coordinate ring $S = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ is stably free but not a free S-module.

One of the major breakthroughs in 1970s was the establishment of an affirmative answer to Question 1' for the case $n = 2$. This was proved over a field of characteristic zero by Fujita, Miyanishi, and Sugie [43,70] and over perfect fields of arbitrary characteristic by Russell [74]. Later, it has been shown that even the hypothesis of perfect field can be dropped [20]. A simplified proof of the cancellation property of $k[X, Y]$ for an algebraically closed field k is given by Crachiola and Makar-Limanov in [22].

Around 1989, Danielewski [26] constructed explicit two-dimensional affine domains over the field of complex numbers $\mathbb{C}$ which are not cancellative over $\mathbb{C}$. New examples of noncancellative varieties over any field k have been studied in [9,32,49]. This addresses the Cancellation Problem, as formulated in Question 1, for all dimensions.

In [45] and [47], the author settled the Zariski Cancellation Problem (Question 1') completely for affine spaces in positive characteristic. She has first shown in [45] that a certain threefold constructed by Asanuma is a counterexample to the ZCP in positive characteristic for the affine three space. Later in [46], she studied a general threefold of the form $x^m y = F(x, z, t)$, which includes the Asanuma threefold as well as the famous Russell cubic defined below. A major theorem of [46] is stated as Theorem 5.4 of this article. In [47], using a modification of the theory developed in [46], she constructed a family of examples which are counterexamples to the ZCP in positive characteristic in all dimensions greater than 2. The ZCP is still a challenging problem in characteristic zero. A few candidate counterexamples are discussed below.

The Russell cubic. Let $A = \mathbb{C}[X, Y, Z, T]/(X^2 Y + X + Z^2 + T^3)$, $V = \operatorname{Spec} A$ and let x denote the image of X in A. The ring A, known as the Russell cubic, is one of the simplest examples of the Koras–Russell threefolds, a family of threefolds which arose in the context of the problem of determining whether there exist nonlinearizable $\mathbb{C}^*$-actions on $\mathbb{C}^3$. It was an exciting open problem for some time whether $A \cong \mathbb{C}^{[3]}$. It was first observed that the ring

A (respectively the variety V) has several properties in common with $\mathbb{C}^{[3]}$ (respectively $\mathbb{C}^3$), for instance,

(i) A is a regular UFD.

(ii)   There exists an injective $\mathbb{C}$-algebra homomorphism from A to $\mathbb{C}^{[3]}$. Note that $\mathbb{C}^{[3]} \hookrightarrow A$.

(iii)   The variety V is homeomorphic (in fact, diffeomorphic) to $\mathbb{R}^6$.

(iv) V has logarithmic Kodaira dimension $-\infty$.

These properties appeared to provide evidence in favor of the surmise that $A \cong \mathbb{C}^{[3]}$. The establishment of an isomorphism between A and $\mathbb{C}^{[3]}$ would have led to counterexamples to the "Linearization Conjecture" on $\mathbb{C}^3$ (stated in [58]) and the Abhyankar–Sathaye Conjecture for $n = 3$ (stated in Section 5 of the present article). Indeed, if A were isomorphic to $\mathbb{C}^{[3]}$, as was then suspected, it would have shown the existence of nonlinearizable $\mathbb{C}^*$-actions on $\mathbb{C}^3$. Moreover, note that

(v) $A/(x - \lambda) = \mathbb{C}^{[2]}$ for every $\lambda \in \mathbb{C}^*$.

(vi) $A/(x) \neq \mathbb{C}^{[2]}$.

Therefore, if A were isomorphic to $\mathbb{C}^{[3]}$, then property (vi) would show that $x - \lambda$ cannot be a coordinate in A for any λ and then, by property (v), it would have yielded a counterexample to the Abhyankar–Sathaye Conjecture for $n = 3$.

However, Makar-Limanov proved [65] that $A \neq \mathbb{C}^{[3]}$; for this result, he introduced a new invariant which distinguished between A and $\mathbb{C}^{[3]}$. This invariant, which he had named AK-invariant, is now named Makar-Limanov invariant and is denoted by ML. It is defined in Section 3. Makar-Limanov proved that

(vii) $\mathrm{ML}(A) = \mathbb{C}[x]$ (Makar-Limanov [65]).

However, the Makar-Limanov invariant of $\mathbb{C}^{[n]}$ is $\mathbb{C}$ for any integer $n \geq 1$. Thus $A \not\cong \mathbb{C}^{[3]}$. Subsequently, other Koras–Russell threefolds were shown to be not isomorphic to the polynomial ring. Eventually, Kaliman–Koras–Makar-Limanov–Russell proved that every $\mathbb{C}^*$-action on $\mathbb{C}^3$ is linearizable (cf. [55]).

Now for ZCP in characteristic zero, a crucial question, still open, is whether $A^{[1]} = \mathbb{C}^{[4]}$. Because if $A^{[1]} = \mathbb{C}^{[4]}$, then A would be a counterexample to the ZCP in characteristic zero for $n = 3$. In this context, the following results have been proved:

(viii) $\mathrm{ML}(A^{[1]}) = \mathbb{C}$ (Dubouloz [30]).

(ix) V is $\mathbb{A}^1$-contractible (Dubouloz–Fasel [31], also see [33,54]).

Note that $A^{[1]} = \mathbb{C}^{[4]}$ would imply that $\mathrm{ML}(A^{[1]}) = \mathbb{C}$ and Dubouloz's result (viii) shows that the latter indeed holds. On the other hand, Asok had suggested a program for showing that the variety V is not $\mathbb{A}^1$-contractible and hence A is not a stably polynomial

ring (see [54]). However, Hoyois, Krishna, and Østvær have proved [54] that a step in his program does not hold for V. They had further shown that V is stably $\mathbb{A}^1$-contractible. In a remarkable paper [31], Dubouloz and Fasel have established that V is in fact $\mathbb{A}^1$-contractible, which seems to provide further evidence in favor of $A^{[1]} = \mathbb{C}^{[4]}$. The variety V is in fact the first example of an $\mathbb{A}^1$-contractible threefold which is not algebraically isomorphic to $\mathbb{C}^3$.

Nonrectifiable epimorphisms and Asanuma's rings. Let $m \leq n$ be two integers. A k-algebra epimorphism $\phi : k[X_1, \ldots, X_n] \twoheadrightarrow k[Y_1, \ldots, Y_m]$ is said to be *rectifiable* if there exists a k-algebra automorphism ψ of $k[X_1, \ldots, X_n]$ such that $\phi \circ \psi(X_i) = Y_i$ for $1 \leq i \leq m$ and $\phi \circ \psi(X_j) = 0$ for $m + 1 \leq j \leq n$. Equivalently, over an algebraically closed field k, a k-embedding $\Phi : \mathbb{A}_k^m \hookrightarrow \mathbb{A}_k^n$ is said to be *rectifiable* if there exists an automorphism Ψ of $\mathbb{A}_k^n$ such that $\Psi \circ \Phi$ is the canonical embedding mapping $(y_1, \ldots, y_m) \to (y_1, \ldots, y_m, 0, \ldots, 0)$.

A famous theorem of Abhyankar–Moh and Suzuki proves that any epimorphism $\phi : k[X, Y] \to k[T]$ is rectifiable in characteristic zero [5, 86]. On the other hand, in positive characteristic, there exist nonrectifiable epimorphisms from $k[X, Y]$ to $k[T]$ (see Segre [83], Nagata [71]). It is an open problem whether there exist nonrectifiable epimorphisms over the field of complex numbers (see [38]).

Asanuma has described an explicit method for constructing affine rings which are stably polynomial rings, by making use of nonrectifiable epimorphisms ([7], also see [38, **PROPOSITION 3.7**]). Such rings are considered to be potential candidates for counterexamples to the ZCP. For instance, when k is of positive characteristic, nonrectifiable epimorphisms from $k[X, Y]$ to $k[T]$ yield counterexamples to the ZCP.

Let $\phi : \mathbb{R}[X, Y, Z] \to \mathbb{R}[T]$ be defined by

$$\phi(X) = T^3 - 3T, \quad \phi(Y) = T^4 - 4T^2, \quad \phi(Z) = T^5 - 10T.$$

Shastri constructed the above epimorphism ϕ and proved that it defines a nonrectifiable (polynomial) embedding of the trefoil knot in $\mathbb{A}_{\mathbb{R}}^3$ [84]. Using a result of Serre [63, **THEOREM 1, P. 281**], one knows that $\ker(\phi) = (f, g)$ for some $f, g \in k[X, Y, Z]$. Using f and g, Asanuma constructed the ring $B = \mathbb{R}[T][X, Y, Z, U, V]/(T^d U - f, T^d V - g)$ and proved that $B^{[1]} = \mathbb{R}[T]^{[4]} = \mathbb{R}^{[5]}$ (cf. [7, **COROLLARY 4.2**]). He asked [7, **REMARK 7.8**]:

Question 2. Is $B = \mathbb{R}^{[4]}$?

The interesting aspect of the question is that once the problem gets solved, irrespective of whether the answer is "Yes" or "No," that is, either way, one would have solved a major problem in Affine Algebraic Geometry. Indeed:

If $B = \mathbb{R}^{[4]}$, then there exist nonlinearizable $\mathbb{R}^*$-actions on the affine four-space $\mathbb{A}_{\mathbb{R}}^4$.

If $B \neq \mathbb{R}^{[4]}$, then clearly B is a counterexample to the ZCP!!

3. CHARACTERIZATION PROBLEM

The Characterization Problem in affine algebraic geometry seeks a "useful characterization" of the polynomial ring or, equivalently (when the ground field is algebraically

closed), an affine n-space. For instance, the following two results give respectively an algebraic and a topological characterization of $k^{[1]}$ (or $\mathbb{A}^1_{\mathbb{C}}$).

Theorem 3.1. *Let k be an algebraically closed field of characteristic zero. Then the polynomial ring $k^{[1]}$ is the only one-dimensional affine UFD with $A^* = k^*$.*

Theorem 3.2. *Let k be the field of complex numbers $\mathbb{C}$. Then the affine line $\mathbb{A}^1_{\mathbb{C}}$ is the only acyclic normal curve.*

While the Characterization Problem is one of the most important problems in affine algebraic geometry in its own right, it is also closely related to some of the challenging open problems on the affine space like the "Cancellation Problem." For instance, each of the above characterizations of $k^{[1]}$ immediately solves the Cancellation Problem in dimension one: $A^{[1]} = k^{[2]} \implies A = k^{[1]}$. The complexity of the Characterization Problem increases with the dimension of the rings.

In his attempt to solve the Cancellation Problem for the affine plane, Ramanujam obtained a remarkable topological characterization of the affine plane $\mathbb{C}^2$ in 1971 [72]. He proved that

Theorem 3.3. *$\mathbb{C}^2$ is the only contractible smooth surface which is simply connected at infinity.*

Ramanujam also constructed contractible surfaces which are not isomorphic to $\mathbb{C}^2$. Soon, in 1975, Miyanishi [67] obtained an algebraic characterization of the polynomial ring $k^{[2]}$. He proved that

Theorem 3.4. *Let k be an algebraically closed field of characteristic zero and A be a two-dimensional affine factorial domain over k. Then $A = k^{[2]}$ if and only if it satisfies the following:*

(i) $A^* = k^*$.

(ii) *There exists an element $f \in A$ and a subring B of A such that $A[f^{-1}] = B[f^{-1}]^{[1]}$.*

This algebraic characterization was used by Fujita, Miyanishi, and Sugie [43,70] to solve the Cancellation Problem for $k[X, Y]$. In 2002 [50], using methods of Mumford and Ramanujam, Gurjar gave a topological proof of the cancellation property of $\mathbb{C}[X, Y]$.

Remarkable characterizations of the affine three space were obtained by Miyanishi [68] and Kaliman [56] (also see [69] for a beautiful survey). We state below the version of Kaliman.

Theorem 3.5. *Let A be a three-dimensional smooth factorial affine domain over the field of complex numbers $\mathbb{C}$. Let $X = \mathrm{Spec}\, A$. Then $A = \mathbb{C}^{[3]}$ if and only if it satisfies the following:*

(i) $A^* = \mathbb{C}^*$.

(ii) $H_3(X, \mathbb{Z}) = 0$, *or X is contractible.*

(iii) *X contains a cylinder-like open set V such that $V \cong U \times \mathbb{A}^2$ for some curve U*
and each irreducible component of the complement $X \setminus V$ has at most isolated
singularities.

When $A^{[1]} = \mathbb{C}^{[4]}$, it is easy to see that A possesses properties (i) and (ii) of Theorem 3.5. Thus, by Theorem 3.5, the ZCP for $\mathbb{C}^{[3]}$ reduces to examining whether condition (iii) necessarily holds for a $\mathbb{C}$-algebra A satisfying $A^{[1]} = \mathbb{C}^{[4]}$.

In [29], we have obtained another characterization of the affine three-space using certain invariants of an affine domain defined by locally nilpotent derivations. We state it below.

Locally nilpotent derivations and a characterization of $\mathbb{C}^{[3]}$. Let B be an affine domain over a field k of characteristic zero. A k-linear derivation D on B is said to be a locally nilpotent derivation if, for any $a \in B$ there exists an integer n (depending on a) satisfying $D^n(a) = 0$. Let $\mathrm{LND}(B)$ denote the set of all locally nilpotent k-derivations of B and let

$$\mathrm{LND}^*(B) = \big\{ D \in \mathrm{LND}(B) \mid Ds = 1 \text{ for some } s \in B \big\}.$$

Then we define

$$\mathrm{ML}(B) := \bigcap_{D \in \mathrm{LND}(B)} \ker D \quad \text{and} \quad \mathrm{ML}^*(B) := \bigcap_{D \in \mathrm{LND}^*(B)} \ker D.$$

The above $\mathrm{ML}(B)$, introduced by Makar-Limanov, is now called the Makar-Limanov invariant of B; $\mathrm{ML}^*(B)$ was introduced by Freudenburg in [41, P. 237]. We call it the Makar-Limanov–Freudenburg invariant or ML-F invariant. If $\mathrm{LND}^*(B) = \emptyset$, we define $\mathrm{ML}^*(B)$ to be B. We have obtained the following theorem [29, THEOREM 4.6].

Theorem 3.6. *Let A be a three-dimensional affine factorial domain over an algebraically closed field k of characteristic zero. Then the following are equivalent:*

(I) $A = k^{[3]}$.

(II) $\mathrm{ML}^*(A) = k$.

(III) $\mathrm{ML}(A) = k$ *and* $\mathrm{ML}^*(A) \neq A$.

A similar result has also been proved in dimension two under weaker hypotheses [29, THEOREM 3.8]. The above characterization of the affine three-space does not extend to higher dimensions [29, EXAMPLE 5.6]. So far, no suitable characterization of the affine n-space for $n \geq 4$ is known to the author.

4. AFFINE FIBRATIONS

Let R be a commutative ring. A fundamental theorem of Bass–Connell–Wright and Suslin [10, 85] on the structure of locally polynomial algebras states that:

Theorem 4.1. *Let A be a finitely presented algebra over a ring R. Suppose that for each maximal ideal $\mathfrak{m}$ of R, $A_{\mathfrak{m}} = R_{\mathfrak{m}}^{[n]}$ for some integer $n \geq 0$. Then $A \cong \mathrm{Sym}_R(P)$ for some finitely generated projective R-module P of rank n.*

Now for a prime ideal P of R, let $k(P)$ denote the residue field R_P/PR_P. The area of affine fibrations seeks to derive information about the structure and properties of an R-algebra A from the information about the fiber rings $A \otimes_R k(P)(= A_P/PA_P)$ of A at the points P of the prime spectrum of R, i.e., at the prime ideals P of R.

An R-algebra A is said to be an $\mathbb{A}^n$-fibration over R if A is a finitely generated flat R-algebra and for each prime ideal P of R, $A \otimes_R k(P) = k(P)^{[n]}$.

The most important problem on $\mathbb{A}^n$-fibrations, due to Veĭsfeĭler and Dolgačev [87], can be formulated as follows:

Question 3. Let R be a Noetherian domain of dimension d and A be an $\mathbb{A}^n$-fibration over R.

(i) If R is regular, is $A \cong \mathrm{Sym}_R(Q)$ for some projective module Q over R? (In particular, if R is regular local, is then $A = R^{[n]}$?)

(ii) In general, what can one say about the structure of A?

Question 3 is considered a hard problem. When $n = 1$, it has an affirmative answer for all d. This has been established in the works of Kambayashi, Miyanishi, and Wright [59, 60]. Their results were further refined by Dutta who showed that it is enough to assume the fiber conditions only on generic and codimension-one fibers ([34]; also see [14, 17, 40]).

In case $n = 2$, $d = 1$, and R contains the field of rational numbers, an important theorem of Sathaye [81] gives an affirmative answer to Question 3 (i). To prove this theorem, Sathaye first generalized the Abhyankar–Moh expansion techniques originally developed over $k[[x]]$ to $k[[x_1, \ldots, x_n]]$ [80]. The expansion techniques were used by Abhyankar–Moh to prove their famous epimorphism theorem. The generalized expansion techniques were further developed by Sathaye [82] to prove a conjecture of Daigle and Freudenburg. The result was a crucial step in Daigle–Freudenburg's theorem that the kernel of any triangular derivation of $k[X_1, X_2, X_3, X_4]$ is a finitely generated k-algebra [23].

When the residue field of R is of positive characteristic, Asanuma has shown in [6, THEOREM 5.1] that Question 3 (i) has a negative answer for $n = 2$, $d = 1$, and the author has generalized Asanuma's ring [47] to give a negative answer to Question 3 (i) for $n = 2$ and any $d > 1$ (also see [48]). In Theorem 5.4, the author proved that in a special situation $\mathbb{A}^2$-fibration is indeed trivial.

However, if $n = 2$, $d = 2$, and R contains the field of rational numbers, Question 3 (i) is an open problem. A candidate counterexample is discussed in Section 7.

In the context of Question 3 (ii), a deep work of Asanuma [6] provides a stable structure theorem for A. As a consequence of Asanuma's structure theorem, it follows that if R is regular local, then there exists an integer $m \geq 0$ such that $A^{[m]} = R^{[m+n]}$. Thus it is very tempting to look for possible counterexamples to the affine fibration problem in order to

obtain possible counterexamples to the ZCP in characteristic zero. One can see [12,24,36,37] and [38, SECTION 3.1] for more results on affine fibrations.

So far we have considered affine fibrations where the fibre rings are polynomial rings. Bhatwadekar and Dutta have obtained some nice results on rings whose fiber rings are of the form $k[X, 1/X]$ [15,16]. Later Bhatwadekar, the author, and A. Abhyankar studied rings whose fiber rings are Laurent polynomial algebras or rings of the form $k[X, 1/f(X)]$, or of the form $k[X, Y, 1/(aX + b), 1/(cY + d)]$ for some $a, b, c, d \in k$ [1, 2, 18, 19, 44]. One of the results of Bhatwadekar and the author provides a Laurent polynomial analogue of Theorem 4.1 and the affine fibration problem Question 3. More generally, we have [19, THEOREMS A AND C]:

Theorem 4.2. *Let R be a Noetherian normal domain with field of fractions K and A be a faithfully flat R-algebra such that*

(i) $A \otimes_R K \cong K[X_1, \frac{1}{X_1}, \ldots, X_n, \frac{1}{X_n}]$,

(ii) *for each height-one prime ideal P of R, $A \otimes_R k(P) \cong k(P)[X_1, \frac{1}{X_1}, \ldots, X_n, \frac{1}{X_n}]$.*

Then A is a locally Laurent polynomial algebra in n variables over R, i.e.,

$$A_m = R_m[X_1, \frac{1}{X_1}, \ldots, X_n, \frac{1}{X_n}]$$

and is of the form $B[I^{-1}]$, where B is the symmetric algebra of a projective R-module Q of rank n, Q is a direct sum of finitely generated projective R-modules of rank one, and I is an invertible ideal of B.

5. EPIMORPHISM PROBLEM

The Epimorphism Problem for hypersurfaces asks the following fundamental question:

Question 4. Let k be a field and $f \in B = k^{[n]}$ for some integer $n \geq 2$. Suppose

$$B/(f) \cong k^{[n-1]}.$$

Does this imply that $B = k[f]^{[n-1]}$, i.e., is f a coordinate in B?

This problem is generally known as the *Epimorphism Problem*. It is an open problem and is regarded as one of the most challenging and celebrated problems in the area of affine algebraic geometry (see [38,69,75,77] for useful surveys).

The first major breakthrough on Question 4 was achieved during 1974–1975, independently, by Abhyankar–Moh and Suzuki [5,86]. They showed that Question 4 has an affirmative answer when k is a field of characteristic zero and $n = 2$. Over a field of positive characteristic, explicit examples of nonrectifiable epimorphisms from $k[X, Y]$ to $k[T]$ (referred to in Section 2) and hence explicit examples of nontrivial lines had already been demonstrated by Segre [83] in 1957 and Nagata [71] in 1971. However, over a field of characteristic zero, we have the following conjecture:

Abhyankar–Sathaye Conjecture. *Let k be a field of characteristic zero and $f \in B = k^{[n]}$ for some integer $n \geq 2$. Suppose that $B/(f) \cong k^{[n-1]}$. Then $B = k[f]^{[n-1]}$.*

In case $n = 3$, some special cases have been solved by Sathaye, Russell, and Wright [73,76,79,89]. In [79], Sathaye proved the conjecture for the linear planes, i.e., polynomials F of the form $aZ - b$, where $a, b \in k[X, Y]$. This was further extended by Russell over fields of any characteristic. They proved that

Theorem 5.1. *Let $F \in k[X, Y, Z]$ be such that $F = aZ - b$, where $a(\neq 0)$, $b \in k[X, Y]$, and $k[X, Y, Z]/(F) = k^{[2]}$. Then there exist $X_0, Y_0 \in k[X, Y]$ such that $k[X, Y] = k[X_0, Y_0]$ with $a \in k[X_0]$ and $k[X, Y, Z] = k[X_0, F]^{[1]}$.*

When k is an algebraically closed field of characteristic $p \geq 0$, Wright [89] proved the conjecture for polynomials F of the form $aZ^m - b$ with $a, b \in k[X, Y]$, $m \geq 2$ and $p \nmid m$. Das and Dutta showed [28, THEOREM 4.5] that Wright's result extends to any field k. They proved that

Theorem 5.2. *Let k be any field with $\mathrm{ch}\, k = p\ (\geq 0)$ and $F = aZ^m - b \in k[X, Y, Z]$ be such that $a(\neq 0)$, $b \in k[X, Y]$, $m \geq 2$ and $p \nmid m$. Suppose that $k[X, Y, Z]/(F) = k^{[2]}$. Then there exists $X_0 \in k[X, Y]$ such that $k[X, Y] = k[X_0, b]$ with $a \in k[X_0]$ and $k[X, Y, Z] = k[F, Z, X_0]$.*

The condition that $p \nmid m$ is necessary in Theorem 5.2 (cf. [28, REMARK 4.6]).

Most of the above cases are covered by the following generalization due to Russell and Sathaye [76, THEOREM 3.6]:

Theorem 5.3. *Let k be a field of characteristic zero and let*

$$F = a_m Z^m + a_{m-1} Z^{m-1} + \cdots + a_1 Z + a_0 \in k[X, Y, Z]$$

where $a_0, \ldots, a_m \in k[X, Y]$ are such that $\mathrm{GCD}\,(a_1, \ldots, a_m) \notin k$. Suppose that

$$k[X, Y, Z]/(F) = k^{[2]}.$$

Then there exists $X_0 \in k[X, Y]$ such that $k[X, Y] = k[X_0, b]$ with $a_m \in k[X_0]$. Further, $k[X, Y, Z] = k[F]^{[2]}$.

Thus, for $k[X, Y, Z]$, the Abhyankar–Sathaye conjecture remains open for the case when $\mathrm{GCD}(a_1, \ldots, a_m) = 1$.

A common theme in most of the partial results proved in the Abhyankar–Sathaye conjecture for $k[X, Y, Z]$ is that, if F is considered as a polynomial in Z, then the coordinates of $k[X, Y]$ can be so chosen that the coefficient of Z becomes a polynomial in X. The Abhyankar–Sathaye conjecture for $k[X, Y, Z]$ can now be split into two parts.

Question 4A. Let k be a field of characteristic zero and let

$$F = a_m Z^m + a_{m-1} Z^{m-1} + \cdots + a_1 Z + a_0 \in k[X, Y, Z]$$

where $a_0, \ldots, a_m \in k[X, Y]$. Suppose that $k[X, Y, Z]/(F) = k^{[2]}$. Does there exist $X_0 \in k[X, Y]$ such that $k[X, Y] = k[X_0]^{[1]}$ with $a_m \in k[X_0]$?

Question 4B. Let k be a field of characteristic zero and suppose

$$F = a_m(X)Z^m + a_{m-1}Z^{m-1} + \cdots + a_1 Z + a_0 \in k[X, Y, Z]$$

where $a_0, \ldots, a_{m-1} \in k[X, Y]$ and $a_m \in k[X]$. Suppose that $k[X, Y, Z]/(F) = k^{[2]}$. Does this imply that $k[X, Y, Z] = k[F]^{[2]}$?

Sangines Garcia in his PhD thesis [78] answered Question 4A affirmatively for the case $m = 2$. In [21], Bhatwadekar and the author have given an alternative proof of this result of Garcia.

When k is any field, as a partial generalization of Theorem 5.1 and Question 4B in four variables, the author proved the Abhyankar–Sathaye conjecture for a polynomial F of the form $X^m Y - F(X, Z, T) \in k[X, Y, Z, T]$. This was one of the consequences of her general investigation on the ZCP [46]. In the process, she related it with other central problems on affine spaces like the affine fibration problem and the ZCP. The author has proved equivalence of ten statements, some of which involve an invariant introduced by Derksen, which is called the Derksen invariant.

The Derksen invariant of an integral domain B, denoted by $\mathrm{DK}(B)$, is defined as the smallest subring of B generated by the kernel of D, where D varies over the set of all locally nilpotent derivations of B.

Theorem 5.4. *Let k be a field of any characteristic and A an integral domain defined by*

$$A = k[X, Y, Z, T]/\big(X^m Y - F(X, Z, T)\big), \quad \text{where } m > 1.$$

Let x, y, z, and t denote, respectively, the images of X, Y, Z, and T in A. Set $f(Z, T) := F(0, Z, T)$ and $G := X^m Y - F(X, Z, T)$. Then the following statements are equivalent:

(i) $k[X, Y, Z, T] = k[X, G]^{[2]}$.

(ii) $k[X, Y, Z, T] = k[G]^{[3]}$.

(iii) $A = k[x]^{[2]}$.

(iv) $A = k^{[3]}$.

(v) $A^{[\ell]} \cong_k k^{[\ell+3]}$ *for some integer* $\ell \geq 0$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.

(vi) A *is an* $\mathbb{A}^2$-*fibration over* $k[x]$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.

(vii) A *is geometrically factorial over* k, $\mathrm{DK}(A) \neq k[x, z, t]$ *and the canonical map* $k^* \to K_1(A)$ *(induced by the inclusion* $k \hookrightarrow A$*) is an isomorphism.*

(viii) A *is geometrically factorial over* k, $\mathrm{DK}(A) \neq k[x, z, t]$ *and* $(A/xA)^* = k^*$.

(ix) $k[Z, T] = k[f]^{[1]}$.

(x) $k[Z, T]/(f) = k^{[1]}$ *and* $\mathrm{DK}(A) \neq k[x, z, t]$.

The equivalence of (ii) and (iv) provides an answer to Question 4 for the special case of the polynomial $X^m Y - F(X, Z, T)$. The equivalence of (i) and (iii) provides an answer

to a special case of Question $4'$ (stated below) for the ring $R = k[x]$. The equivalence of (iii) and (vi) answers Question 3 in a special situation. For more discussions, see [48].

In a remarkable paper Kaliman proved the following result over the field of complex numbers [56]. Later, Daigle and Kaliman extended it over any field k of characteristic zero [25].

Theorem 5.5. *Let k be a field of characteristic zero. Let $F \in k[X, Y, Z]$ be such that $k[X, Y, Z]/(F - \lambda) = k^{[2]}$ for almost every $\lambda \in k$. Then $k[X, Y, Z] = k[F]^{[2]}$.*

A general version of Question 4 can be asked as:

Question $4'$. Let R be a ring and $f \in A = R^{[n]}$ for some integer $n \geq 2$. Suppose

$$A/(f) \cong R^{[n-1]}.$$

Does this imply that $A = R[f]^{[n-1]}$, i.e., is f a coordinate in A?

There have been affirmative answers to Question $4'$ in special cases by Bhatwadekar, Dutta, and Das [11, 13, 28]. Bhatwadekar and Dutta had considered linear planes, i.e., polynomials F of the form $aZ - b$, where $a, b \in R[X, Y]$ over a discrete valuation ring R and proved that special cases of the linear planes are actually variables. Bhatwadekar–Dutta have also shown [12] that a negative answer to Question $4'$ in the case when $n = 3$ and R is a discrete valuation ring containing $\mathbb{Q}$ will give a negative answer to the affine fibration problem (Question 3 (i)) for the case $n = 2$ and $d = 2$. An example of a case of linear planes which remains unsolved is discussed in Section 7.

6. $\mathbb{A}^n$-FORMS

Let A be an algebra over a field k. We say that A is an $\mathbb{A}^n$-form over k if $A \otimes_k L = L^{[n]}$ for some finite algebraic extension L of k. Let A be an $\mathbb{A}^n$-form over a field k.

When $n = 1$, it is well known that if $L|_k$ is a separable extension, then $A = k^{[1]}$ (i.e., trivial) and that if $L|_k$ is purely inseparable then A need not be $k^{[1]}$. An extensive study of such purely inseparable algebras was made by Asanuma in [8]. Over any field of positive characteristic, the nontrivial purely inseparable $\mathbb{A}^1$-forms can be used to give examples of nontrivial $\mathbb{A}^n$-forms for any integer $n > 1$.

When $n = 2$ and $L|_k$ is a separable extension, then Kambayashi established that $A = k^{[2]}$ [57]. However, the problem of existence of nontrivial separable $\mathbb{A}^3$-forms is open in general. A few recent partial results on the triviality of separable $\mathbb{A}^3$-forms are mentioned below.

Let A be an $\mathbb{A}^3$-form over a field k of characteristic zero and $\bar{k}$ be an algebraic closure of k. Then $A = k^{[3]}$ if it satisfies any one of the following:

(1) A admits a fixed point free locally nilpotent derivation D (Daigle and Kaliman [25, COROLLARY 3.3]).

(2) A contains an element f which is a coordinate of $A \otimes_k \bar{k}$ (Daigle and Kaliman [25, **PROPOSITION 4.9**]).

(3) A admits an effective action of a reductive algebraic k-group of positive dimension (Koras and Russell [61, **THEOREM C**]).

(4) A admits either a fixed point free locally nilpotent derivation or a nonconfluent action of a unipotent group of dimension two (Gurjar, Masuda, and Miyanishi [51]).

(5) A admits a locally nilpotent derivation D such that $\mathrm{rk}(D \otimes 1_{\bar{k}}) \leq 2$ (Dutta, Gupta, and Lahiri [39]).

Now let R be a ring containing a field k. An R-algebra A is said to be an $\mathbb{A}^n$-form over R with respect to k if $A \otimes_k \bar{k} = (R \otimes_k \bar{k})^{[n]}$, where $\bar{k}$ denotes the algebraic closure of k. A few results on triviality of separable A^n-forms over a ring R are listed below.

Let A be an $\mathbb{A}^n$-form over a ring R containing a field k of characteristic 0. Then:

(1) If $n = 1$, then A is isomorphic to the symmetric algebra of a finitely generated rank one projective module over R [35, **THEOREM 7**].

(2) If $n = 2$ and R is a PID containing $\mathbb{Q}$, then $A = R^{[2]}$ [35, **REMARK 8**].

(3) If $n = 2$, then A is an $\mathbb{A}^2$-fibration over R.

(4) If $n = 2$ and R is a one-dimensional Noetherian domain, then there exists a finitely generated rank-one projective R-module Q such that $A \cong (\mathrm{Sym}_R(Q))^{[1]}$ [39, **THEOREM 3.7**].

(5) If $n = 2$ and A admits has a fixed point free locally nilpotent R-derivation over any ring R, then there exists a finitely generated rank one projective R-module Q such that $A \cong (\mathrm{Sym}_R(Q))^{[1]}$ [39, **THEOREM 3.8**].

The result (3) above shows that an affirmative answer to the $\mathbb{A}^2$-fibration problem (Question 3 (i)) will ensure an affirmative answer to the problem of $\mathbb{A}^2$-forms over general rings. Over a field F of any characteristic, Das has shown [27] that any factorial $\mathbb{A}^1$-form A over a ring R containing F is trivial if there exists a retraction map from A to R.

We cannot say much about $\mathbb{A}^3$-forms over general rings till the time we solve it over fields.

7. AN EXAMPLE OF BHATWADEKAR AND DUTTA

The following example arose from the study of linear planes over a discrete valuation ring by Bhatwadekar and Dutta [12]. Question 5 stated below is an open problem for at least three decades. Let

$$A = \mathbb{C}[T, X, Y, Z] \quad \text{and} \quad R = \mathbb{C}[T, F] \subset A,$$

where $F = TX^2Z + X + T^2Y + TXY^2$.

Let

$$P := XZ + Y^2,$$
$$G := TY + XP,$$

and

$$H := T^2Z - 2TYP - XP^2$$

Then, we can see that

$$XH + G^2 = T^2P$$

and $F = X + TG$. Clearly, $\mathbb{C}[T, T^{-1}][F, G, H] \subseteq \mathbb{C}[T, T^{-1}][X, Y, Z]$.

Then the following statements hold:

(i) $\mathbb{C}[T, T^{-1}][X, Y, Z] = \mathbb{C}[T, T^{-1}, F, G, H] = \mathbb{C}[T, T^{-1}][F]^{[2]}$.

(ii) $\mathbb{C}[T, X, Y, Z]$ is an $\mathbb{A}^2$-fibration over $\mathbb{C}[T, F]$.

(iii) $\mathbb{C}[T, X, Y, Z]^{[1]} = \mathbb{C}[T, F]^{[3]}$.

(iv) $\mathbb{C}[T, X, Y, Z]/(F) = \mathbb{C}[T]^{[2]} = \mathbb{C}^{[3]}$.

(v) $\mathbb{C}[T, X, Y, Z]/(F - f(T)) = \mathbb{C}[T]^{[2]}$ for every polynomial $f(T) \in \mathbb{C}[T]$.

(vi) $\mathbb{C}[T, X, Y, Z][1/F] = \mathbb{C}[T, F, 1/F, G]^{[1]}$.

(vii)   For any $u \in (T, F)R$, $A[1/u] = R[1/u]^{[2]}$, i.e., $\mathbb{C}[T, X, Y, Z][1/u] = \mathbb{C}[T, F, 1/u]^{[2]}$.

Question 5.   (a) Is $A = \mathbb{C}[T, F]^{[2]}(= R^{[2]})$?

(b) At least is $A = \mathbb{C}[F]^{[3]}$?

If the answer is "No" to (a), then it is a counterexample to the following problems:

(1) $\mathbb{A}^2$-fibration Problem over $\mathbb{C}^{[2]}$ by (ii).

(2) Cancellation Problem over $\mathbb{C}^{[2]}$ by (iii).

(3) Epimorphism problem over the ring $\mathbb{C}[T]$ (see Question 4′) by (iv).

If the answer is "No" to (b) and hence to (a), then it is a counterexample also to the Epimorphism Problem for $\mathbb{C}^{[4]} \twoheadrightarrow \mathbb{C}^{[3]}$.

Though the above properties have been proved in several places, a proof is presented below. A variant of the Bhatwadekar–Dutta example was also constructed by Vénéreau in his thesis [88]; for a discussion on this and related examples, see [24, 41, 64].

Proof. (i) We show that

$$\mathbb{C}[T, T^{-1}][X, Y, Z] = \mathbb{C}[T, T^{-1}][F, G, H]. \tag{1}$$

Note that

$$X = F - TG, \quad P = \frac{XH + G^2}{T^2},$$
$$Y = (G - XP)/T,$$

and

$$Z = (H + 2TYP + XP^2)/T^2,$$

and hence equation (1) follows.

(ii) Clearly, A is a finitely generated R-algebra. It can be shown by standard arguments that A is a flat R-algebra [66, THEOREM 20.H]. We now show that $A \otimes_R k(p) = k(p)^{[2]}$ for every prime ideal p of R. We note that $F - X \in TA$ and hence the image of F in A/TA is same as that of X. Now let p be a prime ideal of R. Then either $T \in p$ or $T \notin p$. If $T \in p$, then $A \otimes_R k(p) = k(p)[Y, Z] = k(p)^{[2]}$. If $T \notin p$, then image of T in $k(p)$ is a unit and the result follows from (i).

(iii) Let $D = A[W] = \mathbb{C}[T, X, Y, Z, W] = \mathbb{C}^{[5]}$. We shall show that

$$D = \mathbb{C}[T, F]^{[3]} = R^{[3]}.$$

Let

$$W_1 := TW + P,$$
$$G_1 := \frac{(G - FW_1)}{T} = Y - XW - (TY + XP)(TW + P) = Y - XW - GW_1,$$
$$H_1 := \frac{\{H + 2GW_1 - (F - GT)W_1^2\}}{T^2} = Z + 2YW - XW^2.$$

Now let

$$G_2 := G_1 + FW_1^2 = (Y - XW) - TW_1(Y - XW - GW_1) = Y - XW - TW_1G_1$$

and

$$W_2 := \frac{W_1 - (H_1F + G_2^2)}{T} = W + 2G_1W_1(Y - XW) - GH_1 - TG_1^2W_1^2$$

Then, it is easy to see that

$$\begin{aligned} D[T^{-1}] &= \mathbb{C}[T, T^{-1}][X, Y, Z, W] \\ &= \mathbb{C}[T, T^{-1}][F, G, H, W_1] \\ &= \mathbb{C}[T, T^{-1}][F, G_1, H_1, W_1] \\ &= \mathbb{C}[T, T^{-1}][F, G_2, H_1, W_2] \end{aligned}$$

and that $\mathbb{C}[T, F, G_2, H_1, W_2] \subseteq D$. Let $D/TD = \mathbb{C}[x, y, z, w]$, where x, y, z, w denote the images of X, Y, Z, W in D/TD. We now show that $D \subseteq \mathbb{C}[T, F, G_2, H_1, W_2]$. For this, it is enough to show that the kernel of the natural map $\phi : \mathbb{C}[T, F, G_2, H_1, W_2] \to D/TD$ is generated by T. We note that the image of ϕ is

$$\mathbb{C}[x, y - xw, z + 2yw - xw^2, w + 2p(y - xw - xp^2)(y - xw) - xp(z + 2yw - xw^2)],$$

which is of transcendence degree 4 over $\mathbb{C}$. Hence the kernel of ϕ is a prime ideal of height one and is generated by T. Therefore, $D = \mathbb{C}[T, F, G_2, H_1, W_2]$.

(iv)–(v) Let $B = \mathbb{C}[T, X, Y, Z]/(F - f(T))$ for some polynomial $f \in \mathbb{C}[T]$ and $S = \mathbb{C}[T]$. By (ii), it follows that B is an $\mathbb{A}^2$-fibration over S. Hence, by Sathaye's theorem [81], B is locally a polynomial ring over S and hence by Theorem 4.1, B is a polynomial ring over S.

(vi) Let $H_1 := \frac{FH+G^2}{T}$. Then

$$H_1 = \frac{(X + TG)(T^2 Z - 2TYP - XP^2) + (TY + XP)^2}{T} = TP + GH.$$

Let $H_2 := \frac{FH_1 + G^3}{T}$. Then

$$
\begin{aligned}
H_2 &= \frac{(X + TG)(TP + GH) + G^3}{T} \\
&= \frac{T(G^2 H + TGP + XP) + G(XH + G^2)}{T} \\
&= \frac{T(G^2 H + TGP + XP) + GT^2 P}{T} \\
&= G^2 H + XP + 2TGP.
\end{aligned}
$$

Let $H_3 := \frac{F(H_2 - G) + G^4}{T}$. Then

$$
\begin{aligned}
H_3 &= \frac{F(G^2 H + XP + 2TGP - XP - TY) + G^4}{T} \\
&= \frac{F(2TGP - TY) + G^2(FH + G^2)}{T} \\
&= \frac{TF(2GP - Y) + TH_1 G^2}{T} \\
&= F(2GP - Y) + H_1 G^2.
\end{aligned}
$$

Now it is easy to see that

$$
\begin{aligned}
\mathbb{C}[T, X, Y, Z, F^{-1}][T^{-1}] &= \mathbb{C}[T, T^{-1}][F, F^{-1}, G, H] \\
&= \mathbb{C}[T, T^{-1}][F, F^{-1}, G, H_1] \\
&= \mathbb{C}[T, T^{-1}][F, F^{-1}, G, H_2] \\
&= \mathbb{C}[T, T^{-1}][F, F^{-1}, G, H_3],
\end{aligned}
$$

and that the image of $\mathbb{C}[T, F, F^{-1}, G, H_2]$ in $A[F^{-1}]/TA[F^{-1}]$ is of transcendence degree 3. Hence $A[F^{-1}] = \mathbb{C}[T, F, F^{-1}, G, H_3] = \mathbb{C}[T, F, F^{-1}, G]^{[1]}$.

(vii) Let m be any maximal ideal of R other than (T, F). Then either $T \notin m$ or $F \notin m$. Thus, in either case, from (i) and (vi), we have $A_m = R_m^{[2]}$.

Let $u \in (T, F)R$. Then a maximal ideal of $R[1/u]$ is an extension of a maximal ideal of R other than $(T, F)R$. Hence $A[1/u]$ is a locally polynomial ring in two variables over $R[1/u]$. Further any projective module over $R[1/u]$ is free. Thus, by Theorem 4.1, we have $A[1/u] = R[1/u]^{[2]}$. ■

ACKNOWLEDGMENTS

REFERENCES

[1]   A. M. Abhyankar and S. M. Bhatwadekar, Generically Laurent polynomial algebras over a D.V.R. which are not quasi Laurent polynomial algebras. *J. Pure Appl. Algebra* **218** (2014), no. 4, 651–660.

[2]   A. M. Abhyankar and S. M. Bhatwadekar, A note on quasi Laurent polynomial algebras in *n* variables. *J. Commut. Algebra* **6** (2014), no. 2, 127–147.

[3]   S. Abhyankar, P. Eakin, and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring. *J. Algebra* **23** (1972), 310–342.

[4]   S. S. Abhyankar, Polynomials and power series. In *Algebra, Arithmetic and Geometry with Applications*, edited by C. Christensen et al., pp. 783–784, Springer, 2004.

[5]   S. S. Abhyankar and T. T. Moh, Embeddings of the line in the plane. *J. Reine Angew. Math.* **276** (1975), 148–166.

[6]   T. Asanuma, Polynomial fibre rings of algebras over Noetherian rings. *Invent. Math.* **87** (1987), 101–127.

[7]   T. Asanuma, Non-linearizable algebraic k^*-actions on affine spaces. *Invent. Math.* **138** (1999), no. 2, 281–306.

[8]   T. Asanuma, Purely inseparable k-forms of affine algebraic curves. In *Affine algebraic geometry*, pp. 31–46, Contemp. Math. 369, Amer. Math. Soc., Providence, RI, 2005.

[9]   T. Asanuma and N. Gupta, On 2-stably isomorphic four dimensional affine domains. *J. Commut. Algebra* **10** (2018), no. 2, 153–162.

[10]   H. Bass, E. H. Connell, and D. L. Wright, Locally polynomial algebras are symmetric algebras. *Invent. Math.* **38** (1977), 279–299.

[11]   S. M. Bhatwadekar, Generalized epimorphism theorem. *Proc. Indian Acad. Sci.* **98** (1988), no. 2–3, 109–166.

[12]   S. M. Bhatwadekar and A. K. Dutta, On affine fibrations. In *Commutative algebra: Conf. Comm. Alg. ICTP (1992)*, edited by A. Simis, N. V. Trung, and G. Valla, pp. 1–17, World Sc., 1994.

[13]   S. M. Bhatwadekar and A. K. Dutta, Linear planes over a discrete valuation ring. *J. Algebra* **166** (1994), no. 2, 393–405.

[14]   S. M. Bhatwadekar and A. K. Dutta, On $\mathbb{A}^1$-fibrations of subalgebras of polynomial algebras. *Compos. Math.* **95** (1995), no. 3, 263–285.

[15]   S. M. Bhatwadekar and A. K. Dutta, Structure of $\mathbb{A}^*$-fibrations over one-dimensional seminormal semilocal domains. *J. Algebra* **220** (1999), 561–573.

[16]   S. M. Bhatwadekar and A. K. Dutta, On $\mathbb{A}^*$-fibrations. *J. Pure Appl. Algebra* **149** (2000), 1–14.

[17]   S. M. Bhatwadekar, A. K. Dutta, and N. Onoda, On algebras which are locally $\mathbb{A}^1$ in codimension-one. *Trans. Amer. Math. Soc.* **365** (2013), no. 9, 4497–4537.

[18]   S. M. Bhatwadekar and N. Gupta, On locally quasi $\mathbb{A}^*$ algebras in codimension-one over a Noetherian normal domain. *J. Pure Appl. Algebra* **215** (2011), 2242–2256.

[19]   S. M. Bhatwadekar and N. Gupta, The structure of a Laurent polynomial fibration in n variables. *J. Algebra* **353** (2012), no. 1, 142–157.

[20]   S. M. Bhatwadekar and N. Gupta, A note on the cancellation property of $k[X, Y]$. *J. Algebra Appl. (special issue in honour of Prof. Shreeram S. Abhyankar)* **14** (2015), no. 9, 15400071, 5 pp.

[21]   S. M. Bhatwadekar and N. Gupta, On quadratic planes. Preprint.

[22]   A. J. Crachiola and L. Makar-Limanov, An algebraic proof of a cancellation theorem for surfaces. *J. Algebra* **320** (2008), no. 8, 3113–3119.

[23]   D. Daigle and G. Freudenburg, Triangular derivations of $k[X_1, X_2, X_3, X_4]$. *J. Algebra* **241** (2001), no. 1, 328–339.

[24]   D. Daigle and G. Freudenburg, Families of affine fibrations. In *Symmetry and spaces*, edited by H. E. A. Campbell et al., pp. 35–43, Progr. Math. 278, Birkhäuser, 2010.

[25]   D. Daigle and S. Kaliman, A note on locally nilpotent derivations and variables of $k[X, Y, Z]$. *Canad. Math. Bull.* **52** (2009), no. 4, 535–543.

[26]   W. Danielewski, On a Cancellation Problem and automorphism groups of affine algebraic varieties. Preprint, Warsaw, 1989. (Appendix by K. Fieseler).

[27]   P. Das, A note on factorial $\mathbb{A}^1$-forms with retractions. *Comm. Algebra* **40** (2012), no. 9, 3221–3223.

[28]   P. Das and A. K. Dutta, Planes of the form $b(X, Y)Z^n - a(X, Y)$ over a DVR. *J. Commut. Algebra* **3** (2011), no. 4, 491–509.

[29]   N. Dasgupta and N. Gupta, An algebraic characterisation of the affine three space. 2019, arXiv:1709.00169. To appear in *J. Commut. Algebra*. https://projecteuclid. org/journals/jca/journal-of-commutative-algebra/DownloadAcceptedPapers/ 20301.pdf.

[30]   A. Dubouloz, The cylinder over the Koras–Russell cubic threefold has a trivial Makar-Limanov invariant. *Transform. Groups* **14** (2009), no. 3, 531–539.

[31]   A. Dubouloz and J. Fasel, Families of $\mathbb{A}^1$-contractible affine threefolds. *Algebr. Geom.* **5** (2018), no. 1, 1–14.

[32]   A. Dubouloz, Affine surfaces with isomorphic $\mathbb{A}^2$-cylinders. *Kyoto J. Math.* **59** (2019), no. 1, 181–193.

[33]   A. Dubouloz, S. Pauli, and P. A. Østvær, $\mathbb{A}^1$-contractibility of affine modifications. *Int. J. Math.* **14** (2019), no. 30, 1950069.

[34]   A. K. Dutta, On $\mathbb{A}^1$-bundles of affine morphisms. *J. Math. Kyoto Univ.* **35** (1995), no. 3, 377–385.

[35] A. K. Dutta, On separable $\mathbb{A}^1$-forms. *Nagoya Math. J.* **159** (2000), 45–51.

[36] A. K. Dutta, Some results on affine fibrations. In *Advances in algebra and geometry*, edited by C. Musili, pp. 7–24, Hindustan Book Agency, India, 2003.

[37] A. K. Dutta, Some results on subalgebras of polynomial algebras. In *Commutative algebra and algebraic geometry*, edited by S. Ghorpade et al., pp. 85–95 390, Contemp. Math., 2005.

[38] A. K. Dutta and N. Gupta, The epimorphism theorem and its generalisations. *J. Algebra Appl. (special issue in honour of Prof. Shreeram S. Abhyankar)* **14** (2015), no. 9, 15400010, 30 pp.

[39] A. K. Dutta, N. Gupta, and A. Lahiri, On Separable $\mathbb{A}^2$ and $\mathbb{A}^3$-forms. *Nagoya Math. J.* **239** (2020), 346–354.

[40] A. K. Dutta and N. Onoda, Some results on codimension-one $\mathbb{A}^1$-fibrations. *J. Algebra* **313** (2007), 905–921.

[41] G. Freudenburg, Algebraic theory of locally nilpotent derivations. In *Invariant Theory and Algebraic Transformation Groups VII*, Encyclopaedia Math. Sci. 136, Springer, Berlin, 2006.

[42] G. Freudenburg and P. Russell, Open problems in affine algebraic geometry, In *Affine algebraic geometry*, pp. 1–30, Contemp. Math. 369, 2005.

[43] T. Fujita, On Zariski problem. *Proc. Jpn. Acad.* **55** (1979), no. A, 106–110.

[44] N. Gupta, On faithfully flat fibrations by a punctured line. *J. Algebra* **415** (2014), 13–34.

[45] N. Gupta, On the Cancellation Problem for the affine space $\mathbb{A}^3$ in characteristic p. *Invent. Math.* **195** (2014), 279–288.

[46] N. Gupta, On the family of affine threefolds $x^m y = F(x, z, t)$. *Compos. Math.* **150** (2014), no. 6, 979–998.

[47] N. Gupta, On Zariski's Cancellation Problem in positive characteristic. *Adv. Math.* **264** (2014), 296–307.

[48] N. Gupta, A survey on Zariski Cancellation Problem. *Indian J. Pure Appl. Math.* **46** (2015), no. 6, 865–877.

[49] N. Gupta and S. Sen, On double Danielewski surfaces and the Cancellation Problem. *J. Algebra* **533** (2019), 25–43.

[50] R. V. Gurjar, A topological proof of cancellation theorem for $\mathbb{C}^2$. *Math. Z.* **240** (2002), no. 1, 83–94.

[51] R. V. Gurjar, K. Masuda, and M. Miyanishi, Affine space fibrations. In *Springer Proceedings in Mathematics and Statistics: Polynomial rings and Affine Algebraic Geometry, (PRAAG) 2018, Tokyo, Japan, February 12–16*, pp. 151–194, Springer, 2018.

[52] E. Hamann, On the R-invariance of $R[x]$. *J. Algebra* **35** (1975), 1–16.

[53] M. Hochster, Non-uniqueness of the ring of coefficients in a polynomial ring. *Proc. Amer. Math. Soc.* **34** (1972), no. 1, 81–82.

[54] M. Hoyois, A. Krishna, and P. A. Østvær, $\mathbb{A}^1$-contractibility of Koras–Russell threefolds. *Algebr. Geom.* **3** (2016), no. 4, 407–423.

[55]   S. Kaliman, M. Koras, L. Makar-Limanov, and P. Russell, $\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable. *Electron. Res. Announc. Am. Math. Soc.* **3** (1997), 63–71.

[56]   Sh. Kaliman, Polynomials with general $\mathbb{C}^{[2]}$-fibers are variables. *Pacific J. Math.* **203** (2002), no. 1, 161–190.

[57]   T. Kambayashi, On the absence of nontrivial separable forms of the affine plane. *J. Algebra* **35** (1975), 449–456.

[58]   T. Kambayashi, Automorphism group of a polynomial ring and algebraic group actions on affine space. *J. Algebra* **60** (1979), 439–451.

[59]   T. Kambayashi and M. Miyanishi, On flat fibrations by affine line. *Illinois J. Math.* **22** (1978), no. 4, 662–671.

[60]   T. Kambayashi and D. Wright, Flat families of affine lines are affine line bundles. *Illinois J. Math.* **29** (1985), no. 4, 672–681.

[61]   M. Koras and P. Russell, Separable forms of $\mathbb{G}_m$-actions on $\mathbb{A}^3_k$. *Transform. Groups* **18** (2013), no. 4, 1155–1163.

[62]   H. Kraft, Challenging problems on affine n-space. *Séminaire Bourbaki* **802** (1995), 295–317.

[63]   T. Y. Lam, *Serre's problem on projective modules*. Springer, Berlin–Heidelberg, 2006.

[64]   D. Lewis, Vénéreau-type polynomials as potential counterexamples. *J. Pure Appl. Algebra* **217** (2013), no. 5, 946–957.

[65]   L. Makar-Limanov, On the hypersurface $x + x^2 y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$. *Israel J. Math.* **96** (1996), no. B, 419–429.

[66]   H. Matsumura, *Commutative algebra. 2nd edn*. Benjamin, 1980.

[67]   M. Miyanishi, An algebraic characterization of the affine plane. *J. Math. Kyoto Univ.* **15** (1975), 169–184.

[68]   M. Miyanishi, An algebro-topological characterization of the affine space of dimension three. *Amer. J. Math.* **106** (1984), 1469–1486.

[69]   M. Miyanishi, Recent developments in affine algebraic geometry: from the personal viewpoints of the author. In *Affine algebraic geometry*, pp. 307–378, Osaka Univ. Press, Osaka, 2007.

[70]   M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets. *J. Math. Kyoto Univ.* **20** (1980), 11–42.

[71]   M. Nagata, *On automorphism group of $k[X, Y]$*. Kyoto Univ. Lect. Math. 5, Kinokuniya, Tokyo, 1972.

[72]   C. P. Ramanujam, A topological characterization of the affine plane as an algebraic variety. *Ann. of Math.* **94** (1971), 69–88.

[73]   P. Russell, Simple birational extensions of two dimensional affine rational domains. *Compos. Math.* **33** (1976), no. 2, 197–208.

[74]   P. Russell, On affine-ruled rational surfaces. *Math. Ann.* **255** (1981), 287–302.

[75]   P. Russell, Embedding problems in affine algebraic geometry. In *Polynomial automorphisms and related topics*, edited by H. Bass et al., pp. 113–135, Pub. House for Sc. and Tech., Hanoi, 2007.

[76] P. Russell and A. Sathaye, On finding and cancelling variables in $k[X, Y, Z]$. *J. Algebra* **57** (1979), no. 1, 151–166.

[77] P. Russell and A. Sathaye, Forty years of the epimorphism theorem. *Eur. Math. Soc. Newsl.* **90** (2013), 12–17.

[78] L. M. Sangines Garcia, *On quadratic planes*. Ph.D. Thesis, McGill Univ., 1983.

[79] A. Sathaye, On linear planes. *Proc. Amer. Math. Soc.* **56** (1976), 1–7.

[80] A. Sathaye, Generalized Newton–Puiseux expansion and Abhyankar–Moh semigroup theorem. *Invent. Math.* **74** (1983), no. 1, 149–157.

[81] A. Sathaye, Polynomial ring in two variables over a D.V.R.: a criterion. *Invent. Math.* **74** (1983), 159–168.

[82] A. Sathaye, An application of generalized Newton Puiseux expansions to a conjecture of D. Daigle and G. Freudenburg. In *Algebra, arithmetic and geometry with applications*, edited by C. Christensen et al., pp. 687–701, Springer, 2004.

[83] B. Segre, Corrispondenze di Möbius e trasformazioni cremoniane intere. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **91** (1956/1957), 3–19.

[84] A. R. Shastri, Polynomial representations of knots. *Tohoku Math. J. (2)* **44** (1992), no. 1, 11–17.

[85] A. A. Suslin, Locally polynomial rings and symmetric algebras. *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), no. 3, 503–515 (Russian).

[86] M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbb{C}^2$. *J. Math. Soc. Japan* **26** (1974), 241–257.

[87] B. Veĭsfeĭler and I. V. Dolgačev, Unipotent group schemes over integral rings. *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 757–799.

[88] S. Vénéreau, *Automorphismes et variables de ĺanneau de polynómes $A[y_1, \ldots, y_m]$*. Ph.D. Thesis, Institut Fourier, Grenoble, 2001.

[89] D. Wright, Cancellation of variables of the form $bT^n - a$. *J. Algebra* **52** (1978), no. 1, 94–100.

NEENA GUPTA

Statistics and Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700 108, India, neenag@isical.ac.in, rnanina@gmail.com