CHARACTER ESTIMATES FOR FINITE SIMPLE GROUPS AND APPLICATIONS

MICHAEL J. LARSEN

ABSTRACT

Let G be a finite simple group, χ an irreducible complex character, and g an element of G. It is often desirable to have upper bounds for $|\chi(g)|$ in terms of $\chi(1)$ and some measure of the regularity of g. This paper reviews what is known in this direction and presents typical applications of such bounds: to proving certain products of conjugacy classes cover G, to solving word equations over G, and to counting homomorphisms from a Fuchsian group to G.

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1. INTRODUCTION

Let G be a finite group, χ the character of an irreducible complex representation ρ of G, and g an element of G. As the eigenvalues of $\rho(g)$ are roots of unity, the bound $|\chi(g)| \leq \chi(1)$ is trivial. For central elements g, no stronger upper bound than $\chi(1)$ is possible. However, according to Schur, we know that

$$\sum_{g \in G} \chi(g)\overline{\chi(g)} = |G|,$$

and since $\chi(x) = \chi(g)$ for all x in the conjugacy class g^G, we obtain the *centralizer bound*

$$\left|\chi(g)\right| \leq \sqrt{\frac{|G|}{|g^G|}} = \sqrt{|C_G(g)|}.$$

Other known upper bounds typically hold only for special classes of groups.

This paper reviews what is known about character bounds when G is a finite simple group or is closely related to such a group. There is a substantial literature on upper bounds for character ratios $\frac{|\chi(g)|}{\chi(1)}$; see Martin Liebeck's survey **[29]** for recent results and applications in the case of groups of Lie type. These bounds are typically weakest for characters χ of low degree, which points to the desirability of *exponential bounds*, that is, bounds of the form $|\chi(g)| \leq \chi(1)^{\alpha(g)}$, where the size of $\alpha(g)$ is typically related to the size of the centralizer of g compared to $|G|$. The next two sections focus on alternating groups and groups of Lie type, respectively. The remaining sections give some applications of these results and present some open problems.

2. SYMMETRIC AND ALTERNATING GROUPS

Motivated by questions in probability theory, a number of people have considered character ratio bounds for symmetric groups. In this series of groups, unlike groups of Lie type, character ratios for nontrivial elements and nontrivial characters can be arbitrarily close to 1. The worst case for $G = \mathsf{S}_n$ is the ratio $\frac{n-3}{n-1}$, achieved when g is a transposition and χ is a character of degree $n-1$. Persi Diaconis and Mehrdad Shahshahani considered the case that g is a transposition and χ is any irreducible character, proving in **[4]** that if both the first row and the first column of the Young diagram for $\chi = \chi_\lambda$ have length $\leq n/2$, then the character ratio is less than $1/2$, while if, for instance, the first row satisfies $\lambda_1 > n/2$, then

$$0 < \frac{\chi(g)}{\chi(1)} \leq \frac{\lambda_1(\lambda_1 - 1) + (n - \lambda_1 - 1)(n - \lambda_1 - 2) - 2}{n(n-1)}.$$

A similar bound was given by Leopold Flatto, Andrew Odlyzko, and David Wales **[8, THEOREM 5.2]**.

Yuval Roichman **[39]** gave a character bound of the form

$$\frac{|\chi(g)|}{\chi(1)} \leq \max\bigl(\lambda_1/n, \lambda_1'/n, c\bigr)^{\mathrm{supp}(g)},$$

where $\mathrm{supp}(g)$ denotes the number of elements of $\{1, \ldots, n\}$ not fixed by g, and $c < 1$ is an absolute constant. This reflects the fact that elements with high support tend to have small centralizers. The bound is quite good when χ has small degree. However, for large n, most characters of S_n have degree greater than A^n for any fixed A, and for such characters, Roichman's bound is weaker than the centralizer bound for most elements $g \in G$.

Philippe Biane [3] gave character ratio bounds for elements of bounded support and "balanced" characters, namely those where $\lambda_1/\sqrt{n}$ and $\lambda_1'/\sqrt{n}$ are bounded. By the work of Logan–Shepp [34] and Veršik–Kerov [44], high degree characters are typically balanced. To be more precise, this is true for characters chosen randomly, weighted by the Plancherel measure. Amarpreet Rattan and Piotr Śniady [38] generalized Biane's character bound so it applies whenever $\mathrm{supp}(g)$ is small enough compared to n; if g cannot be expressed as the product of less than π transpositions, then

$$\frac{|\chi(g)|}{\chi(1)} \leq \left(\frac{D\max(1,\pi^2/n)}{\sqrt{n}}\right)^{\pi},$$

where D depends on the sizes of $\lambda_1/\sqrt{n}$ and $\lambda_1'/\sqrt{n}$. Valentin Féray and Śniady [7] proved a bound of the form

$$\frac{|\chi(g)|}{\chi(1)} \leq \left(\frac{a\max(\lambda_1,\lambda_1',\pi)}{n}\right)^{\pi},$$

which simultaneously improves on the results of [39] and [38].

Thomas Müller and Jan-Christoph Schlage-Puchta gave a character bound of exponential type [37, THEOREM B] which is good in a wide variety of situations. They proved that $|\chi(g)| \leq \chi(1)^{\alpha(g)}$, where

$$\alpha(g) = 1 - \left(\left(1-(1/\log n)\right)^{-1}\frac{12\log n}{\log(n/\mathrm{fix}(g))} + 18\right)^{-1}.$$

Being exponential, it works well whether $\chi(1)$ is large or small. The exponent is optimal, up to a multiplicative constant, for elements g consisting of many cycles, for instance, for involutions. However, it can be greatly improved upon for elements consisting of few cycles. In particular, $\alpha(g)$ is no smaller when g is an n-cycle than when it is of shape $2^{n/2}$.

Sergey Fomin and Nathan Lulov [9] gave a bound specifically for elements g of the shape $r^{n/r}$. For fixed r and varying n, it takes the form

$$\left|\chi(g)\right| = O\left(n^{\frac{r-1}{2r}}\chi(1)^{1/r}\right),$$

so it is essentially a bound of exponential type. Aner Shalev and I gave an exponential bound [22] for elements g of arbitrary shape $1^{a_1}2^{a_2}\cdots$ which is roughly comparable in strength to the Fomin–Lulov bound. Define the sequence $e_1, e_2, \ldots$ such that for all $k \geq 1$,

$$n^{e_1+\cdots+e_k} = \sum_{i=1}^{k} i a_i.$$

Then

$$\left|\chi(g)\right| \leq \chi(1)^{\sum_{i=1}^{n} e_i/i + o(1)}.$$

This result is stronger than the exponential bound of Müller–Schlage-Puchta for almost all elements but inferior to it when the number of fixed points of g is very large.

None of these bounds can compete with the centralizer bound for elements consisting of very few cycles, for instance, for n-cycles, where the centralizer bound gives $|\chi(g)| \le \sqrt{n}$. For such elements, the Murnaghan–Nakayama rule asserts $|\chi(g)| \le 1$, which is obviously optimal.

From symmetric group bounds, we easily obtain alternating group bounds of comparable strength. Recall that for $\lambda \ne \lambda'$, the characters χ_λ and $\chi_{\lambda'}$ restrict to the same irreducible character of A_n. All other irreducible characters of A_n arise from partitions satisfying $\lambda = \lambda'$; for each such λ, the restriction of χ_λ to A_n decomposes as a sum of two irreducibles χ_λ^1 and χ_λ^2. The χ_λ^i take the character value $\chi_\lambda(g)/2$ for all $g \in \mathsf{S}_n \setminus C$, where C is a single S_n-conjugacy class which decomposes into two A_n-conjugacy classes. For elements of C, a theorem of Frobenius gives character values, which are of the form

$$\frac{1 \pm \sqrt{\pm n_1 \cdots n_k}}{2},$$

where $n_i = \lambda_i - i$ for $1 \le i \le k$. Character degree estimates, like those in [22], now imply that $|\chi_\lambda^i(g)| \le \chi_\lambda(1)^\varepsilon$ whenever n is sufficiently large compared to $\varepsilon > 0$.

3. GROUPS OF LIE TYPE

Character estimates for finite simple groups of Lie type go back to the work of David Gluck [13–15]. Unlike in the case of alternating and symmetric groups, there is a uniform bound [15] on character ratios for nontrivial characters and nontrivial g, namely

$$\frac{|\chi(g)|}{\chi(1)} \le \frac{19}{20}.$$

When the cardinality q of the field of definition of G is large, this upper bound can be improved; Gluck [14] gives an upper bound of the form $C/\sqrt{q}$ for large q. The q-exponent is optimal, since for odd q, $\mathrm{PSL}_2(q)$ has characters of degree $\frac{q+1}{2}$ or $\frac{q-1}{2}$, and the value of such a character at a nontrivial unipotent element g is $\frac{\pm 1 \pm \sqrt{(-1)^{\frac{q-1}{2}} q}}{2}$.

If G is a finite simple group of bounded rank, then $\chi(1) < |G| = O(q^D)$, where D denotes the dimension associated to the Lie type of G. Therefore, the Gluck bound $C/\sqrt{q}$ can be converted to an exponential bound $|\chi(g)| \le \chi(1)^\alpha$, where $\alpha < 1$ depends only on the rank. To achieve exponential bounds in general, therefore, it suffices to limit our attention to the case that G is a classical group, that is, one of $\mathrm{PSL}_{r+1}(q)$, $\mathrm{PSU}_{r+1}(q)$, $\mathrm{P\Omega}_{2r}^{\pm}(q)$, $\mathrm{PSp}_{2r}(q)$, or $\mathrm{P\Omega}_{2r+1}(q)$.

We cannot expect that character ratios go to 0 as the order of a classical group goes to infinity. For instance, let $G = \mathrm{PSL}_{r+1}(q)$. The permutation representation associated with the action of G on $\mathbb{P}\mathbb{F}_q^r$ can be expressed as $\chi + 1$, for χ irreducible. Let g be the image of a transvection in $\mathrm{SL}_{r+1}(\mathbb{F}_q)$ in G. Then the fixed points of g form a hyperplane in $\mathbb{P}\mathbb{F}_q^n$, and $\chi(g) = q^{n-1} + q^{n-2} + \cdots + q$. Thus,

$$\lim_{n\to\infty} \frac{\chi(g)}{\chi(1)} = \frac{1}{q}.$$

Defining the support supp(g) as the smallest codimension of any eigenspace of g for the natural projective representation of G, the elements g in the above example have constant support 1 even as the rank of G goes to infinity. Shalev, Pham Huu Tiep, and I proved [24, THEOREM 4.3.6] that as the support goes to infinity, the character ratio goes to 0:

$$\frac{|\chi(g)|}{\chi(1)} \leq q^{-\sqrt{\mathrm{supp}(g)}/481}.$$

This falls well short of a uniform exponential character bound, even for elements of maximal support. Robert Guralnick, Tiep, and I found uniform exponential bounds for elements g whose centralizer is small compared to the order of G. For instance, we proved [16, THEOREM 1.4] that if G is of the form $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ and $|C_G(g)| \leq q^{n^2/12}$, then $|\chi(g)| \leq \chi(1)^{8/9}$. More generally, but less explicitly, we proved [17, THEOREM 1.3] that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|C_G(g)| \leq |G|^\delta$ implies $|\chi(g)| \leq \chi(1)^\varepsilon$. However, the method of these papers applies only to elements with small centralizer, for instance, it does not give any bound at all for involutions.

This defect was remedied in the sequel [28], which proved that for all positive $\delta < 1$ there exists $\varepsilon < 1$ such that $|C_G(g)| \leq |G|^\delta$ implies $|\chi(g)| \leq \chi(1)^\varepsilon$. More precisely, $|\chi(g)| \leq \chi(1)^{\alpha(g)}$ where

$$\alpha(g) = 1 - c + c\frac{\log |C_G(g)|}{\log |G|},$$

and $c > 0$ is an absolute constant, which can be made explicit (but is, unfortunately, extremely small). This theorem holds more generally for quasisimple finite groups of Lie type.

For many elements g in a classical group of rank r, much better exponents are available, thanks to the work of Roman Bezrukavnikov, Liebeck, Shalev, and Tiep [2]. For q odd, if the centralizer of g is a proper split Levi subgroup, then $|\chi(g)| \leq f(r)\chi(1)^{\alpha(g)}$, where $\alpha(g)$ is an explicitly computable rational number which is known to be optimal in many cases. This idea was further developed by Jay Taylor and Tiep, who proved [43], among other things, that for every nontrivial element $g \in \mathrm{PSL}_n(q)$,

$$\left|\chi(g)\right| \leq h(r)\chi(1)^{\frac{n-1}{n-2}}.$$

All of these estimates are poor for elements with small centralizers, such as regular elements. A general result, due to Shelly Garion, Alexander Lubotzky, and myself, which sometimes gives reasonably good bounds for regular elements, is the following [10, THEOREM 3]. Let G be a finite group, not necessarily simple, and g an element of G whose centralizer A is abelian. Suppose $A_1, \dots, A_n$ are subgroups of A not containing g such that the centralizer of every element of $A \setminus \bigcup_i A_i$ is A. Then, for every irreducible character of G,

$$\left|\chi(g)\right| \leq (4/\sqrt{3})^n\left[N_G(A) : A\right].$$

For example, this gives an upper bound of $2(n-1)^2/\sqrt{3}$ for $|\chi(g)|$ when $G = \mathrm{PSL}_n(q)$ and g is the image of an element with irreducible characteristic polynomial. It would be nice to have optimal upper bounds for $|\chi(g)|$ for general regular semisimple elements g.

4. PRODUCTS OF CONJUGACY CLASSES

If $C_1, \dots, C_n$ are conjugacy classes of a finite group G, then the number N of n-tuples $(g_1, \dots, g_n) \in C_1 \times \cdots \times C_n$ satisfying $g_1 g_2 \cdots g_n = 1$ is given by the Frobenius formula

$$N = \frac{|C_1| \cdots |C_n|}{|G|} \sum_{\chi} \frac{\chi(C_1) \cdots \chi(C_n)}{\chi(1)^{n-2}},$$

where χ ranges over all irreducible characters of G. In conjunction with upper bounds for the $|\chi(C_i)|$, this can sometimes be used to prove that $N \neq 0$, as the contribution from $\chi = 1$ often dominates the sum. Exponential bounds for the $\chi(C_i)$ are especially convenient, since results of Liebeck and Shalev [32] give a great deal of information about when we can expect

$$\sum_{\chi \neq 1} \chi(1)^{-s} < 1.$$

A well-known conjecture attributed to Thompson asserts that for every finite simple group G, there exists a conjugacy class C such that $C^2 = G$. Thanks to work of Erich Ellers and Nikolai Gordeev [6], we know that this is true except for a list of possible counterexamples, all finite simple groups of Lie type with $q \leq 8$. Tiep and I used our uniform exponential bounds to show that several of the infinite families on this list, in particular, the symplectic groups for all $q \geq 2$, can be eliminated in sufficiently high rank [28, THEOREM 7.7]. It would be interesting if these results could be extended to the remaining families on the list, giving an asymptotic version of Thompson's conjecture.

Andrew Gleason and Cheng-hao Xu [18,19] proved Thompson's conjecture for alternating groups, using the conjugacy class of an n-cycle if n is odd or a permutation of shape $2^1(n-2)^1$ if n is even. In [22, THEOREM 1.13], Shalev and I proved that in the limit $n \to \infty$ the probability that a randomly chosen $g \in \mathsf{A}_n$ belongs to a conjugacy class with $C^2 = \mathsf{A}_n$ rapidly approaches 1.

The analogous claim cannot be true for all finite simple groups since $C^2 = G$ implies that $C = C^{-1}$, and for, e.g., $\mathrm{PSL}_3(q)$ as $q \to \infty$, the probability that a random element is real goes to 0. However, there are several variants of this question which do not have an obvious counterexample. As the order of G tends to infinity, does the probability that a random *real* element belongs to a conjugacy class with $C^2 = G$ approach 1? Does the probability that a random element g belongs to a conjugacy class C with $C^2 \cup \{1\} = G$ approach 1? Also, as the order of G tends to infinity, does the probability that a random element belongs to a conjugacy class with $CC^{-1} = G$ approach 1?

The weaker claim that every element $g \in G$ lies in CC^{-1} for some conjugacy class (depending, perhaps, on g) is equivalent to the statement that every element of G is a commutator. This was was an old conjecture of Ore and is now a theorem of Liebeck, Eamonn O'Brien, Shalev, and Tiep [30].

One can also ask about S^2 where S is an arbitrary conjugation-invariant subset of G. On naive probabilistic grounds, it might seem plausible that given $\varepsilon > 0$ fixed, for G sufficiently large, every normal subset of G with at least $\varepsilon|G|$ elements satisfies $S^2 = G$. However, a moment's reflection shows that, unless $\varepsilon > \frac{1}{2}$, there is no reason to expect $1 \in S^2$.

Is it true, for G sufficiently large, that $S^2 \cup \{1\} = G$? For alternating groups and for groups of Lie type in bounded rank, the answer is affirmative [26], but we do not know in general.

In a different direction, given a conjugacy class C, how large must n be so that the nth power C^n is all of G? More generally, given conjugacy classes $C_1, \dots, C_n$ with sufficiently strong character bounds, the Frobenius formula can be used to show that each element of G is represented as a product $g_1 \cdots g_n$, with $g_i \in C_i$, in approximately $\frac{|C_1|\cdots|C_n|}{|G|}$ ways. For instance, it follows from the exponential character bounds given above that there exists an absolute constant k such that if G is a finite simple group of Lie type and $C_1, \dots, C_n$ are conjugacy classes in G satisfying $|C_1| \cdots |C_n| > |G|^k$, then for each $g \in G$,

$$\bigl|\{(g_1, \dots, g_n) \in C_1 \times \cdots \times C_n \mid g_1 \cdots g_n = g\}\bigr| = \bigl(1 + o(1)\bigr)\frac{|C_1| \cdots |C_n|}{|G|}.$$

Via Lang–Weil estimates, this further implies that if $\underline{C}_1, \dots, \underline{C}_n$ are conjugacy classes of a simple algebraic group $\underline{G}$, and

$$\dim \underline{C}_1 + \cdots + \dim \underline{C}_n > k \dim \underline{G},$$

then the product morphism of varieties $\underline{C}_1 \times \cdots \times \underline{C}_n \to \underline{G}$ has the property that every fiber is of dimension $\dim \underline{C}_1 + \cdots + \dim \underline{C}_n - \dim \underline{G}$.

In the special case that $C_1 = \cdots = C_n = C$, the question of the distribution of products $g_1 \cdots g_n$, $g_i \in C$, can be expressed in terms of the mixing time of the random walk on the Cayley graph of (G, C). A consequence of the exponential character bounds [28] is that for groups of Lie type, the mixing time of such a random walk is $O(\log |G| / \log |C|)$. This is the same order of growth as the diameter of the Cayley graph, thus settling conjectures of Lubotzky [35, P. 179] and Shalev [42, CONJECTURE 4.3].

The situation is different for alternating groups $G = \mathsf{A}_n$. For instance, if C is the class of 3-cycles and $n \geq 6$, then $\log |G| / \log |C| < n$, and $C^{\lfloor n/2 \rfloor} = G$ [5, THEOREM 9.1]. However, for any fixed k, the probability that the product of kn random 3-cycles g_i fixes 1 is at least the probability that each individual g_i fixes 1, which goes to e^{-3k} as $n \to \infty$. Thus the expected number of fixed points of $g_1 \cdots g_n$ grows linearly with n. It would be interesting to know, for general $C \subset \mathsf{A}_n$, what the mixing time is.

5. WARING'S PROBLEM

Waring's problem for finite simple groups originally meant the following question. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers n and all sufficiently large finite simple groups G (in terms of n), every element of G is a product of $f(n)$ nth powers? Positive solutions were given by Martinez–Zelmanov [36] and Saxl–Wilson [40].

This can be extended as follows. Let w denote a nontrivial element in any free group F_d. For every finite simple group G, w determines a function $G^d \to G$. We replace the nth powers with word values, that is, elements of G in the image of w. Liebeck and Shalev proved [31] that for G sufficiently large (in terms of w), every element of G can be written as a product of a bounded number of word values (where the bound may depend on w, just as in the classical version of Waring's problem, the minimum number of the nth powers needed to represent a given integer may depend on n).

It was therefore, perhaps, surprising when Shalev proved [41] that the Waring number for finite simple groups is uniform in w and is, in fact, at most three. This has now been improved to the optimal bound, two [23,24]. More generally, for any two nontrivial words w_1 and w_2, if G is a sufficiently large finite simple group, every element of G is a product of their word values. In fact, it is even possible [27] to choose subsets S_1 and S_2 of the sets of word values of w_1 and w_2 such that $S_1S_2 = G$ and $|S_i| = O(|G|^{1/2} \log^{1/2} |G|)$. The set of values of any word is a union of conjugacy classes, and the basic strategy of the proof is to try to find conjugacy classes C_1 and C_2 contained in the word values of w_1 and w_2, respectively, such that $C_1C_2 = G$ and very few elements of G have significantly fewer representations as such products than one would expect. Then a random choice of subsets $S_i \subset C_i$ of suitable size can almost always be slightly modified to work.

In general, the probability distribution on the word values of w obtained by evaluation at a uniformly distributed random element of G^d is far from uniform. For instance, for $g \in \mathsf{A}_{3n}$ uniformly distributed, the probability that $g^3 = 1$ is at least $|\mathsf{A}_{3n}|^{-1}$ times the number of elements of shape 3^n, i.e.,

$$(3n-1)(3n-2)\cdot(3n-4)(3n-5)\cdots(2)(1) > (3n-1)!^{\frac{2}{3}} > |\mathsf{A}_{3n}|^{\frac{2}{3}-\frac{1}{3n}}$$

for n sufficiently large. Thus, setting $w_1 = w_2 = x^3$, the probability that the product of cubes of two randomly chosen elements is 1 is at least $|\mathsf{A}_{3n}|^{-2/3-2/3n}$, which, for large n, makes the distribution far from uniform, at least in the L^∞ sense.

Using exponential character estimates, Shalev, Tiep, and I proved [25, THEOREM 4] that for any word w, there exists k such that as $|G| \to \infty$, the L^∞-deviation from uniformity in the product of k independent randomly generated values of w goes to 0. The dependence of k on w is unavoidable, as the above example suggests. On the other hand, the L^1-deviation from uniformity goes to 0 in the product of two independent randomly generated values of w, for any nontrivial word w [25, THEOREM 1]. I do not know what to expect for L^p-deviation for $1 < p < \infty$.

6. FUCHSIAN GROUPS

For $g, m \geq 0$, let $d_1, \ldots, d_m \geq 2$ be integers. For

$$\Gamma = \big\langle x_1, \ldots, x_m, y_1, \ldots, y_g, z_1, \ldots, z_g \mid x_1^{d_1}, \ldots, x_m^{d_m}, \\ x_1 \cdots x_m [y_1, z_1] \cdots [y_g, z_g] \big\rangle,$$

define the Euler characteristic

$$e = 2 - 2g - \sum_{i=1}^{m} \big(1 - d_i^{-1}\big).$$

Assume $e < 0$, so Γ is an oriented, cocompact Fuchsian group. Let G be a finite group, and let $C_1, \ldots, C_m$ denote conjugacy classes in G of elements whose orders divide $d_1, \ldots, d_m$, respectively. The Frobenius formula can be regarded as the $g = 0$ case of a more general formula for the number of homomorphisms $\Gamma \to G$ mapping x_i to an element of C_i for all i,

$$\left|\mathrm{Hom}_{\{C_i\}}(\Gamma, G)\right| = |G|^{2g-1}|C_1|\cdots|C_m|\sum_{\chi}\frac{\chi(C_1)\cdots\chi(C_m)}{\chi(1)^{m+2g-2}}.$$

In favorable situations, one can prove that the $\chi = 1$ term dominates all the others combined, in which case one has a good estimate for the number of such homomorphisms. Using this, Liebeck and Shalev proved [32, THEOREM 1.5] that if $g \geq 2$, and G is a simple of Lie type group of rank r, then

$$\left|\mathrm{Hom}(\Gamma, G)\right| = |G|^{1-e+O(1/r)}.$$

By the same method, employing the character bounds of [28], one obtains the same estimate whenever e is less than some absolute constant, regardless of the value of g. It would be interesting to know whether this is true in general for $e < 0$. Some evidence in favor of this idea is given in [21,33], but for small q the problem is open.

An interesting geometric consequence of the method of Liebeck–Shalev is that if $\underline{G}$ is a simple algebraic group of rank r and $g \geq 2$, the morphism $\underline{G}^{2g} \to \underline{G}$ given by the word $[y_1, z_1]\cdots[y_g, z_g]$ has all fibers of the same dimension, $(2g-1)\dim \underline{G}$. This has been refined by Avraham Aizenbud and Nir Avni, who proved [1] that for $g \geq 373$, the fibers of this morphism are reduced and have rational singularities. It would be interesting to extend this to the case of general Fuchsian groups. For instance, does there exist an absolute constant k such that for all simple algebraic groups $\underline{G}$ and conjugacy classes $\underline{C}_1, \ldots, \underline{C}_m$ with $\dim \underline{C}_1 + \cdots + \dim \underline{C}_m > k \dim \underline{G}$, all fibers of the multiplication morphism $\underline{C}_1 \times \cdots \times \underline{C}_m \to \underline{G}$ are reduced with rational singularities The ideas of Glazer–Hendel [11,12] may be applicable.

For $g = 1$, we can no longer hope for equidimensional fibers, since the generic fiber dimension is $\dim \underline{G}$, while the fiber over the identity element has dimension $r + \dim \underline{G}$. However, Zhipeng Lu and I proved [20] that for $\underline{G} = \mathrm{SL}_n$, all fibers over noncentral elements have dimension $\underline{G}$. It would be interesting to know whether this is true for general simple algebraic groups $\underline{G}$.

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REFERENCES

[1] A. Aizenbud and N. Avni, Representation growth and rational singularities of the moduli space of local systems. *Invent. Math.* **204** (2016), no. 1, 245–316.

[2] R. Bezrukavnikov, M. Liebeck, A. Shalev, and P. H. Tiep, Character bounds for finite groups of Lie type. *Acta Math.* **221** (2018), no. 1, 1–57.

[3] P. Biane, Representations of symmetric groups and free probability. *Adv. Math.* **138** (1998), no. 1, 126–181.

[4] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete* **57** (1981), no. 2, 159–179.

[5] Y. Dvir, Covering properties of permutation groups. In *Products of conjugacy classes in groups*, pp. 197–221, Lecture Notes in Math. 1112, Springer, Berlin, 1985.

[6] E. W. Ellers and N. Gordeev, On the conjectures of J. Thompson and O. Ore. *Trans. Amer. Math. Soc.* **350** (1998), no. 9, 3657–3671.

[7] V. Féray and P. Śniady, Asymptotics of characters of symmetric groups related to Stanley character formula. *Ann. of Math.* **173** (2011), no. 2, 887–906.

[8] L. Flatto, A. M. Odlyzko, and D. B. Wales, Random shuffles and group representations. *Ann. Probab.* **13** (1985), no. 1, 154–178.

[9] S. Fomin and N. Lulov, On the number of rim hook tableaux. *J. Math. Sci. (N. Y.)* **87** (1997), no. 6, 4118–4123.

[10] S. Garion, M. Larsen, and A. Lubotzky, Beauville surfaces and finite simple groups. *J. Reine Angew. Math.* **666** (2012), 225–243.

[11] I. Glazer and Y. I. Hendel (with an appendix joint with G. Kozma), On singularity properties of convolutions of algebraic morphisms—the general case. 2018, arXiv:1811.09838.

[12] I. Glazer and Y. I. Hendel, On singularity properties of word maps and applications to probabilistic Waring type problems. 2019, arXiv:1912.12556.

[13] D. Gluck, Character value estimates for groups of Lie type. *Pacific J. Math.* **150** (1991), 279–307.

[14] D. Gluck, Character value estimates for nonsemisimple elements. *J. Algebra* **155** (1993), no. 1, 221–237.

[15] D. Gluck, Sharper character value estimates for groups of Lie type. *J. Algebra* **174** (1995), no. 1, 229–266.

[16] R. M. Guralnick, M. Larsen, and P. H. Tiep, Character levels and character bounds. *Forum Math. Pi* **8** (2020), e2, 81 pp.

[17] R. M. Guralnick, M. Larsen, and P. H. Tiep, Character levels and character bounds for finite classical groups. 2019, arXiv:1904.08070.

[18] C. Hsü, The commutators of the alternating group. *Sci. Sinica* **14** (1965), 339–342.

[19] D. M. W. H. Husemoller, Ramified coverings of Riemann surfaces. *Duke Math. J.* **29** (1962), 167–174.

[20] M. Larsen and Z. Lu, Flatness of the commutator map over SL_n. *Int. Math. Res. Not. IMRN* **2021**, no. 8, 5605–5622.

[21] M. Larsen and A. Lubotzky, Representation varieties of Fuchsian groups. In *From Fourier analysis and number theory to Radon transforms and geometry*, pp. 375–397, Dev. Math. 28, Springer, New York, 2013.

[22] M. Larsen and A. Shalev, Characters of symmetric groups: sharp bounds and applications. *Invent. Math.* **174** (2008), no. 3, 645–687.

[23] M. Larsen and A. Shalev, Word maps and Waring type problems. *J. Amer. Math. Soc.* **22** (2009), no. 2, 437–466.

[24] M. Larsen, A. Shalev, and P. H. Tiep, The Waring problem for finite simple groups. *Ann. of Math.* **174** (2011), no. 3, 1885–1950.

[25] M. Larsen, A. Shalev, and P. H. Tiep, Probabilistic Waring problems for finite simple groups. *Ann. of Math.* **190** (2019), no. 2, 561–608.

[26] M. Larsen, A. Shalev, and P. H. Tiep, Products of normal subgroups and derangements. 2020, arXiv:2003.12882.

[27] M. Larsen and P. H. Tiep, A refined Waring problem for finite simple groups. *Forum Math. Sigma* **3** (2015), Paper No. e6, 22 pp.

[28] M. Larsen and P. H. Tiep, Uniform characters bounds for finite classical groups (submitted).

[29] M. W. Liebeck, Character ratios for finite groups of Lie type, and applications. In *Finite simple groups: thirty years of the Atlas and beyond*, pp. 193–208, Contemp. Math. 694, Amer. Math. Soc., Providence, RI, 2017.

[30] M. W. Liebeck, E. A. O'Brien, A. Shalev, and P. H. Tiep, The Ore conjecture. *J. Eur. Math. Soc.* **12** (2010), no. 4, 939–1008.

[31] M. W. Liebeck and A. Shalev, Diameters of simple groups: sharp bounds and applications. *Ann. of Math.* **154** (2001), 383–406.

[32] M. W. Liebeck and A. Shalev, Fuchsian groups, finite simple groups and representation varieties. *Invent. Math.* **159** (2005), no. 2, 317–367.

[33] M. W. Liebeck, A. Shalev, and P. H. Tiep, Character ratios, representation varieties and random generation of finite groups of Lie type. *Adv. Math.* **374** (2020), 107386, 39 pp.

[34] B. F. Logan and L. A. Shepp, A variational problem for random Young tableaux. *Adv. Math.* **26** (1977), no. 2, 206–222.

[35] A. Lubotzky, Cayley graphs: eigenvalues, expanders and random walks. In *Surveys in combinatorics, 1995 (Stirling)*, pp. 155–189, London Math. Soc. Lecture Note Ser. 218, Cambridge Univ. Press, Cambridge, 1995.

[36] C. Martinez and E. Zelmanov, Products of powers in finite simple groups. *Israel J. Math.* **96** (1996), part B, 469–479.

[37] T. W. Müller and J-C. Schlage-Puchta, Character theory of symmetric groups, subgroup growth of Fuchsian groups, and random walks. *Adv. Math.* **213** (2007), no. 2, 919–982.

[38] A. Rattan and P. Śniady, Upper bound on the characters of the symmetric groups for balanced Young diagrams and a generalized Frobenius formula. *Adv. Math.* **218** (2008), 673–695.

[39] Y. Roichman, Upper bound on the characters of the symmetric groups. *Invent. Math.* **125** (1996), 451–485.

[40] J. Saxl and J. S. Wilson, A note on powers in simple groups. *Math. Proc. Cambridge Philos. Soc.* **122** (1997), no. 1, 91–94.

[41] A. Shalev, Word maps, conjugacy classes, and a noncommutative Waring-type theorem. *Ann. of Math. (2)* **170** (2009), no. 3, 1383–1416.

[42] A. Shalev, Conjugacy classes, growth and complexity. In *Finite simple groups: thirty years of the atlas and beyond*, pp. 209–221, Contemp. Math. 694, Amer. Math. Soc., Providence, RI, 2017.

[43] J. Taylor and P. H. Tiep, Lusztig induction, unipotent supports, and character bounds. *Trans. Amer. Math. Soc.* **373** (2020), no. 12, 8637–8676.

[44] A. M. Veršik and S. V. Kerov, Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux (Russian). *Dokl. Akad. Nauk SSSR* **233** (1977), no. 6, 1024–1027.

MICHAEL J. LARSEN

Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A., mjlarsen@indiana.edu