FINITE APPROXIMATIONS AS A TOOL FOR STUDYING TRIANGULATED CATEGORIES

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ABSTRACT

Small, finite entities are easier and simpler to manipulate than gigantic, infinite ones. Consequently, huge chunks of mathematics are devoted to methods reducing the study of big, cumbersome objects to an analysis of their finite building blocks. The manifestation of this general pattern, in the study of derived and triangulated categories, dates back almost to the beginnings of the subject—more precisely to articles by Illusie in SGA6, way back in the early 1970s.

What is new, at least new in the world of derived and triangulated categories, is that one gets extra mileage from analyzing more carefully and quantifying more precisely just how efficiently one can estimate infinite objects by finite ones. This leads one to the study of metrics on triangulated categories, and of how accurately an object can be approximated by finite objects of bounded size.

1. INTRODUCTION

In every branch of mathematics, we try to solve complicated problems by reducing to simpler ones, and from antiquity people have used finite approximations to study infinite objects. Naturally, whenever a new field comes into being, one of the first developments is to try to understand what should be the right notion of finiteness in the discipline. Derived and triangulated categories were introduced by Verdier in his PhD thesis in the mid-1960s (although the published version only appeared much later in [38]). Not surprisingly, the idea of studying the finite objects in these categories followed suit soon after, see Illusie [13–15].

Right from the start there was a pervasive discomfort with derived and triangulated categories—the intuition that had been built up, in dealing with concrete categories, mostly fails for triangulated categories. In case the reader is wondering: in the previous sentence the word "concrete" has a precise, technical meaning, and it is an old theorem of Freyd [10, 11] that triangulated categories often are not concrete. Further testimony, to the strangeness of derived and triangulated categories, is that it took two decades before the intuitive notion of finiteness, which dates back to Illusie's articles [13–15], was given its correct formal definition. The following may be found in [23, DEFINITION 1.1].

Definition 1.1. Let $\mathscr{T}$ be a triangulated category with coproducts. An object $C \in \mathscr{T}$ is called *compact* if $\mathrm{Hom}(C, -)$ commutes with coproducts. The full subcategory of all compact objects will be denoted by $\mathscr{T}^c$.

Remark 1.2. I have often been asked where the name "compact" came from. In the preprint version of [23], these objects went by a different name, but the (anonymous) referee did not like it. I was given a choice: I was allowed to baptize them either "compact" or "small."

Who was I to argue with a referee?

Once one has a good working definition of what the finite objects ought to be, the next step is to give the right criterion which guarantees that the category has "enough" of them. For triangulated categories, the right definition did not come until [24, DEFINITION 1.7].

Definition 1.3. Let $\mathscr{T}$ be a triangulated category with coproducts. The category $\mathscr{T}$ is called *compactly generated* if every nonzero object $X \in \mathscr{T}$ admits a nonzero map $C \to X$, with $C \in \mathscr{T}$ a compact object.

As the reader may have guessed from the name, compactly generated triangulated categories are those in which it is often possible to reduce general problems to questions about compact objects—which tend to be easier.

All of the above nowadays counts as "classical," meaning that it is two or more decades old and there is already a substantial and diverse literature exploiting the ideas. This article explores the recent developments that arose from trying to understand how efficiently one can approximate arbitrary objects by compact ones. We first survey the results obtained to date. This review is on the skimpy side, partly because there already are other, more expansive published accounts in the literature, but mostly because we want to leave ourselves space to suggest possible directions for future research. Thus the article can be

thought of as having two components: a bare-bone review of what has been achieved to date, occupying Sections 2 to 6, followed by Section 7 which comprises suggestions of avenues that might merit further development.

Our review presents just enough detail so that the open questions, making up Section 7, can be formulated clearly and comprehensibly, and so that the significance and potential applications of the open questions can be illuminated. This has the unfortunate side effect that we give short shrift to the many deep, substantial contributions, made by numerous mathematicians, which preceded and inspired the work presented here. The author apologizes in advance for this omission, which is the inescapable corollary of page limits. The reader is referred to the other surveys of the subject, where more care is taken to attribute the ideas correctly to their originators, and give credit where credit is due.

We permit ourselves to gloss over difficult technicalities, nonchalantly skating by nuances and subtleties, with only an occasional passing reference to the other surveys or to the research papers for more detail.

The reader wishing to begin with examples and applications, to keep in mind through the forthcoming abstraction, is encouraged to first look at the Introduction to [31].

2. APPROXIMABLE TRIANGULATED CATEGORIES—THE FORMAL DEFINITION AS A VARIANT ON FOURIER SERIES

It is now time to start our review, offering a glimpse of the recent progress that was made by trying to measure how "complicated" an object is, in other words, how far it is from being compact. What follows is sufficiently new for there to be much room for improvement: the future will undoubtedly see cleaner, more elegant, and more general formulations. What is presented here is the current crude state of this emerging field.

Discussion 2.1. This section is devoted to defining approximable triangulated categories, and the definition is technical and at first sight could appear artificial, maybe even forbidding. It might help therefore to motivate it with an analogy.

Let $\mathbb{S}^1$ be the circle, and let $M(\mathbb{S}^1)$ be the set of all complex-valued, Lebesgue-measurable functions on $\mathbb{S}^1$. As usual we view $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ as the quotient of its universal cover $\mathbb{R}$ by the fundamental group $\mathbb{Z}$; this identifies functions on $\mathbb{S}^1$ with periodic functions on $\mathbb{R}$ with period 1. In particular the function $g(x) = e^{2\pi i x}$ belongs to $M(\mathbb{S}^1)$. And, for each $\ell \in \mathbb{Z}$, we have that $g(x)^\ell = e^{2\pi i \ell x}$ also belongs to $M(\mathbb{S}^1)$. Given a norm on the space $M(\mathbb{S}^1)$, for example, the L^p-norm, we can try to approximate arbitrary $f \in M(\mathbb{S}^1)$ by Laurent polynomials in g, that is, look for complex numbers $\{\lambda_\ell \in \mathbb{C} \mid -n \le \ell \le n\}$ such that

$$\left\| f(x) - \sum_{\ell=-n}^n \lambda_\ell g(x)^\ell \right\|_p = \left\| f(x) - \sum_{\ell=-n}^n \lambda_\ell e^{2\pi i \ell x} \right\|_p < \varepsilon$$

with $\varepsilon > 0$ small. This leads us to the familiar territory of Fourier series.

Now imagine trying to do the same, but replacing $M(\mathbb{S}^1)$ by a triangulated category. Given a triangulated category $\mathscr{T}$, which we assume to have coproducts, we would like to pretend to do Fourier analysis on it. We would need to choose:

(1) Some analog of the function $g(x) = e^{2\pi i x}$. Our replacement for this will be to choose a compact generator $G \in \mathscr{T}$. Recall that a *compact generator* is a compact object $G \in \mathscr{T}$ such that every nonzero object $X \in \mathscr{T}$ admits a nonzero map $G[i] \to X$ for some $i \in \mathbb{Z}$.

(2) We need to choose something like a metric, the analog of the L^p-norm on $M(\mathbb{S}^1)$. For us this will be done by picking a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ on $\mathscr{T}$. The heuristic is that we will view a morphism $E \to F$ in $\mathscr{T}$ as "short" if, in the triangle $E \to F \to D$, the object D belongs to $\mathscr{T}^{\leq -n}$ for large n. We will come back to this in Discussion 6.10.

(3) We need to have an analog of the construction that passes, from the function $g(x) = e^{2\pi i x}$ and the integer $n > 0$, to the vector space of trigonometric Laurent polynomials $\sum_{\ell=-n}^{n} \lambda_\ell e^{2\pi i \ell x}$.

As it happens our solution to (3) is technical. We need a recipe that begins with the object G and the integer $n > 0$, and proceeds to cook up a collection of more objects. We ask the reader to accept it as a black box, with only a sketchy explanation just before Remark 2.3.

Black Box 2.2. Let $\mathscr{T}$ be a triangulated category and let $G \in \mathscr{T}$ be an object. Let $n > 0$ be an integer. We will have occasion to refer to the following four full subcategories of $\mathscr{T}$:

(1) The subcategory $\langle G \rangle_n \subset \mathscr{T}$ is defined unconditionally, and if $\mathscr{T}$ has coproducts one can also define the larger subcategory $\overline{\langle G \rangle}_n$. Both of these subcategories are classical, the reader can find the subcategory $\langle G \rangle_n$ in Bondal and Van den Bergh [6, THE DISCUSSION BETWEEN LEMMA 2.2.2 AND DEFINITION 2.2.3], and the subcategory $\overline{\langle G \rangle}_n$ in [6, THE DISCUSSION BETWEEN DEFINITION 2.2.3 AND PROPOSITION 2.2.4].

(2) If the category $\mathscr{T}$ has coproducts, we will also have occasion to consider the full subcategory $\overline{\langle G \rangle}^{(-\infty, n]}$. Once again this category is classical (although the name is not). The reader can find it in Alonso, Jeremías, and Souto [1], where it would go by the name "the cocomplete pre-aisle generated by $G[-n]$".

(3) Once again assume that $\mathscr{T}$ has coproducts. Then we will also look at the full subcategory $\overline{\langle G \rangle}_n^{[-n, n]}$. This construction is relatively new.

Below we give a vague description of what is going on in these constructions; but when it comes to the technicalities, we ask the reader to either accept these as black boxes, or refer to [29, REMINDER 0.8 (VII), (XI) AND (XII)] for detail. We mention that there is a slight clash of notation in the literature: what we call $\overline{\langle G \rangle}_n$ in (1), following Bondal and Van den Bergh,

goes by a different name in **[29, REMINDER 0.8 (XI)]**. The name it goes by there is the case $A = -\infty$ and $B = \infty$ of the more general subcategory $\overline{\langle G \rangle}_n^{[A,B]}$.

Now for the vague explanation of what goes on in (1), (2), and (3) above: in a triangulated category $\mathscr{T}$, there are not many ways to build new objects out of old ones. One can shift objects, form direct summands, form finite direct sums (or infinite ones if coproducts exist), and one can form extensions. In the categories $\langle G \rangle_n$ and $\overline{\langle G \rangle}_n$ of (1), there is a bound on the number of allowed extensions, and the difference between the two is whether infinite coproducts are allowed. In the category $\overline{\langle G \rangle}^{(-\infty,n]}$ of (2), the bound is on the permitted shifts. And in the category $\overline{\langle G \rangle}_n^{[-n,n]}$ of (3), both the shifts allowed and the number of extensions permitted are restricted.

Remark 2.3. The reader should note that an example would not be illuminating, the categories $\langle G \rangle_n$, $\overline{\langle G \rangle}_n$, $\overline{\langle G \rangle}^{(-\infty,n]}$, and $\overline{\langle G \rangle}_n^{[-n,n]}$ are not usually overly computable. For example, let R be an associative ring, and let $\mathscr{T} = \mathbf{D}(R)$ be the unbounded derived category of complexes of left R-modules. The object $R \in \mathscr{T}$, that is, the complex which is R in degree zero and vanishes in all other degrees, is a compact generator for $\mathscr{T} = \mathbf{D}(R)$.

But if we wonder what the categories $\langle R \rangle_n$, $\overline{\langle R \rangle}_n$, $\overline{\langle R \rangle}^{(-\infty,n]}$, and $\overline{\langle R \rangle}_n^{[-n,n]}$ might turn out to be, only the category $\overline{\langle R \rangle}^{(-\infty,n]}$ is straightforward: it is the category of all cochain complexes whose cohomology vanishes in degrees $> n$. The three categories $\langle R \rangle_n$, $\overline{\langle R \rangle}_n$, and $\overline{\langle R \rangle}_n^{[-n,n]}$ are mysterious in general. In fact, the computation of $\langle G \rangle_n$ is the subject of conjectures that have attracted much interest. We will say a tiny bit about theorems in this direction in Section 4, and will mention one of the active, open conjectures in the discussion between Definition 7.7 and Problem 7.8.

Remark 2.4. In the definition of approximable triangulated categories, which is about to come, the category $\overline{\langle G \rangle}_n^{[-n,n]}$ will play the role of the replacement for the vector space of trigonometric Laurent polynomials of degree $\leq n$, which came up in the desiderata of Discussion 2.1(3). The older categories $\langle G \rangle_n$, $\overline{\langle G \rangle}_n$, and $\overline{\langle G \rangle}^{(-\infty,n]}$ will be needed later in the article.

Remark 2.5. Let us return to the heuristics of Discussion 2.1. Assume we have chosen the t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ as in Discussion 2.1(2), which we think of as our replacement for the L^p-norm on $M(\mathbb{S}^1)$. And we have also chosen a compact generator $G \in \mathscr{T}$ as in Discussion 2.1(1), which we think of as the analog of the exponential function $g(x) = e^{2\pi i x}$. We have declared that the subcategories $\overline{\langle G \rangle}_n^{[-n,n]}$ will be our replacement for the vector space of trigonometric Laurent polynomials of degree $\leq n$, as in Discussion 2.1(3). It is now time to start approximating functions by trigonometric Laurent polynomials.

Let us therefore assume we start with some object $F \in \mathscr{T}$, and find a good approximation of it by the object $E \in \overline{\langle G \rangle}_m^{[-m,m]}$, meaning that we find a morphism $E \to F$ such that, in the triangle $E \to F \to D$, the object D belongs to $\mathscr{T}^{\leq -M}$ for some suitably large M.

Now we can try to iterate, and find a good approximation for D. Thus we can look for a morphism $E'' \to D$, with $E'' \in \overline{\langle G \rangle}_n^{[-n,n]}$, and such that in the triangle $E'' \to D \to D'$ the

object D' belongs to $\mathscr{T}^{\leq -N}$, with $N > M$ even more enormous than M. Can we combine these to improve our initial approximation of F?

To do this, let us build up the octahedron on the composable morphisms $F \to D \to D'$. We end up with a diagram where the rows and columns are triangles

$$
\begin{array}{ccccc}
E & \longrightarrow & E' & \longrightarrow & E'' \\
\| & & \downarrow & & \downarrow \\
E & \longrightarrow & F & \longrightarrow & D \\
& & \downarrow & & \downarrow \\
& & D' & = & D'
\end{array}
$$

and in particular the triangle $E' \to F \to D'$ gives that E' is an even better approximation of F than E was. We are therefore interested in knowing if the triangle $E \to E' \to E''$, coupled with the fact that $E \in \overline{\langle G \rangle}_m^{[-m,m]}$ and $E'' \in \overline{\langle G \rangle}_n^{[-n,n]}$, gives any information about where E' might lie with respect to the construction of Black Box 2.2(3). Hence it is useful to know the following.

Facts 2.6. Let $\mathscr{T}$ be a triangulated category with coproducts. The construction of Black Box 2.2(3) satisfies

(1) If E is an object of $\overline{\langle G \rangle}_n^{[-n,n]}$, then the shifts $E[1]$ and $E[-1]$ both belong to $\overline{\langle G \rangle}_{n+1}^{[-n-1,n+1]}$.

(2) Given an exact triangle $E \to E' \to E''$, with $E \in \overline{\langle G \rangle}_m^{[-m,m]}$ and $E'' \in \overline{\langle G \rangle}_n^{[-n,n]}$, it follows that $E' \in \overline{\langle G \rangle}_{m+n}^{[-m-n,m+n]}$.

Combining Remark 2.5 with Facts 2.6 allows us to improve approximations through iteration. Hence part (2) of the definition below becomes natural, it iterates to provide arbitrarily good approximations.

Definition 2.7. Let $\mathscr{T}$ be a triangulated category with coproducts. It is *approximable* if there exist a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$, a compact generator $G \in \mathscr{T}$, and an integer $n > 0$ such that

(1) G belongs to $\mathscr{T}^{\leq n}$ and $\mathrm{Hom}(G, \mathscr{T}^{\leq -n}) = 0$;

(2) Every object $X \in \mathscr{T}^{\leq 0}$ admits an exact triangle $E \to X \to D$ with $E \in \overline{\langle G \rangle}_n^{[-n,n]}$ and with $D \in \mathscr{T}^{\leq -1}$.

Remark 2.8. While part (2) of Definition 2.7 comes motivated by the analogy with Fourier analysis, part (1) of the definition seems random. It requires the t-structure, which is our replacement for the L^p-norm, to be compatible with the compact generator, which is the analog of $g(x) = e^{2\pi i x}$. As the reader will see in Proposition 5.5, this has the effect of uniquely specifying the t-structure (up to equivalence). So maybe a better parallel would be to fix our norm to be a particularly nice one, for example, the L^2-norm on $M(\mathbb{S}^1)$.

Let me repeat myself: as with all new mathematics, Definition 2.7 should be viewed as provisional. In the remainder of this survey, we will discuss the applications as they now

stand, to highlight the power of the methods. But I would not be surprised in the slightest if future applications turn out to require modifications, and/or generalizations, of the definitions and of the theorems that have worked so far.

3. EXAMPLES OF APPROXIMABLE TRIANGULATED CATEGORIES

In Section 1 we gave the definition of approximable triangulated categories. The definition combines old, classical ingredients (t-structures and compact generators) with a new construction, the category $\overline{\langle G \rangle}_n^{[-n,n]}$ of Black Box 2.2(3). The first thing to show is that the theory is nonempty: we need to produce examples, categories people care about which satisfy the definition of approximability. The current section is devoted to the known examples of approximable triangulated categories. We repeat what we have said before: the subject is in its infancy, there could well be many more examples out there.

Example 3.1. Let $\mathscr{T}$ be a triangulated category with coproducts. If $G \in \mathscr{T}$ is a compact generator such that $\mathrm{Hom}(G, G[i]) = 0$ for all $i > 0$, then the category $\mathscr{T}$ is approximable.

This example turns out to be easy, the reader is referred to [**29, EXAMPLE 3.3**] for the (short) proof. Special cases include

(1) $\mathscr{T} = \mathbf{D}(R\text{–Mod})$, where R is a dga with $H^i(R) = 0$ for $i > 0$;

(2) The homotopy category of spectra.

Example 3.2. If X is a quasicompact, separated scheme, then the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable. We remind the reader of the traditional notation being used here: the category $\mathbf{D}(X)$ is the unbounded derived category of complexes of sheaves of $\mathscr{O}_X$-modules, and the full subcategory $\mathbf{D}_{\mathbf{qc}}(X) \subset \mathbf{D}(X)$ has for objects the complexes with quasicoherent cohomology.

The proof of the approximability of $\mathbf{D}_{\mathbf{qc}}(X)$ is not trivial. The category has a standard t-structure, that part is easy. The existence of a compact generator G needs proof, it may be found in Bondal and Van den Bergh [**6, THEOREM 3.1.1(II)**]. Their proof is not constructive, it is only an existence proof, but it does give enough information to deduce that part (1) of Definition 2.7 is satisfied by every compact generator (indeed, it is satisfied by every compact object). See [**6, THEOREM 3.1.1(I)**]. But it is a challenge to show that we may choose a compact generator G and an integer $n > 0$ in such a way that Definition 2.7(2) is satisfied.

If we further assume that X is of finite type over a noetherian ring R, then the (relatively intricate) proof of the approximability of $\mathbf{D}_{\mathbf{qc}}(X)$ occupies [**33, SECTIONS 4 AND 5**]. The little trick, that extends the result to all quasicompact and separated X, was not observed until later: it appears in [**29, LEMMA 3.5**].

Example 3.3. It is a theorem that, under mild hypotheses, the recollement of any two approximable triangulated categories is approximable. To state the "mild hypotheses" precisely: suppose we are given a recollement of triangulated categories

$$\mathscr{R} \rightrightarrows \mathscr{S} \rightrightarrows \mathscr{T}$$

with $\mathscr{R}$ and $\mathscr{T}$ approximable. Assume further that the category $\mathscr{S}$ is compactly generated, and any compact object $H \in \mathscr{S}$ has the property that $\operatorname{Hom}(H, H[i]) = 0$ for $i \gg 0$. Then the category $\mathscr{S}$ is also approximable.

The reader can find the proof in [**7, THEOREM 4.1**], it is the main result in the paper. The bulk of the article is devoted to developing the machinery necessary to prove the theorem—hence it is worth noting that this machinery has since demonstrated usefulness in other contexts, see the subsequent articles [**27,28**].

There is a beautiful theory of noncommutative schemes, and a rich literature studying them. And many of the interesting examples of such schemes are obtained as recollements of ordinary schemes, or of admissible pieces of them. Thus the theorem that recollements of approximable triangulated categories are approximable gives a wealth of new examples of approximable triangulated categories.

Since this ICM is being held in St. Petersburg, it would be remiss not to mention that the theory of noncommutative algebraic geometry, in the sense of the previous paragraph, is a subject to which Russian mathematicians have contributed a vast amount. The seminal work of Bondal, Kontsevich, Kuznetsov, Lunts, and Orlov immediately springs to mind. For a beautiful introduction to the field, the reader might wish to look at the early sections of Orlov [**34**]. The later sections prove an amazing new theorem, but the early ones give a lovely survey of the background. In fact, the theory sketched in this survey was born when I was trying to read and understand Orlov's beautiful article.

4. APPLICATIONS: STRONG GENERATION

We begin by reminding the reader of a classical definition, going back to Bondal and Van den Bergh [**6**].

Definition 4.1. Let $\mathscr{T}$ be triangulated category. An object $G \in \mathscr{T}$ is called a *strong generator* if there exists an integer $\ell > 0$ with $\mathscr{T} = \langle G \rangle_\ell$, where the notation is as in Black Box 2.2(1). The category $\mathscr{T}$ is called *regular* or *strongly generated* if it contains a strong generator.

The first application of approximability is the proof of the following two theorems.

Theorem 4.2. *Let X be a quasicompact, separated scheme. The derived category of perfect complexes on X, denoted here by $\mathbf{D}^{\mathbf{perf}}(X)$, is regular if and only if X has a cover by open subsets $\operatorname{Spec}(R_i) \subset X$, with each R_i of finite global dimension.*

Remark 4.3. If X is noetherian and separated, then Theorem 4.2 specializes to saying that $\mathbf{D}^{\mathbf{perf}}(X)$ is regular if and only if X is regular and finite-dimensional. Hence the terminology.

Theorem 4.4. *Let X be a noetherian, separated, finite-dimensional, quasiexcellent scheme. Then the category $\mathbf{D}^b(\operatorname{Coh}(X))$, the bounded derived category of coherent sheaves on X, is always regular.*

Remark 4.5. The reader is referred to [33] and to Aoki [4] for the proofs of Theorems 4.2 and 4.4. More precisely, for Theorem 4.2 see [33, THEOREM 0.5]. About Theorem 4.4: if we add the assumption that every closed subvariety of X admits a regular alteration then the result may be found in [33, THEOREM 0.15], but Aoki [4] found a lovely argument that allowed him to extend the statement to all quasiexcellent X.

There is a rich literature on strong generation, with beautiful papers by many authors. In the introduction to [33], as well as in [26] and [31, SECTION 7], the reader can find an extensive discussion of (some of) this fascinating work and of the way Theorems 4.2 and 4.4 compare to the older literature. For a survey taking an entirely different tack, see Minami [22], which places in historical perspective a couple of the key steps in the proofs that [33] gives for Theorems 4.2 and 4.4.

Since all of this is now well documented in the published literature, let us focus the remainder of the current survey on the other applications of approximability. Those are all still in preprint form, see [27–29], although there are (published) surveys in [31, SECTIONS 8 AND 9] and in [30]. Those surveys are fuller and more complete than the sketchy one we are about to embark on. As we present the material, we will feel free to refer the reader to the more extensive surveys whenever we deem it appropriate.

5. THE FREEDOM IN THE CHOICE OF COMPACT GENERATOR AND T-STRUCTURE

Definition 2.7 tells us that a triangulated category $\mathscr{T}$ with coproducts is approximable if there exist, in $\mathscr{T}$, a compact generator G and a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ satisfying some properties. The time has come to explore just how free we are in the choice of the compact generator and of the t-structure. To address this question we begin by formulating:

Definition 5.1. Let $\mathscr{T}$ be a triangulated category. Then two t-structures $(\mathscr{T}_1^{\leq 0}, \mathscr{T}_1^{\geq 0})$ and $(\mathscr{T}_2^{\leq 0}, \mathscr{T}_2^{\geq 0})$ are declared *equivalent* if there exists an integer $n > 0$ such that

$$\mathscr{T}_1^{\leq -n} \subset \mathscr{T}_2^{\leq 0} \subset \mathscr{T}_1^{\leq n}.$$

Discussion 5.2. Let $\mathscr{T}$ be a triangulated category with coproducts. If $G \in \mathscr{T}$ is a compact object and $\overline{\langle G \rangle}^{(-\infty,0]}$ is as in Black Box 2.2(2), then Alonso, Jeremías, and Souto [1, THEOREM A.1], building on the work of Keller and Vossieck [16], teaches us that there is a unique t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ with $\mathscr{T}^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,n]}$. We will call this the *t-structure generated by G,* and denote it $(\mathscr{T}_G^{\leq 0}, \mathscr{T}_G^{\geq 0})$.

In Black Box 2.2(2) we asked the reader to accept, as a black box, the construction passing from an object $G \in \mathscr{T}$ to the subcategory $\overline{\langle G \rangle}^{(-\infty,0]}$. If G is compact, then [1, THEOREM A.1] allows us to express this as $\mathscr{T}_G^{\leq 0}$ for a unique t-structure. We ask the reader to accept on faith that:

Lemma 5.3. *If G and H are two compact generators for the triangulated category $\mathscr{T}$, then the two t-structures $(\mathscr{T}_G^{\leq 0}, \mathscr{T}_G^{\geq 0})$ and $(\mathscr{T}_H^{\leq 0}, \mathscr{T}_H^{\geq 0})$ are equivalent as in Definition 5.2.*

As it happens, the proof of Lemma 5.3 is easy, the interested reader can find it in **[29, REMARK 0.15]**. And Lemma 5.3 leads us to:

Definition 5.4. Let $\mathcal{T}$ be a triangulated category in which there exists a compact generator. We define the *preferred equivalence class of t-structures* as follows: a t-structure belongs to the preferred equivalence class if it is equivalent to $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$ for some compact generator $G \in \mathcal{T}$, and by Lemma 5.3 it is equivalent to $(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0})$ for every compact generator H.

The following is also not too hard, and may be found in **[29, PROPOSITIONS 2.4 AND 2.6]**.

Proposition 5.5. *Let $\mathcal{T}$ be an approximable triangulated category. Then for any t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, and for any compact generator $H \in \mathcal{T}$, there exists an integer $n > 0$ (which may depend on H and on the t-structure), satisfying*

(1) *H belongs to $\mathcal{T}^{\leq n}$ and $\mathrm{Hom}(H, \mathcal{T}^{\leq -n}) = 0$;*

(2) *Every object $X \in \mathcal{T}^{\leq 0}$ admits an exact triangle $E \to X \to D$ with $E \in \overline{\langle H \rangle}_n^{[-n,n]}$ and with $D \in \mathcal{T}^{\leq -1}$.*

Moreover, if H is a compact generator, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure, and there exists an integer $n > 0$ satisfying (1) *and* (2) *above, then the t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ must belong to the preferred equivalence class.*

Remark 5.6. Strangely enough, the value of Proposition 5.5 can be that it allows us to find an explicit t-structure in the preferred equivalence class.

Consider the case where X is a quasicompact, separated scheme. By Bondal and Van den Bergh **[6, THEOREM 3.1.1(II)]**, we know that the category $\mathbf{D}_{\mathbf{qc}}(X)$ has a compact generator, but in Example 3.2 we mentioned that the existence proof is not overly constructive, it does not give us a handle on any explicit compact generator. Let G be some compact generator. From Alonso, Jeremías, and Souto **[1, THEOREM A.1]**, we know that the subcategory $\overline{\langle G \rangle}^{(-\infty,0]}$ of Black Box 2.2(2) is equal to $\mathcal{T}_G^{\leq 0}$ for a unique t-structure $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$ in the preferred equivalence class. But this does not leave us a whole lot wiser—the compact generator G is not explicit, hence neither is the t-structure.

However, the combination of **[33, THEOREM 5.8]** and **[29, LEMMA 3.5]** tells us that the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable, and it so happens that the t-structure *used in the proof,* that is, the t-structure for which a compact generator H and an integer $n > 0$ satisfying (1) and (2) of Proposition 5.5 are shown to exist, happens to be the standard t-structure. From Proposition 5.5, we now deduce that the standard t-structure is in the preferred equivalence class.

6. STRUCTURE THEOREMS IN APPROXIMABLE TRIANGULATED CATEGORIES

An approximable triangulated category $\mathcal{T}$ must have a compact generator G, and Definition 5.4 constructed for us a preferred equivalence class of t-structures—namely all

those equivalent to $(\mathscr{T}_G^{\leq 0}, \mathscr{T}_G^{\geq 0})$. Recall that, for any t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$, it is customary to define

$$\mathscr{T}^- = \bigcup_{n=1}^{\infty} \mathscr{T}^{\leq n}, \quad \mathscr{T}^+ = \bigcup_{n=1}^{\infty} \mathscr{T}^{\geq -n}, \quad \mathscr{T}^b = \mathscr{T}^- \cap \mathscr{T}^+.$$

It is an easy exercise to show, directly from Definition 5.1, that equivalent t-structures give rise to identical $\mathscr{T}^-$, $\mathscr{T}^+$, and $\mathscr{T}^b$. Therefore triangulated categories with a single compact generator, and in particular approximable triangulated categories, have preferred subcategories $\mathscr{T}^-$, $\mathscr{T}^+$, and $\mathscr{T}^b$, which are intrinsic—they are simply those corresponding to any t-structure in the preferred equivalence class. In the remainder of this survey, we will assume that $\mathscr{T}^-$, $\mathscr{T}^+$, and $\mathscr{T}^b$ always stand for the preferred ones.

In the heuristics of Discussion 2.1(2), we told the reader that a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is to be viewed as a metric on $\mathscr{T}$. In Definition 6.1 below, the heuristic is that we construct a full subcategory $\mathscr{T}_c^-$ to be the closure of $\mathscr{T}^c$ with respect to any of the (equivalent) metrics that come from t-structures in the preferred equivalence class.

Definition 6.1. Let $\mathscr{T}$ be an approximable triangulated category. The full subcategory $\mathscr{T}_c^-$ is given by

$$\mathrm{Ob}(\mathscr{T}_c^-) = \left\{ F \in \mathscr{T} \left| \begin{array}{c} \text{For every integer } n > 0 \text{ and for every t-structure} \\ \left(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0}\right) \text{ in the preferred equivalence class,} \\ \text{there exists an exact triangle } E \to F \to D \text{ in } \mathscr{T} \\ \text{with } E \in \mathscr{T}^c \text{ and } D \in \mathscr{T}^{\leq -n} \end{array} \right. \right\}.$$

The full subcategory $\mathscr{T}_c^b$ is defined to be $\mathscr{T}_c^b = \mathscr{T}_c^- \cap \mathscr{T}^b$.

Remark 6.2. Let $\mathscr{T}$ be an approximable triangulated category. Aside from the classical, full subcategory $\mathscr{T}^c$ of compact objects, which we encountered back in Definition 1.1, we have in this section concocted five more intrinsic, full subcategories of $\mathscr{T}$: they are $\mathscr{T}^-$, $\mathscr{T}^+$, $\mathscr{T}^b$, $\mathscr{T}_c^-$, and $\mathscr{T}_c^b$. It can be proved that all six subcategories, that is, the old $\mathscr{T}^c$ and the five new ones, are thick subcategories of $\mathscr{T}$. In particular, each of them is a triangulated category.

Example 6.3. It becomes interesting to figure out what all these categories come down to in examples.

Let X be a quasicompact, separated scheme. From Example 3.2, we know that the category $\mathscr{T} = \mathbf{D}_{\mathbf{qc}}(X)$ is approximable, and in Remark 5.6 we noted that the standard t-structure is in the preferred equivalence class. This can be used to show that, for $\mathscr{T} = \mathbf{D}_{\mathbf{qc}}(X)$, we have

$$\mathscr{T}^- = \mathbf{D}_{\mathbf{qc}}^-(X), \quad \mathscr{T}^+ = \mathbf{D}_{\mathbf{qc}}^+(X), \quad \mathscr{T}^b = \mathbf{D}_{\mathbf{qc}}^b(X),$$
$$\mathscr{T}^c = \mathbf{D}^{\mathbf{perf}}(X), \quad \mathscr{T}_c^- = \mathbf{D}_{\mathbf{coh}}^-(X), \quad \mathscr{T}_c^b = \mathbf{D}_{\mathbf{coh}}^b(X),$$

where the last two equalities assume that the scheme X is noetherian, and all six categories on the right of the equalities have their traditional meanings.

The reader can find an extensive discussion of the claims above in [31], more precisely in the paragraphs between [31, **PROPOSITION 8.10**] and [31, **THEOREM 8.16**]. That discussion

goes beyond the scope of the current survey, it analyzes the categories $\mathscr{T}_c^b \subset \mathscr{T}_c^-$ in the generality of non-noetherian schemes, where they still have a classical description—of course, not involving the category of coherent sheaves. After all coherent sheaves do not behave well for non-noetherian schemes.

Remark 6.4. In this survey we spent some effort introducing the notion of approximable triangulated categories. In Example 3.2 we told the reader that it is a theorem (and not a trivial one) that, as long as a scheme X is quasicompact and separated, the derived category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable. In this section we showed that every approximable triangulated category comes with canonically defined, intrinsic subcategories $\mathscr{T}^-$, $\mathscr{T}^+$, $\mathscr{T}^b$, $\mathscr{T}^c$, $\mathscr{T}_c^-$, and $\mathscr{T}_c^b$, and in Example 6.3 we informed the reader that, in the special case where $\mathscr{T} = \mathbf{D}_{\mathbf{qc}}(X)$, these turn out to be $\mathbf{D}_{\mathbf{qc}}^-(X)$, $\mathbf{D}_{\mathbf{qc}}^+(X)$, $\mathbf{D}_{\mathbf{qc}}^b(X)$, $\mathbf{D}^{\mathbf{perf}}(X)$, $\mathbf{D}_{\mathbf{coh}}^-(X)$, and $\mathbf{D}_{\mathbf{coh}}^b(X)$, respectively.

Big deal. This teaches us that the traditional subcategories $\mathbf{D}_{\mathbf{qc}}^-(X)$, $\mathbf{D}_{\mathbf{qc}}^+(X)$, $\mathbf{D}_{\mathbf{qc}}^b(X)$, $\mathbf{D}^{\mathbf{perf}}(X)$, $\mathbf{D}_{\mathbf{coh}}^-(X)$, and $\mathbf{D}_{\mathbf{coh}}^b(X)$ of the category $\mathbf{D}_{\mathbf{qc}}(X)$ all have intrinsic descriptions. This might pass as a curiosity, unless we can actually use it to prove something we care about that we did not use to know.

Discussion 6.5. To motivate the next theorem, it might help to think of the parallel with functional analysis.

Let $M(\mathbb{R})$ be the vector space of Lebesgue-measurable, real-valued functions on $\mathbb{R}$. Given any two functions $f, g \in M(\mathbb{R})$, we can pair them by integrating the product, that is, we form the pairing

$$\langle f, g \rangle = \int fg \, d\mu,$$

where μ is Lebesgue measure. This gives us a map

$$M(\mathbb{R}) \times M(\mathbb{R}) \xrightarrow{\langle -, - \rangle} \mathbb{R} \cup \{\infty\},$$

where the integral is declared to be infinite if it does not converge.

We can restrict this pairing to subspaces of $M(\mathbb{R})$. For example, if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$ then the integral converges, that is, $\langle f, g \rangle \in \mathbb{R}$, and we deduce a map

$$L^p(\mathbb{R}) \longrightarrow \mathrm{Hom}\big(L^q(\mathbb{R}), \mathbb{R}\big)$$

which turns out to be an isometry of Banach spaces.

The category-theoretic version is that on any category $\mathscr{T}$ there is the pairing sending two objects $A, B \in \mathscr{T}$ to $\mathrm{Hom}(A, B)$. Of course, this pairing is not symmetric, we have to keep track of the position of A and of B in $\mathrm{Hom}(A, B)$. If R is a commutative ring and $\mathscr{T}$ happens to be an R-linear category, then $\mathrm{Hom}(A, B)$ is an R-module and the pairing delivers a map

$$\mathscr{T}^{\mathrm{op}} \times \mathscr{T} \xrightarrow{\mathrm{Hom}(-,-)} R\text{--Mod},$$

where the op keeps track of the variable in the first position. And now we are free to restrict to subcategories of $\mathscr{T}$.

If $\mathscr{T}$ happens to be approximable and R-linear, we have just learned that it comes with six intrinsic subcategories $\mathscr{T}^-$, $\mathscr{T}^+$, $\mathscr{T}^b$, $\mathscr{T}^c$, $\mathscr{T}_c^-$, and $\mathscr{T}_c^b$. We are free to restrict the Hom pairing to any couple of them. This gives us 36 possible pairings, and each of those yields two maps from a subcategory to the dual of another. There are 72 cases we could study, and the theorem below tells us something useful about four of those.

Theorem 6.6. *Let R be a noetherian ring, and let $\mathscr{T}$ be an R-linear, approximable triangulated category. Suppose there exists in $\mathscr{T}$ a compact generator G so that $\mathrm{Hom}(G, G[n])$ is a finite R-module for all $n \in \mathbb{Z}$. Consider the two functors*

$$\mathscr{Y} : \mathscr{T}_c^- \to \mathrm{Hom}_R\big([\mathscr{T}^c]^{\mathrm{op}}, R\text{–Mod}\big), \quad \widetilde{\mathscr{Y}} : [\mathscr{T}_c^-]^{\mathrm{op}} \to \mathrm{Hom}_R\big(\mathscr{T}_c^b, R\text{–Mod}\big)$$

defined by the formulas $\mathscr{Y}(B) = \mathrm{Hom}(-, B)$ and $\widetilde{\mathscr{Y}}(A) = \mathrm{Hom}(A, -)$, as in Discussion 6.5. Now consider the following composites:

$$\mathscr{T}_c^b \overset{i}{\longrightarrow} \mathscr{T}_c^- \overset{\mathscr{Y}}{\longrightarrow} \mathrm{Hom}_R\big([\mathscr{T}^c]^{\mathrm{op}}, R\text{–Mod}\big),$$

$$[\mathscr{T}^c]^{\mathrm{op}} \overset{\tilde{\imath}}{\longrightarrow} [\mathscr{T}_c^-]^{\mathrm{op}} \overset{\widetilde{\mathscr{Y}}}{\longrightarrow} \mathrm{Hom}_R\big(\mathscr{T}_c^b, R\text{–Mod}\big).$$

We assert:

(1) *The functor $\mathscr{Y}$ is full, and the essential image consists of the locally finite homological functors (see Explanation 6.7 for the definition of locally finite functors). The composite $\mathscr{Y} \circ i$ is fully faithful, and the essential image consists of the finite homological functors (again, see Explanation 6.7 for the definition).*

(2) *With the notation as in Black Box 2.2(1), assume[1] that $\mathscr{T} = \overline{\langle H \rangle}_n$ for some integer $n > 0$ and some object $H \in \mathscr{T}_c^b$. Then the functor $\widetilde{\mathscr{Y}}$ is full, and the essential image consists of the locally finite homological functors. The composite $\widetilde{\mathscr{Y}} \circ \tilde{\imath}$ is fully faithful, and the essential image consists of the finite homological functors.*

Explanation 6.7. In the statement of Theorem 6.6, the *locally finite* functors, either of the form $H : [\mathscr{T}^c]^{\mathrm{op}} \to R\text{–Mod}$ or of the form $H : \mathscr{T}_c^b \to R\text{–Mod}$, are the functors such that

(1) $H(A[i])$ is a finite R-module for every $i \in \mathbb{Z}$ and every A in either $\mathscr{T}^c$ or $\mathscr{T}_c^b$;

(2) For fixed A, in one of $\mathscr{T}^c$ or $\mathscr{T}_c^b$, we have $H(A[i]) = 0$ if $i \ll 0$.

The *finite* functors are those for which we also have

(3) $H(A[i]) = 0$ for all $i \gg 0$.

Remark 6.8. The proof of part (1) of Theorem 6.6 may be found in [29], while the proof of part (2) of Theorem 6.6 occupies [28]. These are not easy theorems.

Let $\mathscr{T} = \mathbf{D_{qc}}(X)$, with X a scheme proper over a noetherian ring R. Then the hypotheses of Theorem 6.6(1) are satisfied. We learn (among other things) that the natural

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[1] What's important for the current survey is that, if X is a noetherian, separated scheme, then $\mathscr{T} = \mathbf{D_{qc}}(X)$ satisfies this hypothesis provided X is finite-dimensional and quasiexcellent.

functor, taking an object $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$ to the R-linear functor $\mathrm{Hom}(-, B) : \mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}} \to$ Mod–R, is a fully faithful embedding

$$\mathbf{D}^b_{\mathbf{coh}}(X) \xrightarrow{\quad \mathscr{Y} \circ i \quad} \mathrm{Hom}_R\big(\mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}}, \; R\text{–Mod}\big)$$

whose essential image is precisely the finite homological functors.

If we further assume that the scheme X is finite-dimensional and quasiexcellent then the hypotheses of Theorem 6.6(2) are also satisfied. We learn that the functor, taking an object $A \in \mathbf{D}^{\mathbf{perf}}(X)$ to the R-linear functor $\mathrm{Hom}(A, -)$, is a fully faithful embedding

$$\mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathscr{Y} \circ i} \quad} \mathrm{Hom}_R\big(\mathbf{D}^b_{\mathbf{coh}}(X), \; R\text{–Mod}\big)$$

whose essential image is also the finite homological functors.

In [**31, HISTORICAL SURVEY 8.2**] the reader can find a discussion of the (algebro-geometric) precursors of Theorem 6.6. As for the applications, let us go through one of them.

Remark 6.9. Let X be a scheme proper over the field $\mathbb{C}$ of complex numbers, and let X^{an} be the underlying complex analytic space. The analytification induces a functor we will call $\mathscr{L} : \mathbf{D}^b_{\mathbf{coh}}(X) \to \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$, it is the functor taking a bounded complex of coherent algebraic sheaves on X to the analytification, which is a bounded complex of coherent analytic sheaves on X^{an}. The pairing sending an object $A \in \mathbf{D}^{\mathbf{perf}}(X)$ and an object $B \in \mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}})$ to $\mathrm{Hom}(\mathscr{L}(A), B)$ delivers a map

$$\mathbf{D}^b_{\mathbf{coh}}(X^{\mathrm{an}}) \xrightarrow{\qquad\qquad} \mathrm{Hom}_R\big(\mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}}, \; \mathbb{C}\text{–Mod}\big).$$

Since the image lands in the finite homological functors, Theorem 6.6(1) allows us to factor this uniquely through the inclusion $\mathscr{Y} \circ i$, that is, there exists (up to canonical natural isomorphism) a unique functor $\mathscr{R}$ rendering commutative the triangle

And proving Serre's GAGA theorem reduces to the easy exercise of showing that $\mathscr{L}$ and $\mathscr{R}$ are inverse equivalences, the reader can find this in the (short) [**29, SECTION 8 AND APPENDIX A**].

The brilliant inspiration underpinning the approach is due to Jack Hall [**12**], he is the person who came up with the idea of using the pairing above, coupled with representability theorems, to prove GAGA. The representability theorems available to Jack Hall at the time were not powerful enough, and Theorem 6.6 was motivated by trying to find a direct path from the ingenious, simple idea to a fullblown proof.

Discussion 6.10. In preparation for the next theorem, we give a very brief review of metrics in triangulated categories. The reader is referred to the survey article [**30**] for a much fuller and more thorough account.

Given a triangulated category $\mathscr{T}$, a *metric* on $\mathscr{T}$ assigns a length to every morphism. In this article the only metrics we consider are the ones arising from t-structures. If $\mathscr{T}$ is an approximable triangulated category we choose a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ in the preferred equivalence class, and this induces a metric as follows. Given a morphism $f : X \to Y$, we may complete to an exact triangle $X \xrightarrow{f} Y \to D$, and the length of f is given by the formula

$$\text{Length}\,(f) = \inf\left\{ \frac{1}{2^n} \,\middle|\, n \in \mathbb{Z} \text{ and } D \in \mathscr{T}^{\leq -n} \right\}.$$

In this survey we allow the length of a morphism to be infinite; if the set on the right is empty then we declare $\text{Length}\,(f) = \infty$.

This metric depends on the choice of a t-structure, but not a lot. As all t-structures in the preferred equivalence class are equivalent, any two preferred t-structures will give rise to equivalent metrics (with an obvious definition of equivalence of metrics).

Note that if $\mathscr{T}$ is a triangulated category and $\mathscr{S}$ is a triangulated subcategory, then a metric on $\mathscr{T}$ restricts to a metric on $\mathscr{S}$. In particular, if $\mathscr{T}$ is approximable, the metric on $\mathscr{T}$ of the previous paragraph restricts to give metrics on the full subcategories $\mathscr{T}^c$ and $\mathscr{T}_c^b$. Once again these metrics are only defined up to equivalence. And, of course, a metric on $\mathscr{S}$ is also a metric on $\mathscr{S}^{\text{op}}$, thus we have specified (up to equivalence) canonical metrics on $\mathscr{T}^c$, $\mathscr{T}_c^b$, $[\mathscr{T}^c]^{\text{op}}$, and $[\mathscr{T}_c^b]^{\text{op}}$.

Suppose $\mathscr{S}$ is a triangulated category with a metric. A *Cauchy sequence* in $\mathscr{S}$ is a sequence of morphisms $E_1 \to E_2 \to E_3 \to \cdots$ which eventually become arbitrarily short. If $\mathscr{A}\mathscr{e}$ is the category of abelian groups, then the Yoneda embedding $Y : \mathscr{S} \to \text{Mod–}\mathscr{S}$ embeds $\mathscr{S}$ into the category $\text{Mod–}\mathscr{S}$ of additive functors $\mathscr{S}^{\text{op}} \to \mathscr{A}\mathscr{e}$. In the category $\text{Mod–}\mathscr{S}$ colimits exist, allowing us to define

(1) The category $\mathfrak{L}(\mathscr{S})$ is the full subcategory of $\text{Mod–}\mathscr{S}$, whose objects are the colimits of Yoneda images of Cauchy sequences in $\mathscr{S}$;

(2) The full subcategory $\mathfrak{S}(\mathscr{S}) \subset \mathfrak{L}(\mathscr{S})$ has for objects those functors $F \in \mathfrak{L}(\mathscr{S}) \subset \text{Mod–}\mathscr{S}$ which take sufficiently short morphisms to isomorphisms. In symbols, $F \in \mathfrak{L}(\mathscr{S})$ belongs to $\mathfrak{S}(\mathscr{S})$ if there exists an $\varepsilon > 0$ such that

$$\{\text{Length}\,(f) < \varepsilon\} \Rightarrow \{F(f) \text{ is an isomorphism}\};$$

(3) The exact triangles in $\mathfrak{S}(\mathscr{S})$ are the colimits in $\text{Mod–}\mathscr{S}$ of Yoneda images of Cauchy sequences of exact triangles in $\mathscr{S}$, where the colimits happen to lie in $\mathfrak{S}(\mathscr{S})$.

A word of caution about (3): if we are given in $\mathscr{S}$ a Cauchy sequence of exact triangles, we can form the colimit in $\text{Mod–}\mathscr{S}$ of its Yoneda image. This colimit is guaranteed to lie in $\mathfrak{L}(\mathscr{S})$, but will not usually lie in the smaller $\mathfrak{S}(\mathscr{S})$. If it happens to lie in $\mathfrak{S}(\mathscr{S})$ then (3) declares it to be an exact triangle in $\mathfrak{S}(\mathscr{S})$.

And now we are ready for the theorem.

Theorem 6.11. *Let $\mathscr{S}$ be a triangulated category with a metric. Assume the metric is* good; *this is a technical term, see* **[30, DEFINITION 10]** *for the precise formulation. Then*

(1) *The category $\mathfrak{S}(\mathscr{S})$ of Discussion 6.10(2), with the exact triangles as defined in Discussion 6.10(3), is a triangulated category.*

Now let $\mathscr{T}$ be an approximable triangulated category. In Discussion 6.10 we constructed (up to equivalence) a metric on $\mathscr{T}$, and hence on its subcategories $\mathscr{T}^c$ and $[\mathscr{T}_c^b]^{\mathrm{op}}$. Those metrics are all good, and the theorem goes on to give natural, exact equivalences of triangulated categories

(2) $\mathfrak{S}(\mathscr{T}^c) \cong \mathscr{T}_c^b$. *This equivalence is unconditional.*

(3) *If the approximable triangulated category $\mathscr{T}$ happens to be* noetherian *as in* **[27, DEFINITION 5.1]**, *then $\mathfrak{S}([\mathscr{T}_c^b]^{\mathrm{op}}) \cong [\mathscr{T}^c]^{\mathrm{op}}$.*

Remark 6.12. First of all, in Theorem 6.11(3) we assumed that the approximable triangulated $\mathscr{T}$ is noetherian as in **[27, DEFINITION 5.1]**. The only observation we want to make here is that if X is a noetherian, separated scheme then the approximable triangulated category $\mathscr{T} = \mathbf{D}_{\mathbf{qc}}(X)$ is noetherian. Thus, for noetherian, separated schemes X, Theorem 6.11 gives exact equivalences of triangulated categories

$$\mathfrak{S}\big(\mathbf{D}^{\mathbf{perf}}(X)\big) \cong \mathbf{D}_{\mathbf{coh}}^b(X), \quad \mathfrak{S}\big(\mathbf{D}_{\mathbf{coh}}^b(X)^{\mathrm{op}}\big) \cong \mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}}.$$

The research paper **[27]** contains the proofs of the assertions in Theorem 6.11. The reader can find a skimpy survey in **[31, SECTION 9]** and a more extensive one in **[30]**. In **[31, HISTORICAL SURVEY 9.1]** there is a discussion of precursors of the results.

7. FUTURE DIRECTIONS

New scientific developments are tentative and unpolished; only with the passage of time do they acquire the gloss and elegance of a refined, varnished theory. And there is nothing more difficult to predict than the future. My colleague Neil Trudinger used to joke that my beard makes me look like a biblical prophet—the reader should not be deceived, appearances are notoriously misleading, the abundance of facial hair is not a reliable yardstick for measuring the gift of foresight that marks out a visionary, and I am certifiably not a clairvoyant. All I do in this section is offer a handful of obvious questions that spring to mind. The list is not meant to be exhaustive, and might well be missing major tableaux of the overall picture. It is entirely possible that the future will see this theory flourish in directions orthogonal to those sketched here.

Let us begin with what is freshest in our minds: we have just seen Theorem 6.11, part (1) of which tells us that, given a triangulated category $\mathscr{S}$ with a good metric, there is a recipe producing another triangulated category $\mathfrak{S}(\mathscr{S})$, which, as it happens, comes with an induced good metric. We can ask:

Problem 7.1. Can one formulate reasonable sufficient conditions, on the triangulated category $\mathscr{S}$ and on its good metric, to guarantee that $\mathfrak{S}(\mathfrak{S}(\mathscr{S})^{\mathrm{op}}) = \mathscr{S}^{\mathrm{op}}$? Who knows, maybe even necessary and sufficient conditions?

Motivating Example 7.2. Let $\mathscr{T}$ be an approximable triangulated category and put $\mathscr{S} = \mathscr{T}^c$, with the metric of Discussion 6.10. Theorem 6.11(2) computes for us that $\mathfrak{S}(\mathscr{T}^c) \cong \mathscr{T}_c^b$. I ask the reader to believe that the natural, induced metric on $\mathfrak{S}(\mathscr{T}^c)$ agrees with the metric on $\mathscr{T}_c^b \subset \mathscr{T}$ given in Discussion 6.10. Now Theorem 6.11(3) goes on to tell us that, as long as the approximable triangulated category $\mathscr{T}$ is noetherian, we also have that $\mathfrak{S}([\mathscr{T}_c^b]^{\mathrm{op}}) \cong [\mathscr{T}^c]^{\mathrm{op}}$; as it happens, the induced good metric on $\mathfrak{S}([\mathscr{T}_c^b]^{\mathrm{op}})$ also agrees, up to equivalence, with the metric that Discussion 6.10 created on $[\mathscr{T}^c]^{\mathrm{op}}$. Combining these we have many examples of exact equivalences of triangulated categories $\mathfrak{S}(\mathfrak{S}(\mathscr{S})^{\mathrm{op}}) \cong \mathscr{S}^{\mathrm{op}}$, which are homeomorphisms with respect to the metrics. Thus Problem 7.1 asks the reader to find the right generalization.

Next one can wonder about the functoriality of the construction. Suppose $F : \mathscr{S} \to \mathscr{T}$ is a triangulated functor, and that both $\mathscr{S}$ and $\mathscr{T}$ have good metrics. What are reasonable sufficient conditions which guarantee the existence of an induced functor $\mathfrak{S}(F)$, either from $\mathfrak{S}(\mathscr{S})$ to $\mathfrak{S}(\mathscr{T})$ or in the other direction? So far there is one known result of this genre, the reader can find the statement below in Sun and Zhang [**37, THEOREM 1.1(3)**].

Theorem 7.3. *Suppose we are given two triangulated categories $\mathscr{S}$ and $\mathscr{T}$, both with good metrics. Suppose we are also given a pair of functors $F : \mathscr{S} \rightleftarrows \mathscr{T} : G$ with $F \dashv G$, meaning that F is left adjoint to G. Assume further that both F and G are continuous with respect to the metrics, in the obvious sense.*

Then the functor $\hat{F} : \mathrm{Mod}\text{–}\mathscr{T} \to \mathrm{Mod}\text{–}\mathscr{S}$ induced by composition with F, that is, the functor taking the $\mathscr{T}$–module $H : \mathscr{T}^{\mathrm{op}} \to \mathscr{A}\ell$ to the $\mathscr{S}$–module $(H \circ F) : \mathscr{S}^{\mathrm{op}} \to \mathscr{A}\ell$, restricts to a functor which we will denote $\mathfrak{S}(F) : \mathfrak{S}(\mathscr{T}) \to \mathfrak{S}(\mathscr{S})$. That is, the functor $\mathfrak{S}(F)$ is defined to be the unique map making the square below commute

$$
\begin{array}{ccc}
\mathfrak{S}(\mathscr{T}) & \xrightarrow{\ \mathfrak{S}(F)\ } & \mathfrak{S}(\mathscr{S}) \\
\Big\uparrow & & \Big\uparrow \\
\mathrm{Mod}\text{–}\mathscr{T} & \xrightarrow{\ \hat{F}\ } & \mathrm{Mod}\text{–}\mathscr{S}
\end{array}
$$

where the vertical inclusions are given by the definition of $\mathfrak{S}(?) \subset \mathfrak{L}(?) \subset \mathrm{Mod}\text{–}?$ of Discussion 6.10 (1) and (2).

Furthermore, the functor $\mathfrak{S}(F)$ respects the exact triangles as defined in Discussion 6.10(3).

Sun and Zhang go on to study recollements. Suppose we are given a recollement of triangulated categories

$$
\mathscr{R} \underset{I_\rho}{\overset{I_\lambda}{\rightleftarrows}} \xrightarrow{\ I\ } \mathscr{S} \underset{J_\rho}{\overset{J_\lambda}{\rightleftarrows}} \xrightarrow{\ J\ } \mathscr{T}.
$$

If all three triangulated categories come with good metrics, and if all six functors are continuous, then the following may be found in [**37**, **THEOREM 1.2**].

Theorem 7.4. *Under the hypotheses above, applying $\mathfrak{S}$ yields a right recollement*

$$\mathfrak{S}(\mathscr{R}) \underset{\mathfrak{S}(I)}{\overset{\mathfrak{S}(I_\lambda)}{\rightleftarrows}} \mathfrak{S}(\mathscr{S}) \underset{\mathfrak{S}(J)}{\overset{\mathfrak{S}(J_\lambda)}{\rightleftarrows}} \mathfrak{S}(\mathscr{T}).$$

In the presence of enough continuous adjoints, we deduce that a semiorthogonal decomposition of $\mathscr{S}$ gives rise to a semiorthogonal decomposition of $\mathfrak{S}(\mathscr{S})$. In view of the fact that there are metrics on $\mathbf{D}^{\mathbf{perf}}(X)$ and $\mathbf{D}^b_{\mathbf{coh}}(X)$ such that

$$\mathfrak{S}\big(\mathbf{D}^{\mathbf{perf}}(X)\big) = \mathbf{D}^b_{\mathbf{coh}}(X), \quad \mathfrak{S}\big(\mathbf{D}^b_{\mathbf{coh}}(X)^{\mathrm{op}}\big) = \mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}},$$

it is natural to wonder how the recent theorem of Sun and Zhang [**37**, **THEOREM 1.2**] compares with the older work of Kuznetsov [**19**, **SECTION 2.5**] and [**20**, **SECTION 4**].

The above shows that, subject to suitable hypotheses, the construction taking $\mathscr{S}$ to $\mathfrak{S}(\mathscr{S})$ can preserve (some of) the internal structure on the category $\mathscr{S}$—for example, semiorthogonal decompositions. This leads naturally to

Problem 7.5. What other pieces of the internal structure of $\mathscr{S}$ are respected by the construction that passes to $\mathfrak{S}(\mathscr{S})$? Under what conditions are these preserved?

Problem 7.5 may sound vague, but it can be made precise enough. For example, there is a huge literature dealing with the group of autoequivalences of the derived categories $\mathbf{D}^b_{\mathbf{coh}}(X)$. Now, as it happens, the metrics for which Remark 6.12 gives the equivalences

$$\mathfrak{S}\big(\mathbf{D}^{\mathbf{perf}}(X)\big) \cong \mathbf{D}^b_{\mathbf{coh}}(X), \quad \mathfrak{S}\big(\mathbf{D}^b_{\mathbf{coh}}(X)^{\mathrm{op}}\big) \cong \mathbf{D}^{\mathbf{perf}}(X)^{\mathrm{op}}$$

can be given (up to equivalence) intrinsic descriptions. Note that the way we introduced these metrics, in Discussion 6.10, was to use a preferred t-structure on $\mathscr{T} = \mathbf{D}_{\mathbf{qc}}(X)$ to give on $\mathscr{T}$ a metric, unique up to equivalence, and hence induced metrics on $\mathscr{T}^c = \mathbf{D}^{\mathbf{perf}}(X)$ and on $\mathscr{T}^b_c = \mathbf{D}^b_{\mathbf{coh}}(X)$ which are also unique up to equivalence. But this description seems to depend on an embedding into the large category $\mathscr{T}$. What I am asserting now is that there are alternative descriptions of the same equivalence classes of metrics on $\mathscr{T}^c$ and on $\mathscr{T}^b_c$, which do not use the embedding into $\mathscr{T}$. The interested reader can find this in the later sections of [**27**]. Anyway, a consequence is that any autoequivalence, of either $\mathbf{D}^{\mathbf{perf}}(X)$ or of $\mathbf{D}^b_{\mathbf{coh}}(X)$, must be continuous with a continuous inverse. Hence the group of autoequivalences of $\mathbf{D}^b_{\mathbf{coh}}(X)$ must be isomorphic to the group of autoequivalences of $\mathbf{D}^{\mathbf{perf}}(X)$. Or more generally, assume $\mathscr{T}$ is a noetherian, approximable triangulated category, where noetherian has the meaning of [**27**, **DEFINITION 5.1**]. Then the group of exact autoequivalences of $\mathscr{T}^c$ is canonically isomorphic to the group of exact autoequivalences of $\mathscr{T}^b_c$.

Are there similar theorems about t-structures in $\mathscr{S}$ going to t-structures in $\mathfrak{S}(\mathscr{S})$? Or about stability conditions on $\mathscr{S}$ mapping to stability conditions on $\mathfrak{S}(\mathscr{S})$?

We should note that any such theorem will have to come with conditions. After all, the category $\mathbf{D}^b_{\mathbf{coh}}(X)$ always has a bounded t-structure, while Antieau, Gepner, and Heller [**3**,

THEOREM 1.1] show that $\mathbf{D}^{\mathbf{perf}}(X)$ does not in general. Thus it is possible for $\mathscr{S}$ to have a bounded t-structure but for $\mathfrak{S}(\mathscr{S})$ not to. And in this particular example, the equivalence class of the metric has an intrinsic description, in the sense mentioned above.

Perhaps we should remind the reader that the article [3], by Antieau, Gepner, and Heller, finds a K-theoretic obstruction to the existence of bounded t-structures, more precisely if an appropriate category $\mathscr{E}$ has a bounded t-structure then $K_{-1}(\mathscr{E}) = 0$. Hence the reference to [3] immediately raises the question of how the construction passing from $\mathscr{S}$ to $\mathfrak{S}(\mathscr{S})$ might relate to K-theory, especially to negative K-theory. Of course, one has to be a little circumspect here. While there is a K-theory for triangulated categories (see [25] for a survey), this K-theory has only been proved to behave well for "nice" triangulated categories, for example, for triangulated categories with bounded t-structures. Invariants like negative K-theory have never been defined for triangulated categories, and might well give nonsense. In what follows we will assume that all the K-theoretic statements are for triangulated categories with chosen enhancements, and that K-theory means the Waldhausen K-theory of the enhancement. We recall in passing that the enhancements are unique for many interesting classes of triangulated categories, see Lunts and Orlov [21], Canonaco and Stellari [9], Antieau [2] and Canonaco, Neeman, and Stellari [8].

With the disclaimers out of the way, what do the results surveyed in this article have to do with negative K-theory?

Let us begin with Schlichting's conjecture [36, **CONJECTURE 1 OF SECTION 10**]; this conjecture, now known to be false [32], predicted that the negative K-theory of any abelian category should vanish. But Schlichting also proved that (1) $K_{-1}(\mathscr{A}) = 0$ for any abelian category $\mathscr{A}$, and (2) $K_{-n}(\mathscr{A}) = 0$ whenever $\mathscr{A}$ is a noetherian abelian category and $n > 0$. Now note that the $K(\mathscr{A}) = K(\mathscr{A}^{\mathrm{op}})$, hence the negative K-theory of any artinian abelian category must also vanish. And playing with extensions of abelian categories, we easily deduce the vanishing of the negative K-theory of a sizeable class of abelian categories. So while Schlichting's conjecture is false in the generality in which it was stated, there is some large class of abelian categories for which it is true. The challenge is to understand this class.

It becomes interesting to see what relation, if any, the results surveyed here have with this question.

Let us begin with Theorems 4.4 and 4.2. Theorem 4.4 tells us that, when X is a quasiexcellent, finite-dimensional, separated noetherian scheme, the category $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated. This category has a unique enhancement whose K-theory agrees with the K-theory of the noetherian abelian category $\mathrm{Coh}(X)$, hence the negative K-theory vanishes. Theorem 4.2 and Remark 4.3 tell us that the category $\mathbf{D}^{\mathbf{perf}}(X)$ has a strong generator if and only if X is regular and finite-dimensional—in which case it is equivalent to $\mathbf{D}^b_{\mathbf{coh}}(X)$ and its unique enhancement has vanishing negative K-theory. This raises the question:

Problem 7.6. If $\mathscr{T}$ is a triangulated category with a strong generator, does it follow that any enhancement of $\mathscr{T}$ has vanishing negative K-theory?

Let us refine this question a little. In Definition 4.1 we learned that a *strong generator,* for a triangulated category $\mathscr{T}$, is an object $G \in \mathscr{T}$ such that there exists an integer $\ell > 0$

with $\mathscr{T} = \langle G \rangle_\ell$. Following Rouquier, we can ask for estimates on the integer ℓ. This leads us to:

Definition 7.7. Let $\mathscr{T}$ be a triangulated category. The *Rouquier dimension* of $\mathscr{T}$ is the smallest integer $\ell \geq 0$ (we allow the possibility $\ell = \infty$), for which there exists an object G with $\mathscr{T} = \langle G \rangle_{\ell+1}$. See Rouquier [35] for much more about this fascinating invariant.

There is a rich and beautiful literature estimating this invariant and its various cousins—see Rouquier [35] for the origins of the theory, and a host of other places for subsequent developments. For this survey we note only that, for $\mathbf{D}^b_{\mathrm{coh}}(X)$, the Rouquier dimension is conjectured to be equal to the Krull dimension of X. But by a conjecture of Weibel [39], now a theorem of Kerz, Strunk, and Tamme [18], the Krull dimension of X also has a K-theoretic description: the groups K_n of the unique enhancement of $\mathbf{D}^{\mathrm{perf}}(X)$ vanish for all $n < -\dim(X)$. Recalling that $\mathscr{S} = \mathbf{D}^b_{\mathrm{coh}}(X)$ is related to $\mathbf{D}^{\mathrm{perf}}(X)$ by the fact that the construction $\mathfrak{S}$ interchanges them (up to passing to opposite categories, which has no effect on K-theory), this leads us to ask:

Problem 7.8. Let $\mathscr{S}$ be a regular (= strongly generated) triangulated category as in Definition 4.1, and let $N < \infty$ be its Rouquier dimension. Is it true that K_n vanishes on any enhancement of $\mathfrak{S}(\mathscr{S})$, for any metric on $\mathscr{S}$ and whenever $n < -N$?

In an entirely different direction, we know that the construction $\mathfrak{S}$ interchanges $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^b_{\mathrm{coh}}(X)$, and that these categories coincide if and only if X is regular. This leads us to ask:

Problem 7.9. Is there a way to measure the "distance" between $\mathscr{S}$ and $\mathfrak{S}(\mathscr{S})$, in such a way that resolution of singularities can be viewed as a process reducing this distance? Who knows, maybe there is even a good metric on $\mathscr{S} = \mathbf{D}^{\mathrm{perf}}(X)$ and/or on $\mathscr{S}' = \mathbf{D}^b_{\mathrm{coh}}(X)$ such that the construction $\mathfrak{S}$ takes either $\mathscr{S}$ or $\mathscr{S}'$ to an $\mathfrak{S}(\mathscr{S})$ or $\mathfrak{S}(\mathscr{S}')$ which is $\mathbf{D}^{\mathrm{perf}}(Y) = \mathbf{D}^b_{\mathrm{coh}}(Y)$ for some resolution of singularities Y of X.

While on the subject of regularity (= strong generation):

Problem 7.10. Is there some way to understand which are the approximable triangulated categories $\mathscr{T}$ for which $\mathscr{T}^c$ and/or $\mathscr{T}^b_c$ are regular?

Theorems 4.2 and 4.4 deal with the case $\mathscr{T} = \mathbf{D}_{\mathrm{qc}}(X)$. Approximability is used in the proofs given in [33] and [4], but only to ultimately reduce to the case of $\mathscr{T}^c = \mathbf{D}^{\mathrm{perf}}(X)$ with X an affine scheme—this case turns out to be classical, it was settled already in Kelly's 1965 article [17]. And the diverse precursors of Theorems 4.2 and 4.4, which we have hardly mentioned in the current survey, are also relatively narrow in scope. But presumably there are other proofs out there, yet to be discovered. And new approaches might well lead to generalizations that hold for triangulated categories having nothing to do with algebraic geometry.

Next let us revisit Theorem 6.6, the theorem identifying each of $[\mathscr{T}^c]^{\mathrm{op}}$ (respectively $\mathscr{T}^b_c$) as the finite homological functors on the other. In view of the motivating application, discussed in Remark 6.9, and of the generality of Theorem 6.6, it is natural to wonder:

Problem 7.11. Do GAGA-type theorems have interesting generalizations to other approximable triangulated categories? The reader is invited to check [**29**, SECTION 8 AND APPENDIX A]: except for the couple of paragraphs in [**29**, EXAMPLE A.2] everything is formulated in gorgeous generality and might be applicable in other contexts.

In the context of $\mathbf{D}^b_{\mathbf{coh}}(X)$, where X is a scheme proper over a noetherian ring R, there was a wealth of different-looking GAGA-statements before Jack Hall's lovely paper [**12**] unified them into one. In other words, the category $\mathbf{D}^b_{\mathbf{coh}}(X) = \mathscr{T}^b_c$ had many different-looking incarnations, and it was not until Hall's paper that it was understood that there was one underlying reason why they all coincided.

Hence Problem 7.11 asks whether this pattern is present for other $\mathscr{T}^b_c$, in other words for $\mathscr{T}^b_c \subset \mathscr{T}$ where $\mathscr{T}$ are some other R-linear, approximable triangulated categories.

And finally:

Problem 7.12. Is there a version of Theorem 6.6 that holds for non-noetherian rings?

There is evidence that something might be true, see Ben-Zvi, Nadler, and Preygel [**5**, SECTION 3]. But the author has no idea what the right statement ought to be, let alone how to go about proving it.

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