SYZYGIES OVER A POLYNOMIAL RING

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ABSTRACT

We discuss results and open problems on graded minimal free resolutions over polynomial rings.

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1. INTRODUCTION

Research on free resolutions is a core and beautiful area in Commutative Algebra. It contains a number of challenging conjectures and open problems; some of them are discussed in the book [101].

For simplicity, we will work throughout over the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$, which is *standard graded* by $\deg(x_i) = 1$ for every i. Many of the results work in much bigger generality; for example, over any field, or over some graded quotient rings of S. We leave it to the interested reader to look for the precise generality of the results using the references. We focus on some main ideas about finite resolutions which are present over polynomial rings.

The idea to describe the structure of a module by a free resolution was introduced by Hilbert in his famous paper [76]; this approach was present in the work of Cayley [35] as well. Every finitely generated S-module T has a free resolution. If T is graded, there exists a minimal free resolution $\mathbf{F}_T$ which is unique up to an isomorphism and is contained in any free resolution of T. Hilbert's insight was that the properties of the minimal free resolution $\mathbf{F}_T$ are closely related to the invariants of the resolved module T. The key point is that the resolution can be interpreted as an exact complex of finitely generated free modules F_i so that

$$\cdots \to F_2 \xrightarrow{d_2 = \begin{pmatrix}\text{minimal}\\ \text{relations}\\ \text{on the}\\ \text{relations}\\ \text{in } d_1\end{pmatrix}} F_1 \xrightarrow{d_1 = \begin{pmatrix}\text{minimal}\\ \text{relations}\\ \text{on the}\\ \text{generators}\\ \text{of } T\end{pmatrix}} F_0 \xrightarrow{d_0 = \begin{pmatrix}\text{minimal}\\ \text{generators}\\ \text{of } T\end{pmatrix}} T \to 0. \tag{1.1}$$

Thus, the resolution is a way of describing the structure of T.

The condition of minimality is important. The mere existence of free resolutions suffices for computing Hilbert functions and for foundational issues such as the definition of Ext and Tor. However, without minimality, resolutions are not unique, and the uniformity of constructions of nonminimal resolutions (like the Bar resolution) implies that they give little insight into the structure of the resolved modules. In contrast, the minimal free resolution $\mathbf{F}_T$ encodes a lot of properties of T; for example, the Auslander–Buchsbaum formula expresses the depth of T in terms of the length (called projective dimension) of $\mathbf{F}_T$, while nonminimal resolutions do not measure depth.

Free resolutions have applications in mathematical fields as diverse as Algebraic Geometry, Combinatorics, Computational Algebra, Invariant Theory, Mathematical Physics, Noncommutative Algebra, Number Theory, and Subspace Arrangements. For many years, they have been both central objects and fruitful tools in Commutative Algebra.

The connections of resolutions to Algebraic Geometry are especially rich, and the book [51] is focussed on that. One of the most challenging open problems in this area, which remains open to this date, is Green's conjecture; see the recent paper by Aprodu–Farkas–Papadima–Raicu–Weyman [5] for more details on this problem.

It should be noted that the world of minimal free resolutions is much wider and diverse than graded resolutions over polynomial rings. Resolutions are studied in other major situations, and there are many important and exciting results and open problems there. For example, there is an extensive research in the multigraded case, which contains resolutions of monomial ideals, resolutions of toric ideals, and resolutions of binomial edge ideals. Another fascinating and important area is the study of minimal free resolutions over quotient rings; such resolutions are usually infinite (by a theorem of Serre) and so their properties are quite different than what we see in finite resolutions over a polynomial ring. An interesting new idea is the recent introduction of virtual resolutions by Berkesch–Erman–Smith **[13]**.

2. FREE RESOLUTIONS

A *free resolution* of a finitely generated S-module T is an exact sequence

$$\mathbf{F}: \quad \ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} T \to 0$$

of homomorphisms of free finitely generated S-modules F_i. The maps d_i are called *differentials*.

If T is graded, there exists a minimal free resolution $\mathbf{F}_T$ which is unique up to an isomorphism and is contained in any free resolution of T (see **[101, THEOREM 7.5]**, **[101, THEOREM 3.5]**). Minimality can be characterized in the following simple way: $\mathbf{F}$ is *minimal* if

$$d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i \quad \text{for all } i \geq 0,$$

that is, no invertible elements appear in the differential matrices.

Hilbert's intuition was that the properties of the minimal free resolution $\mathbf{F}_T$ are closely related to the invariants of the resolved module T. The key point is that the map $d_0 : F_0 \to T$ sends a basis of F_0 to a minimal system $\mathcal{G}$ of generators of T, the first differential d_1 describes the minimal relations $\mathcal{R}$ among the generators $\mathcal{G}$, the second differential d_2 describes the minimal relations on the relations $\mathcal{R}$, etc.; see (1.1). Hilbert's Syzygy Theorem 4.1 is a fundamental result on the structure of such resolutions and leads to many applications. It shows that every finitely generated graded S-module has a finite free resolution (that is, $F_j = 0$ for $j \gg 0$).

The submodule $\mathrm{Im}(d_i) = \mathrm{Ker}(d_{i-1})$ of F_{i-1} is called the ith *syzygy module* of T, and its elements are called ith *syzygies*.

3. BETTI NUMBERS

Let T be a graded finitely generated S-module. The differentials in the minimal free resolution $\mathbf{F}_T$ of T are often very intricate, and so it may be more fruitful to focus on numerical invariants. The rank of the free module F_i in $\mathbf{F}_T$ is called the ith *Betti number* and is denoted by $b_i(T)$. It may be expressed as

$$b_i(T) = \dim \mathrm{Tor}_i^S(T, \mathbb{C}) = \dim \mathrm{Ext}_S^i(T, \mathbb{C}).$$

The Betti numbers are extensively studied numerical invariants of T, and they encode a lot of information about the module.

Note that in the graded case we have graded Betti numbers $b_{i,j}(T)$: Since T is graded, it has a graded minimal free resolution, that is, the differentials preserve degree (they are homogeneous maps of degree 0). Thus, we have graded Betti numbers

$$b_{i,j}(T) = \dim \operatorname{Tor}_i^S(T, \mathbb{C})_j = \dim \operatorname{Ext}_S^i(T, \mathbb{C})_j.$$

Hilbert showed how to use them in order to compute the *Hilbert series* $\sum_{i=0} t^i \dim_{\mathbb{C}}(T_i)$ which measures the size of the module T; see [101, THEOREM 16.2].

The graded Betti numbers can be assembled in the *Betti table* $\beta(T)$, which has entry $b_{i,i+j} = b_{i,i+j}(T)$ in position i, j. Following the conventions in the computer algebra system Macaulay2 [68], the columns of $\beta(T)$ are indexed from left to right by homological degree, and the rows are indexed increasingly from top to bottom. For example, if T is generated in nonnegative degrees then the Betti table $\beta(T)$ has the form:

	0	1	2	$\dots$
0:	$b_{0,0}$	$b_{1,1}$	$b_{2,2}$	$\dots$
1:	$b_{0,1}$	$b_{1,2}$	$b_{2,3}$	$\dots$
2:	$b_{0,2}$	$b_{1,3}$	$b_{2,4}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The main general open-ended question on Betti numbers is:

Question 3.1. How do the properties of the (graded) Betti numbers relate to the structure of the minimal free resolution of T and/or the structure of T?

The BEH Conjecture is a long-standing open conjecture on Betti numbers:

BEH Conjecture 3.2 (Buchsbaum–Eisenbud, Horrocks, [20,73]). *If T is a finitely generated graded artinian S-module (artinian means that the module has finite length), then*

$$b_i(T) \geq \binom{n}{i} \quad \textit{for } i \geq 0.$$

Essentially, the conjecture states that the Koszul resolution (see [101, SECTION 14]) of the residue field $\mathbb{C}$ is the smallest minimal free resolution of an artinian module.

If the above conjecture holds, then it easily follows that we get a lower bound on the Betti numbers for any module (not necessarily artinian) in terms of its codimension; see [17]. The expository papers [17] by Boocher–Grifo and [36] by Charalambous–Evans provide nice overviews on the scarce positive results that are known so far; for example, Herzog–Kühl [74] proved the desired inequalities for linear resolutions. The best currently known result is:

Theorem 3.3 (Walker, [108]). *If T is a finitely generated graded artinian S-module, then*

$$\sum_{i=0}^{n} b_i(T) \geq 2^n,$$

and equality holds if and only if T is a complete intersection.

People have wondered how sharp the above bound is when the module is not a complete intersection (that is, T is not a quotient ring by a regular sequence):

Question 3.4 (Charalambous–Evans–Miller, [37]). If T is a finitely generated graded artinian S-module that is not a complete intersection, then do we have

$$\sum_{i=0}^{n} b_i(T) \geq 2^n + 2^{n-1}?$$

There are many questions that one may ask and study about Betti numbers when restricted to special classes of modules; most ambitiously, we would like to have a characterization of the sequences that are Betti numbers. A recent result of this kind is the Boij–Söderberg theory, which was conjectured by Boij–Söderberg [16], and proved soon after that. Eisenbud–Fløystad–Weyman proved the characteristic-zero case in [52] and then Eisenbud–Schreyer dealt with any characteristic in [56]. Later, efficient methods for such constructions were given by Berkesch, Kummini, Erman, Sam in [14] and by Fløystad in [61, SECTION 3]. The expository papers [60, 62] provide nice overviews of this topic.

4. PROJECTIVE DIMENSION

Projective dimension and regularity are the main numerical invariants that measure the complexity of a minimal free resolution. We will discuss regularity in the next section.

The *projective dimension* of a graded finitely generated S-module T is

$$\operatorname{pd}(T) = \max\{i \mid b_i(T) \neq 0\},$$

and it is the index of the last nonzero column of the Betti table $\beta(T)$, so it measures the width of the table.

Hilbert's Syzygy Theorem 4.1 (see [101, THEOREM 15.2]). *The minimal graded free resolution of a finitely generated graded S-module is finite, and its projective dimension is at most n (recall that n is the number of variables in the polynomial ring S).*

Hilbert's Syzygy Theorem 4.1 provides a nice upper bound on the projective dimension in terms of the number of variables in the polynomial ring. One may wonder if the number of minimal generators of an ideal can be used to get another nice upper bound on projective dimension. The answer turns out to be negative. A construction of Burch [21] and Kohn [79] produces ideals with 3 generators whose projective dimension is arbitrarily large. Later Bruns [18] showed that the minimal free resolutions of three-generated ideals capture all the pathology of minimal free resolutions of modules. However, the degrees of the generators in these constructions are forced to grow large. Motivated by computational complexity issues, Stillman raised the following question:

Question 4.2 (Stillman, [102, PROBLEM 3.14]). Fix an $m \geq 1$ and a sequence of natural numbers $a_1, \ldots, a_m$. Is there a number p such that $\operatorname{pd}(I) \leq p$ for every homogeneous ideal I with a minimal system of generators of degrees $a_1, \ldots, a_m$ in a polynomial ring? Note that the number of variables in the polynomial ring is not fixed.

A positive answer is provided by:

Theorem 4.3 (Ananyan–Hochster, [4]). *Stillman's Question 4.2 has a positive answer.*

Other proofs were later given by Erman–Sam–Snowden [58] and Draisma–Lasoń–Leykin [50]. Yet, there are many open questions motivated by a desire to get better upper bounds since the known bounds are quite large. See the recent paper by Caviglia–Liang [29] for some explicit bounds.

Families of ideals with large projective dimension were constructed by McCullough in [89] and by Beder, McCullough, Núñez-Betancourt, Seceleanu, Snapp, Stone in [12]. Such constructions indicate that finding tight bounds could be difficult. Many results dealing with special cases are known in this direction. The expository papers [62, 94] provide nice overviews of this topic.

5. REGULARITY

Let L be a homogeneous ideal in S. The height of the Betti table of L is measured by the index of the last nonzero row, and is called the *(Castelnuovo–Mumford) regularity* of L, so

$$\operatorname{reg}(L) = \max\{j \mid \text{there exists an } i \text{ such that } b_{i,i+j}(L) \neq 0\}.$$

Note that $\operatorname{reg}(L) < \infty$ by Hilbert's Syzygy Theorem 4.1. An important role of regularity is that it measures the complexity of the minimal free resolution of L, in the sense that it shows up to what degree we have nonvanishing Betti numbers. It has several other important roles.

The definition of regularity implies that it provides an upper bound on the generating degree, namely

$$\operatorname{reg}(L) \geq \operatorname{maxdeg}(L),$$

where $\operatorname{maxdeg}(L)$ is the maximal degree of an element in a minimal system of homogeneous generators of L.

Another role of regularity is that it identifies how high we have to truncate an ideal in order to get a linear resolution; we say that a graded ideal has an r-*linear resolution* if the ideal is generated in degree r and the entries in the differential maps in its minimal free resolution are linear.

Theorem 5.1 (see [101, THEOREM 19.7]). *Let L be a graded ideal in S. If $r \geq \operatorname{reg}(L)$ then*

$$L_{\geq r} := L \cap \left(\bigoplus_{i \geq r} S_i\right)$$

has an r-linear minimal free resolution, equivalently,

$$\operatorname{reg}(L_{\geq r}) = r.$$

Another role of regularity is related to Gröbner basis computation. Many computer computations in Commutative Algebra and Algebraic Geometry are based on Gröbner basis theory. It is used, for example, in the computer algebra systems Cocoa [1], Macaulay2 [68], Singular [49]. It is proved by Bayer–Stillman [9] that in generic coordinates and with respect to revlex order, one has to compute up to degree $\operatorname{reg}(L)$ in order to compute a Gröbner basis of L. This means that $\operatorname{reg}(L)$ is the degree-complexity of the Gröbner basis computation.

Yet another role of regularity is that it can be defined in terms of vanishing of local cohomology modules. See the expository paper [19] for a detailed discussion.

The expository papers [38,39] provide nice overviews of the properties of regularity.

In the rest of this section, we discuss bounds on regularity.

The projective dimension $\operatorname{pd}(L)$ of L is bounded above by the number of variables n in S by Hilbert's Syzygy Theorem 4.1. This bound is very nice in several ways: it is small, involves only one parameter, and is given by a simple formula. One may hope that similarly, a nice upper bound on regularity exists. In contrast, the upper bound on regularity involving n is doubly exponential. Bayer–Mumford (see [8, THEOREM 3.7]) and Caviglia–Sbarra [32] proved:

Theorem 5.2 (Bayer–Mumford [8], Caviglia–Sbarra [32]). *Let L be a graded ideal in S. Then*

$$\operatorname{reg}(L) \leq \big(2 \operatorname{maxdeg}(L)\big)^{2^{n-2}},$$

where $\operatorname{maxdeg}(L)$ *is the maximal degree of an element in a minimal system of homogeneous generators of L.*

This bound is nearly sharp. The Mayr–Meyer construction [88] leads to examples of families of ideals attaining high regularity. The following three types of families of ideals attaining doubly exponential regularity were constructed by Bayer–Mumford [8], Bayer–Stillman [10], and Koh [78]:

Theorem 5.3. (1) (Bayer–Stillman, [10, THEOREM 2.6]) *For $r \geq 1$, there exists a homogeneous ideal I_r (using $d = 3$ in their notation) in a polynomial ring with* $10r + 11$ *variables for which*

$$\operatorname{maxdeg}(I_r) = 5,$$
$$\operatorname{reg}(I_r) \geq 3^{2^{r-1}}.$$

(2) (Bayer–Mumford, [8, PROPOSITION 3.11]) *For $r \geq 1$, there exists a homogeneous ideal I_r in* $10r + 1$ *variables for which*

$$\operatorname{maxdeg}(I_r) = 4,$$
$$\operatorname{reg}(I_r) \geq 2^{2^r}.$$

(3) (Koh, [78]) *For $r \geq 1$, there exists a homogeneous I_r generated by $22r - 2$ quadrics in a polynomial ring with $22r$ variables for which*

$$\operatorname{maxdeg}(I_r) = 2,$$
$$\operatorname{reg}(I_r) \geq 2^{2^{r-1}}.$$

Further examples of ideals with high regularity were produced by Beder et. al. [12], Caviglia [23], Chardin–Fall [41], and Ullery [107].

Despite these examples of high regularity, there are many important and interesting cases where regularity is bounded by (or equal to) a nice formula and is not dramatically large. As always, the following open-ended problem is of high interest:

Problem 5.4. Find important and interesting cases where regularity is bounded by (or equal to) a nice formula and is not dramatically large.

6. REGULARITY OF PRIME IDEALS

Regularity was studied in Algebraic Geometry as well. In that setting, much better bounds than the doubly-exponential bound discussed in Theorem 5.2, are expected for the regularity of the defining ideals of geometrically nice projective varieties. Lazarsfeld's book [86, SECTION 1.8] and the introduction of the paper [84] by Kwak–Park provide nice overviews of that point of view. In fact, the concept of regularity was introduced by Mumford [98] and generalizes ideas of Castelnuovo. The relation between the definitions of regularity of a coherent sheaf and regularity of a graded ideal (or module) is given in Eisenbud–Goto [53], and may be also found in [51, PROPOSITION 4.16].

Consider a *nondegenerate* projective variety $X \subset \mathbb{P}^{n-1}$, that is, X does not lie on a hyperplane in $\mathbb{P}^{n-1}$.

Some nice bounds were proved in the smooth case. The following bound follows from a more general result by Bertram–Ein–Lazarsfeld [15]:

Theorem 6.1 (Bertram–Ein–Lazarsfeld, [15]). *Let $X \subset \mathbb{P}^{n-1}$ be a smooth irreducible projective variety. If X is cut out scheme-theoretically by hypersurfaces of degree $\leq s$, then*

$$\operatorname{reg}(X) \leq 1 + (s-1)\operatorname{codim}(X).$$

This result was generalized in [42] and [48]. See also [38] for an overview.

Theorem 6.2 (Mumford, [8, THEOREM 3.12]). *If $X \subset \mathbb{P}^{n-1}$ is a nondegenerate smooth projective variety, then*

$$\operatorname{reg}(X) \leq \big(\dim(X) + 1\big)\big(\deg(X) - 2\big) + 2.$$

This bound was improved by Kwak–Park as follows:

Theorem 6.3 (Kwak–Park, [84, THEOREM C]). *If $X \subset \mathbb{P}^{n-1}$ is a nondegenerate smooth projective variety with* $\operatorname{codim}(X) \geq 2$, *then*

$$\operatorname{reg}(X) \leq \dim(X)\big(\deg(X) - 2\big) + 1.$$

In the influential paper [8], Bayer and Mumford wrote:

"...the main missing piece of information between the general case and the geometrically nice smooth case is that we do not have yet a reasonable bound on the regularity of all reduced equidimensional ideals."

Note that the bounds in the above theorems involve two parameters; for example, $\dim(X)$ and $\deg(X)$ are used in Theorem 6.2. The following bound involving only $\deg(X)$ was first considered in the smooth case:

$$\operatorname{reg}(X) \leq \deg(X).$$

It was conjectured by Eisenbud–Goto [53] for any reduced and irreducible nondegenerate variety, and they expected that it might even hold for reduced equidimensional X which are connected in codimension 1 [8]. In fact, they conjectured the more refined bound

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1,$$

which is sharp as equality holds for the twisted cubic curve. This is called the Regularity Conjecture. In particular, it yields the following regularity conjecture for prime ideals:

Conjecture 6.4 (Eisenbud–Goto [53], 1984). *If L is a homogeneous prime ideal in S, and $L \subset (x_1, \ldots, x_n)^2$, then*

$$\operatorname{reg}(L) \leq \deg(L).$$

In particular, L is generated in degrees $\leq \deg(L)$.

The condition $L \subset (x_1, \ldots, x_n)^2$ is equivalent to requiring that the projective variety $V(L)$ is not contained in a hyperplane in $\mathbb{P}^{n-1}$. Prime ideals that satisfy this condition are called *nondegenerate*.

The Regularity Conjecture is proved for curves by Gruson–Lazarsfeld–Peskine [69], completing fundamental work of Castelnuovo [22]; see also [67]. It is also proved for smooth surfaces by Lazarsfeld [85] and Pinkham [103]. In the smooth case, Kwak [81–83] gives bounds for regularity in dimensions 3 and 4 that are only slightly worse than the optimal ones. The conjecture also holds in the Cohen–Macaulay case by a result of Eisenbud–Goto [53]. Many other special cases and related bounds have been proved as well.

In [92] Jason McCullough and I construct counterexamples to the Regularity Conjecture. We provide a family of prime ideals P_r, depending on a parameter r, whose degree is singly exponential in r and whose regularity is doubly exponential in r. Our main theorem is much stronger:

Theorem 6.5 (McCullough–Peeva, [92]). *The regularity of nondegenerate homogeneous prime ideals is not bounded by any polynomial function of the degree (multiplicity), i.e., for any polynomial $f(x) \in \mathbb{R}[x]$ there exists a nondegenerate homogeneous prime ideal Y in a standard graded polynomial ring over $\mathbb{C}$ such that* $\operatorname{reg}(Y) > f(\deg(Y))$.

For this purpose, we introduce in [92] an approach which, starting from a homogeneous ideal I, produces a prime ideal P whose projective dimension, regularity, degree, dimension, depth, and codimension are expressed in terms of numerical invariants of I. Our approach involves two new concepts:

(1) Rees-like algebras (inspired by an example by Hochster published in [11]) which, unlike the standard Rees algebras, have well-structured defining equations and minimal free resolutions;

(2) A step-by-step homogenization technique which, unlike classical homogenization, preserves graded Betti numbers.

Further research in this direction was carried out by Caviglia–Chardin–McCullough–Peeva–Varbaro in [24]. Our expository paper [93] provides an overview of counterexamples and the techniques used to prove them.

The bound in the Regularity Conjecture is very elegant, so it is reasonable to expect that work will continue on whether it holds when we impose extra conditions on the prime ideal: for example, for smooth varieties or for toric ideals (in the sense of the definition in [101, SECTION 65]).

Instead of trying to repair the Regularity Conjecture by imposing extra conditions, one may wonder:

Question 6.6 (McCullough–Peeva, [93]). What is an optimal function $f(x)$ such that $\mathrm{reg}(L) \le f(\deg(L))$ for any nondegenerate homogeneous prime ideal L in a standard graded polynomial ring over $\mathbb{C}$?

Since Theorem 5.2 gives a doubly exponential bound on regularity for all homogeneous ideals, and in view of Theorem 6.5, the following question is of interest:

Question 6.7 (McCullough–Peeva, [93]). Does there exist a singly exponential bound for regularity of homogeneous nondegenerate prime ideals in a standard graded polynomial ring over $\mathbb{C}$, in terms of the multiplicity alone?

In [8, COMMENTS AFTER THEOREM 3.12] Bayer and Mumford wrote:

"We would conjecture that if a linear bound doesn't hold, at the least a single exponential bound, i.e. $\mathrm{reg}(L) \le \mathrm{maxdeg}(L)^{\mathcal{O}(n)}$*, ought to hold for any reduced equidimensional ideal. This is an essential ingredient in analyzing the worst-case behavior of all algorithms based on Gröbner bases."*

For prime ideals, their conjecture is:

Conjecture 6.8 (Bayer–Mumford, [8, COMMENTS AFTER THEOREM 3.12]). *If L is a homogeneous non-degenerate prime ideal in $S = \mathbb{C}[x_1, \dots, x_n]$, then*

$$\mathrm{reg}(L) \le \mathrm{maxdeg}(L)^{\mathcal{O}(n)},$$

where $\mathrm{maxdeg}(L)$ *is the maximal degree of an element in a minimal system of homogeneous generators of L.*

7. REGULARITY OF THE RADICAL

Ravi [104] proved that in some cases the regularity of the radical of an ideal is no greater than the regularity of the ideal itself. For a long time, there was a folklore conjecture that this would hold for every homogeneous ideal. However, counterexamples were constructed by Chardin–D'Cruz [40]. They obtained examples where regularity of the radical is nearly the square (or the cube) of that of the ideal.

Theorem 7.1 (Chardin–D'Cruz, [40, EXAMPLE 2.5]). *For $m \geq 1$ and $r \geq 3$, the ideal*

$$J_{m,r} = (y^m u^2 - x^m zv,\ z^{r+1} - xu^r,\ u^{r+1} - xv^r,\ y^m v^r - x^{m-1} zu^{r-1} v)$$

in the polynomial ring $\mathbb{C}[x, y, z, u, v]$ has

$$\begin{aligned} \operatorname{reg}(J_{m,r}) &= m + 2r + 1, \\ \operatorname{reg}(\sqrt{J_{m,r}}) &= m(r^2 - 2r - 1) + 1. \end{aligned}$$

The existence of a polynomial bound is very unclear, so perhaps it is reasonable to focus on the following folklore question which is currently open:

Question 7.2. Is there a singly exponential bound on $\operatorname{reg}(\sqrt{I})$ in terms of $\operatorname{reg}(I)$ (and possibly $\operatorname{codim}(I)$ or n) for every homogeneous ideal I in a standard graded polynomial ring over $\mathbb{C}$?

In order to form reasonable conjectures, it would be very helpful to develop methods for producing interesting examples. In [86, REMARK 1.8.33] Lazarsfeld wrote:

> *"...the absence of systematic techniques for constructing examples is one of the biggest lacunae in the current state of the theory."*

8. SHIFTS

Let T be a graded finitely generated S-module. The (upper) shifts are refinements of the numerical invariant regularity. The *(upper) shift* at step i is

$$t_i(T) = \max\{j \mid b_{i,j}(T) \neq 0\}$$

and the *adjusted shift* is

$$r_i(T) = \max\{j \mid b_{i,i+j}(T) \neq 0\},$$

so

$$r_i(T) = t_i(T) - i.$$

Note that $r_0(T)$ is the maximal degree of an element in a minimal system of generators of T, and

$$\operatorname{reg}(T) = \max_i\{r_i(T)\}.$$

Let L be a graded ideal in S. The a, b*-subadditivity condition* for L is

$$t_{a+b}(S/L) \leq t_a(S/L) + t_b(S/L). \tag{8.1}$$

Note that it is equivalent to

$$r_{a+b}(S/L) \leq r_a(S/L) + r_b(S/L).$$

We say that L satisfies the *general subadditivity condition* if (8.1) holds for every a, b. We say that L satisfies the *initial subadditivity condition* if (8.1) holds for $b = 1$ and every a. We say that L satisfies the *closing subadditivity condition* if (8.1) holds for every a, b with $a + b = \mathrm{pd}(L)$. Gorenstein ideals failing the subadditivity condition were constructed by McCullough–Seceleanu in [95].

Problem 8.1. (1) (McCullough, [91]) It is expected that the general subadditivity condition holds for every monomial ideal L.

(2) (Avramov–Conca–Iyengar, [7]) It is conjectured that the general subadditivity condition holds if S/L is a Koszul algebra.

(3) (McCullough, [91]) It is expected that the general subadditivity condition holds for every toric ideal L.

There are supporting results in special cases; the expository paper [91] provides a nice overview of the current state of these problems. For monomial ideals, Herzog–Srinivasan [75] proved that the initial subadditivity condition holds.

Another interesting direction of using shifts is:

Problem 8.2. Find good upper bounds on regularity using the shifts in part of the minimal free resolution.

The following result shows how this may work:

Theorem 8.3 (McCullough, [90]). *Let L be a homogeneous ideal in S. Set $c = \lceil \frac{n}{2} \rceil$. Then*

$$\mathrm{reg}(S/L) \leq \sum_{i=1}^{c} t_i(S/I) + \frac{\prod_{i=1}^{h} t_i(S/I)}{(c-1)!}.$$

9. THE EGH CONJECTURE

We start with a brief introduction to Hilbert functions and lex ideals. If I is a homogeneous ideal in S, then the quotient $R := S/I$ inherits the grading by $R_i = S_i/I_i$ for all i. The size of a homogeneous ideal J in R is measured by its *Hilbert function*

$$\mathrm{Hilb}_{R/J}(i) = \dim_{\mathbb{C}}(R_i/J_i) \quad \text{for } i \in \mathbf{Z}.$$

Hilbert's insight was that $\mathrm{Hilb}_{R/J}$ is determined by finitely many of its values. He proved that there exists a polynomial (called the *Hilbert polynomial*) $g(t) \in \mathbf{Q}[t]$ such that

$$\mathrm{Hilb}_{R/J}(i) = g(i) \quad \text{for } i \gg 0.$$

If S/J (here $R = S$) is the coordinate ring of a projective algebraic variety X, then the degree of the Hilbert polynomial equals the dimension of X; the leading coefficient of the Hilbert polynomial determines another important invariant – the degree (multiplicity) of X. Hilbert functions for monomial ideals in the ring $\mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ have been extensively studied in Combinatorics since each such Hilbert function counts the number of faces in a simplicial complex.

Lex ideals are fruitful tools in the study of Hilbert functions. They are monomial ideals defined in a simple way: Denote by $>_{lex}$ the lexicographic order on the monomials in S extending $x_1 > \cdots > x_n$. A monomial ideal L in S is *lex* if the following property holds: if $m \in L$ is a monomial and $q >_{lex} m$ is a monomial of the same degree, then $q \in L$ (that is, for each $i \geq 0$ the vector space L_i is either zero or is spanned by lex-consecutive monomials of degree i starting with x_1^i).

A core result in Commutative Algebra is Macaulay's Theorem 9.1, which characterizes the Hilbert functions of homogeneous ideals in the polynomial ring S:

Theorem 9.1 (Macaulay, [87]). *For every homogeneous ideal in S there exists a lex ideal with the same Hilbert function.*

The Hilbert function of a lex ideal is easy to count. This leads to an equivalent formulation of Macaulay's Theorem 9.1 which characterizes numerically (by certain inequalities) the Hilbert functions of homogeneous ideals; see [101, SECTION 49].

Lex ideals also play an important role in the study of Hilbert schemes. Grothendieck introduced the Hilbert scheme $\mathcal{H}_{r,g}$ that parametrizes subschemes of $\mathbf{P}^r$ with a fixed Hilbert polynomial g. The structure of the Hilbert scheme is known to be very complicated. In [71] Harris and Morrison state Murphy's Law for Hilbert schemes:

> *"There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme."*

The main structural result on $\mathcal{H}_{r,g}$ is Hartshorne's Theorem:

Theorem 9.2 (Hartshorne, [72]). *The Hilbert scheme $\mathcal{H}_{r,g}$ is connected.*

The situation is that every homogeneous ideal with a fixed Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h. A deformation connects two ideals $J_{t=0}$ and $J_{t=1}$ in the sense that we have a family of homogeneous ideals J_t varying with the parameter $t \in [0, 1]$ so that the Hilbert function is preserved; in this case, the ideals J_t form a path on the Hilbert scheme. Hartshorne's proof [72] relies on deformations called "distractions" which use generic change of coordinates and polarization. Analyzing the paths on a Hilbert scheme may shed light on whether there exists an object with maximal Betti numbers.

Theorem 9.3 (Bigatti–Hulett–Pardue, see [100]). *A lex ideal in S has the greatest Betti numbers among all homogeneous ideals in S with the same Hilbert function.*

This result was quite surprising when it was discovered since counterexamples were known in which no ideal with a fixed Hilbert function attains minimal Betti numbers. It yields numerical upper bounds on Betti numbers as follows: the minimal free resolution of a lex ideal is the Eliahou–Kervaire resolution [57] (see [101, SECTION 28]), and it provides numerical formulas for the Betti numbers of a lex ideal.

It is natural to ask if similar results hold over other rings. For starters, we need rings over which Theorem 9.1 holds. It actually fails over most graded quotient rings of S. For example, there is no lex ideal in the ring $\mathbb{C}[x,y]/(x^2y, xy^2)$ with the same Hilbert function as the ideal (xy).

Theorem 9.4. *Macaulay's Theorem 9.1 holds over the following rings:*

(1) (Kruskal, Katona, [77,80]) *an exterior algebra E over $\mathbb{C}$.*

(2) (Clements–Lindström, [45]) *a Clements–Lindström ring $C := \mathbb{C}[x_1,\dots,x_n]/P$, where P is an ideal generated by powers of the variables.*

(3) (Gasharov–Murai–Peeva, [66]) *a Veronese ring $V := S/J$, where J is the defining ideal of a Veronese toric variety.*

Proving analogues of Theorem 9.3 for the above rings is difficult since minimal resolutions over exterior algebras, Clements–Lindström rings, or Veronese rings are infinite (in contrast, Theorem 9.3 is about finite resolutions) and so they are considerably more intricate. It was proved that every lex ideal has the greatest Betti numbers among all homogeneous ideals with the same Hilbert function over the following rings: over E by Aramova–Herzog–Hibi [6], over C by Murai–Peeva [99], and over V by Gasharov–Murai–Peeva [66].

Hilbert functions of ideals containing $(x_1^2,\dots,x_n^2)$ are characterized numerically (by certain inequalities) by Kruskal–Katona's Theorem [77,80], which is a natural analogue of Macaulay's Theorem; see Theorem 9.4(1,2). Eisenbud–Green–Harris conjectured that the same numerical inequalities for the Hilbert function hold for all ideals in S containing a quadratic regular sequence:

Conjecture 9.5 (Eisenbud–Green–Harris, [54]). *Let $L \subset S$ be a homogeneous ideal containing a regular sequence of n quadratic forms. There exists an ideal N containing $x_1^2,\dots,x_n^2$ with the same Hilbert function as L.*

Kruskal–Katona's Theorem was generalized by Clements–Lindström [45] to a characterization of the Hilbert functions of ideals containing powers of the variables; see Theorem 9.4(2). In view of this, Eisenbud–Green–Harris noted in [54] that Conjecture 9.5 can be extended to cover all complete intersections as follows:

Conjecture 9.6 (Eisenbud–Green–Harris, [54]). *Let $L \subset S$ be a homogeneous ideal containing a regular sequence of forms of degrees $a_1 \le \dots \le a_n$. There exists an ideal N containing $x_1^{a_1},\dots,x_n^{a_n}$ with the same Hilbert function as L.*

Conjecture 9.5 is considered to be the main case of the Eisenbud–Green–Harris Conjectures, called the EGH Conjectures.

In their original form in [54], the EGH Conjectures are stated in terms of numerical inequalities for the Hilbert function. We give an equivalent form, which follows immediately from the Clements–Lindström Theorem 9.4(2).

Eisenbud, Green, and Harris were led to the EGH Conjectures by extending a series of results and conjectures in Castelnuovo Theory in [54]. After that, they made the connection to the Cayley–Bacharach Theory in [55]. They provide in [55] a nice survey of the long history of Cayley–Bacharach theory in Algebraic Geometry.

The EGH Conjectures turned out to be very challenging. Some special cases, applications, and related results are proved in [2,3,25,27,30,33,34,43,44,46,47,54,55,59,70,96,105,106]. One of the strongest results is the recent paper [26] by Caviglia–DeStefani.

We now focus on Betti numbers related to the EGH Conjectures. Let $L \subset S$ be a homogeneous ideal containing a regular sequence of forms of degrees $a_1 \leq \cdots \leq a_n$. The concept of a lex ideal can be generalized to the concept of a *lex-plus-powers ideal* which is a monomial ideal containing $x_1^{a_1}, \ldots, x_n^{a_n}$ and otherwise is like a lex ideal. G. Evans conjectured the more general Lex-Plus-Powers Conjecture that, among all graded ideals with a fixed Hilbert function and containing a homogeneous regular sequence of degrees $a_1 \leq \cdots \leq a_n$, the lex-plus-powers ideal (which exists according to the EGH Conjectures) has the greatest Betti numbers. This conjecture was inspired by Theorem 9.3.

Theorem 9.7 (Mermin–Murai, [97]). *The Lex-Plus-Powers Conjecture holds for ideals containing pure powers.*

The general Lex-Plus-Powers Conjecture (for ideals containing a homogeneous regular sequence) is very difficult. Some special cases are proved in [31,59,63,64,105,106]. The expository papers by Caviglia–DeStefani–Sbarra [28] and by Francisco-Richert [65] provide nice overviews of this challenging topic.

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