# SURVEY LECTURE ON ARITHMETIC DYNAMICS

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**ABSTRACT**

Arithmetic dynamics is a relatively new field in which classical problems from number theory and algebraic geometry are reformulated in the setting of dynamical systems. Thus, for example, rational points on algebraic varieties become rational points in orbits, and torsion points on abelian varieties become points having finite orbits. Moduli problems also appear, where, for example, the complex multiplication points in the moduli space of abelian varieties correspond to the postcritically finite points in the moduli space of rational maps. In this article we give a survey of some of the major problems motivating the field of arithmetic dynamics, and some of the progress that has been made during the past 20 years.

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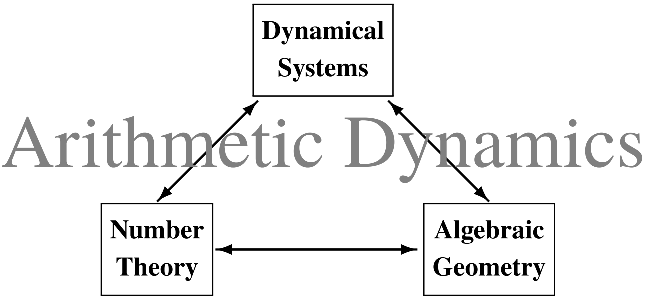
## 1. INTRODUCTION

This article is a survey of the comparatively new field of *Arithmetic Dynamics*, a field where arithmetic and dynamics join forces.[1] But the word "arithmetic" in "arithmetic dynamics" is itself short for "arithmetic geometry," a field where the venerable subjects of number theory and algebraic geometry meet. Thus arithmetic dynamics is a melting pot filled with ingredients from three classical areas of mathematics.




In this article we will discuss arithmetic dynamics over global fields, which for the sake of exposition we will generally take to be number fields, i.e., finite extensions of $\mathbb{Q}$. Our primary focus will be dynamical analogues and generalizations of famous theorems and conjectures in arithmetic geometry, centered around the following five major topics that have helped drive the development of arithmetic dynamics over the past few decades:

- Topic #1: Dynamical Uniform Boundedness
- Topic #2: Dynamical Moduli Spaces
- Topic #3: Dynamical Unlikely Intersections
- Topic #4: Dynatomic and Arboreal Representations
- Topic #5: Dynamical and Arithmetic Complexity

**Remark 1.1.** Of course, our chosen five topics do not fully cover the varied problems that fall under the rubric of arithmetic dynamics over global fields. And there are also highly active areas of arithmetic dynamics in which people study dynamical systems defined over non-archimedean fields such as $\mathbb{Q}_p$ and $\mathbb{C}_p$ and over finite fields $\mathbb{F}_q$. We refer the interested reader to the survey article [10] for a more extensive discussion.

1 As Jung might have said: "The meeting of two mathematical fields is like the contact of two chemical substances: if there is any reaction, both are transformed."

## 2. DEFINITIONS AND TERMINOLOGY

An abstract *dynamical system* is simply an object $X$ and an endomorphism (self-map)[2]

$$f : X \to X.$$

The *iterates of* $f$ are denoted by

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies of } f},$$

and the (*forward*) $f$*-orbit of an element* $x \in X$ is its image for the iterates of $f$,[3]

$$\mathcal{O}_f(x) = \{f^n(x) : n \geq 0\}.$$

We say that $x \in X$ is $f$*-periodic* if

$$f^n(x) = x \quad \text{for some } n \geq 1,$$

in which case the smallest such $n$ is the (*exact*) *period of* $x$. A point $x \in X$ is $f$*-preperiodic* if its $f$-orbit $\mathcal{O}_f(x)$ is finite, or equivalently, if $f^m(x)$ is periodic for some $m \geq 0$.

Two dynamical systems $f_1, f_2 : X \to X$ are *isomorphic* if there is an automorphism $\varphi \in \operatorname{Aut}(X)$ such that

$$f_2 = f_1^\varphi = \varphi^{-1} \circ f_1 \circ \varphi. \tag{2.1}$$

Note that (2.1) is a good notion of isomorphism for dynamics, since it respects iteration,

$$(f^\varphi)^n = (\varphi^{-1} \circ f \circ \varphi)^n = \varphi^{-1} \circ f^n \circ \varphi = (f^n)^\varphi.$$

In particular, orbits and (pre)periodic points of the isomorphic dynamical systems $f$ and $f^\varphi$ are more-or-less identical, since

$$\mathcal{O}_{f^\varphi}(x) = \varphi^{-1}\big(\mathcal{O}_f(\varphi(x))\big), \quad \operatorname{Per}(f^\varphi) = \varphi^{-1}\big(\operatorname{Per}(f)\big), \quad \operatorname{PrePer}(f^\varphi) = \varphi^{-1}\big(\operatorname{PrePer}(f)\big).$$

We conclude this section with a brief discussion of endomorphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ i.e., rational functions of one variable. For $P \in \mathbb{P}^1$, we choose a local parameter $z_P$ at $P$ and define $P$ to be a *critical point of* $f$ if

$$\frac{df}{dz_P}(P) = 0. \tag{2.2}$$

The vanishing condition (2.2) is independent of the choice of $z_P$, and counted with appropriate multiplicities, the map $f$ has $2\deg(f) - 2$ critical points.[4]

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**2** To avoid complications, we always work in a subcategory of the category of sets, i.e., all of our objects are sets.

**3** More generally, let $\mathcal{F} = \{f_1, \ldots, f_r\}$ be a set of endomorphisms of $X$, and let $\langle \mathcal{F} \rangle$ be the semigroup of maps generated by arbitrary composition of elements of $\mathcal{F}$. Then the $\mathcal{F}$*-orbit of* $x$ is the set $\mathcal{O}_{\mathcal{F}}(x) = \{f(x) : f \in \langle \mathcal{F} \rangle\}$.

**4** More precisely, this is true as long as $f$ is separable, so in particular it is always true in characteristic 0.

| Arithmetic Geometry | | Dynamical Systems |
| --- | --- | --- |
| rational and integral points on varieties | ⟷ | rational and integral points in orbits |
| torsion points on abelian varieties | ⟷ | periodic and preperiodic points of rational maps |
| abelian varieties with complex multiplication | ⟷ | postcritically finite rational maps |

**TABLE 1**
A dictionary for Arithmetic Dynamics **[82, §6.5]**

The critical points of an endomorphism $f$ of $\mathbb{P}^1$ are the points at which $f$ fails to be locally bijective. Their location crucially affects the dynamics of $f$.

**Definition 2.1.** A (separable) endomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is *postcritically finite* (PCF) if all of its critical points are preperiodic. PCF maps play a key role in the study of dynamics on $\mathbb{P}^1$.

## 3. A DICTIONARY FOR ARITHMETIC DYNAMICS

Table 1 gives three fundamental analogies that are used to travel between the worlds of arithmetic geometry and dynamical systems. The associations described in the first two lines of Table 1 are fairly tight, in the sense that they may be used to reformulate many standard results and conjectures in arithmetic geometry as dynamical statements. The following two examples illustrate these connections.

**Example 3.1.** Let $A$ be an abelian group, let $P \in A$, and let $f_P : A \to A$ be the translation-by-$P$ map, i.e., $f_P(Q) = Q + P$. Then the subgroup of $A$ generated by $P$ is the union of two orbits

$$\mathbb{Z}P = \mathcal{O}_{f_P}(0) \cup \mathcal{O}_{f_{-P}}(0).$$

More generally, for any finite set of elements $P_1, \dots, P_r \in A$, we let $\mathcal{P} = \{\pm P_1, \dots, \pm P_r\}$, and then the subgroup $\langle \mathcal{P} \rangle$ generated by $P_1, \dots, P_r$ is the generalized orbit

$$\langle \mathcal{P} \rangle = \mathcal{O}_{\mathcal{P}}(0) = \bigl\{ f_P(0) : P \in \langle \mathcal{P} \rangle \bigr\}.$$

In this way, statements about finitely generated subgroups of abelian varieties may be reformulated as statements about orbits.

**Example 3.2.** Let $G$ be a group, let $d \geq 2$, and let $f_d : G \to G$ be the $d$-power map $f_d(g) = g^d$. Then it is an easy exercise to check that

$$\operatorname{PrePer}(f) = G_{\text{tors}},$$

i.e., the elements of $G$ that are preperiodic for the $d$-power map are exactly the elements of $G$ having finite order. In this way statements about torsion points on abelian varieties may be reformulated as statements about preperiodic points for the multiplication-by-$d$ map.

**Remark 3.3.** Examples 3.1 and 3.2 help to justify the associations described in the first two lines of Table 1. The third line is a bit more nebulous. It is a rough analogy based on the following reasoning:[5]

- The CM points in the moduli space $\mathcal{A}_g$ of abelian varieties of dimension $g$ are associated to abelian varieties that have a special algebraic property, namely their endomorphism ring is unusually large. The set of CM points is a countable, Zariski-dense set of points in $\mathcal{A}_g$ whose coordinates are algebraic numbers.
- The PCF points in the moduli space $\mathcal{M}_d^1$ of endomorphisms of $\mathbb{P}^1$ are associated to maps that have a special dynamical property, namely the orbits of their critical points are unusually small. The set of PCF points is a countable, Zariski-dense set of points in $\mathcal{M}_d^1$ whose coordinates are algebraic numbers.

Section 6 describes some progress that helps to justify the third analogy in Table 1. But we must also note that the analogy is not perfect. In particular, CM abelian varieties are abundant in all dimensions, i.e., CM points are Zariski-dense in $\mathcal{A}_g$ for all $g \geq 1$. However, evidence suggests that for $N \geq 2$, PCF maps are not Zariski dense in the moduli space $\mathcal{M}_d^N$ of endomorphisms of $\mathbb{P}^N$; cf. **[34]**.

## 4. TOPIC #1: DYNAMICAL UNIFORM BOUNDEDNESS

The prototype and motivation for the dynamical uniform boundedness conjecture is the following famous theorem.

**Theorem 4.1** (**[54]**). *Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$. Then*

$$\#E(\mathbb{Q})_{\text{tors}} \leq 16.$$

**Remark 4.2.** Mazur's theorem was generalized by Kamienny **[37]** to number fields of small degree, and then by Merel **[58]**, who proved that for all number fields $K/\mathbb{Q}$ and for all elliptic curves $E/K$, there is a uniform bound

$$\#E(K)_{\text{tors}} \leq C, \quad \text{where } C \text{ depends only on the degree } [K : \mathbb{Q}].$$

A long-standing conjecture says that the same should be true for abelian varieties $A/K$ of any dimension, where the upper bound depends on $[K : \mathbb{Q}]$ and $\dim(A)$.

Using the dictionary in Table 1, the theorems of Mazur–Kamienny–Merel and the conjectural abelian variety generalization lead us to a major motivating problem in arithmetic dynamics.

5 See Section 5 for the construction of the moduli space $\mathcal{M}_d^N$.

**Conjecture 4.3** (Dynamical uniform boundedness conjecture, [62]). *Fix integers $N \geq 1$, $d \geq 2$, and $D \geq 1$. There is a constant $C(N, d, D)$ such that for all degree-$d$ morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ defined over a number field $K$ of degree $[K : \mathbb{Q}] = D$, the number of $K$-rational preperiodic points is uniformly bounded,*

$$\#\operatorname{PrePer}\bigl(f, \mathbb{P}^N(K)\bigr) \leq C(N, d, D).$$

**Remark 4.4.** See also [79] for an earlier dynamical uniform boundedness conjecture for K3 surfaces admitting noncommuting involutions.

**Remark 4.5.** Although Conjecture 4.3 only deals with preperiodic points in projective space, it can be used to prove the uniform boundedness conjecture for abelian varieties alluded to in Remark 4.2 [21].

**Remark 4.6.** Conjecture 4.3 has been generalized to cover quite general families of dynamical systems; see [72, QUESTION 3.2].

Conjecture 4.3 seems out of reach at present. Indeed, even quite special cases present challenges that have not been overcome. We briefly summarize what is known and conjectured in the simplest nontrivial case, which is quadratic polynomials over $\mathbb{Q}$.

**Theorem/Conjecture 4.7.** *For $c \in \mathbb{Q}$, let $f_c(x) = x^2 + c$.*

(a) **Theorem**. *For each $n \in \{1, 2, 3\}$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x)$ has a $\mathbb{Q}$-rational point of period $n$.*

(b) **Theorem**. *For all $c \in \mathbb{Q}$, the polynomial $f_c(x)$ does not have a $\mathbb{Q}$-rational point …*

- *of order* 4 [60];
- *of order* 5 [25];
- *of order* 6*, conditional on the Birch–Swinnerton-Dyer conjecture* [86].

(c) **Conjecture**. *For all $n \geq 4$, the polynomial $f_c(x)$ does not have a $\mathbb{Q}$-rational point of period $n$; see* [91][6] *and* [25].

**Remark 4.8.** Just as there are elliptic modular curves $X_1^{\mathrm{ell}}(n)$ whose points classify pairs $(E, P)$ consisting of an elliptic curve $E$ and an $n$-torsion point $P$, there are so-called dynatomic modular curves $X_1^{\mathrm{dyn}}(n)$ whose points classify pairs $(c, \alpha)$ such that $\alpha$ is a point of period $n$ for the polynomial $f_c(x) = x^2 + c$. Mazur's method for proving Theorem 4.1 is to show that $X_1^{\mathrm{ell}}(n)$ has no (noncuspidal) $\mathbb{Q}$-rational points by mapping $X_1^{\mathrm{ell}}(n)$ into a carefully chosen quotient $A$ of its Jacobian variety and showing that the group $A(\mathbb{Q})$ is

6 Although in fairness it should be noted that [91] suggests the opposite conclusion, stating: "Are there any rational periodic orbits of a quadratic $x^2 + c$ of period greater than 3? The results for periods 1, 2, and 3 would lead one to suspect that there must be."

finite. The proof of Theorem 4.7 starts similarly using $X_1^{\mathrm{dyn}}(n)$ instead of $X_1^{\mathrm{ell}}(n)$, but in this situation, the Jacobian generally does not have a quotient whose group of rational points is finite. Current methods, such as Chabauty–Coleman, for explicitly determining the rational points on curves of high genus (barely) suffice to handle $X_1^{\mathrm{dyn}}(n)$ for $n \le 6$. The difficulty, or more concretely the difference, between the elliptic curve and dynamical settings centers around the lack of a theory of Hecke correspondences in the dynamical case. (Mea culpa: This simplified explanation is not entirely accurate, but it is meant to convey the overall strategy of the proofs.)

**Remark 4.9.** Contingent on an appropriate version of the $abcd$-conjecture, the uniform boundedness conjecture has been proven for the family of polynomials $x^d + c$ [47], and more recently for all polynomials [46]. An alternative proof, also using the $abc$-conjecture and only valid over $\mathbb{Q}$, says that if $d$ is sufficiently large and $c \neq -1$, then $x^d + c$ has no $\mathbb{Q}$-rational periodic points other than fixed points [68].[7]

**Remark 4.10.** A function field analogue of the uniform boundedness conjecture for $x^d + c$ is proven in [17,18]. In the function field setting, the uniformity in the degree $[K:\mathbb{Q}]$ described in Conjecture 4.3 is replaced by a bound that depends on the gonality[8] of the field extension.

## 5. TOPIC #2: DYNAMICAL MODULI SPACES

We fix a field $K$ and consider parameter and moduli spaces for the set of rational self-maps of $\mathbb{P}_K^N$. A rational map $f : \mathbb{P}_K^N \dashrightarrow \mathbb{P}_K^N$ of degree-$d$ is specified by an $(N+1)$-tuple of degree-$d$ homogeneous polynomials,

$$f = [f_0, \ldots, f_N], \quad f_0, \ldots, f_N \in K[X_0, \ldots, X_n],$$

such that $f_0, \ldots, f_N$ have no common factors. The map $f$ is a morphism if $f_0, \ldots, f_N$ have no common roots in $\mathbb{P}^N(\bar{K})$. We label the coefficients of $f_0, \ldots, f_N$ in some specified order as $a_1(f), a_2(f), \ldots, a_\nu(f)$, where $\nu = \nu(N,d) := \binom{N+d}{d}(N+1)$. Then each such $f$ determines a point

$$f = \big[a_1(f), \ldots, a_\nu(f)\big] \in \mathbb{P}^{\nu-1}.$$

There is a homogeneous polynomial $\mathcal{R} \in \mathbb{Z}[a_1, \ldots, a_\nu]$ called the Macaulay resultant having the property that

$$f = [f_0, \ldots, f_N] \text{ is a morphism} \iff \mathcal{R}\big(a_1(f), \ldots, a_\nu(f)\big) \neq 0.$$

The parameter space of degree-$d$ endomorphisms of $\mathbb{P}^N$ is

$$\mathrm{End}_d^N = \big\{ f \in \mathbb{P}^{\nu-1} : \mathcal{R}(f) \neq 0 \big\}.$$

7 We remark that it is easy to prove uniform boundedness for $x^d + c$ over $\mathbb{Q}$ when $d$ is odd, and more generally over any field $K/\mathbb{Q}$ with a real embedding. Indeed, it is an elementary fact that if $f : \mathbb{R} \to \mathbb{R}$ is any nondecreasing function, then $f$ has no nonfixed periodic points; cf. [64].

8 The *gonality* of an algebraic curve $X$, or its function field, is the minimal degree of a non-constant map $X \to \mathbb{P}^1$.

The isomorphism class of dynamical systems associated to $f$ is the set of all conjugates, i.e., the set of all

$$f^{\varphi} = \varphi^{-1} \circ f \circ \varphi, \quad \text{where } \varphi \in \operatorname{Aut}(\mathbb{P}^N) = \operatorname{PGL}_{N+1}.$$

Conjugation gives an algebraic action of $\operatorname{PGL}_{N+1}$ on the parameter space $\operatorname{End}_d^N$ via

$$\operatorname{PGL}_{N+1} \times \operatorname{End}_d^N \to \operatorname{End}_d^N, \quad (\varphi, f) \mapsto f^{\varphi}, \tag{5.1}$$

and this action extends naturally to $\mathbb{P}^{\nu}$.

**Definition 5.1.** The *moduli space of degree-d dynamical systems on* $\mathbb{P}^N$ is the quotient space of $\operatorname{End}_d^N$ for the conjugation action (5.1),

$$\mathcal{M}_d^N = \operatorname{End}_d^N / \operatorname{PGL}_{N+1}. \tag{5.2}$$

It is natural to ask whether the quotient (5.2) can be given some nice sort of structure. Geometric invariant theory (GIT) [63] provides a powerful tool for studying quotients of a variety (or scheme) $X$ by an infinite algebraic group $G$. GIT says that there are stable and semistable loci $X^{\mathrm{s}} \subseteq X^{\mathrm{ss}} \subseteq X$ such that there exist quotient varieties (or schemes) $X^{\mathrm{s}}/\!/G$ and $X^{\mathrm{ss}}/\!/G$ having many agreeable properties.[9]

**Theorem 5.2.** *Let $N \geq 1$ and $d \geq 2$.*

(a) *The quotient space $\mathcal{M}_d^N(\mathbb{C}) = \operatorname{End}_d^N(\mathbb{C})/\operatorname{PGL}_{N+1}(\mathbb{C})$ has a natural structure as an orbifold over $\mathbb{C}$* [59].

(b) *The quotient space $\mathcal{M}_d^N = \operatorname{End}_d^N / \operatorname{PGL}_{N+1}$ has a natural structure as a GIT quotient scheme over $\mathbb{Z}$; see* [80] *for $N = 1$ and* [44,69] *for $N \geq 1$.*[10]

It is clear that $\mathcal{M}_d^N$ is unirational, i.e., it is rationally finitely covered by a projective space, since $\operatorname{End}_d^N$ is itself an open subset of a projective space. A subtler question is whether $\mathcal{M}_d^N$ is rational.

**Theorem 5.3.** *Let $d \geq 2$.*

(a) *There is an isomorphism $\mathcal{M}_2^1 \cong \mathbb{A}^2$, and the semi-stable GIT compactification of $\mathcal{M}_2^1$ as the quotient of the semi-stable locus in $\mathbb{P}^5$ is isomorphic to $\mathbb{P}^2$* [59,80].

(b) *The space $\mathcal{M}_d^1$ is rational, i.e., there exists a birational map $\mathbb{P}^{2d-2} \dashrightarrow \mathcal{M}_d^1$* [44].

**Question 5.4.** Is $\mathcal{M}_d^N$ rational for all $d \geq 2$ and all $N \geq 1$?

9 For example, over $\mathbb{C}$ the stable GIT quotient satisfies $(X^{\mathrm{s}}/\!/G)(\mathbb{C}) = X^{\mathrm{s}}(\mathbb{C})/G(\mathbb{C})$, i.e., the geometric points of the stable quotient $X^{\mathrm{s}}/\!/G$ are the $G(\mathbb{C})$-orbits of the geometric points of $X$. And the semistable GIT quotient has the property that $(X^{\mathrm{ss}}/\!/G)(\mathbb{C})$ is proper, i.e., it is compact, so it provides a natural compactification of the stable quotient.

10 More precisely, the parameter space $\operatorname{End}_d^N$ is in the GIT stable locus for the action of $\operatorname{SL}_{N+1}$ on $\mathbb{P}^{\nu}$ linearized relative to the line bundle $\mathcal{O}_{\mathbb{P}^{\nu}}(1)$, and thus the quotient $\mathcal{M}_d^N$ exists as a GIT quotient scheme over $\mathbb{Z}$.

Just as is done with the moduli space of abelian varieties, it is advantageous to add level structure to dynamical moduli spaces by specifying maps together with points or cycles of various shapes. We start with the case of a single periodic point, and then consider more complicated level structures.

**Definition 5.5.** For $N \geq 1$, $n \geq 1$, and $d \geq 2$, we write

$$\mathrm{End}_d^N[n] = \{(f, P) \in \mathrm{End}_d^N \times \mathbb{P}^N : P \text{ has exact } f\text{-period } n\}.$$

Thus the points of $\mathrm{End}_d^N[n]$ classify maps with a marked point of exact period $n$.

More generally, we define a (*preperiodic*) *portrait* $\mathcal{P}$ to be the directed graph of a self-map of a finite set of points. (See Figure 1 for an example of a portrait.) Then for a portait $\mathcal{P}$ having $k$ vertices, we let[11]

$$\mathrm{End}_d^N[\mathcal{P}] = \left\{ (f, P_1, \dots, P_k) \in \mathrm{End}_d^N \times (\mathbb{P}^N)^k : \begin{array}{l} P_1, \dots, P_k \text{ are } f\text{-preperiodic and} \\ f : \{P_1, \dots, P_k\} \to \{P_1, \dots, P_k\} \\ \text{is a model for the portrait } \mathcal{P} \end{array} \right\}.$$

There is a natural action of $\varphi \in \mathrm{PGL}_{N+1}$ on $\mathrm{End}_d^N[\mathcal{P}]$ given by

$$(f, P_1, \dots, P_k)^\varphi = \left(f^\varphi, \varphi^{-1}(P_1), \dots, \varphi^{-1}(P_k)\right).$$

We denote the resulting quotient space by

$$\mathcal{M}_d^N[\mathcal{P}] = \mathrm{End}_d^N[\mathcal{P}] / \mathrm{PGL}_{N+1}. \tag{5.3}$$

If $\mathcal{C}_n$ is a portrait consisting of a single $n$-cycle, then $\mathrm{End}_d^N[\mathcal{C}_n] \cong \mathrm{End}_d^N[n]$, and we write $\mathcal{M}_d^N[n]$ for $\mathcal{M}_d^N[\mathcal{C}_n]$.

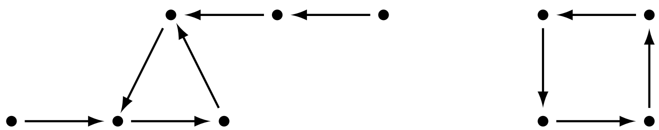

**FIGURE 1**
A portrait consisting of a 3-cycle, a 4-cycle, and three other preperiodic points

**Theorem 5.6** ([20]). *Let $\mathcal{P}$ be a preperiodic portrait.*[12] *Then the quotient space $\mathcal{M}_d^N[\mathcal{P}]$ described in (5.3) exists*[13] *as a GIT geometric quotient scheme over $\mathbb{Z}$.*

**11** This definition of $\mathrm{End}_d^N[\mathcal{P}]$ conveys the right idea; see [20] for a rigorous definition.
**12** More generally, one can construct the moduli space $\mathcal{M}_d^N[\mathcal{P}]$ associated to a portrait $\mathcal{P}$ that includes nonpreperiodic points and/or whose vertices are assigned multiplicities.
**13** There is a precise combinatorial-geometric characterization of the portraits $\mathcal{P}$ for which $\mathcal{M}_d^1[\mathcal{P}](\mathbb{C}) \neq \emptyset$, but analogous characterizations for $N \geq 2$ and/or in positive characteristic are not currently known.

It is known [87] that the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties is of general type for all $g \geq 9$. Analogous results for dynamical moduli spaces are still unknown, but our dictionary yields some conjectures.[14]

**Conjecture 5.7.** *Let $N \geq 1$ and $d \geq 2$.*

(a) *For all $n \geq 1$, the moduli space $\mathcal{M}_d^N[n]$ is irreducible.*

(b) *For all sufficiently large $n$, depending on $N$ and $d$, the moduli space $\mathcal{M}_d^N[n]$ is a variety of general type.*

**Remark 5.8.** The moduli space $\mathcal{M}_2^1[n]$ of degree-2 endomorphisms of $\mathbb{P}^1$ is a finite cover of $\mathcal{M}_2^1 \cong \mathbb{A}^2$, so it is a surface. It is known to be irreducible for all $n \geq 1$ [48]. For $1 \leq n \leq 5$, the surface $\mathcal{M}_2^1[n]$ is a rational surface, while $\mathcal{M}_2^1[6]$ is a surface of general type [12].

**Remark 5.9.** Tai's proof [87] that $\mathcal{A}_g$ is of general type relies on the theory of theta functions, which are used to create sections of the canonical bundle. There are similarly naturally defined functions on $\mathcal{M}_d^N$, and more generally on $\mathcal{M}_d^N[n]$, that are defined using multiplier systems.[15] For $N = 1$, it is known that a multiplier system of sufficiently high degree gives a map $\mathcal{M}_d^1 \to \mathbb{A}^r$ that is (essentially) finite-to-one[16] onto its image [55]. So although the analogy between theta functions on $\mathcal{A}_g$ and multiplier system functions on $\mathcal{M}_d^N$ is tenuous at best, the latter currently provide one of the most natural ways to create dynamically defined functions on dynamical moduli spaces.

A map $f \in \mathrm{End}_d^N(K)$ defined over $K$ with a $K$-rational $n$-periodic point $P \in \mathbb{P}^N(K)$ gives a $K$-rational point $\langle f, P\rangle \in \mathcal{M}_d^N[n](K)$. The dynamical uniform boundedness conjecture (Conjecture 4.3) is thus closely related to the question of $K$-rational points on dynamical moduli spaces. We formulate a uniform boundedness conjecture for such spaces.

**Conjecture 5.10** (Dynamical uniform boundedness conjecture: version 2). *Fix integers $N \geq 1$, $d \geq 2$, and $D \geq 1$. There is a constant $C'(N, d, D)$ such that for all number fields $K$ of degree $[K : \mathbb{Q}] = D$ and all preperiodic portraits $\mathcal{P}$,*

$$\big(\#\{\text{vertices of } \mathcal{P}\} \geq C'(N, d, D)\big) \Longrightarrow \mathcal{M}_d^N[\mathcal{P}](K) = \emptyset.$$

14 See [10, CONJECTURE 10.13] for a generalization of Conjecture 5.7 that deals with quite general dynamical moduli spaces that classify families of maps with marked periodic points of large period, including bounds on their number of components, Kodaira dimension, and gonality.

15 Briefly, for $N = 1$, let $k \geq 1$, let $f \in \mathrm{End}_d^1$, and let $P_1, \ldots, P_r$ be the periodic points of $f$ with period dividing $k$. The derivatives $(f^k)'(P_i)$ are $\mathrm{PGL}_2$-conjugate independent, and the $k$-level multiplier system of $f$ is the list $\Lambda_k(f)$ of the elementary symmetric functions of $(f^k)'(P_1), \ldots, (f^k)'(P_r)$. Then $\Lambda_k(f)$ gives a well-defined morphism $\Lambda_k(f) : \mathcal{M}_d^1 \to \mathbb{A}^r$.

16 More precisely, the map is finite-to-one unless $n$ is a square, in which case it maps the $j$-line of flexible Lattès maps to a single point. This is thus one of those results that's "true except in the obvious cases where it is false."

**Remark 5.11.** It is clear that Conjecture 5.10 implies Conjecture 4.3. The opposite implication is also true, but the proof is more difficult due to the *Field-of-Moduli versus Field-of-Definition Problem*. The key step, proven in **[19]** and **[20, SECTIONS 16–17]**, is to show that every point in $\mathcal{M}_d^N[\mathcal{P}](K)$ is represented by a point in $\operatorname{End}_d^N[\mathcal{P}](L)$ defined over an extension $L/K$ whose degree is bounded solely by $d$ and $N$. When $N = 1$, one can take $[L : K] \le 2$ **[32]**, but for $N \ge 2$ it is an open question whether $[L : K]$ needs to depend on $d$.

Within $\mathcal{M}_d^N$ and its GIT semistable compactification $\overline{\mathcal{M}}_d^N$ lie many interesting subvarieties. For example:

- The space of polynomial maps[17]
$$\operatorname{Poly}_d^N = \{f \in \mathcal{M}_d^N : f \text{ comes from a morphism } \mathbb{A}^N \to \mathbb{A}^N\}$$
is a subvariety of $\mathcal{M}_d^N$ satisfying $\dim(\operatorname{Poly}_d^1) = \frac{N}{N+1}\dim(\mathcal{M}_d^N)$.
- Iteration of dominant rational maps presents its own interesting challenges; see Section 8 for some examples. The set of degree $d$ dominant rational maps $\mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a Zariski open subvariety of $\mathbb{P}^{\nu-1}(\mathbb{C})$ **[81, PROPOSITION 7]**, but the locus of points in $(\overline{\mathcal{M}}_d^N \smallsetminus \mathcal{M}_d^N)(\mathbb{C})$ arising from dominant rational maps is not well understood; cf. **[42]**.

The spaces of polynomial maps and dominant rational maps have large dimension. At the other extreme are various 1-parameter families of maps that have been much studied, starting with the ubiquitous family of quadratic polynomials
$$f_c(x) = x^2 + c$$
that gives a line $\mathbb{A}^1$ in $\mathcal{M}_2^1 \cong \mathbb{A}^2$. Adding level structure leads to a dynamical analogue of the classical elliptic modular curve $X_1^{\mathrm{ell}}(n)$ that classifies pairs $(E, P)$ consisting of an elliptic curve $E$ and an $n$-torsion point $P$. In the dynamical setting, we replace the $n$-torsion point with a point of period $n$, but the following example shows that some care is needed.

**Example 5.12.** The polynomial $f(x) = x^2 - \frac{3}{4}$ has no points of exact period 2, since
$$f(x) - x = (2x+1)(2x-3) \quad \text{and} \quad f^2(x) - x = (2x+1)^3(2x-3).$$
But since $\frac{f^2(x)-x}{f(x)-x} = (2x+1)^2$, we say that $x = -\frac{1}{2}$ is a point of *formal period* 2 for $f(x)$.[18]

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**17** For example, the space $\operatorname{Poly}_d^1 \subset \mathcal{M}_d^1$ is the space of polynomials $x^d + a_2x^{d-2} + \cdots + a_d$ modulo the conjugation $x \to \zeta x$ for a primitive $(d-1)$-root of unity $\zeta$, so $\operatorname{Poly}_d^1$ is a quotient of $\mathbb{A}^{d-1}$ by a finite group.

**18** In general, points of *formal period* $n$ for the polynomial $f(x)$ are roots of the *dynatomic polynomial*
$$\Phi_f(x) := \prod_{d|n} (f^d(x) - x)^{\mu(n/d)},$$
where $\mu$ is the Möbius function. Dynatomic polynomials are thus dynamical analogues of classical cyclotomic polynomials, but with the caveat that $\Phi_f(x)$ may have roots of multiplicity greater than 1, even in characteristic 0. In higher dimension, the points of formal period $n$ give a *dynatomic* 0-*cycle* whose effectivity is proven in **[33]**.

**Definition 5.13.** The *level n dynatomic curve*[19] (*associated to* $x^2+c$) is the affine curve

$$Y_1^{\mathrm{dyn}}(n) = \{(c,\alpha) \in \mathbb{A}^2 : \alpha \text{ is a point of formal period } n \text{ for } f_c(x) = x^2 + c\}.$$

The desingularized projective completion of $Y_1^{\mathrm{dyn}}(n)$ is denoted $X_1^{\mathrm{dyn}}(n)$. The points in the complement $X_1^{\mathrm{dyn}}(n) \setminus Y_1^{\mathrm{dyn}}(n)$, which correspond to degenerate maps, are called *cusps*.[20]

Points in $Y_1^{\mathrm{dyn}}(n)(K)$ classify quadratic polynomials defined over $K$ having a $K$-rational point of period $n$, so a version of Theorem/Conjecture 4.7 says that

$$\begin{array}{rll} \textbf{Exercise} & X_1^{\mathrm{dyn}}(n) \cong \mathbb{P}^1 & \text{for } n \in \{1,2,3\}, \\ \textbf{Theorem} & X_1^{\mathrm{dyn}}(n)(\mathbb{Q}) = \{\text{cusps}\} & \text{for } n \in \{4,5,6\}, \\ \textbf{Conjecture} & X_1^{\mathrm{dyn}}(n)(\mathbb{Q}) = \{\text{cusps}\} & \text{for all } n \geq 4. \end{array}$$

Much is known about the geometry of $X_1^{\mathrm{dyn}}(n)$, as summarized in the next result, although we note that even the proof that $X_1^{\mathrm{dyn}}(n)$ is geometrically irreducible relies on dynamical properties of $x^2+c$ as reflected in the geometry of the Mandelbrot set.

**Theorem 5.14.** *Let $X_1^{\mathrm{dyn}}(n)$ be the smooth projective dynatomic curve associated to $x^2+c$.*

(a) *The dynatomic modular curve $X_1^{\mathrm{dyn}}(n)$ is geometrically irreducible over $\mathbb{C}$* **[13, 41, 76]**.[21]

(b) *There is an explicit, but rather complicated, formula for the genus of $X_1^{\mathrm{dyn}}(n)$* **[61]**. *In any case,* $\mathrm{genus}(X_1^{\mathrm{dyn}}(n)) \to \infty$ *as $n \to \infty$.*

(c) *The gonality*[22] *of $X_1^{\mathrm{dyn}}(n)$ goes to $\infty$ as $n \to \infty$* **[18]**.

## 6. TOPIC #3: DYNAMICAL UNLIKELY INTERSECTIONS

The guiding philosophy of unlikely intersections in arithmetic geometry is the following general, albeit somewhat vague, principle.

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**19** There are dynatomic curves associated to many other interesting 1-parameter families of maps, including, for example, the family of degree-$d$ *unicritical polynomials* $f_{d,c}(x) = x^d + c$ and the family of degree-2 rational maps $g_b(x) = x/(x^2+b)$ that admit a nontrivial automorphism $g_b(-x) = -g_b(x)$.

**20** We mention that there is a natural action of $f$ on $Y_1^{\mathrm{dyn}}(n)$ defined by $(c,\alpha) \mapsto (c, f(\alpha))$, and that the quotient curve $Y_0^{\mathrm{dyn}}(n) = Y_1^{\mathrm{dyn}}(n)/\langle f \rangle$ and its completion $X_0^{\mathrm{dyn}}(n)$ provide analogues of the elliptic modular curve $X_0^{\mathrm{ell}}(n)$.

**21** More generally, the dynatomic modular curves associated to the family of unicritical polynomials $x^d + c$ are irreducible. However, the dynatomic modular curves associated to the family $x/(x^2+b)$ turn out to be reducible for even $n$; see **[48]**.

**22** The *gonality* of an algebraic curve $X$ is the minimal degree of a nonconstant map $X \to \mathbb{P}^1$.

**The Tao of Unlikely Intersections**
Let $X$ be an algebraic variety, let $Y \subseteq X$ be an algebraic subvariety of $X$, and let $\Gamma \subset X$ be an "interesting" countable subset of $X$. Then

$$\Gamma \cap Y \text{ is sparse (except when it is "obviously" not).}$$

Slightly more precisely, if $\Gamma \cap Y$ is Zariski dense in $Y$, then there should be a geometric reason that explains its density.

We recall two famous unlikely intersection theorems from arithmetic geometry, which we initially state in an intuitively appealing, though somewhat whimsical, manner.

**Theorem 6.1** (Mordell–Lang conjecture, [23,90]). *Let $A/\mathbb{C}$ be an abelian variety, let $Y \subseteq A$ be a subvariety of $A$, and let $\Gamma \subset A(\mathbb{C})$ be a finitely generated subgroup of $A$. Then*[23]

$$\Gamma \cap Y \textit{ is not Zariski dense in } Y \textit{ (except when it "obviously" is).}$$

**Theorem 6.2** (Manin–Mumford conjecture, [73,74]). *Let $A/\mathbb{C}$ be an abelian variety, and let $Y \subseteq A$ be a subvariety of $A$. Then*

$$A_{\text{tors}} \cap Y \textit{ is not Zariski dense in } Y \textit{ (except when it "obviously" is).}$$

The actual statements of Theorems 6.1 and 6.2 explain quite precisely that if $Y$ is saturated with special points, then there is a geometric reason for that saturation.

**Theorem 6.3** (Rigorous formulation of Theorems 6.1 and 6.2). *If $\Gamma \cap Y$ or $A_{\text{tors}} \cap Y$ is Zariski dense in $Y$, then $Y$ is necessarily a translate of an abelian subvariety of $A$ by a torsion point of $A$.*

**Remark 6.4.** Theorems 6.1 and 6.2 may be combined and strengthened by replacing the abelian variety $A$ with a semi-abelian variety and by replacing $\Gamma$ with its divisible subgroup $\bigcup_{n\geq 0}[n]^{-1}(\Gamma)$; see [56].

Theorem 6.1 says that points in a finitely generated subgroup $\Gamma$ generally do not lie on a subvariety. According to Table 1, for the dynamical analogue of Theorem 6.1 we should replace the group $\Gamma$ with the points in an orbit. This leads to our first dynamical unlikely intersection conjecture.

**Conjecture 6.5** (Dynamical Mordell–Lang conjecture). *Let $X/\mathbb{C}$ be a smooth quasiprojective variety, let $f : X \to X$ be a regular self-map of $X$, let $P \in X(\mathbb{C})$ be a point with infinite $f$-orbit, and let $Y \subseteq X$ be a subvariety of $X$. Then*

$$\mathcal{O}_f(P) \cap Y \textit{ is not Zariski dense in } Y \textit{ (except when it "obviously" is).}$$

***Rigorous Formulation* #1.** *If $\mathcal{O}_f(P) \cap Y$ is Zariski dense, then $Y$ is $f$-periodic.*[24]

23 The proof of Theorem 6.1 uses methods from Diophantine approximation. An earlier proof in the case that $Y$ is a curve of genus at least 2 used moduli-theoretic techniques [22].

24 We says that $Z$ is *$f$-periodic* if there is an integer $n > 0$ such that $f^n(Z) = Z$.

***Rigorous Formulation #2.*** *The set*

$$\{n \geq 0 : f^n(P) \in Y\}$$

*is a finite union of one-sided arithmetic progressions* [29].[25]

**Example 6.6.** Among the known cases of the dynamical Mordell–Lang conjecture, we cite the following:

> **Unramified maps.** Conjecture 6.5 is true for étale morphisms of quasiprojective varieties [7]. See the monograph [8] for additional information.
>
> **Endomorphisms of $\mathbb{A}^2$.** Conjecture 6.5 is true for all endomorphisms of $\mathbb{A}^2$ defined over $\bar{\mathbb{Q}}$ [92].
>
> **Split endomorphisms.** Conjecture 6.5 is true for split endomorphisms of $(\mathbb{P}^1)^n$, which are maps of the form $f_1(P_1) \times \cdots \times f_n(P_n)$ [11], and more generally for certain skew-split endomorphisms [31].

**Remark 6.7.** The dynamical Mordell–Lang conjecture has also been investigated in characteristic $p$, although the statement may need a tweak. For example, if $f$ is a projective surface automorphism or a birational endomorphism of $\mathbb{A}^2$ whose dynamical degree (see Section 8) satisfies $\delta_f > 1$, then Conjecture 6.5 is true in all characteristics [94]. For other results in finite characteristic, see, for example, [8, 14, 26].

We now turn to Theorem 6.2, which asserts that torsion points generally do not lie on a subvariety. According to Table 1, for the dynamical analogue we should replace the torsion points with preperiodic points, leading to our second dynamical unlikely intersection conjecture.

**Conjecture 6.8** (Dynamical Manin–Mumford conjecture). *Let $X/\mathbb{C}$ be a smooth quasi-projective variety, let $f : X \to X$ be a regular self-map of $X$, and let $Y \subseteq X$ be a subvariety of $X$. Then*

$$\operatorname{PrePer}(f) \cap Y \text{ is not Zariski dense in } Y \text{ (except when it “obviously” is).}$$

Unfortunately, the following natural rigorous formulation of Conjecture 6.8 turns out to be false.

| |
|---|
| ~~**Incorrect Rigorous Formulation of Conjecture 6.8.**~~ |
| ~~If $\operatorname{PrePer}(f) \cap Y$ is Zariski dense in $Y$, then $Y$ is $f$-preperiodic.~~ |

See [30] for a counterexample, and for an alternative formulation of Conjecture 6.8 that requires more stringent hypotheses on $f$ and $Y$.

Both the Mumford–Manin and Mordell–Lang conjectures concern how special points lie on subvarieties of a given variety. The André–Oort conjecture has a similar flavor,

25 A *one-sided arithmetic progression* is a set of integers of the form $\{ak + b : k \in \mathbb{N}\}$ for some fixed $a, b \in \mathbb{N}$. N.B. We allow $a = 0$.

but the ambient variety is a moduli space and the specialness of the points comes from the properties of the objects that they represent. The André–Oort conjecture is easy to state as long as we are willing to sweep some quite technical definitions under the rug![26]

**Conjecture 6.9** (André–Oort conjecture). *Let $\mathcal{S}$ be a Shimura variety, let $\Gamma \subset \mathcal{S}$ be a set of special points of $\mathcal{S}$, and let $Y \subset \mathcal{S}$ be an irreducible subvariety such that $\Gamma \cap Y$ is Zariski dense in $Y$. Then $Y$ is a special subvariety of $\mathcal{S}$.*

The rough idea is that $\mathcal{S}$ is a moduli space whose points classify a certain class of abelian varieties, a collection of special points $\mathcal{T} \subset \mathcal{S}$ consists of points whose associated abelian varieties have an additional special structure, and a special subvariety is one in which every associated abelian variety has the $\mathcal{T}$ property for geometric reasons. The André–Oort conjecture has been proven in many cases, including for $\mathcal{S} = \mathcal{A}_1^d$ [70] and for $\mathcal{S} = \mathcal{A}_g$ [71,88].

We describe two sample dynamical unlikely intersection theorems that take place in the moduli space of unicritical polynomials, which are polynomials of the form $x^d + c$. We view the first as a mixed unlikely intersection, because it involves one moduli parameter and two orbit parameters.

**Theorem 6.10** ([3]). *Let $d \geq 2$, and let $a, b \in \mathbb{C}$ be complex numbers with $a^2 \neq b^2$. Then*

$$\Big\{ \underbrace{c \in \mathbb{C}}_{\text{moduli parameter}} : \underbrace{a \text{ and } b \text{ are both preperiodic}}_{\text{special orbit parameters}} \text{ for } x^d + c \Big\} \quad \text{is a finite set.}$$

The second result has more of the flavor of the André–Oort conjecture in that it involves only moduli parameters and follows the dictionary in Table 1 by replacing complex multiplication abelian varieties with postcritically finite rational maps.

**Theorem 6.11** ([28]). *Let $d \geq 2$, and let $Y \subset \mathbb{A}^2$ be an irreducible curve that is not a line of one of the following forms*:

$$\text{vertical line } \{(a,t) : t \in \mathbb{A}^1\}; \quad \text{horizontal line } \{(t,b) : t \in \mathbb{A}^1\};$$
$$\text{shifted diagonal line } \{(t,\zeta t) : t \in \mathbb{A}^1\}, \text{ where } \zeta^{d-1} = 1.$$

*Then*

$$\underbrace{\Big\{(a,b) \in Y : x^2 + a \text{ and } x^2 + b \text{ are both PCF}\Big\}}_{\text{special moduli parameters}} \quad \text{is a finite set.}$$

A conjectural generalization of Theorem 6.10 allows both the map $x^2 + c$ and the points $a$ and $b$ to vary simultaneously.

**Conjecture 6.12** ([15,27]). *Let $d \geq 2$, let $T$ be an irreducible curve, and let*

$$\alpha : T \to \mathbb{P}^1, \quad \beta : T \to \mathbb{P}^1, \quad \text{and} \quad f : T \to \mathrm{End}_d^1$$

*be morphisms, i.e., $\alpha$ and $\beta$ are 1-parameter families of points in $\mathbb{P}^1$ and $f$ is a 1-parameter*

26 See, for example, [89] for the precise definition of *Shimura variety*, *special point*, and *special subvariety*.

*family of degree-d endomorphisms of* $\mathbb{P}^1$. *Assume that the families* $\alpha$ *and* $\beta$ *are not* $f$*-dynamically related.*[27] *Then*

$$\{t \in T : \alpha_t \text{ and } \beta_t \text{ are both preperiodic for } f_t\} \quad \text{is a finite set.}$$

Formulating a general dynamical André–Oort conjecture is more complicated. The first step is to construct an appropriate moduli space of rational maps with marked critical points:[28]

$$\mathcal{M}_d^{\text{crit}} := \left\{ (f, P_1, \ldots, P_{2d-2}) : \begin{array}{l} f \in \text{End}_d^1 \text{ and } P_1, \ldots, P_{2d-2} \\ \text{are critical points of } f \end{array} \right\} / \text{PGL}_2 .$$

**Conjecture 6.13** (Dynamical André–Oort Conjecture, **[4,82]**). *Let* $Y \subseteq \mathcal{M}_d^{\text{crit}}$ *be an algebraic subvariety such that the PCF maps in* $Y$ *are Zariski dense in* $Y$. *Then* $Y$ *is cut out by "*critical orbit relations.*"*

Formulas of the form $f^n(P_i) = f^m(P_j)$ define critical point relations,[29] but other relations may arise from symmetries of $f$, and even subtler relations may come from "hidden relations" due to subdynamical systems. See **[82, REMARK 6.58]** for an example due to Ingram. Thus for now we do not have a good geometric description of the phrase "critical orbit relations" in general, but there is such a description for 1-dimensional families, i.e., for Conjecture 6.13 with $\dim(Y) = 1$ **[4]**. In this case the conjecture has been proven for 1-dimensional families of polynomials **[24]**, but it remains open for rational maps.

## 7. TOPIC #4: DYNATOMIC AND ARBOREAL REPRESENTATIONS

The focus of this section is on the arithmetic of fields generated by the coordinates of dynamically interesting points. We let $K/\mathbb{Q}$ be a number field, and we start with a motivating result from arithmetic geometry. Let $E/K$ be an elliptic curve, and let

$$\rho_{E/K,\ell}^{\text{ell}} : \text{Gal}(\bar{K}/K) \to \text{Aut}\big(T_\ell(E)\big) \cong \text{GL}_2(\mathbb{Z}_\ell) \tag{7.1}$$

be the representation that describes the action of the Galois group on the $\ell$-power torsion points of $E$. A famous theorem characterizes the image.[30]

**27** Intuitively, the families $\alpha$ and $\beta$ are $f$-dynamically related if there is a relationship between the $f$-orbits of $\alpha$ and $\beta$ that holds identically for all parameter values in $T$. However, there are some subtleties; see **[10, DEFINITION 11.2]** for a discussion and the precise, albeit somewhat technical, definition.

**28** It is easy to construct the GIT quotient for maps $f$ having $2d - 2$ *distinct* marked critical points, but some care is needed to handle maps having higher multiplicity critical points; see **[20]**.

**29** One might view these $f^n(P_i) = f^m(P_j)$ relations as dynamical analogues of Hecke correspondences, although the analogy is somewhat tenuous.

**30** A 19th century precursor to Serre's theorem is a fundamental result on cyclotomic fields. It says that the cyclotomic representation $\rho^{\text{cyclo}} : \text{Gal}(\bar{K}/K) \to \text{Aut}(\mu_{\ell^\infty}) \cong \mathbb{Z}_\ell^*$ describing the action of the Galois group on $\ell$-power roots of unity is surjective when $K = \mathbb{Q}$, and that the image of $\rho^{\text{cyclo}}$ has finite index in $\mathbb{Z}_\ell^*$ for all $K$.

**Theorem 7.1** (Serre's Image-of-Galois Theorem, [77, 78]). *Assume that $E$ does not have complex multiplication.*

(a) *For all sufficiently large primes $\ell$, the Galois representation $\rho^{\mathrm{ell}}_{E/K,\ell}$ is surjective.*

(b) *For all primes $\ell$, the image of the Galois representation $\rho^{\mathrm{ell}}_{E/K,\ell}$ is a subgroup of finite index in $\mathrm{GL}_2(\mathbb{Z}_\ell)$.*

There are analogous conjectures, and some theorems, for the Galois representations associated to higher-dimensional abelian varieties. We consider two analogues in arithmetic dynamics.

### 7.1. Topic #4(a): Dynatomic representations

Let

$$f : \mathbb{P}^N \to \mathbb{P}^N$$

be a morphism of degree $d \geq 2$ defined over $K$, and let

$$\mathrm{Per}^*_n(f) = \{P \in \mathbb{P}^N(\bar{K}) : P \text{ is } f\text{-periodic with exact period } n\}.$$

The action of $f$ on $\mathrm{Per}^*_n(f)$ splits it into a disjoint union of directed $n$-cycles, and the action of $\mathrm{Gal}(\bar{K}/K)$ on $\mathrm{Per}^*_n(f)$ respects the cycle structure. The analogue of $\mathrm{GL}_2$ in (7.1) is thus the group of automorphisms of the graph

$$\mathcal{P}_{n,\nu} = \text{a disjoint union of } \nu \text{ directed } n\text{-gons}.$$

The abstract automorphism group of the directed graph $\mathcal{P}_{n,\nu}$ is naturally described as a wreath product in which an automorphism of $\mathcal{P}_{n,\nu}$ is characterized as a permutation of the $\nu$ polygons combined with a rotation of each polygon:

$$\mathrm{Aut}(\mathcal{P}_{n,\nu}) \cong (\mathbb{Z}/n\mathbb{Z}) \wr \mathcal{S}_\nu \cong (\mathbb{Z}/n\mathbb{Z})^\nu \rtimes \mathcal{S}_\nu.$$

**Definition 7.2.** Let $f \in \mathrm{End}^d_N(K)$. The *$n$-level dynatomic representation of $f$ over $K$* is the homomorphism

$$\rho^{\mathrm{dyn}}_{K,n,f} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}(\mathcal{P}_{n,\nu(f)}), \quad \text{where } \nu(f) = \frac{1}{n}\#\mathrm{Per}^*_n(f).$$

The analogue of Serre's theorem would assert that if $f$ has no automorphisms,[31] then $\rho^{\mathrm{dyn}}_{K,n,f}$ is surjective for sufficiently large $n$. It seems too much to ask that this be true for all maps, so we pose the following challenge:

**Question 7.3** (Dynatomic Image-of-Galois Problem). Let $K/\mathbb{Q}$ be a number field, let $N \geq 1$, and let $d \geq 2$. Characterize the maps $f \in \mathrm{End}^N_d(K)$ for which there is a constant $C(f)$ such that for all $n \geq 1$,

$$\mathrm{Image}(\rho^{\mathrm{dyn}}_{K,f,n}) \text{ has index at most } C(f) \text{ in } \mathrm{Aut}(\mathcal{P}_{n,\nu(f)}).$$

31 The automorphism group of $f$ is $\mathrm{Aut}(f) = \{\varphi \in \mathrm{PGL}_{N+1} : \varphi^{-1} \circ f \circ \varphi = f\}$. The elements of $\mathrm{Gal}(\bar{K}/K)$ commute with the action of $\mathrm{Aut}_K(f)$, so if $\mathrm{Aut}_K(f) \neq (1)$, then the image of $\rho^{\mathrm{dyn}}_{K,n,f}$ is restricted, just as the image of $\rho^{\mathrm{ell}}_{E/K,\ell}$ is restricted if $E$ has CM.

### 7.2. Topic #4(b): Arboreal representations

The dynatomic extensions described in Section 7.1 are generated by points with finite orbits. In this section we consider arboreal extensions, which are extensions generated by backward orbits.

**Example 7.4.** We illustrate with the map $f(x) = x^d$.

$$\begin{bmatrix} \textit{Dynatomic extension.} & \text{Field generated by roots of } x^{d^n} = x \text{ for } n \geq 1. \\ \textit{Arboreal extension.} & \text{Field generated by roots of } x^{d^n} = a \text{ for } n \geq 1. \end{bmatrix} \tag{7.2}$$

Thus (7.2) suggests that dynatomic extensions resemble cyclotomic extensions, while the arboreal extensions resemble Kummer extensions; although we readily admit that this is far from a perfect analogy.

**Definition 7.5.** Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over $K$, and let $P \in \mathbb{P}^N(K)$. The *inverse image tree of $f$ rooted at $P$* is the (disjoint) union of the inverse images of $P$ by the iterates of $f$:

$$\mathcal{T}_{f,P} = \bigcup_{n \geq 0} f^{-n}(P) = \bigcup_{n \geq 0} \{Q \in \mathbb{P}^N(\bar{K}) : f^n(Q) = P\}.$$

We say that $f$ is *arboreally complete at $P$* if $\#f^{-n}(P) = d^{nN}$ for all $n \geq 0$, in which case $\mathcal{T}_{f,P}$ is a complete rooted $d^N$-ary tree, where $f$ maps the points in $f^{-n-1}(P)$ to the points in $f^{-n}(P)$. Figure 2 illustrates a complete inverse image tree for a degree-2 map $f : \mathbb{P}^1 \to \mathbb{P}^1$.

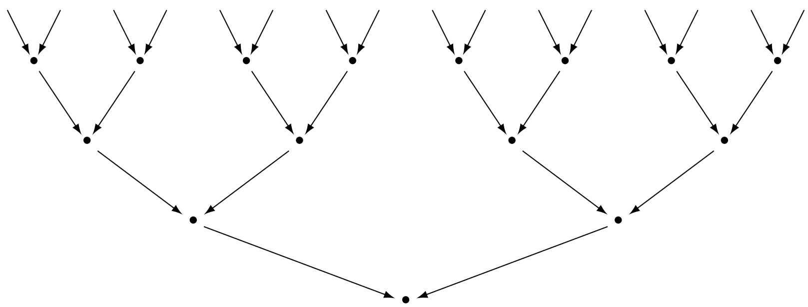

**FIGURE 2**
A complete binary inverse image tree

The points in the iterated inverse image of $P$ generate a (generally infinite) algebraic extension of $K$, so the Galois group $\text{Gal}(\bar{K}/K)$ acts on the points in $\mathcal{T}_{f,P}$. And since the action of the Galois group commutes with the map $f$, the action of $\text{Gal}(\bar{K}/K)$ on $\mathcal{T}_{f,P}$ preserves the tree structure. Thus in this case, the analogue of $\text{GL}_2$ in (7.1) is the group of automorphisms of the tree $\mathcal{T}_{f,P}$, which leads us to our primary object of study.

**Definition 7.6.** Let $f \in \operatorname{End}_N^d(K)$, and let $P \in \mathbb{P}^N(K)$. The *arboreal representation (over $K$) of $f$ rooted at $P$* is the homomorphism

$$\rho_{K,f,P}^{\text{dyn}} : \operatorname{Gal}(\bar{K}/K) \to \operatorname{Aut}(\mathcal{T}_{f,P}).$$

The *Odoni*[32] *index over $K$ of $f$ at $P$* is the index of the image in the full tree automorphism group,

$$\iota_K(f,P) = \left[\operatorname{Aut}(\mathcal{T}_{f,P}) : \operatorname{Image}(\rho_{K,f,P}^{\text{dyn}})\right].$$

As in the dynatomic case, it is again too much to hope that the image of $\rho_{K,f,P}^{\text{dyn}}$ has finite index in $\operatorname{Aut}(\mathcal{T}_{f,P})$ for all $f$, but we might expect this to be true for most $f$. This leads to a number of fundamental questions.

**Question 7.7** (Arboreal Image-of-Galois Problem).

(a) Let $K/\mathbb{Q}$ be a number field, and let $N \geq 1$ and $d \geq 2$. Characterize the maps $f \in \operatorname{End}_d^N(K)$ and points $P \in \mathbb{P}^N(K)$ whose Odoni index $\iota_K(f,P)$ is finite, especially when $f$ is arboreally complete at $P$.

(b) (Generalized Odoni conjecture) For all number fields $K/\mathbb{Q}$ and all $N \geq 1$ and $d \geq 2$, does there exist a point $P \in \mathbb{P}^N(K)$ and a map $f \in \operatorname{End}_d^N(K)$ that is arboreally complete at $P$ such that $\iota_K(f,P) = 1$?

(c) Fix a number field $K/\mathbb{Q}$ and integers $N \geq 1$ and $d \geq 2$. Is it true that $\iota_K(f,P) = 1$ for "almost all" pairs $(f,P)$ in $\operatorname{End}_d^N(K) \times \mathbb{P}^N(K)$ for some appropriate sense of density?

**Remark 7.8.** Odoni's original conjecture was both more restrictive and stronger than Question 7.7(b) in that he considered only $N = 1$ and polynomial maps. Odoni asked if for all $K/\mathbb{Q}$ and all $d \geq 2$, there exists a degree-$d$ monic polynomial $f(x) \in K[x]$ and a point $\alpha \in K$ such that $\mathcal{T}_{f,\alpha}$ is a complete $d$-ary tree and such that $\iota_K(f,\alpha) = 1$. Odoni's conjecture was proven over $\mathbb{Q}$ for prime values of $d$ in **[45]**, and then in full generality in **[85]**. We mention that Odoni originally conjectured that the statement should hold for all Hilbertian fields, but this was recently resolved in the negative **[36]**.

**Remark 7.9.** We close with the well-known observation that the automorphism group of an $n$-level complete rooted regular tree (labeling the levels $0, 1, 2, \ldots, n$) is an $n$-fold wreath product of the symmetric group. Hence if $f$ is arboreally complete at $P$, then the automorphism group of $\mathcal{T}_{f,P}$ is the inverse limit

$$\operatorname{Aut}(\mathcal{T}_{f,P}) \cong \varprojlim \underbrace{\mathcal{S}_{d^N} \wr \mathcal{S}_{d^N} \wr \cdots \wr \mathcal{S}_{d^N}}_{n\text{-fold iterated wreath product with } n \to \infty}.$$

**32** Named in honor of R. W. K. Odoni, who appears to have been the first to seriously study such problems in a series of papers **[65–67]**, in one of which he proves that $\iota_{\mathbb{Q}}(x^2 - x + 1, 0) = 1$.

The profinite group $G(\bar{K}/K)$ then acts continuously on the profinite group $\mathrm{Aut}(\mathcal{T}_{f,P})$, just as in arithmetic geometry $G(\bar{K}/K)$ acts continuously on the Tate module $T_\ell(A) = \varprojlim A[\ell^n]$ of an abelian variety $A/K$.

## 8. TOPIC #5: DYNAMICAL AND ARITHMETIC COMPLEXITY

We informally define the *complexity* of a mathematical object to be a rough estimate for how much space it takes to store the object:

$$\begin{aligned} h(\mathcal{X}) &= \text{complexity of object } \mathcal{X} \\ &\asymp \text{\# of basic storage units (e.g., bits, scalars) required to describe } \mathcal{X}. \end{aligned}$$

**Example 8.1.** The complexity of a nonzero integer $c \in \mathbb{Z}$ is the number of bits needed to describe $c$, so roughly $\log |c|$.

**Example 8.2.** The complexity of a nonzero polynomial $f(x) \in K[x]$ is the number of coefficients needed to describe $f$, so roughly $\deg(f)$.

For a sequence of objects $\boldsymbol{\mathcal{X}} = (\mathcal{X}_n)_{n\geq 1}$ whose complexity is expected to grow exponentially, we define the *sequential complexity of* $\boldsymbol{\mathcal{X}}$ to be the limit[33]

$$\sigma(\boldsymbol{\mathcal{X}}) = \lim_{n\to\infty} h(\mathcal{X}_n)^{1/n}.$$

**Example 8.3.** Let $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a degree-$d$ dominant rational map, i.e., a map given by homogeneous degree-$d$ polynomials $[f_0, \ldots, f_N]$ in $\mathbb{C}[x_0, \ldots, x_N]$ having no common factors. Then $h(f) = \deg(f) = d$. The sequential complexity of the sequence of iterates $f^n$ is called the *dynamical degree of* $f$ and is denoted

$$\delta_f = \lim_{n\to\infty} (\deg f^n)^{1/n}. \tag{8.1}$$

**Example 8.4.** Let $P = [c_0, \ldots, c_N] \in \mathbb{P}^N(\mathbb{Q})$ be a point written with relatively prime integer coordinates. Then

$$h(P) = \log \max |c_i|. \tag{8.2}$$

More generally, if $K/\mathbb{Q}$ is a number field, then there is a well-defined Weil height function[34]

$$h : \mathbb{P}^N(K) \to [0, \infty) \tag{8.3}$$

33 In cases where the limit is not known to exist, we may consider the *upper and lower sequential complexities*

$$\overline{\sigma}(\boldsymbol{\mathcal{X}}) = \limsup_{n\to\infty} h(\mathcal{X}_n)^{1/n} \quad \text{and} \quad \underline{\sigma}(\boldsymbol{\mathcal{X}}) = \liminf_{n\to\infty} h(\mathcal{X}_n)^{1/n}.$$

34 The Weil height of a point $P = [a_0, \ldots, a_N] \in \mathbb{P}^N(K)$ may be defined as follows: Let $d = [K : \mathbb{Q}]$, write the fractional ideal generated by $a_0, \ldots, a_N$ as $\mathfrak{A}\mathfrak{B}^{-1}$ with relatively prime integral ideals $\mathfrak{A}$ and $\mathfrak{B}$, and let $\sigma_1, \ldots, \sigma_d : K \hookrightarrow \mathbb{C}$ be the distinct complex embeddings of $K$. Then

$$h(P) = \frac{1}{d} \log |\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{B})| + \frac{1}{d} \sum_{i=1}^{d} \log \max_{0\leq j\leq N} |\sigma_i(a_j)|.$$

that generalizes (8.2). The height of a point $P \in \mathbb{P}^N(K)$ measures the complexity of the coordinates of $P$.

Now let $K/\mathbb{Q}$ be a number field, let $P \in \mathbb{P}^N(K)$, and let $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map defined over $K$. Then the sequential complexity of the orbit $\mathcal{O}_f(P)$ is called the *arithmetic degree of the $f$-orbit of $P$* and is denoted

$$\alpha_f(P) = \lim_{n\to\infty} h\big(f^n(P)\big)^{1/n}. \tag{8.4}$$

The notation in Table 2 will be used throughout the remainder of this section. We will generalize the complexity measures from Examples 8.3 and 8.4 and describe a number of results and questions.

**Definition 8.5.** The (*first*) *dynamical degree* of a dominant rational map $f : X \dashrightarrow X$ is

$$\delta_f = \lim_{n\to\infty} \big(\deg_X(f^n)\big)^{1/n}. \tag{8.5}$$

The limit (8.5) converges and is independent of the choice of the ample divisor $H$ used to define $\deg_X$ **[16]**.[35] Dynamical degrees on $\mathbb{P}^N$ were first studied in the 1990s **[2, 9, 75]**. A long-standing question concerning the algebraicity of the dynamical degree was recently answered in the negative.

**Theorem 8.6** (**[5, 6]**). *For all $N \ge 2$, there exist dominant rational maps $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ defined over $\mathbb{Q}$ such that $\delta_f$ is a transcendental number. For $N \ge 3$, there exist such maps that are birational automorphisms of $\mathbb{P}^N$.*

| | |
|---|---|
| $K$ | a number field with algebraic closure $\bar{K}$ |
| $X$ | a smooth projective variety of dimension $d$ defined over $K$ |
| $f$ | a dominant rational map $f : X \dashrightarrow X$ defined over $K$ |
| $X_f$ | $= \{P \in X(\bar{K}) : f$ is well-defined at $f^n(P)$ all $n \ge 0\}$ |
| $\deg_X(f)$ | $= (f^*H)\cdot H^{d-1}$, where $H$ is an ample divisor on $X$, and this formula is a $d$-fold intersection index |
| $h_X$ | the height on $X$ coming from a projective embedding $\iota : X \hookrightarrow \mathbb{P}^N$, i.e., $h_X = h \circ \iota$, where $h$ is the Weil height (8.3) on $\mathbb{P}^N$ |
| $h_X^+$ | $= \max\{1, h_X\}$ |

**TABLE 2**
Notation for Section 8

35 The convergence of (8.5) when $X = \mathbb{P}^N$ is a fun exercise using $\deg(f \circ g) \le (\deg f)(\deg g)$.

There is an arithmetic analogue of the dynamical degree that measures the average arithmetic complexity of the algebraic points in an orbit. But since rational maps may not be defined everywhere, the next definition must restrict attention to $X_f$, the points in $X$ where the full forward orbit of $f$ is well defined.[36]

**Definition 8.7.** Let $f : X \dashrightarrow X$ be a dominant rational map defined over $K$, and let $P \in X_f(\bar{K})$. The *arithmetic degree of the $f$-orbit of $P$* is

$$\alpha_f(P) = \lim_{n\to\infty} h_X^+\big(f^n(P)\big)^{1/n}. \tag{8.6}$$

**Question 8.8.** Does the limit (8.6) always exist?

In any case, we may consider the upper and lower arithmetic degrees $\underline{\alpha}_f(P)$ and $\overline{\alpha}_f(P)$ defined using, respectively, the liminf and the limsup. It is not hard to show that these quantities are independent of the choice of the complexity function $h_X^+$. It is also easy to show that $\overline{\alpha}_f(P)$ is finite, but more difficult to show that there is a uniform geometric bound, as in the next result.

**Theorem 8.9** ([50]). *Let $f : X \dashrightarrow X$ be a dominant rational map defined over $K$, and let $P \in X_f(\bar{K})$. Then*

$$\overline{\alpha}_f(P) \le \delta_f.$$

| |
|---|
| **Moral of Theorem 8.9.** The arithmetic complexity of an orbit is no worse than the dynamical complexity of the map. |

Theorem 8.9 suggests a natural question. When do the arithmetic and dynamical complexities coincide?

**Conjecture 8.10** ([39, 40]). *Let $f : X \dashrightarrow X$ be a dominant rational map defined over $K$, and let $P \in X_f(\bar{K})$. Then*

$$\mathcal{O}_f(P) \textit{ is Zariski dense in } X \implies \overline{\alpha}_f(P) = \delta_f.$$

| |
|---|
| **Moral of Conjecture 8.10.** An orbit with maximal geometric complexity also has maximal arithmetic complexity. |

**Question 8.11.** Does $X(\bar{K})$ always contain a point with Zariski dense $f$-orbit? The answer is clearly no. For example, if there exists a dominant rational map $\varphi : X \dashrightarrow \mathbb{P}^1$ satisfying $\varphi \circ f = \varphi$, then each $f$-orbit lies in a fiber of $\varphi$. Xie asks whether this is the only obstruction. An affirmative answer for certain maps in dimension 2 is given in [35, 93].

36 The complement $X \setminus X_f$ is a countable union of proper subvarieties, so cardinality considerations show that $X_f(\mathbb{C})$ is nonempty; but the situation is less clear for a countable field such as $\bar{\mathbb{Q}}$. It is shown in [1] that $X_f(\bar{\mathbb{Q}})$ is Zariski dense in $X$.

**Example 8.12.** It is easy to prove Conjecture 8.10 for morphisms $f$ of $\mathbb{P}^N$, since in that case $\delta_f = \deg(f)$, and the theory of canonical heights implies that

$$\alpha_f(P) = \begin{cases} \deg(f) & \text{if } \#\mathcal{O}_f(P) = \infty, \\ 1 & \text{if } P \text{ is } f\text{-prepediodic.} \end{cases}$$

More generally, a similar argument works for endomorphisms of any smooth projective variety whose Néron–Severi group has rank 1 [38]. But the conjecture is still open for dominant rational maps of $\mathbb{P}^N$, and for morphisms of more general varieties.

**Example 8.13.** The past decade has been significant progress on various cases of Conjecture 8.10, especially in the case of morphisms, using an assortment of tools ranging from linear-forms-in-logarithms to canonical heights for nef divisors to the minimal model program in algebraic geometry. In particular, Conjecture 8.10 has been proven for

- group endomorphisms (homomorphisms composed with translations) of semi-abelian varieties (extensions of abelian varieties by algebraic tori) [39, 52, 83, 84],
- endomorphisms of (not necessarily smooth) projective surfaces [38, 53, 57],
- extensions to $\mathbb{P}^N$ of regular affine automorphisms of $\mathbb{A}^N$ [38],
- endomorphisms of hyperkähler varieties [43],
- endomorphisms of degree greater than 1 of smooth projective threefolds of Kodaira dimension 0 [43],
- endomorphisms of normal projective varieties such that $\mathrm{Pic}^0 \otimes \mathbb{Q} = 0$ and with nef cone generated by finitely many semi-ample integral divisors [49], and
- smooth projective threefolds having at least one int-amplified[37] endomorphism, and surjective endomorphisms of smooth rationally connected projective varieties [51].

**Remark 8.14.** Various generalizations of Conjecture 8.10 have been proposed. We mention in particular the *Small Arithmetic Non-Density Conjecture* [51], which says that points of small arithmetic degree are not Zariski dense when $f$ is a morphism. However, as the authors observe, their conjecture is only for morphisms, since it may fail for dominant rational maps. The authors of [51] prove the SAND conjecture for many of the cases listed in Example 8.13.

Conjecture 8.10 is a relatively coarse estimate for the height growth of points in Zariski-dense orbits. An affirmative answer to the following question would yield a quantitative version of the conjecture.

**37** A morphism $f : X \to X$ is *int-amplified* if there exists an ample Cartier divisor $H$ such that $f^*H - H$ is also ample.

**Question 8.15** ([10, QUESTION 14.5]). Let $f : X \dashrightarrow X$ be a dominant rational map defined over $K$, and let $P \in X_f(\bar{K})$ be a point whose orbit $\mathcal{O}_f(P)$ is Zariski dense in $X$. Do there exist (integers) $0 \le \ell_f \le N$ and $k_f \ge 0$ such that

$$h\big(f^n(P)\big) \asymp \delta_f^n \cdot n^{\ell_f} \cdot (\log n)^{k_f},$$

where the implied constants depend on $f$ and $P$, but not on $n$? If $\delta_f > 1$, is it further true that $k_f = 0$?


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