RELATIVE TRACE FORMULAE AND THE GAN–GROSS–PRASAD CONJECTURES

RAPHAËL BEUZART-PLESSIS

ABSTRACT

This paper reports on some recent progress that have been made on the so-called Gan–Gross–Prasad conjectures through the use of relative trace formulae. In their global aspects, these conjectures, as well as certain refinements first proposed by Ichino–Ikeda, give precise relations between the central values of some higher-rank L-functions and automorphic periods. There are also local counterparts describing branching laws between representations of classical groups. In both cases, approaches through relative trace formulae have shown to be very successful and have even lead to complete proofs, at least in the case of unitary groups. However, the works leading to these definite results have also been the occasion to develop further and gain new insights on these fundamental tools of the still emerging relative Langlands program.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 11F70; Secondary 22E50, 22E55, 11F72

KEYWORDS

Gan-Gross-Prasad conjecture, relative trace formula, periods of automorphic forms, L-functions, fundamental lemma, endoscopy

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 3, pp. 1712–1743
DOI 10.4171/ICM2022/182

Published by EMS Press

In broad terms, the Gan–Gross–Prasad conjectures concern two interrelated questions in the fields of representation theory and automorphic forms. On the one hand, these conjectures predict highly-sophisticated descriptions of some branching laws between representations of classical groups (that is, orthogonal, symplectic/metaplectic, or unitary groups) over local fields which can be seen as direct descendants of classical results of H. Weyl on similar branching problems for compact Lie groups. The predictions are given in terms of the recently established local Langlands correspondence for these groups that provides a parameterization of the irreducible representations in terms of data of arithmetic nature. On the other hand, the Gan–Gross–Prasad conjectures also give far-reaching higher-rank generalizations of certain celebrated relations between special values of L-functions and period integrals. We start this paper by discussing two, by now well-known, examples of the former kind of relations.

First, we briefly review Hecke's integral representation for L-functions of modular forms. Let $S_2(\Gamma_1(N))$ be the space of cuspidal modular form of weight 2 for the group

$$\Gamma_1(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

It consists in the holomorphic functions $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$ is Poincaré upper half-plane, satisfying the functional equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \tag{0.1}$$

and that are "vanishing at the cusps," a condition imposing in some sense means that f is rapidly decreasing modulo the above symmetries. Another more geometric way to describe $S_2(\Gamma_1(N))$ is as a space of holomorphic differential forms: for $f \in S_2(\Gamma_1(N))$, the form $\omega_f = f(z)dz$ descends to the open modular curve $Y_1(N) = \Gamma_1(N)\backslash\mathbb{H}$ (a Riemann surface as soon as $N > 3$) and the vanishing at the cusps condition translates to the fact that this form extends holomorphically to the canonical compactification $X_1(N)$ of $Y_1(N)$. Moreover, the map $f \mapsto \omega_f$ yields an isomorphism $S_2(\Gamma_1(N)) \simeq \Omega^1(X_0(N))$.

It follows from the functional equation (0.1) that every $f \in S_2(\Gamma_1(N))$ is periodic of period 1 and thus admits a Fourier expansion

$$f = \sum_{n \geqslant 1} a_n q^n, \quad q = e^{2i\pi z}, \tag{0.2}$$

where the fact that the sum is restricted to positive integers is part of the assumption that f vanishes at the cusps. The Hecke L-function of f is then defined as the Dirichlet series

$$L(s, f) = \sum_{n \geqslant 1} \frac{a_n}{n^s},$$

converging absolutely in the range $\Re(s) > 2$. Hecke has shown that this can also essentially be expressed as the Mellin transform of the restriction of f to the imaginary line,

$$(2\pi)^{-s}\Gamma(s)L(s, f) = \int_0^\infty f(iy)y^{s-1}dy. \tag{0.3}$$

This formula implies at once two essential analytic properties of $L(s, f)$: its analytic continuation to the complex plane and a functional equation of the form $s \leftrightarrow 2-s$. Moreover, it also has interesting arithmetic content: when specialized to the central value $s=1$ and combined with a theorem of Drinfeld and Manin, it allows showing that the ratio between the central value of the L-function of a (modular) elliptic curve and its (unique) real period is always rational as predicted by a refinement of the Birch–Swinnerton-Dyer conjecture.

The above formula of Hecke can be reformulated (and slightly generalized) in the language of adelic groups and automorphic forms as follows. Let π be a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$, where $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p$ denotes the adele ring of the rationals. This roughly means that π is an irreducible representation realized in a space of smooth and rapidly decreasing functions on $\mathrm{PGL}_2(\mathbb{Q})\backslash \mathrm{PGL}_2(\mathbb{A})$. Then, for every $\varphi \in \pi$ we have an identity of the following shape:

$$\int_{A(\mathbb{Q})\backslash A(\mathbb{A})} \varphi(a)da \sim L\left(\frac{1}{2}, \pi\right), \tag{0.4}$$

where $A = \begin{pmatrix} \star & \\ & \star \end{pmatrix}$ is the standard split torus in PGL_2 and $L(s,\pi)$ is the L-function of π, a particular instance of the notion of automorphic L-functions defined by Langlands. For specific π's, this recovers Hecke's formula (0.3) for $s=1$, although $L(s,\pi)$ then coincides with the L-functions of a modular form only up to a renormalization that moves its center of symmetry to $1/2$. Moreover, the $\sim$ sign means that the equality only holds up to other, arguably more elementary, multiplicative factors.

Let $E/\mathbb{Q}$ be a quadratic extension. In the 1980s, Waldspurger [46] has established another striking formula for the central value of the base-change L-function

$$L(s, \pi_E) = L(s,\pi)L(s, \pi \otimes \chi_E)$$

where $\chi_E : \mathbb{A}^\times/\mathbb{Q}^\times$ is the idele class character associated to the extension $E/\mathbb{Q}$. Waldspurger's formula roughly takes the following shape:

$$\left| \int_{T(\mathbb{Q})\backslash T(\mathbb{A})} \varphi(t)dt \right|^2 \sim L\left(\frac{1}{2}, \pi_E\right) \tag{0.5}$$

for $\varphi \in \pi$, where T is a torus in PGL_2 isomorphic to $\mathrm{Res}_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$ ($\mathrm{Res}_{E/F}$ denoting Weil's restriction of scalars). This result has lead in the subsequent years to striking arithmetic applications such as to the Birch–Swinnerton-Dyer conjecture or to p-adic L-functions.

Although of a similar shape, the two formulas (0.4) and (0.5) also have important differences, e.g., the left-hand side of (0.5) is typically far more algebraic in nature, and indeed sometimes just reduces to a finite sum, whereas the formula (0.4) can be deformed to all complex number s, giving an integral representation of the L-function $L(s,\pi)$ as Hecke's original formula, and typically carries information that is more transcendental.

The left-hand sides of (0.4) and (0.5) are particular instances of *automorphic periods* that can be informally defined as the integral of an automorphic form over a subgroup. We can also consider these two period integrals in a more representation-theoretic way as giving explicit $A(\mathbb{A})$- or $T(\mathbb{A})$-invariant linear forms on π. This point of view rapidly leads to a local related problem which, given a place v of $\mathbb{Q}$, aims to describe the irreducible representations of $\mathrm{PGL}_2(\mathbb{Q}_v)$ admitting a nonzero $A(\mathbb{Q}_v)$- or $T(\mathbb{Q}_v)$-invariant linear form. It turns out that for the torus A the answer is always positive except for some degenerate one-dimensional representations. On the other hand, the answer for the torus T is far more subtle and involves local ε-factors as shown by Tunnell and Saito **[44]**.

A natural generalization of Hecke's formula (0.4) is given by the theory of so-called Rankin–Selberg convolutions as developed by Jacquet Piatetski-Shapiro and Shalika **[31]**. On the other hand, the Gan–Gross–Prasad conjectures **[23]** aim to give far-reaching higher-rank generalizations of the above result of Waldspurger as well as of the theorem of Tunnell–Saito.

There has been a lot of progress on these conjectures, as well as some refinements thereof, in recent years, in particular in the case of unitary groups. In this paper, we will survey some of these developments with a particular emphasis on the use of (various forms of) *relative trace formulae*. Actually, a point I will try to advocate here is that the long journey towards the Gan–Gross–Prasad conjectures was also the occasion to develop and discover new features of these trace formulae.

The content is roughly divided as follows. In the first section, we review the local conjectures of Gan–Gross–Prasad and discuss their proofs in some cases based on some local trace formulae. Then, in Section 2, we introduce the global conjectures for unitary groups, as well as their refinements by Ichino–Ikeda, and describe an approach to both of them through a comparison of global relative trace formulae proposed by Jacquet and Rallis. The next two sections, Sections 3 and 4, are devoted to explaining the various ingredients needed to carry out this comparison effectively. In the final Section 5, we offer few thoughts about possible future developments.

1. THE LOCAL CONJECTURES AND MULTIPLICITY FORMULAE

1.1. The branching problem

Let F be a local field (of any characteristic) and E be either a separable quadratic extension of F or F itself. In the case where $[E : F] = 2$, we let c denote the nontrivial F-automorphism of E and otherwise, to obtain uniform notation, we simply set $c = 1$. Let V be a Hermitian or quadratic space over E i.e. a finite dimensional E-vector space equipped with a nondegenerate c-sesquilinear form

$$h : V \times V \to E$$

satisfying $h(v, w) = h(w, v)^c$ for every $v, w \in V$. Let $W \subset V$ be a nondegenerate subspace and let $U(V)$ (resp. $U(W)$) be the group of E-linear automorphisms $g \in \mathrm{GL}_E(V)$ (resp. $g \in \mathrm{GL}_E(W)$) that preserve the form h and are of determinant one when $E = F$. In other words, $U(V)$, $U(W)$ are the unitary groups associated of the Hermitian spaces V, W when $[E : F] = 2$ and the special orthogonal groups of the quadratic spaces V, W when $E = F$. Note that there is a natural embedding $U(W) \hookrightarrow U(V)$ given by extending the action of $g \in U(W)$ trivially on the orthogonal complement $Z = W^\perp$ of W in V. We assume that

$$Z \text{ is odd-dimensional and } U(Z) \text{ is quasisplit.} \tag{1.1}$$

Concretely, this means that there exists a basis $(z_i)_{-r \leqslant i \leqslant r}$ of Z and $\nu \in F^\times$ such that $h(z_i, z_j) = \nu\delta_{i,-j}$ for $-r \leqslant i, j \leqslant r$. Let $N \subset U(V)$ be the unipotent radical of a parabolic subgroup $P \subset U(V)$ stabilizing a maximal flag of isotropic subspaces in Z, e.g., with a basis as before, we can take the flag $Ez_r \subset Ez_r \oplus Ez_{r-1} \subset \cdots \subset Ez_r \oplus \cdots \oplus Ez_1$. Then, $U(W)$ normalizes N and Gan–Gross–Prasad construct a certain conjugacy class of $U(W)$-invariant characters $\xi : N \to \mathbb{C}^\times$. Concretely, we can take

$$\xi(u) = \psi\left(\sum_{i=0}^{r-1} h(uz_i, z_{-i-1})\right), \quad u \in N,$$

where $\psi : F \to \mathbb{C}^\times$ is a nontrivial character.

The local GGP conjectures roughly address the following branching problems: for smooth irreducible complex representations (π, V_π) and (σ, V_σ) of $U(V)$ and $U(W)$ respectively, determine the dimension (also called *multiplicity*) of the following intertwining space:

$$m(\pi, \sigma) = \dim \operatorname{Hom}_{U(W)\ltimes N}(V_\pi, V_\sigma \otimes \xi). \tag{1.2}$$

Here, when F is Archimedean by a *smooth* representation we actually mean an admissible smooth Fréchet representation of moderate growth in the sense of Casselman–Wallach [19]. Moreover, in this case V_π, V_σ are Fréchet spaces and by definition $\operatorname{Hom}_{U(W)\ltimes N}(V_\pi, V_\sigma \otimes \xi)$ only consists in the *continuous* $U(W) \ltimes N$-equivariant intertwining maps.

By deep theorems of Aizenbud–Gourevitch–Rallis–Schiffmann [2] in the p-adic case and Sun–Zhu [42] in the Archimedean case, the branching multiplicity $m(\pi, \sigma)$ is known to always be at most 1 (at least when F is of characteristic 0, see [37] for the case of positive characteristic). Thus, the question reduces to determine when $m(\pi, \sigma)$ is nonzero.

Gan, Gross, and Prasad formulated a precise answer to this question, under some restrictions on the representations π and σ, based on the so-called *Langlands correspondences* for the groups $U(V)$ and $U(W)$. More precisely, these give ways to parametrize smooth irreducible representations of those groups in terms of *L-parameters* which are certain kind of morphisms

$$\phi : \mathcal{L}_F \to {}^L U(V) \quad \text{or} \quad {}^L U(W)$$

from a group $\mathcal{L}_F$ which can be taken to be either the Weil group W_F (in the Archimedean case) or a product $W_F \times \mathrm{SL}_2(\mathbb{C})$ (in the non-Archimedean case) to a semidirect product ${}^L U(V) = \widehat{U(V)} \rtimes W_F$ or ${}^L U(W) = \widehat{U(W)} \rtimes W_F$ known as the *L-group*. In the cases at hand, the connected components $\widehat{U(V)}$ and $\widehat{U(W)}$ turn out to be either complex general linear groups (in the unitary case) or complex special orthogonal/symplectic groups (in the orthogonal case) and the relevant sets of L-parameters can be more concretely described as sets of complex representations of $\mathcal{L}_E$ of fixed dimension and satisfying certain properties of (conjugate-)self-duality. We refer the reader to [23, §8] for details and content ourself to briefly sketch this alternative description for unitary groups: the L-parameters for $U(V)$ can be equivalently described as isomorphism classes of $n = \dim(V)$-dimensional complex semisimple representations $\phi : \mathcal{L}_E \to \mathrm{GL}(M)$ which are conjugate-self-dual of sign $(-1)^{n-1}$. Here, ϕ is said to be *conjugate-self-dual* if there is an isomorphism $T : M \to M^{\vee\sigma}$

with its conjugate-contragredient $\phi^{\vee\sigma} : \mathcal{L}_E \to \mathrm{GL}(M^{\vee\sigma})$ obtained by twisting the contragredient by any chosen lift $\sigma \in \mathcal{L}_F \setminus \mathcal{L}_E$ of c and it is, moreover, said to be of sign $\varepsilon \in \{\pm\}$ if the isomorphism T can be chosen so that ${}^tT\phi(\sigma^2) = \varepsilon T$. Besides these L-parameters ϕ, the local Langlands correspondence is also supposed to associate to irreducible representations irreducible characters of the finite group of components

$$S_\phi = \pi_0\big(\mathrm{Cent}_{\widehat{U(V)}}(\phi)\big)$$

of the centralizer of the image of ϕ in $\widehat{U(V)}$. For the group considered here, S_ϕ is always a 2-group and moreover, once again, it also admits a more concrete description, e.g., if $U(V)$ is a unitary group and we identify ϕ with a $(-1)^{n-1}$-conjugate-self-dual representation of $\mathcal{L}_E$ as before, this can be decomposed into irreducible representations as follows:

$$\phi = \bigoplus_{i\in I} n_i\phi_i \bigoplus_{j\in J} m_j\phi_j \bigoplus_{k\in K} l_k(\phi_k \oplus \phi_k^{\vee\sigma}) \tag{1.3}$$

where the ϕ_i's (resp. ϕ_j's) are irreducible conjugate-self-dual of the same sign $(-1)^{n-1}$ (resp. of opposite sign $(-1)^n$) whereas the ϕ_k's are irreducible but not conjugate-self-dual and using this decomposition we have

$$S_\phi = \bigoplus_{i\in I} \mathbb{Z}/2\mathbb{Z}e_i. \tag{1.4}$$

We are now ready to state a version of the local Langlands correspondence, including an essential refinement by Vogan [45], necessary for the local Gan–Gross–Prasad conjecture. It turns out to be more easily described if we consider more than one group at the same time: besides $U(V)$ [1] itself, we need to consider its *pure inner forms* which here consist of the groups $U(V')$ where V' runs over the isomorphism classes of Hermitian/quadratic spaces of the same dimension as V and of the same discriminant in the orthogonal case. If F is non-Archimedean, and provided V is not an hyperbolic quadratic plane, there are always two such isomorphism classes of Hermitian/quadratic spaces and thus as many pure inner forms whereas if F is Archimedean, by their classification using signatures there are $\dim(V)+1$ (resp. $\frac{\dim(V)+1}{2}$ for $\dim(V)$ odd, $\frac{\dim(V)+\mathrm{disc}(V)+1}{2}$ for $\dim(V)$ even) pure inner forms in the unitary case (resp. orthogonal case). For such a pure inner form, let us denote by $\mathrm{Irr}(U(V'))$ the set of isomorphism classes of smooth irreducible representations of $U(V')$. Then, modulo the auxilliary choice of a quasisplit pure inner form $U(V')$ and a *Whittaker datum* on it[2] that we will suppress from the notation, the local Langlands correspondence posits the existence of a natural decomposition into finite sets called *L-packets*

$$\bigsqcup_{V'} \mathrm{Irr}\big(U(V')\big) = \bigsqcup_{\phi} \Pi(\phi),$$

where the left union runs over all pure inner forms whereas the right union is over all L-parameters $\phi : \mathcal{L}_F \to {}^LU(V)$ (the pure inner forms all share the same L-group) together

1 Of course, the following discussion also applies to $U(W)$.

2 A Whittaker datum of $U(V')$ is a pair (N,θ) consisting of a maximal unipotent subgroup $N \subset U(V')$ and a generic character $\theta : N \to \mathbb{C}^\times$. This datum only matters up to conjugacy.

with bijections

$$\begin{aligned} \Pi(\phi) &\simeq \widehat{S_\phi}, \\ \pi(\phi,\chi) &\leftarrow\!\dashv \chi, \end{aligned} \tag{1.5}$$

with the character group $\widehat{S_\phi}$ of S_ϕ. Thus, in a sense the correspondence gives a way to parameterize the admissible duals of all the pure inner forms of $U(V)$ at the same time. However, there is a precise recipe for the characters $\widehat{S_\phi}^{V'}$ corresponding to the intersection $\Pi^{V'}(\phi) = \Pi(\phi) \cap \mathrm{Irr}(U(V'))$ and therefore this also induces a parameterization of the individual admissible duals $\mathrm{Irr}(U(V'))$. Moreover, the naturality condition can be made precise through the so-called *endoscopic relations* that characterize the Langlands parameterization uniquely in terms of the known correspondence for GL_n.[3] For real groups, the correspondence was constructed long ago by Langlands and is known to satisfy the endoscopic relations thanks to the work of Shelstad. In his monumental work [7], Arthur has established, among other things, the existence of this correspondence for quasisplit special orthogonal or symplectic p-adic groups (with an important technical caveat for even special orthogonal groups $\mathrm{SO}(2n)$ where the correspondence is only proven up to conjugation by the full orthogonal group $O(2n)$). This work was subsequently extended in [39] and [34] to include unitary groups (not necessarily quasisplit).

For the purpose of stating the local Gan–Gross–Prasad conjecture, we will also need to vary the two groups $U(V)$, $U(W)$. However, we will need these to vary in a compatible way in order for the multiplicity (1.2) to still be well-defined. More precisely, the *relevant* pure inner forms of $U(V) \times U(W)$ are defined by varying the small Hermitian/quadratic space W while keeping the orthogonal complement $Z = W^\perp$ fixed: these are the groups of the form $U(V') \times U(W')$ where W' is a Hermitian/quadratic space of the same dimension as W, and same discriminant in the orthogonal case, whereas V' is given by the orthogonal sum $V' = W' \oplus^\perp Z$. Since the orthogonal complement Z is the same, for each relevant pure inner form $U(V') \times U(W')$ we can define as before a multiplicity function $(\pi,\sigma) \in \mathrm{Irr}(U(V')) \times \mathrm{Irr}(U(W')) \mapsto m(\pi,\sigma)$.

We are now ready to formulate the local Gan–Gross–Prasad conjecture except for one technical but important detail: as alluded to above, the local Langlands correspondences, and more particularly the bijections (1.5), depend on the choice of Whittaker data on some pure inner forms of $U(V)$ and $U(W)$. Actually, it turns out that there exists a unique relevant pure inner form $U(V_{qs}) \times U(W_{qs})$ which is quasisplit and on which we can fix a Whittaker datum through the choice of a nontrivial character $\psi : E \to \mathbb{C}^\times$ that is, moreover, trivial for F in the unitary case (see [23, §12] for details). With these prerequisites in place, we can now state:

Conjecture 1.1 (Gan–Gross–Prasad). *Let $\phi : \mathcal{L}_F \to {}^L U(V)$ and $\phi' : \mathcal{L}_F \to {}^L U(W)$ be L-parameters for $U(V)$ and $U(W)$, respectively. Assume that the corresponding L-packets*

3 This situation is peculiar to classical groups because those can be realized as *twisted endoscopic groups* of some GL_N.

$\Pi(\phi)$, $\Pi(\phi')$ are generic, that is, they contain one representation which is generic with respect to each Whittaker datum. Then:

(1) *There exists a unique pair*

$$(\pi, \sigma) \in \bigsqcup_{W'} \Pi^{V'}(\phi) \times \Pi^{W'}(\phi'),$$

where the union runs over relevant pure inner forms, such that $m(\pi, \sigma) = 1$.

(2) *The unique characters $\chi \in \widehat{S_\phi}$ and $\chi' \in \widehat{S_{\phi'}}$ such that $\pi = \pi(\phi, \chi)$ and $\sigma = \pi(\phi', \chi')$ are given by explicit formulas involving local root numbers, e.g., in the unitary case, identifying ϕ, ϕ' with conjugate-self-dual representations of $\mathcal{L}_E$ and using the description (1.4) of $S_{\phi'}$ in terms of the decomposition (1.3), we have*

$$\chi(e_i) = \varepsilon(\phi_i \otimes \phi', \psi_{2\delta}), \quad \text{for all } i \in I. \tag{1.6}$$

Here δ stands for the discriminant of the odd dimensional Hermitian space among (V_{qs}, W_{qs}), $\psi_{2\delta}(z) := \psi(2\delta z)$ and $\varepsilon(\phi_i \otimes \phi', \psi_{2\delta})$ denotes the local root number of the Weil or Weil–Deligne representation $\phi_i \otimes \phi'$ associated to this additive character [43].

When $(\dim(V), \dim(W)) = (3, 2)$ (quadratic case) or $(\dim(V), \dim(W)) = (2, 1)$ (Hermitian case), the above conjecture essentially reduces to the result of Tunnell and Saito [44] on restrictions of irreducible representations of GL(2) to a maximal torus mentioned in the introduction. There has been a lot of recent progress towards Conjecture 1.1 and here is the status of what is currently known in the characteristic zero case:

Theorem 1.1. *Assume that F is of characteristic 0. Then:*

(1) *Both (1) and (2) of Conjecture 1.1 hold true in the following cases: if V, W are Hermitian spaces (i.e., in the unitary case) or if these are quadratic spaces and F is p-adic.*

(2) *Conjecture 1.1 (1) is verified when V, W are quadratic spaces and F is Archimedean.*

The first real breakthrough on Conjecture 1.1 was made by Waldspurger who established in a stunning series of papers [38, 47–49], the last one in collaboration with Mœglin, the full conjecture for p-adic special orthogonal groups under the assumption that the local Langlands correspondence is known for those groups and have expected properties. In my PhD thesis [8–10], I extended the method to deal with p-adic unitary groups therefore obtaining the conjecture under the slightly weaker assumption that the parameters ϕ, ϕ' are *tempered* which means that the corresponding L-packets consist of tempered representations. The extension to generic L-packets was carried out in the appendix to [24] using crucially a result of Heiermann. Later, I revisited Waldspurger's method which is based on a novel sort of local trace formulae, putting it on firmer grounds, and in the monograph [12] I established part (1) of the conjecture (sometimes called the *multiplicity one property for L-packets*) for unitary groups over arbitrary fields of characteristic 0, thus reproving part of my thesis in the p-adic case, and still under the assumption that L-parameters are tempered which as we will see is quite natural from the method. In the meantime, H. He [28] has developed a different approach to the conjecture based on the local θ-correspondence and very special features of the representation theory of real unitary groups (in particular, this approach cannot deal with p-adic groups) which allowed him to prove the full conjecture for those groups whenever ϕ and ϕ' are *discrete* parameters (a stronger condition than being tempered). Recently, this technique was enhanced by H. Xue [54] who was able to show the conjecture for real unitary groups without any restriction. Finally, in the recent preprint [36] Z. Luo adapted my previous work to deal with real special orthogonal groups proving the multiplicity one property for tempered L-packets.

1.2. Approach through local trace formulae

Let me give more details on the general structure of the approach taken by Waldspurger which was clarified and then further refined in [12]. It is mainly based on one completely novel ingredient that is a formula expressing the multiplicity $m(\pi,\sigma)$ in terms of the Harish-Chandra characters of π and σ. To be more specific, we recall a deep result of Harish-Chandra asserting that the distribution-character of a smooth irreducible representation π, i.e. the distribution $f \in C_c^\infty(U(V)) \mapsto \operatorname{Trace}\pi(f)$, can be represented by a locally L^1 function Θ_π known as its Harish-Chandra character. The aforementioned formula gives an identity roughly of the form:

$$m(\pi,\sigma) = \int_{\Gamma(V,W)}^{\text{reg}} c_\pi(x) c_\sigma(x^{-1}) dx, \tag{1.7}$$

where $\Gamma(V,W)$ is a certain set of semisimple conjugacy classes in $U(V)$ equipped with some measure dx reminiscent of Weyl integration formula (although it is more singular than measures coming from maximal tori, e.g., singular orbits are typically not negligible for dx), $c_\pi(x)$ and $c_\sigma(x^{-1})$ are renormalized values for the characters Θ_π and Θ_σ, respectively (although these characters are smooth on open dense subsets of regular semisimple elements, they typically blow up at the singular conjugacy classes in $\Gamma(V,W)$; the renormalization is based on further results of Harish-Chandra describing the local behavior of characters near singular elements), and finally the "reg" sign indicates that the integral itself has sometimes to be regularized in a certain way (or put another way, it is *improperly* convergent). Originally, formula (1.7) was only proven to hold for *tempered* representations but through the process of reducing the general conjecture to the tempered case, it was eventually shown a posteriori to hold for every irreducible representations belonging to generic L-packets. In the degenerate case where $U(V)$ is compact, the right-hand side of the integral formula (1.7) reduces to the L^2-scalar product of $\Theta_\pi\mid_{U(W)}$ and Θ_σ and the formula itself is an easy consequence of the orthogonality relations of characters, but in general the formula looks much more mysterious.

Let us sketch very briefly how we can deduce from formula (1.7) the first part of Conjecture 1.1 for tempered parameters (multiplicity one in tempered L-packets). The idea, due to Waldspurger, is to take advantage of inner cancellations in the sum

$$\sum_{W'} \sum_{(\pi,\sigma)} m(\pi,\sigma) = \sum_{W'} \sum_{(\pi,\sigma)\in\Pi^{V'}(\phi)\times\Pi^{W'}(\phi')} \int_{\Gamma(V',W')}^{\mathrm{reg}} c_\pi(x) c_\sigma(x^{-1})dx \tag{1.8}$$

that can be deduced from certain character relations (which are basic instances of the already mentioned endoscopic relations). The first step is to rewrite the sum as

$$\sum_{W'} \int_{\Gamma(V',W')}^{\mathrm{reg}} c_\phi^{V'}(x) c_{\phi'}^{W'}(x^{-1})dx \tag{1.9}$$

where $\Theta_\phi^{V'} = \sum_{\pi\in\Pi^{V'}(\phi)} \Theta_\pi$, $\Theta_{\phi'}^{W'} = \sum_{\sigma\in\Pi^{W'}(\phi')} \Theta_\sigma$ and $c_\phi^{V'}(x)$, $c_{\phi'}^{W'}(x^{-1})$ are renormalized values for those characters as before. The first property of the Langlands correspondence that we need is that $\Theta_\phi^{V'}$, $\Theta_{\phi'}^{W'}$ are *stable*, i.e., are constant on the union of semisimple regular conjugacy classes that become the same over an algebraic closure (a so-called regular stable conjugacy class). It follows from this stability property that the renormalized functions $c_\phi^{V'}$, $c_{\phi'}^{W'}$ are also invariant under a suitable extension of stable conjugation for singular elements. Consequently, the sum of multiplicities can be further rewritten as

$$\sum_{W'} \sum_{(\pi,\sigma)} m(\pi,\sigma) = \sum_{W'} \int_{\Gamma(V',W')/\mathrm{stab}}^{\mathrm{reg}} c_\phi^{V'}(y) c_{\phi'}^{W'}(y^{-1})dy, \tag{1.10}$$

where $\Gamma(V', W')/\mathrm{stab}$ stands for the space of stable conjugacy classes in $\Gamma(V', W')$. At this point, it is convenient to make the simplifying assumption that F is p-adic and W is not a split quadratic space of dimension $\leqslant 2$. Then, there are exactly two relevant pure inner forms $U(V) \times U(W)$ and $U(V') \times U(W')$ with, say, the first one quasisplit. Moreover, the character relations in this case read

$$\Theta_\phi^V(y) = \varepsilon_V \Theta_\phi^{V'}(y') \quad \big(\text{resp. } \Theta_{\phi'}^W(y) = \varepsilon_W \Theta_{\phi'}^{W'}(y')\big)$$

for certain signs $\varepsilon_V, \varepsilon_W \in \{\pm 1\}$ satisfying $\varepsilon_V \varepsilon_W = -1$ and for every regular stable conjugacy classes y, y' in $U(V)$, $U(V')$ (resp. in $U(W)$, $U(W')$) that are related by a certain correspondence (which is just an identity of characteristic polynomials except in the even orthogonal case). This correspondence actually naturally extends to give an embedding $\Gamma(V', W')/\mathrm{stab} \hookrightarrow \Gamma(V, W)/\mathrm{stab}$, $y' \mapsto y$, for which we have

$$c_\phi^V(y) c_{\phi'}^W(y) = -c_\phi^{V'}(y') c_{\phi'}^{W'}(y').$$

This implies that in the right-hand side of (1.10), all the terms indexed by $\Gamma(V', W')/\mathrm{stab}$ can be cancelled with the corresponding terms coming from their images in $\Gamma(V, W)/\mathrm{stab}$. The only remaining contribution, it turns out, is that of the trivial conjugacy class:

$$\sum_{W'} \sum_{(\pi,\sigma)} m(\pi,\sigma) = c_\phi^V(1) c_{\phi'}^W(1) \tag{1.11}$$

which, by a result of Rodier, can be interpreted as the number of representations in the packet $\Pi^V(\phi) \otimes \Pi^W(\phi')$ that are generic with respect to a certain Whittaker datum (actually really an average of such numbers over all Whittaker data in the unitary case). By a third property of tempered L-packets (existence and unicity of a generic representation for a given Whittaker datum), this number is just 1 and this immediately implies the first part of Conjecture 1.1.

The proof of the multiplicity formula (1.7), on the other hand, is much more involved. Set $G = U(W) \times U(V)$ and $H = U(W) \ltimes N$ that we see as a subgroup of G through the natural diagonal embedding. Then, following the approach that I have developed in [12], (1.7) can be deduced from a certain simple trace formula for the "space" $X = (H, \xi)\backslash G$. More precisely, this trace formula is roughly seeking to compute the trace of the convolution operators

$$\phi \in L^2(X, \xi) \mapsto (R(f)\phi)(x) = \int_G f(g)\phi(xg)dg, \quad \text{for } f \in C_c^\infty(G),$$

where $L^2(X, \xi)$ denotes the Hilbert space of measurable functions ϕ on G satisfying $\phi(hg) = \xi(h)\phi(g)$ for $(h, g) \in H \times G$ and $\int_{H\backslash G} |\phi(x)|^2 dx < \infty$. It is classical, and easy to see, that these operators are given by kernels,

$$\big(R(f)\phi\big)(x) = \int_X K_f(x, y)\phi(y)dy, \quad \text{for } (f, \phi) \in C_c^\infty(G) \times L^2(X, \xi),$$

where $K_f(x, y) = \int_H f(x^{-1}hy)\xi(h)dh$. Thus, at a formal level (hence the quotation marks) we have

$$\text{“Trace } R(f) = \int_X K_f(x, x)dx\text{”}.$$

However, neither of the two sides above make sense in general: the convolution operator is not of trace-class and the kernel not integrable over the diagonal. The basic idea is then to restrict oneself to a subspace of test functions for which at least one of the two expressions is meaningful. A convenient such subspace is that of *strongly cuspidal* functions introduced by Waldspurger in [47]: a function $f \in C_c^\infty(G)$ is strongly cuspidal if for every proper parabolic subgroup $P = MU \subsetneq G$, we have

$$\int_U f(mu)du = 0, \quad \forall m \in M.$$

Moreover, as is shown in [12], for $f \in C_c^\infty(G)$ strongly cuspidal, the integral

$$J(f) = \int_X K_f(x, x)dx$$

is absolutely convergent (the argument of [12] is given in the context of Gan–Gross–Prasad for unitary groups but it can be adapted to a much more general context). Then, the aforementioned simple local trace formula expands the distribution $f \to J(f)$ in two different ways:

Theorem 1.2. *For every strongly cuspidal $f \in C_c^\infty(G)$, we have the identities*

$$\int_{\Gamma(V,W)}^{\text{reg}} c_f(x)dx = J(f) = \int_{\mathcal{X}(G)} m(\Pi)\widehat{\theta_f}(\Pi)d\Pi, \tag{1.12}$$

where

- *$c_f(x)$ is the renormalized value of a function $x \mapsto \theta_f(x)$ constructed from weighted orbital integrals of f in the sense of Arthur [3] and whose local behavior is similar to that of Harish-Chandra characters on the group G;*

- *$\mathcal{X}(G)$ is a certain space of virtual representations of G obtained by parabolic induction from the so-called elliptic representations (as defined in* **[6]***) of Levi subgroups and $f \mapsto \widehat{\theta_f}(\Pi)$ is a weighted character in the sense of Arthur* **[4]***;*
- *Finally, for an irreducible representation $\Pi = \pi \otimes \sigma$ of G, $m(\Pi)$ is defined as the multiplicity $m(\pi, \sigma^\vee)$ with $\sigma^\vee$ the smooth contragredient of σ.*

We refer the reader to **[12]** for precise definitions of all the terms and a proof in the case of unitary groups. This was adapted in **[36]** to special orthogonal groups. The deduction of the integral formula (1.7) roughly goes as follows: we first show the multiplicity formula for representations that are properly parabolically induced by expressing both sides in terms of the inducing data and applying an induction hypothesis whereas for the remaining representations, the so-called elliptic representations, the formula can be obtained by applying the trace formula (1.12) to some sort of *pseudocoefficient*.

Finally, let us say a word on how the more refined part (2) of Conjecture 1.1 can be proven using this approach (so far it has only been done for p-adic groups in **[49]** and **[9]**, following the previous slightly different method of Waldspurger, but there is little doubt that the techniques developed in **[12]** should allow to treat the case of real groups in a similar way). For Langlands parameters ϕ, ϕ' as in Conjecture 1.1, as well as characters $\chi \in \widehat{S_\phi}$, $\chi' \in \widehat{S_{\phi'}}$, combining the multiplicity formula (1.7) with the general endoscopic character relations that characterize the Langlands correspondences for $U(V)$ and $U(W)$, we can express $m(\pi(\phi, \chi), \sigma(\phi', \chi'))$ as a sum of integrals of (renormalized) twisted characters on some products $\mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$. The remaining ingredient is to relate these integrals of twisted characters to the epsilon factors of pairs defined by Jacquet–Piatetski-Shapiro–Shalika in **[31]**. More precisely, these expressions involve the twisted characters of tempered irreducible representations $\pi^{\mathrm{GL}}, \sigma^{\mathrm{GL}}$ of general linear groups $\mathrm{GL}_n(E), \mathrm{GL}_m(E)$, with $n \geqslant m$ of distinct parities, which are self-dual (in the orthogonal case) or conjugate-self-dual (in the unitary case). These properties of (conjugate-)self-duality imply that π^{GL} and σ^{GL} extend to representations $\pi^{\mathrm{GL}}, \sigma^{\mathrm{GL}}$ of the nonconnected groups $\mathrm{GL}_n(E) \rtimes \langle\theta_n\rangle$ and $\mathrm{GL}_m(E) \rtimes \langle\theta_m\rangle$, respectively, where θ_k $(k = n, m)$ denotes the automorphism $g \mapsto {}^t(g^c)^{-1}$. The twisted characters in question are then the restrictions $\Theta_{\pi^{\mathrm{GL}}}$ and $\Theta_{\sigma^{\mathrm{GL}}}$ of the Harish-Chandra characters of π^{GL} and σ^{GL} to the nonneutral components $\widetilde{\mathrm{GL}_n}(E) = \mathrm{GL}_n(E)\theta_n$ and $\widetilde{\mathrm{GL}_m}(E) = \mathrm{GL}_m(E)\theta_m$, respectively. Replacing the functions c_π, c_σ by similar suitable renormalizations of these twisted characters at singular semisimple conjugacy classes, there is a formula very analogous to (1.7) for the ε-factor of pair $\varepsilon(\pi^{\mathrm{GL}} \times \sigma^{\mathrm{GL}}, \psi)$.

For p-adic fields, this formula was established in **[48]** in the self-dual case and in **[8]** in the conjugate-self-dual case. The proof follows closely that of (1.7) and is based on a local trace formula very similar to that of Theorem 1.2 for the natural action of $G' := \widetilde{\mathrm{GL}_n}(E) \times \widetilde{\mathrm{GL}_m}(E)$ on the homogeneous space $X' = H' \backslash G'$ where $G' = \mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$ and $H' = \mathrm{GL}_m(E) \ltimes N'$ is the semidirect product with a unipotent subgroup N' whose definition is analogous to that of N. More precisely, there is also a similar unitary character ξ' of N' that is $\mathrm{GL}_m(E)$-invariant and the twisted trace formula we are mentioning is roughly trying to compute the trace of convolution operators $R(f)$ of functions $f \in C_c^\infty(G')$ on

$L^2(X', \xi')$. Rather than describing it in details, let us just explain how the ε-factors show up in the analysis. As in Theorem 1.2, one of the main ingredient on the spectral side of this trace formula is a twisted multiplicity $m(\pi^{\mathrm{GL}} \otimes \sigma^{\mathrm{GL}})$ which computes the trace of a natural operator on the space of intertwiners

$$\mathrm{Hom}_H(\pi^{\mathrm{GL}} \otimes \sigma^{\mathrm{GL}}, \xi). \tag{1.13}$$

The operator in question is given by $\ell \mapsto \ell \circ (\pi^{\mathrm{GL}} \otimes \sigma^{\mathrm{GL}})(\theta)$ where $\theta \in \widetilde{\mathrm{GL}}_n(E) \times \widetilde{\mathrm{GL}}_m(E)$ is a certain element stabilizing the pair (H, ξ) (which is anyway needed to extend the right action of G' on $L^2(X', \xi')$ to an action of G'). Actually, it turns out that the space (1.13) is always one-dimensional and a reformulation of the so-called *local functional equation* from [31] shows that this operator is essentially acting (for suitable normalizations of π^{GL}, σ^{GL} and up to more elementary factors) as multiplication by the ε-factor $\varepsilon(\pi \times \sigma, \psi)$.

2. THE GLOBAL GAN–GROSS–PRASAD CONJECTURES AND ICHINO–IKEDA REFINEMENTS

2.1. Statements and results

We now move to a global setting. Let E/F be a quadratic extension of number fields and $W \subset V$ be Hermitian spaces over E satisfying condition (1.1) (there are similar, and actually prior, conjectures for orthogonal groups, but here we will concentrate on unitary groups for which much more is known). By a construction similar to that from the previous section, we may obtain from these data a triple (G, H, ξ) where $G = U(V) \times U(W)$, $H = U(W) \ltimes N$ is a subgroup of G (which we will this time consider as *algebraic groups* over F) and $\xi : N(\mathbb{A}_F) \to \mathbb{C}^\times$ is a character on the adelic points of N trivial on the subgroup $N(F)$ and that extends to a character of $H(\mathbb{A}_F)$ trivial on $U(W)(\mathbb{A}_F)$.

The global analog of the previous branching problem is that of characterizing the nonvanishing of the *automorphic period* associated to the pair (H, ξ). More precisely, if $\pi = \pi_V \otimes \pi_W \hookrightarrow \mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F))$ is a cuspidal automorphic representation of $G(\mathbb{A}_F)$, we consider the automorphic period

$$\mathcal{P}_{H,\xi} : \pi \to \mathbb{C},$$
$$\mathcal{P}_{H,\xi}(\varphi) = \int_{[H]} \varphi(h)\xi(h)dh, \tag{2.1}$$

where here and throughout the rest of the paper, for a linear algebraic group R over F, we denote by $[R] = R(F)\backslash R(\mathbb{A}_F)$ the corresponding automorphic quotient. On the other hand, let $\pi_E = \pi_{V,E} \otimes \pi_{W,E}$ be the (weak) base-change of π to $\mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_m(\mathbb{A}_E)$ where $(n, m) = (\dim(V), \dim(W))$. Here, $\pi_{V,E}$, $\pi_{W,E}$ are automorphic representations whose Satake parameters at almost every unramified places are the image by the base-change homomorphisms ${}^L U(V) \to {}^L \mathrm{Res}_{E/F} \mathrm{GL}_{n,E}$, ${}^L U(W) \to {}^L \mathrm{Res}_{E/F} \mathrm{GL}_{m,E}$ (where $\mathrm{Res}_{E/F}$ denotes Weil's restriction of scalars) of the local Satake parameters of π_V, π_W, respectively. The existence of these weak base-changes in general is one of the main results of [34,39]. Also, although $\pi_{V,E}$, $\pi_{W,E}$ are not always cuspidal, they are isobaric sums of cuspidal representations which implies, by a result of Jacquet and Shalika, that they are uniquely determined by their Satake parameters at almost all places hence that the weak base-change π_E is unique. We denote by

$$L(s, \pi_E) = L(s, \pi_{V,E} \times \pi_{W,E})$$

the corresponding completed Rankin–Selberg L-function associated to $\pi_{V,E}$ and $\pi_{W,E}$.

Define the *automorphic L-packet* of π as the set of cuspidal automorphic representations π' of the various pure inner form $G' = U(V') \times U(W')$ of G with the same base-change $\pi'_E = \pi_E$ as π. By the Jacquet–Shalika theorem again and injectivity of base-change homomorphisms at the level of conjugacy classes, it is equivalent to asking that π and π' are *nearly equivalent*, that is, $\pi_v \simeq \pi'_v$ for almost all places v (this makes sense since $G_v \simeq G'_v$ for almost all v). Moreover, for a relevant pure inner form G' of G, we can define a pair (H', ξ') in exactly the same way as (H, ξ). The global version of the Gan–Gross–Prasad conjecture can now be stated as follows:

Conjecture 2.1 (Gan–Gross–Prasad **[23]**). *Assume that π_E is generic. Then, the following assertions are equivalent:*

(1) $L(\frac{1}{2}, \pi_E) \neq 0$*;*

(2) *There exists a relevant pure inner form $G' = U(W') \times U(V')$ of G (see Section 1.1 for the definition of a relevant pure inner form), a cuspidal automorphic representation π' of $G'(\mathbb{A}_F)$ in the same automorphic L-packet as π and a form $\varphi' \in \pi'$ such that*

$$\mathcal{P}_{H',\xi'}(\varphi') \neq 0.$$

When $(\dim(V), \dim(W)) = (2, 1)$, the conjecture essentially reduces to the celebrated theorem of Waldspurger **[46]** on toric periods for GL_2. Actually, as explained in the introduction, Waldspurger's result is more precise and gives an explicit identity relating (the square of) $\mathcal{P}_H(\varphi)$ to the central value $L(\frac{1}{2}, \pi_E)$.

There is also a similar conjecture for special orthogonal groups which actually predates the one for unitary groups **[26]** (as well as other conjectures for the so-called Fourier–Jacobi periods on unitary and symplectic/metaplectic groups stated in **[23]**). In **[30]**, Ichino and Ikeda have proposed a refinement of this conjecture for $SO(n) \times SO(n-1)$ in the form of a precise identity generalizing Waldspurger's formula. Subsequently, similar refinements have been proposed by R. N. Harris **[27]**, for $U(n) \times U(n-1)$, and then by Y. Liu **[35]** for general Bessel periods on orthogonal or unitary groups.

In order to state this refinement, we need to introduce two extra ingredients, namely local periods and a certain finite group S_π of endoscopic nature.

We start with the local periods. We endow $H(\mathbb{A}_F)$ with its global Tamagawa measure dh (this is the measure with which we will normalize the period integral (2.1)) and we fix a factorization $dh = \prod_v dh_v$ into a product of local Haar measures. We also fix a decomposition $\pi = \bigotimes'_v \pi_v$ of π into smooth irreducible representations of the localizations

$G_v = G(F_v)$ as well as a factorization $\langle \cdot, \cdot \rangle_{\mathrm{Pet}} = \prod_v \langle \cdot, \cdot \rangle_v$ of the Petersson inner product

$$\langle \varphi, \varphi \rangle_{\mathrm{Pet}} = \int_{G(F)\backslash G(\mathbb{A}_F)} \left|\varphi(g)\right|^2 dg$$

(which we also normalize using the Tamagawa measure on $G(\mathbb{A}_F)$) into local invariant inner products. The local periods are now given by the sesquilinear forms

$$\mathcal{P}_{H,\xi,v} : \varphi_v \otimes \varphi_v' \in \pi_v \otimes \pi_v \mapsto \int_{H_v}^{\mathrm{reg}} \left\langle \pi_v(h_v)\varphi_v, \varphi_v' \right\rangle_v \xi_v(h_v)\, dh_v. \tag{2.2}$$

The above integral of matrix coefficient is actually not convergent in general and has to be regularized (hence the "reg" sign above the integral). This regularization is, moreover, only possible under the extra assumption that the local component π_v is tempered. It is expected (under the Generalized Ramanujan Conjecture) that the hypothesis of the base-change π_E being generic implies that each of the local component π_v is tempered, but this is far from being known in general. Assuming now that π_v is tempered at every place v, an unramified computation shows that for almost all places v, if $\varphi_v \in \pi_v^{G(\mathcal{O}_v)}$ is a spherical vector such that $\langle \varphi_v, \varphi_v \rangle_v = 1$, we have

$$\mathcal{P}_{H,\xi,v}(\varphi_v, \varphi_v) = \Delta_v \frac{L(\frac{1}{2}, \pi_{E,v})}{L(1, \pi_v, Ad)}$$

where $L(\frac{1}{2}, \pi_{E,v})$, $L(1, \pi_v, Ad)$ denote the local Rankin–Selberg and adjoint L-factors of π_E and π, respectively, whereas Δ_v stands for the product of local abelian L-factors

$$\Delta_v = \prod_{i=1}^{n} L(i, \eta_{E_v/F_v}^i)$$

with η_{E_v/F_v} the quadratic character associated to the local extension E_v/F_v and $n = \dim(V)$. The *normalized* local periods are then defined by

$$\mathcal{P}^{\natural}_{H,\xi,v}(\varphi_v, \varphi_v) = \Delta_v^{-1} \frac{L(1, \pi_v, Ad)}{L(\frac{1}{2}, \pi_{E,v})} \mathcal{P}_{H,\xi,v}(\varphi_v, \varphi_v).$$

Finally, writing the base-change π_{V_E} and $\pi_{W,E}$ as isobaric sums

$$\pi_{V,E} = \pi_{V,1} \boxplus \cdots \boxplus \pi_{V,k}, \quad \pi_{W,E} = \pi_{W,1} \boxplus \cdots \boxplus \pi_{W,l}$$

of cuspidal automorphic representations of some general linear groups, we set $S_\pi = (\mathbb{Z}/2\mathbb{Z})^{k+l}$. It serves as a substitute for the centralizer of the, yet nonexistent in general, global Langlands parameter of π.

Conjecture 2.2 (Ichino–Ikeda, N. Harris, Y. Liu). *Assume that for every place v of F, π_v is a tempered representation. Then, for every factorizable vector $\varphi = \bigotimes_v' \varphi_v \in \pi$, we have*

$$\left|\mathcal{P}_{H,\xi}(\varphi)\right|^2 = |S_\pi|^{-1} \Delta \frac{L(\frac{1}{2}, \pi_E)}{L(1, \pi, Ad)} \prod_v \mathcal{P}^{\natural}_{H,\xi,v}(\varphi_v, \varphi_v) \tag{2.3}$$

where $\Delta = \prod_{i=1}^n L(i, \eta_{E/F}^i)$ and $L(s, \pi, Ad) = \prod_v L(s, \pi_v, Ad)$ denotes the completed adjoint L-function of π.

Note that at a formal level, that is, formally expanding L-functions as Euler products outside the range of convergence, the above formula can be rewritten in the more compact way as

$$\left|\mathcal{P}_{H,\xi}(\varphi)\right|^2 = |S_\pi|^{-1} \prod_v{}' \mathcal{P}_{H,\xi,v}(\varphi_v, \varphi_v), \tag{2.4}$$

where the prime symbol on the product sign indicates that it is not convergent and has to be suitably reinterpreted "in the sense of L-functions" as identity (2.3).

Thanks to the work of many authors that we are going to summarize in the next sections, it is now relatively easy to state the current status on these two conjectures:

Theorem 2.1. *Both Conjectures 2.1 and 2.2 hold in full generality.*

The rest of this paper is devoted to reviewing the long series of works leading to the above theorem. They all stem from a strategy originally proposed by Jacquet and Rallis **[32]** of comparing two relative trace formulae. Let us mention here that there has actually been other fruitful approaches to the global Gan–Gross–Prasad conjecture among which one of the most notable has been the method pioneered by Ginzburg–Jiang–Rallis **[25]** using automorphic descent and that has recently seen much development with the work **[33]** of Jiang and L. Zhang proving in full generality the implication (2) $\Rightarrow$ (1) of Conjecture 2.1.

2.2. The approach of Jacquet–Rallis

In **[32]**, Jacquet and Rallis have proposed a way to attack the Gan–Gross–Prasad conjecture for unitary groups through a comparison of relative trace formulae. They only consider the case where $\dim(W) = \dim(V) - 1$ (in which case $H = U(W)$ and the character ξ is trivial) and we assume throughout that this condition is satisfied. The global relative trace formulae considered here are powerful analytic tools introduced originally by Jacquet and that relate automorphic periods to more geometric distributions known as relative orbital integrals.

Let us be more specific in the case at hand. For $f \in C_c^\infty(G(\mathbb{A}_F))$, a global test function, we let

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y), \quad x, y \in G(F)\backslash G(\mathbb{A}_F),$$

be its automorphic kernel which describes the operator $R(f)$ of right convolution by f on the space of automorphic forms. The first trace formula introduced by Jacquet and Rallis is formally obtained by expanding the (usually divergent) expression

$$J(f) = \int_{[H]\times[H]} K_f(h_1, h_2) dh_1 dh_2 \tag{2.5}$$

in two different ways. More precisely, but still at a formal level, this distribution can be expanded as

$$\cdots + \sum_{\delta \in H(F)\backslash G_{\mathrm{rs}}(F)/H(F)} O(\delta, f) = J(f) = \sum_{\varphi \in \mathcal{A}_{\mathrm{cusp}}(G)} \mathcal{P}_H\big(R(f)\varphi\big)\overline{\mathcal{P}_H(\varphi)} + \cdots, \tag{2.6}$$

where the right sum runs over an orthonormal basis for the space of cuspidal automorphic forms whereas the left sum is indexed by the so-called *regular semisimple* double cosets of $H(F)$ in $G(F)$. Here, an element $\delta \in G$ is called (relatively) regular semisimple if its stabilizer under the $H \times H$-action is trivial and the corresponding orbit is (Zariski) closed. We denote by G_{rs} the nonempty Zariski open subset of regular semisimple elements and for $\delta \in G_{\mathrm{rs}}(F)$,

$$O(\delta, f) = \int_{H(\mathbb{A}_F)\times H(\mathbb{A}_F)} f(h_1\delta h_2)dh_1dh_2$$

denotes the corresponding *relative orbital integral* of f at δ. The left suspension points in (2.6) represent the remaining contributions from singular orbits whereas the right suspension points indicate the contribution of the continuous spectrum (both of which are somehow responsible for the divergence of the original expression (2.5)).

The second trace formula introduced by Jacquet and Rallis has to do with the following triple of groups:

$$\begin{aligned} H_1 &= \mathrm{Res}_{E/F}\,\mathrm{GL}_{n,E} \hookrightarrow G' = \mathrm{Res}_{E/F}\,\mathrm{GL}_{n+1,E} \times \mathrm{Res}_{E/F}\,\mathrm{GL}_{n,E} \hookleftarrow H_2 \\ &= \mathrm{GL}_{n+1,F} \times \mathrm{GL}_{n,F}, \end{aligned}$$

where $n = \dim(W)$, the first embedding is the diagonal one and the second embedding is the natural one. Note that G' is the group on which the base-change π_E "lives." For $f' \in C_c^\infty(G'(\mathbb{A}_F))$, we write (again formally)

$$I(f') = \int_{[H_1]\times[H_2]} K_{f'}(h_1, h_2)\eta(h_2)dh_1dh_2 \tag{2.7}$$

where $K_{f'}$ is the automorphic kernel of f' and $\eta : [H_2] \to \{\pm 1\}$ is the automorphic character defined by $\eta(g_n, g_{n+1}) = \eta_{E/F}(\det g_n)^{n+1}\eta_{E/F}(\det g_{n+1})^n$. This formal distribution can be analogously expanded as

$$\cdots + \sum_{\gamma \in H_1(F)\backslash G'_{\mathrm{rs}}(F)/H_2(F)} O_\eta(\gamma, f') = I(f') = \sum_{\varphi \in \mathcal{A}_{\mathrm{cusp}}(G')} \mathcal{P}_{H_1}\big(R(f')\varphi\big)\overline{\mathcal{P}_{H_2,\eta}(\varphi)} + \cdots, \tag{2.8}$$

where G'_{rs} stands for the open subset of regular and semisimple elements under the $H_1 \times H_2$-action, the relative orbital integrals are given by

$$O_\eta(\gamma, f') = \int_{H_1(\mathbb{A}_F)\times H_2(\mathbb{A}_F)} f'(h_1\gamma h_2)\eta(h_2)dh_1dh_2$$

and $\mathcal{P}_{H_1}$, $\mathcal{P}_{H_2,\eta}$ denote the automorphic period integrals over $[H_1]$ and $[H_2]$ twisted by η, respectively.

The discussion so far is, of course, oversimplifying and ignoring serious analytical and convergence issues (we will come back to this later). However, as a motivation for considering this relative trace formula on G', we have the following results on automorphic periods:

- The period $\mathcal{P}_{H_1}$ is a Rankin–Selberg period studied by Jacquet–Piatetskii-Shapiro–Shalika that essentially represents the central value $L(\frac{1}{2}, \Pi)$ on $\Pi \hookrightarrow \mathcal{A}_{\mathrm{cusp}}(G')$;

- The period $\mathcal{P}_{H_2,\eta}$ was studied by Rallis and Flicker who have shown that it detects exactly the cuspidal automorphic Π's that come by base-change from G (i.e., it is nonzero precisely on those cuspidal representations of the form π_E, for π a cuspidal automorphic representation of G).

Thus, on a very formal and sketchy sense, the Gan–Gross–Prasad conjecture implies that the spectral sides of $I(f')$ should somehow "match" that of $J(f)$. The idea of Jacquet and Rallis was to make precise the existence of such a comparison, from which the global Gan–Gross–Prasad conjecture was eventually to be deduced, by equalling the geometric sides term by term. As a first step, they define a *correspondence* of orbits, which here takes the form of a natural embedding between regular semisimple cosets

$$H(k)\backslash G_{\mathrm{rs}}(k)/H(k) \hookrightarrow H_1(k)\backslash G'_{\mathrm{rs}}(k)/H_2(k), \quad \delta \mapsto \gamma, \tag{2.9}$$

for every field extension k/F. Using this correspondence, they then introduced a related notion of local *transfer* (or *matching*): for a place v of F, two test functions $f_v \in C_c^\infty(G_v)$ and $f'_v \in C_c^\infty(G'_v)$ are said to be transfers of each other (simply denoted by $f_v \leftrightarrow f'_v$ for short) if for every $\delta \in H(F_v)\backslash G_{\mathrm{rs}}(F_v)/H(F_v)$ we have an identity

$$O(\delta, f_v) = \Omega_v(\gamma) O_{\eta_v}(\gamma, f'_v), \tag{2.10}$$

where $\gamma \in H_1(F_v)\backslash G'_{rs}(F_v)/H_2(F_v)$ is the image of δ by the above correspondence, $O(\delta, f_v)$ and $O_{\eta_v}(\gamma, f'_v)$ are local relative orbital integrals defined in the same way as their global counterparts (replacing in the domain of integration, adelic groups by the corresponding local groups) and $\gamma \mapsto \Omega_v(\gamma)$ is a certain transfer factor which in particular has the effect of making the right-hand side above $H_1(F_v) \times H_2(F_v)$-invariant in γ.

As in the usual paradigm of endoscopy, to make this notion useful and allow for a global comparison we basically need two local ingredients: first the existence of local transfer (i.e., for every $f_v \in C_c^\infty(G_v)$ there exists $f'_v \in C_c^\infty(G'_v)$ such that $f_v \leftrightarrow f'_v$ and conversely, every f'_v admits a transfer f_v) and then a *fundamental lemma* (saying, at least, that $\mathbf{1}_{G(\mathcal{O}_v)} \leftrightarrow \mathbf{1}_{G'(\mathcal{O}_v)}$ for almost all v).

3. COMPARISON: LOCAL TRANSFER AND FUNDAMENTAL LEMMA

A first breakthrough on the Jacquet–Rallis approach to the Gan–Gross–Prasad conjecture was made in [57] by Wei Zhang who proved the existence of the local transfer at all non-Archimedean places. His strategy for doing so roughly goes as follows:

- The first step is to reduce to a statement on Lie algebras using some avatar of the exponential map (also known as Cayley map): we are then left with proving the existence of a similar transfer between the orbital integrals for the adjoint action of $U(W_v)$ on $\mathfrak{u}(V_v) = \mathrm{Lie}(U(V_v))$ and for the adjoint action of $\mathrm{GL}_n(F_v)$ on $\mathfrak{gl}_{n+1}(F_v)$.
- Then, a crucial ingredient in Zhang's proof is to show that the transfer at the Lie algebra level essentially commutes (i.e., up to some explicit multiplicative

constants) with 3 different partial Fourier transforms $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ that can naturally be defined on the two spaces $C_c^\infty(\mathfrak{u}(V_v))$, $C_c^\infty(\mathfrak{gl}_{n+1}(F_v))$. One of them, that we will denote by $\mathcal{F}_1$, is the Fourier transform with respect to "the last row and column" on $\mathfrak{gl}_{n+1}(F_v)$ or $\mathfrak{u}(V_v)$ when realizing the latter in matrix form using a basis adapted to the decomposition $V_v = W_v \oplus W_v^\perp$. (Recall that we are assuming that $\dim(W_v^\perp) = 1$.) For this, Zhang develops some relative trace formulae for the aforementioned actions on $\mathfrak{gl}_{n+1}(F_v)$ and $\mathfrak{u}(V_v)$ and combines them with a clever induction argument.

- Finally, the proof of the existence of transfer on Lie algebras is obtained by combining the second step with a certain uncertainty principle due to Aizenbud [1], which allows reducing the construction of the transfer to functions that are supported away from the *relative nilpotent cones* (i.e., the set of elements whose orbit closure contains an element of the center of the Lie algebra), as well as a standard descent argument whose essence goes back to Harish-Chandra.

It is noteworthy to mention that this result was subsequently extended, following the same strategy, by H. Xue [53] to Archimedean places, although the final result there is slightly weaker. (More precisely, Xue was only able to show that a *dense* subspace of test functions admit a transfer but also observed that it is sufficient for all expected applications.)

The Jacquet–Rallis fundamental lemma for its part, was proven earlier by Yun [55] in the case of fields of positive characteristic following and adapting the geometric-cohomological approach based on Hitchin fibrations that was developed by Ngô in the context of the endoscopic fundamental lemma. This result was then transferred to fields of characteristic zero, but of sufficiently large residual characteristic, using model-theoretic techniques by Julia Gordon in the appendix of [55].

Later, in [14], I found a completely new and elementary proof of this fundamental lemma. The argument, despite that of Gordon–Yun, works directly in characteristic zero and is purely based on techniques from harmonic analysis. Thus, we have:

Theorem 3.1 (Yun–Gordon, Beuzart-Plessis). *Let v be a place of F of residue characteristic not 2 that is unramified in E and assume that the Hermitian spaces W_v, $W_v^\perp$ both admit self-dual lattices L_v^W and $L_v^{W^\perp}$. Set $L_v = L_v^W \oplus L_v^{W^\perp}$ (a self-dual lattice in V_v) and $K_v = \mathrm{Stab}_{G_v}(L_v \times L_v^W)$ for the stabilizer in $G_v = U(V_v) \times U(W_v)$ of the lattices L_v and L_v^W (a hyperspecial compact subgroup of G_v). Then, setting $K_v' = \mathrm{GL}_{n+1}(\mathcal{O}_{E_v}) \times \mathrm{GL}_n(\mathcal{O}_{E_v})$, we have $\mathbf{1}_{K_v} \leftrightarrow \mathbf{1}_{K_v'}$.*

More precisely, in [14] I proved a Lie algebra analog of the Jacquet–Rallis fundamental lemma (of which the original statement can easily be reduced; at least in residual characteristic not 2) stating that the relative orbital integrals of $\mathbf{1}_{\mathfrak{u}(L_v)}$ *match* those of $\mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O}_{F_v})}$ in a suitable sense (where $\mathfrak{u}(L_v)$ denotes the lattice in $\mathfrak{u}(V_v)$ stabilizing L_v). The argument is based on a hidden SL(2) symmetry involving a Weil representation. More specifically, we consider the Weil representations of $\mathrm{SL}(2, F_v)$ associated to the quadratic form q sending a matrix of size $n+1$,

$$X = \begin{pmatrix} A & b \\ c & \lambda \end{pmatrix},$$

either in $\mathfrak{gl}_{n+1}(F_v)$ or in $\mathfrak{u}(V_v)$, to $q(X) = cb$ (where here, A is a square-matrix, b is a column vector, and c a row vector all of size n). Using the aforementioned result of Zhang that the transfer commutes with the partial Fourier transform $\mathcal{F}_1$, it can be shown that these representations descend to spaces of relative orbital integrals on $C_c^\infty(\mathfrak{u}(V_v))$ and $C_c^\infty(\mathfrak{gl}_{n+1}(F_v))$ and coincide on their intersections (identifying the spaces of regular semisimple orbits through a correspondence similar to (2.9)). Consider then the difference

$$\Phi : X \in \mathfrak{u}(V_v)_{\mathrm{rs}}/U(W_v) \mapsto O(X, \mathbf{1}_{\mathfrak{u}(L_v^V)}) - \omega_v(Y) O_{\eta_v}(Y, \mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O}_{F_v})}),$$

where $\mathfrak{u}(V_v)_{\mathrm{rs}}$ denotes the Lie algebra analog of the relative regular semisimple locus, Y is the image of X by a correspondence of orbits $\mathfrak{u}(V_v)_{\mathrm{rs}}/U(W_v) \hookrightarrow \mathfrak{gl}_{n+1}(F_v)_{\mathrm{rs}}/\mathrm{GL}_n(F_v)$ similar to (2.9) and $\omega_v(Y)$ is the Lie algebra counterpart of the transfer factor. The fundamental lemma then states that Φ is identically zero. The proof proceeds roughly in three steps:

- First, we show that $\Phi(X) = 0$ for $|q(X)| \geqslant 1$. When $|q(X)| = 1$, this requires an inductive argument on n. Moreover, this vanishing can be reformulated by saying that Φ is fixed by the subgroup $\left(\begin{smallmatrix} 1 & \mathfrak{p}_{F_v}^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ through the Weil representation.
- Secondly, we remark that Φ is also fixed by $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This comes from the fact that the action of w descends from the partial Fourier transform $\mathcal{F}_1$ which leaves (for a suitable normalization) the functions $\mathbf{1}_{\mathfrak{u}(L_v^V)}$, $\mathbf{1}_{\mathfrak{gl}_{n+1}(\mathcal{O}_{F_v})}$ invariant.
- Finally, as $\mathrm{SL}_2(F_v)$ is generated by $\left(\begin{smallmatrix} 1 & \mathfrak{p}_{F_v}^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ and w, we infer that Φ is fixed by $\mathrm{SL}_2(F_v)$ from which it is relatively straightforward to deduce $\Phi = 0$.

It is also worth mentioning that in a very interesting work, Jingwei Xiao **[51]** has shown that the Jacquet–Rallis fundamental lemma implies the (usual) endocospic fundamental lemma for unitary groups. Thus, combining his argument with the proof outlined above yields a completely elementary proof of the Langlands–Shelstad fundamental lemma for unitary groups!

The two previous results on smooth transfer and the fundamental lemma are already enough to imply the Gan–Gross–Prasad Conjecture 2.1 under some local restrictions on the cuspidal representation π (originating from the use of simple versions of the Jacquet–Rallis trace formulae, allowing to bypass all convergence issues) as was done by W. Zhang in **[57]**. However, to derive the refinement of Conjecture 2.2 following the same strategy, we need an extra local ingredient relating the local periods of Ichino–Ikeda to similar local distributions associated to the Rankin–Selberg and Flicker–Rallis periods. More precisely, by the work of Jacquet–Piatetskii-Shapiro–Shalika, on the one hand, and Flicker–Rallis, on the other hand, it is known that the two automorphic periods $\mathcal{P}_{H_1}$ and $\mathcal{P}_{H_2,\eta}$ admit factorizations of the form

$$\mathcal{P}_{H_1}(\varphi) = \prod_v{}' \mathcal{P}_{H_1,v}(W_{\varphi,v}), \tag{3.1}$$

$$\mathcal{P}_{H_2,\eta}(\varphi) = \frac{1}{4}\prod_v{}' \mathcal{P}_{H_2,\eta,v}(W_{\varphi,v}) \tag{3.2}$$

for φ a factorizable vector in a given cuspidal automorphic representation $\Pi = \Pi_{n+1} \otimes \Pi_n$ of $G'(\mathbb{A}_F)$, where

$$W_\varphi(g) = \int_{[N']} \varphi(ug)\psi'(u)^{-1}du = \prod_v W_{\varphi,v}(g_v)$$

denotes a factorization of the Whittaker function of φ (here N' stands for the standard maximal unipotent subgroup of G' and ψ' is a nondegenerate character of $[N']$), $\mathcal{P}_{H_1,v}$, $\mathcal{P}_{H_2,\eta,v}$ are explicit linear forms on the local Whittaker model $\mathcal{W}(\Pi_v, \psi'_v)$ of Π_v and the products in (3.1), (3.2) are to be regularized and understood "in the sense of L-functions" in a way similar to (2.4).

Based on the factorizations (3.1) and (3.2), the contribution of Π to the spectral expansion (2.8) can be shown to itself admit a factorization roughly as the product of local distributions (called *relative characters*) I_{Π_v} defined by

$$I_{\Pi_v}(f'_v) = \sum_{W_v \in \mathcal{W}(\Pi_v,\psi'_v)} \mathcal{P}_{H_1,v}\big(\Pi_v(f'_v)W_v\big)\overline{\mathcal{P}_{H_2,\eta,v}(W_v)}, \quad f'_v \in C_c^\infty(G'_v),$$

where the sum runs over a suitable orthonormal basis of the Whittaker model. On the other hand, from the Ichino–Ikeda Conjecture 2.2, we expect the contribution of $\pi \hookrightarrow \mathcal{A}_{\mathrm{cusp}}(G)$ to the spectral expansion of (2.6) to essentially factorize into the product of the local relative characters (where again the sum is taken over an orthonormal basis)

$$J_{\pi_v}(f_v) = \sum_{\varphi_v \in \pi_v} \mathcal{P}_{H,v}\big(\pi_v(f_v)\varphi_v, \varphi_v\big), \quad f_v \in C_c^\infty(G_v).$$

In **[56]**, W. Zhang has conjectured that the local Jacquet–Rallis transfer $f_v \leftrightarrow f'_v$ also satisfies certain precise spectral relations involving the above relative characters. This is exactly the extra local ingredient needed to finish the proof of the Ichino–Ikeda conjecture based on a comparison of the Jacquet–Rallis trace formulae. This conjecture was shown in **[56]** to hold for unramified and supercuspidal representations, and the method was further extended and amplified in **[13]**, allowing to prove the conjecture for all (tempered) representations at non-Archimedean places. Later, in **[15]** I gave a better proof of this conjecture which also has the advantage of working uniformly at all places (including Archimedean ones). To state the result, we introduce some terminology/notation: for a place v of F and a smooth irreducible representation π_v of G_v, we denote by $\pi_{E,v}$ the local *base-change* of π_v, that is, the smooth irreducible representation of G'_v whose L-parameter is given by composing that of π_v with the natural embedding of L-groups ${}^LG_v \to {}^LG'_v$, and, moreover, we say that π_v is *H_v-distinguished* if $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \neq 0$, that is, with the notation of Section 1.1, if the multiplicity $m(\pi_v)$ equals 1.

Theorem 3.2. *There exist explicit local constants $(\kappa_v)_v$ indexed by the set of all places of F and satisfying the product formula $\prod_v \kappa_v = 1$ such that the following property is verified: for every place v, every tempered representation π_v of G_v which is H_v-distinguished and every pair $(f_v, f'_v) \in C_c^\infty(G_v) \times C_c^\infty(G'_v)$ of matching functions (that is, $f_v \leftrightarrow f'_v$), we have*

$$I_{\pi_{v,E}}(f'_v) = \kappa_v J_{\pi_v}(f_v). \tag{3.3}$$

Moreover, the above identities characterize the Jacquet–Rallis transfer, that is, if two functions $f_v \in C_c^\infty(G_v)$, $f'_v \in C_c^\infty(G'_v)$ satisfy (3.3) for every tempered irreducible representation π_v of G_v that is H_v-distinguished, then these functions are transfers of each other.

The proof given in [15] of the above theorem is mainly based on another ingredient of independent interest which is an explicit Plancherel decomposition for the space $G'_v/H_{2,v}$ or rather, decomposing this quotient as a product in a natural way, for the symmetric variety $\mathrm{GL}_n(E_v)/\mathrm{GL}_n(F_v)$. This spectral decomposition is roughly obtained by applying the Plancherel formula for the group $\mathrm{GL}_n(E_v)$ to a family of zeta integrals, depending on a complex parameter s, introduced by Flicker and Rallis [22] and that represents local Asai L-factors and taking the residue at $s = 1$ of the resulting expression. We will not describe the exact process here, but just mention that this settles in the case at hand a general conjecture of Sakellaridis–Venkatesh [41] on the spectral decomposition of spherical varieties. This Plancherel formula is then used to write the explicit spectral expansion for a local analog of the Jacquet–Rallis trace formula (2.8) which is then compared with a local counterpart of the trace formula (2.6) yielding as a consequence Theorem 3.2 above. Moreover, as another by-product of this local comparison, we also get a formula conjectured by Hiraga–Ichino–Ikeda for the formal degree of discrete series [29] in the case of unitary groups.

4. GLOBAL ANALYSIS OF JACQUET–RALLIS TRACE FORMULAE

With all the local ingredients explained in the previous section in place, the only remaining tasks to finish the program initiated by Jacquet and Rallis to prove the Gan–Gross–Prasad and Ichino–Ikeda conjectures are global. More specifically, although simple versions of the Jacquet–Rallis trace formulae have been successfully used to establish these conjectures under some local restrictions [13, 57], in order to detect all the relevant cuspidal representations of unitary groups, we need more refined versions of the geometric and spectral expansions of (2.6) and (2.8).

As a first important step in that direction, Zydor [58, 59] has completely regularized the singular contributions to the geometric sides. We can summarize his main results as follows: for all test functions $f \in C_c^\infty(G(\mathbb{A}_F))$ and $f' \in C_c^\infty(G'(\mathbb{A}_F))$, there exist "canonical" regularization of the (usually divergent) integrals (2.5) and (2.7), that we will still denote by $J(f)$ and $I(f')$, as well as decompositions

$$J(f) = \sum_{\delta \in (H\backslash G//H)(F)} O(\delta, f) \quad \text{and} \quad I(f') = \sum_{\gamma \in (H_1\backslash G'//H_2)(F)} O_\eta(\gamma, f'), \tag{4.1}$$

where $H\backslash G//H$ and $H_1\backslash G'//H_2$ stand for the corresponding categorical quotients and $O(\delta,\cdot)$, $O_\eta(\gamma,\cdot)$ are distributions supported on the union of the adelic double cosets with images δ and γ in $(H\backslash G//H)(\mathbb{A}_F)$ and $(H_1\backslash G'//H_2)(\mathbb{A}_F)$, respectively, which coincide with the previously defined relative orbital integrals when δ and γ are regular semisimple.

Zydor obtains these regularized orbital integrals by adapting a truncation procedure developed by Arthur in the context of the usual trace formula to the relative setting at hand. It should be emphasized that contrary to what happens with Arthur's trace formula, the resulting distributions are directly invariant (in a relative sense, that is, here under the natural action of $H\times H$ or $H_1\times H_2$) and do not depend on any auxiliary choice (such as that of a maximal compact subgroup). It is in this sense that the regularizations of Zydor are really "canonical." It should be mentioned that another, different, approach to such regularization was proposed by Sakellaridis [40] in the context of general relative trace formulae. It is based on analyzing the exponents at infinity of generalized theta series together with a natural procedure to regularize integrals of multiplicative functions when the corresponding character is nontrivial.

Before we even consider the analogous, more subtle, regularization problem on the spectral side, there appears the natural question of how to compare the singular contributions to the refined geometric expansions of (4.1). This issue was completely resolved in a very long paper [20] by Chaudouard and Zydor. To state their main result, it is convenient to again consider the relevant pure inner forms of G (as defined in Section 1.1): for every Hermitian space W' of the same dimension as W, we have a relevant pure inner form $G^{W'}=U(V')\times U(W')$ with its diagonal subgroup $H^{W'}=U(W')$ where $V'=W'\oplus^\perp W^\perp$. Moreover, the correspondence of orbits (2.9) extends to an isomorphism between categorical quotients,

$$H\backslash G//H\simeq H_1\backslash G'//H_2, \tag{4.2}$$

and for every W' as before, $H^{W'}\backslash G^{W'}//H^{W'}$ can naturally be identified with $H\backslash G//H$. With these preliminaries, the main result of Chaudouard and Zydor can now be stated as follows:

Theorem 4.1 (Chaudouard–Zydor). *Assume that $f^{W'}=\prod_v f_v^{W'}\in C_c^\infty(G^{W'}(\mathbb{A}_F))$, where W' runs over all isomorphism classes of Hermitian spaces of dimension n, and $f'=\prod_v f'_v\in C_c^\infty(G'(\mathbb{A}_F))$ are factorizable test functions such that for every place v, and each W', $f_v^{W'}$ and f'_v are Jacquet–Rallis transfers of each other (that is, $f_v^{W'}\leftrightarrow f'_v$). Then, for every $\delta\in(H\backslash G//H)(F)$ with image $\gamma\in(H_1\backslash G'//H_2)(F)$ by (4.2), we have*

$$\sum_{W'} O(\delta,f^{W'})=O_\eta(\gamma,f'). \tag{4.3}$$

It should be noted that when δ, hence also γ, is regular semisimple, the left-hand sum in (4.3) only contains one nonidentically vanishing term but that in general more than one relevant pure inner forms can contribute. Also, the above result extends to nonfactorizable test functions, provided the wording is changed suitably.

The next natural step would be to develop regularized spectral expansions similar to (4.1). As a first result in that direction, Zydor has shown decompositions of the form

$$J(f) = \sum_{\chi \in \mathcal{X}(G)} J_\chi(f) \quad \text{and} \quad I(f') = \sum_{\chi' \in \mathcal{X}(G')} I_{\chi'}(f'), \tag{4.4}$$

where $\mathcal{X}(G)$ and $\mathcal{X}(G')$ stand for the set of cuspidal data of the groups G and G' respectively, that is the sets of pairs (M, σ) where M is a Levi subgroup (of G or G') and σ is a cuspidal automorphic representation of $M(\mathbb{A}_F)$ taken up to conjugacy (by $G(F)$ or $G'(F)$). According to Langlands theory of pseudo-Eisenstein series, these sets index natural equivariant Hilbertian decompositions:

$$L^2([G]) = \widehat{\bigoplus_{\chi \in \mathcal{X}(G)}} L^2_\chi([G]), \quad L^2([G']) = \widehat{\bigoplus_{\chi' \in \mathcal{X}(G')}} L^2_{\chi'}([G']).$$

The automorphic kernels K_f, $K_{f'}$ decompose accordingly into series $K_f = \sum_\chi K_{f,\chi}$, $K_{f'} = \sum_{\chi'} K_{f',\chi'}$ where $K_{f,\chi}$ and $K_{f',\chi}$ are kernel functions representing the restrictions $R_\chi(f)$ and $R_{\chi'}(f')$ of the right convolution operators $R(f)$ and $R(f')$ to $L^2_\chi([G])$ and $L^2_{\chi'}([G'])$, respectively. The distributions $f \mapsto J_\chi(f)$ and $f' \mapsto I_{\chi'}(f')$ are then roughly defined by applying the same regularization procedure that Zydor used for the expressions $J(f)$ and $I(f')$ up to replacing the integrands by $K_{f,\chi}$ and $K_{f',\chi'}$ respectively, that is, in symbolic terms:

$$\begin{aligned} J_\chi(f) &= \int_{[H]\times[H]}^{\mathrm{reg}} K_{f,\chi}(h_1, h_2)\,dh_1\,dh_2, \\ I_{\chi'}(f') &= \int_{[H_1]\times[H_2]}^{\mathrm{reg}} K_{f',\chi'}(h_1, h_2)\eta(h_2)\,dh_1\,dh_2. \end{aligned} \tag{4.5}$$

However, the expansions (4.4) are of little use as they stand and need to be suitably refined to allow for a meaningful comparison of the trace formulae. In Arthur's terminology, (4.4) are *coarse spectral expansions* and we need *refined spectral expansions* for each of the terms $J_\chi(f)$ or $I_{\chi'}(f')$.

This problem has so far proved to be a very difficult for general cuspidal data χ and χ'. However, a recent result of mine in collaboration with Y. Liu, W. Zhang, and X. Zhu **[17]** allows isolating in the coarse spectral expansions (4.4) the only terms that are eventually of interest consequently reducing the problem to some very particular cuspidal data χ' of G'.

The result proved in **[17]** is very general so let us place ourself for one moment in the framework of an arbitrary connected reductive group G over the number field F. Let Σ be a set of non-Archimedean places of F (possibly infinite) such that for each $v \in \Sigma$, the group G_v is unramified and fix a hyperspecial compact subgroup $K_v \subset G_v$ with $K_v = G(\mathcal{O}_v)$ for almost all $v \in \Sigma$. We let $\mathcal{X}_\Sigma(G)$ be the set of Σ-unramified cuspidal data of G, that is, the cuspidal data represented by pairs (M, σ) with σ unramified at all places of $v \in \Sigma$ (with respect to K_v or, rather, the hyperspecial subgroup it induces in M_v). For $\chi \in \mathcal{X}_\Sigma(G)$, we define its Σ-*near equivalence class*, henceforth denoted by $\mathcal{N}_\Sigma(\chi)$, as the set of all cuspidal data $\chi' \in \mathcal{X}_\Sigma(G)$ such that if χ and χ' are represented by pairs (M, σ) and (M', σ') respectively, then there exist automorphic unramified characters λ and λ' of $M(\mathbb{A}_F)$

and $M'(\mathbb{A}_F)$, respectively, with the property that for every $v \in \Sigma$ the Satake parameters of the unique K_v-unramified subquotients in $I_{P_v}^{G_v}(\sigma_v \otimes \lambda_v)$ and $I_{P'_v}^{G_v}(\sigma'_v \otimes \lambda'_v)$ (where P, P' are arbitrary chosen parabolics with Levi components M, M') are isomorphic. We also fix a compact-open subgroup $K = \prod_{v \in S_f} K_v$ of $G(\mathbb{A}_f)$ (where S_f denotes the set of finite places of F and K_v coincides with the previous choice of hyperspecial subgroup when $v \in \Sigma$) and we define the *Schwartz space* of K-biinvariant functions on $G(\mathbb{A}_F)$ as the restricted tensor product

$$\mathcal{S}_K\big(G(\mathbb{A}_F)\big) = \mathcal{S}\big(G(F_\infty)\big) \otimes \bigotimes_{v \in S_f}' C_c(K_v \backslash G_v / K_v),$$

where $C_c(K_v \backslash G_v / K_v)$ denotes the space of bi-K_v-invariant compactly supported functions on G_v (that is the K_v-spherical Hecke algebra when $v \in \Sigma$), F_∞ is the product of the Archimedean completions of F and $\mathcal{S}(G(F_\infty))$ stands for the Schwartz space of the reductive Lie group $G(F_\infty)$ in the sense of [19]. More precisely, $\mathcal{S}(G(F_\infty))$ is the space of smooth functions $f : G(F_\infty) \to \mathbb{C}$ such that for every polynomial differential operator on $G(F_\infty)$, the derivatives Df is bounded or, equivalently, such that for every left- (or right-)invariant differential operator X, Xf is decreasing faster than the inverse of any polynomial on $G(F_\infty)$.

The Schwartz space $\mathcal{S}(G(F_\infty))$ is naturally a Fréchet algebra under the convolution product and we also set

$$\mathcal{M}_\infty(G) = \mathrm{End}_{\mathrm{cont}, \mathcal{S}(G(F_\infty))-\mathrm{bimod}}\Big(\mathcal{S}\big(G(F_\infty)\big)\Big)$$

for the space of continuous endomorphisms of $\mathcal{S}(G(F_\infty))$ seen as a bimodule over itself. This is an algebra acting on any smooth admissible Fréchet representation of moderate growth of $G(F_\infty)$ in the sense of Casselman–Wallach. Moreover, as an application of a form of Schur lemma, for every irreducible Casselman–Wallach representation π_∞ of $G(F_\infty)$ and every $\mu_\infty \in \mathcal{M}_\infty(G)$ there exists a scalar $\mu_\infty(\pi_\infty) \in \mathbb{C}$ such that $\pi_\infty(\mu_\infty) = \mu_\infty(\pi_\infty) Id$. Thus, $\mathcal{M}_\infty(G)$ can be seen as some big algebra of multipliers for $\mathcal{S}(G(F_\infty))$. We also define the algebra of Σ*-multipliers* as the restricted tensor product

$$\mathcal{M}_\Sigma(G) = \mathcal{M}_\infty(G) \bigotimes_{v \in \Sigma}' \mathcal{H}(G_v, K_v),$$

where, for $v \in \Sigma$, $\mathcal{H}(G_v, K_v) = C_c(K_v \backslash G_v / K_v)$ is the spherical Hecke algebra. Then, $\mathcal{M}_\Sigma(G)$ acts naturally on the global Schwartz space $\mathcal{S}_K(G(\mathbb{A}_F))$, and we shall denote this action as the convolution product $*$. One of the main result of [17] can now be stated as follows:

Theorem 4.2 (Beuzart-Plessis–Liu–Zhang–Zhu). *Let $\chi \in \mathfrak{X}_\Sigma(G)$. Then, there exists a multiplier $\mu_\chi \in \mathcal{M}_\Sigma(G)$ such that for every Schwartz function $f \in \mathcal{S}_K(G(\mathbb{A}_F))$ and every other cuspidal datum $\chi' \in \mathfrak{X}_\Sigma(G)$, we have*

$$R_{\chi'}(\mu_\chi * f) = \begin{cases} R_{\chi'}(f) & \text{if } \chi' \in \mathcal{N}_\Sigma(\chi), \\ 0 & \text{otherwise.} \end{cases}$$

The above theorem can be roughly paraphrased by saying that the multiplier μ_χ "isolates" the near-equivalence class $\mathcal{N}_\Sigma(\chi)$ from the other cuspidal data. A large part of the proof given in [17] consists in establishing the existence of a large subalgebra of $\mathcal{M}_\infty(G)$ which admits an explicit spectral description, that is, through its action on irreducible Casselman–Wallach representations of $G(F_\infty)$. The algebra thus constructed generalizes Arthur's multipliers [5] and, moreover, builds on previous work of Delorme [21].

Going back to the setting of the Jacquet–Rallis trace formulae, the above theorem can be applied to isolate in the expansions (4.4) the automorphic L-packet of a given cuspidal automorphic representation π of $G(\mathbb{A}_F)$, on the one hand, and the cuspidal datum χ of G' "supporting" its base-change π_E, on the other hand. Moreover, essentially using the spectral characterization of Theorem 3.2 for the transfer, this can be done by multipliers $\mu_\pi \in \mathcal{M}_\Sigma(G)$ and $\mu_\chi \in \mathcal{M}_\Sigma(G')$ that are compatible with the Jacquet–Rallis transfer in the following sense: if $f = \prod_v f_v \in \mathcal{S}_K(G(\mathbb{A}_F))$ and $f' = \prod_v f'_v \in \mathcal{S}_{K'}(G'(\mathbb{A}_F))$ are transfers of each other then so are $\mu_\pi * f$ and $\mu_\chi * f'$ (where here we take Σ to consist of almost all places that split in E and for K, K' arbitrary compact-open subgroups of $G(\mathbb{A}_f)$, $G'(\mathbb{A}_f)$ that are hyperspecial at places in Σ). All in all, applying these multipliers to global test functions f and f' that are transfers of each other, and comparing the geometric expansions (4.1), we obtain an identity of the following shape:

$$\sum_{W'} \sum_{\substack{\pi' \hookrightarrow \mathcal{A}_{\text{cusp}}(G^{W'}) \\ \pi'_E = \pi_E}} J_{\pi'}(f) = I_\chi(f'),$$

where the outside left sum runs over isomorphism classes of Hermitian spaces of the same dimension as W (or, equivalently, relevant pure inner forms of G). Besides, as a consequence of the local Gan–Gross–Prasad conjecture, when π_E is generic, the left-hand side contains at most one nonzero term. Thus, as a final step to establish the Gan–Gross–Prasad and Ichino–Ikeda conjectures, it only remains to analyze the distribution I_χ. When the base-change π_E is itself cuspidal, that is, when $\chi = \{(G', \pi_E)\}$, by the works of Jacquet–Piatetski-Shapiro–Shalika and Flicker–Rallis already recalled, I_χ essentially factors as the product of the local relative characters $I_{\pi_{E,v}}$ and Theorem 3.2 then allows to conclude. However, in general a similar factorization of I_χ is far from obvious and was actually established in my joint work with Chaudouard and Zydor [16]. It is exactly of the shape predicted by the Ichino–Ikeda conjecture. More precisely:

Theorem 4.3 (Beuzart-Plessis–Chaudouard–Zydor). *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ whose base-change π_E is generic. Let χ be the cuspidal datum of G' such that π_E contributes to the spectral decomposition of $L^2_\chi([G'])$. Then, for every factorizable test function $f' = \prod_v f'_v \in \mathcal{S}(G'(\mathbb{A}_F))$, we have*

$$I_\chi(f') = \frac{1}{|S_\pi|} {\prod_v}' I_{\pi_{E,v}}(f'_v), \tag{4.6}$$

where the product has to be understood, as for (2.4), "in the sense of L-functions."

In [16], two proofs are actually given of the above theorem: one using truncations operators and the other one based on the global theory of Zeta integrals. For both methods, a crucial step is to spectrally expand the restriction of the Flicker–Rallis period (that is, the integral over $[H_2]$) to functions $\varphi \in L^2_\chi([G'])$ that are sufficiently rapidly decreasing. A consequence of this computation is that this period only depends on the π_E-component of φ and it is mainly for this reason that the contribution of χ to the Jacquet–Rallis trace formula $I(f')$ is eventually discrete (although in the case at hand, $L^2_\chi([G'])$ usually has a purely continuous spectrum). For this, the truncation method is based on the work of Jacquet–Lapid–Rogawski who have defined and studied generalizations of Arthur's truncation operator to the setting of Galois periods and proved analogs of the Maass–Selberg relations in this context. On the other hand, the other method starts by expressing the Flicker–Rallis period as a residue of the integral over $[H_2]$ of φ against an Eisenstein series. Unfolding carefully this expression as in the work of Flicker–Rallis, we can rewrite it as a Zeta integral of the sort that represents Asai L-functions. The precise location of the poles of these L-functions, as well as an explicit residue computation of a family of distributions, then allows to conclude.

Finally, let me mention that in work in progress with P.-H. Chaudouard, we are able to analyze the contributions to the Jacquet–Rallis trace formula of more general cuspidal data $\chi \in \mathfrak{X}(G')$ than that appearing in Theorem [16]. The final result is similar to (4.6) except that the right-hand side has to be integrated over a certain family of automorphic representations π of $G(\mathbb{A}_F)$. More precisely, our results include some cuspidal data supporting the base-changes of automorphic representations of $G = U(V) \times U(W)$ that are Eisenstein in the first factor and cuspidal in the second. In this particular case, the contribution of the corresponding cuspidal datum to the trace formula $J(f)$ is absolutely convergent and a refined spectral expansion can readily be obtained as an integral of Gan–Gross–Prasad periods between a cusp form and an Eisenstein series. These last periods are related, by some unfolding, to Bessel periods of cusp forms on smaller unitary groups. For this reason, our extension of Theorem 4.3 with Chaudouard should have similar applications to the Gan–Gross–Prasad and Ichino–Ikeda conjectures for general Bessel periods.

5. LOOKING FORWARD

As illustrated in the previous sections, various trace formula approaches to the Gan–Gross–Prasad conjectures for unitary groups have been very successful. However, despite these favorable and definite results, these developments also raise interesting questions or have lead to fertile new research direction:

- First, there is the question of whether similar techniques can be applied to prove the global Gan–Gross–Prasad conjectures for other groups. Indeed, the original conjectures in [23] also include general Bessel periods on (product of) orthogonal groups $\mathrm{SO}(n) \times \mathrm{SO}(m)$ ($n \not\equiv m$ [2]), as well as so-called Fourier–Jacobi periods on unitary groups $U(n) \times U(m)$ ($n \equiv m$ [2]) or symplectic/metaplectic groups $\mathrm{Mp}(n) \times \mathrm{Sp}(m)$. In the case of $U(n) \times U(n)$, a relative trace formula approach has been proposed by Y. Liu and further developed by H. Xue [52]. However, the situation is not as complete as for the Jacquet–Rallis trace formulae in the case of $U(n+1) \times U(n)$. It would be interesting to see if the latest developments, in particular those from my joint work with Chaudouard and Zydor [16], can be adapted to this setting. This could possibly lead to a proof of the Gan–Gross–Prasad conjecture for general Fourier–Jacobi periods on unitary groups. The situation for orthogonal and symplectic/metaplectic groups is much less satisfactory and there is no clear approach through a comparison of relative trace formulae, yet. This is due in particular to the fact that, instead of the Flicker–Rallis periods, in these cases we are naturally lead to consider period integrals originally studied by Bump–Ginzburg that detect cuspidal automorphic representations of $\mathrm{GL}(n)$ of orthogonal type. These period integrals involve the product of two exceptional theta series on a double cover of $\mathrm{GL}(n)$ and do not have any obvious geometric realizations (except when $n = 2$). This makes the task of writing a geometric expansion for the corresponding trace formulae quite unclear. It would certainly be interesting to see if the recent Hamiltonian duality picture of Ben Zvi–Sakellaridis–Venkatesh can shed some light on this matter (in particular, by associating a Hamiltonian space to the Bump–Ginzburg periods).

- In the local setting, the new trace formulae first discovered by Waldspurger [47] and further developed in [12] seem to be of quite broad applicability to all kind of *distinction problems*. Actually, similar trace formulae have already been established in other contexts [11, 18, 50] with new applications in the spirit of "relative Langlands functorialities" each time. However, all these developments have been made on a case-by-case basis so far and it would be very interesting and instructive to elaborate a general theory. In particular, in view of the proposal by Sakellaridis–Venkatesh [41] of a general framework for the relative Langlands program, we could hope to establish general local relative trace formulae for the L^2 spaces of spherical varieties X and relate those to the dual group construction of Sakellaridis–Venkatesh.

- Finally, in a slightly different direction the general isolation Theorem 4.2 clearly has the potential to be applied in other context, e.g., it would be interesting to see if it can be used as a technical device to simplify some other known comparison of trace formulae. Another intriguing question is to look for a precise spectral description of the (abstract) multiplier algebra $\mathcal{M}_\infty(G)$ and in [17], we actually argue that $\mathcal{M}_\infty(G)$ should be seen as the natural Archimedean analog of the Bernstein center for p-adic groups.

ACKNOWLEDGMENTS

First and foremost, I thank my former PhD advisor Jean-Loup Waldspurger for introducing me to this fertile circle of ideas and for always being so generous in sharing his insights on these problems.

I am also grateful to my colleagues and coauthors Pierre-Henri Chaudouard, Patrick Delorme, Wee-Teck Gan, Yifeng Liu, Dipendra Prasad, Yiannis Sakellaridis, Chen Wan, Wei Zhang, Xinwen Zhu, and Michal Zydor for many inspiring discussions and exchanges over the years.

FUNDING

The author's work was partially supported by the Excellence Initiative of Aix-Marseille University (A*MIDEX), a French "Investissements d'Avenir" program.

REFERENCES

[1] A. Aizenbud, A partial analog of the integrability theorem for distributions on p-adic spaces and applications. *Israel J. Math.* **193** (2013), no. 1, 233–262.

[2] A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann, Multiplicity one theorems. *Ann. of Math. (2)* **172** (2010), no. 2, 1407–1434.

[3] J. Arthur, A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbb{Q})$. *Duke Math. J.* **45** (1978), no. 4, 911–952.

[4] J. Arthur, A trace formula for reductive groups. II. Applications of a truncation operator. *Compos. Math.* **40** (1980), no. 1, 87–121.

[5] J. Arthur, A Paley–Wiener theorem for real reductive groups. *Acta Math.* **150** (1983), no. 1–2, 1–89.

[6] J. Arthur, On elliptic tempered characters. *Acta Math.* **171** (1993), no. 1, 73–138.

[7] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups*. Amer. Math. Soc. Colloq. Publ. 61, American Mathematical Society, Providence, RI, 2013, xviii+590 pp.

[8] R. Beuzart-Plessis, Expression d'un facteur epsilon de paire par une formule intégrale. *Canad. J. Math.* **66** (2014), no. 5, 993–1049.

[9] R. Beuzart-Plessis, Endoscopie et conjecture locale raffinée de Gan–Gross–Prasad pour les groupes unitaires. *Compos. Math.* **151** (2015), no. 7, 1309–1371.

[10] R. Beuzart-Plessis, La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes unitaires. *Mém. Soc. Math. Fr. (N.S.)* **149** (2016), vii+191 pp.

[11] R. Beuzart-Plessis, On distinguished square-integrable representations for Galois pairs and a conjecture of Prasad. *Invent. Math.* **214** (2018), no. 1, 437–521.

[12] R. Beuzart-Plessis, A local trace formula for the Gan–Gross–Prasad conjecture for unitary groups: the Archimedean case. *Astérisque* **418** (2020), viii+299 pp.

[13] R. Beuzart-Plessis, Comparison of local relative characters and the Ichino–Ikeda conjecture for unitary groups. *J. Inst. Math. Jussieu* **20** (2021), no. 6, 1803–1854.

[14] R. Beuzart-Plessis, A new proof of the Jacquet–Rallis fundamental lemma. *Duke Math. J.* **170** (2021), no. 12, 2805–2814.

[15] R. Beuzart-Plessis, Plancherel formula for $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(E)$ and applications to the Ichino–Ikeda and formal degree conjectures for unitary groups. *Invent. Math.* **225** (2021), no. 1, 159–297.

[16] R. Beuzart-Plessis, P.-H. Chaudouard, and M. Zydor, The global Gan–Gross–Prasad conjecture for unitary groups: the endoscopic case. *Publ. Math. Inst. Hautes Études Sci.* (2022).

[17] R. Beuzart-Plessis, Y. Liu, W. Zhang, and X. Zhu, Isolation of cuspidal spectrum, with application to the Gan–Gross–Prasad conjecture. *Ann. of Math. (2)* **194** (2021), no. 2, 519–584.

[18] R. Beuzart-Plessis and C. Wan, A local trace formula for the generalized Shalika model. *Duke Math. J.* **168** (2019), no. 7, 1303–1385.

[19] W. Casselman, Canonical extensions of Harish-Chandra modules to representations of G. *Canad. J. Math.* **XLI** (1989), no. 3, 385–438.

[20] P.-H. Chaudouard and M. Zydor, Le transfert singulier pour la formule des traces de Jacquet–Rallis. *Compos. Math.* **157** (2021), no. 2, 303–434.

[21] P. Delorme, Formules limites et formules asymptotiques pour les multiplicités dans $L^2(G/\Gamma)$. *Duke Math. J.* **53** (1986), no. 3, 691–731.

[22] Y. Z. Flicker, Twisted tensors and Euler products. *Bull. Soc. Math. France* **116** (1988), no. 3, 295–313.

[23] W. T. Gan, B. H. Gross, and D. Prasad, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. In *Sur les conjectures de Gross et Prasad. I*, pp. 1–109, Astérisque 346, Société Mathématique de France, Paris, 2012.

[24] W. T. Gan and A. Ichino, The Gross–Prasad conjecture and local theta correspondence. *Invent. Math.* **206** (2016), no. 3, 705–799.

[25] D. Ginzburg, D. Jiang, and S. Rallis, On the nonvanishing of the central value of the Rankin–Selberg L-functions. *J. Amer. Math. Soc.* **17** (2004), no. 3, 679–722.

[26] B. H. Gross and D. Prasad, On irreducible representations of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$. *Canad. J. Math.* **46** (1994), no. 5, 930–950.

[27] R. N. Harris, The refined Gross–Prasad conjecture for unitary groups. *Int. Math. Res. Not. IMRN* **2** (2014), 303–389.

[28] H. He, On the Gan–Gross–Prasad conjecture for $U(p,q)$. *Invent. Math.* **209** (2017), no. 3, 837–884.

[29] K. Hiraga, A. Ichino, and T. Ikeda, Formal degrees and adjoint γ-factors. *J. Amer. Math. Soc.* **21** (2008), no. 1, 283–304.

[30] A. Ichino and T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gross–Prasad conjecture. *Geom. Funct. Anal.* **19** (2010), no. 5, 1378–1425.

[31] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, Rankin–Selberg convolutions. *Amer. J. Math.* **105** (1983), no. 2, 367–464.

[32] H. Jacquet and S. Rallis, On the Gross–Prasad conjecture for unitary groups. In *On certain L-functions*, pp. 205–264, Clay Math. Proc. 13, Amer. Math. Soc., Providence, RI, 2011.

[33] D. Jiang and L. Zhang, Arthur parameters and cuspidal automorphic modules of classical groups. *Ann. of Math. (2)* **191** (2020), no. 3, 739–827.

[34] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White, Endoscopic classification of representations: inner forms of unitary groups. 2014, arXiv:1409.3731.

[35] Y. Liu, Refined global Gan–Gross–Prasad conjecture for Bessel periods. *J. Reine Angew. Math.* **717** (2016), 133–194.

[36] Z. Luo, The local Gan–Gross–Prasad conjecture for special orthogonal groups over Archimedean local fields. 2021, arXiv:2102.11404.

[37] D. Mezer, Multiplicity one theorems over positive characteristic. 2020, arXiv:2010.16112.

[38] C. Mœglin and J.-L. Waldspurger, La conjecture locale de Gross–Prasad pour les groupes spéciaux orthogonaux: le cas général. In *Sur les conjectures de Gross et Prasad. II*, pp. 167–216, Astérisque 347, Société Mathématique de France, Paris, 2012.

[39] C. P. Mok, Endoscopic classification of representations of quasi-split unitary groups. *Mem. Amer. Math. Soc.* **235** (2015), no. 1108, vi+248 pp.

[40] Y. Sakellaridis, The Schwartz space of a smooth semi-algebraic stack. *Selecta Math. (N.S.)* **22** (2016), no. 4, 2401–2490.

[41] Y. Sakellaridis and A. Venkatesh, Periods and harmonic analysis on spherical varieties. *Astérisque* **396** (2017), viii+360 pp.

[42] B. Sun and C.-B. Zhu, Multiplicity one theorems: the Archimedean case. *Ann. of Math. (2)* **175** (2012), no. 1, 23–44.

[43] J. Tate, Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, pp. 3–26, Proc. Sympos. Pure Math. XXXIII, Amer. Math. Soc., Providence, R.I., 1977.

[44] J. Tunnell, Local ε-factors and characters of GL(2). *Amer. J. Math.* **105** (1983), no. 6, 1277–1307.

[45] D. A. Vogan, The local Langlands conjecture. In *Representation theory of groups and algebras*, pp. 305–379, Contemp. Math. 145, Amer. Math. Soc., Providence, RI, 1993.

[46] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie. *Compos. Math.* **54** (1985), no. 2, 173–242.

[47] J.-L. Waldspurger, Une formule intégrale reliée à la conjecture locale de Gross–Prasad. *Compos. Math.* **146** (2010), no. 5, 1180–1290.

[48] J.-L. Waldspurger, Calcul d'une valeur d'un facteur ε par une formule intégrale. In *Sur les conjectures de Gross et Prasad. II*, pp. 1–102, Astérisque 347, Société Mathématique de France, Paris, 2012.

[49] J.-L. Waldspurger, La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux. In *Sur les conjectures de Gross et Prasad. II*, pp. 103–165, Astérisque 347, Société Mathématique de France, Paris, 2012.

[50] C. Wan, A local relative trace formula for the Ginzburg–Rallis model: the geometric side. *Mem. Amer. Math. Soc.* **261** (2019), no. 1263.

[51] J. Xiao, Endoscopic transfer for unitary Lie algebras. 2018, arXiv:1802.07624.

[52] H. Xue, The Gan–Gross–Prasad conjecture for $U(n) \times U(n)$. *Adv. Math.* **262** (2014), 1130–1191.

[53] H. Xue, On the global Gan–Gross–Prasad conjecture for unitary groups: approximating smooth transfer of Jacquet–Rallis. *J. Reine Angew. Math.* **756** (2019), 65–100.

[54] H. Xue, Bessel models for real unitary groups: the tempered case, 2020, https://www.math.arizona.edu/~xuehang/lggp.pdf.

[55] Z. Yun, The fundamental lemma of Jacquet and Rallis. With an appendix by Julia Gordon. *Duke Math. J.* **156** (2011), no. 2, 167–227.

[56] W. Zhang, Automorphic period and the central value of Rankin–Selberg L-function. *J. Amer. Math. Soc.* **27** (2014), no. 2, 541–612.

[57] W. Zhang, Fourier transform and the global Gan–Gross–Prasad conjecture for unitary groups. *Ann. of Math. (2)* **180** (2014), no. 3, 971–1049.

[58] M. Zydor, La variante infinitésimale de la formule des traces de Jacquet–Rallis pour les groupes unitaires. *Canad. J. Math.* **68** (2016), no. 6, 1382–1435.

[59] M. Zydor, La variante infinitésimale de la formule des traces de Jacquet–Rallis pour les groupes linéaires. *J. Inst. Math. Jussieu* **17** (2018), no. 4, 735–783.

RAPHAËL BEUZART-PLESSIS

Aix Marseille Univ, CNRS, I2M, Marseille, France, raphael.beuzart-plessis@univ-amu.fr