THE NUMBER OF RATIONAL POINTS ON A CURVE OF GENUS AT LEAST TWO

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ABSTRACT

The Mordell Conjecture states that a smooth projective curve of genus at least 2 defined over number field F admits only finitely many F-rational points. It was proved by Faltings in the 1980s and again using a different strategy by Vojta. Despite there being two different proofs of the Mordell Conjecture, many important questions regarding the set of F-rational points remain open. This survey concerns recent developments towards upper bounds on the number of rational points in connection with a question of Mazur.

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1. INTRODUCTION

Mordell's Conjecture asserts the finiteness of the set of rational solutions

$$\{(x,y) \in \mathbb{Q}^2 : P(x,y) = 0\}$$

for certain bivariate polynomials $P \in \mathbb{Q}[X,Y]$.

To make the statement and results precise, we will adopt the language of projective algebraic curves. Indeed, for the study of the zero set, we may assume that P is irreducible, even as a polynomial in $\mathbb{C}[X,Y]$. Moreover, its homogenization defines a projective curve in the projective plane. The classical procedure of normalization allows us to resolve any singularities. The result is an irreducible smooth projective curve defined over $\mathbb{Q}$. Its complex points define a compact Riemann surface of genus $g \in \{0,1,2,\ldots\}$.

Conversely, let us assume we are presented with a smooth projective curve C defined over $\mathbb{Q}$ that is irreducible as a curve taken over $\mathbb{C}$. The genus g of $C(\mathbb{C})$ taken as a Riemann surface has important consequences for arithmetic questions on $C(\mathbb{Q})$. Indeed, Mordell's Conjecture, proved by Faltings [25], states that $\#C(\mathbb{Q})$ is finite if $g \geq 2$.

We begin by formulating the Mordell Conjecture in slightly higher generality. We then discuss the history of results towards this conjecture. Finally, we give an overview of the proof of a joint work by Ziyang Gao, Vesselin Dimitrov, and the author towards a question of Mazur regarding upper bounds for the cardinality $\#C(\mathbb{Q})$. The upper bound will depend on the genus g and the Mordell–Weil rank of the Jacobian of C. For a special case of this result that does not make reference to Jacobians, we refer to Section 6.

1.1. The Mordell Conjecture

We begin by recalling Faltings's Theorem [25], a finiteness statement originally conjectured by Mordell [48]. By a curve we mean a geometrically irreducible projective variety of dimension 1. Throughout, we let F denote a number field and $\overline{F}$ a fixed algebraic closure of F.

Theorem 1.1 (Faltings [25]). *Let C be a smooth curve of genus at least 2 defined over a number field F. Then $C(F)$ is finite.*

If the genus of C is small, then one cannot expect finiteness. Indeed, the set $C(F)$ is nonempty after replacing F by a suitable finite extension. If C has genus 0, then C is isomorphic to the projective line and thus $C(F)$ is infinite. If C has genus 1, then C together with a point in $C(F)$ is an elliptic curve. In particular, we obtain an algebraic group. After possibly extending F again, we may assume that $C(F)$ contains a point of infinite order. So $C(F)$ is infinite.

To prove the Mordell Conjecture, Faltings first proved the Shafarevich Conjecture for abelian varieties. At the time, the latter was known to imply the Mordell Conjecture thanks to a construction of Kodaira–Parshin.

Later, Vojta [62] gave a different proof of the Mordell Conjecture that is rooted in diophantine approximation. Bombieri [8] then simplified Vojta's proof. We will recall Vojta's approach for curves in Section 3. The technical heart is the Vojta inequality which we formulate below as Theorem 3.1.

Faltings generalized Vojta's proof of the Mordell Conjecture to cover subvarieties of any dimension of an abelian variety. Indeed, Faltings [26,27] and Hindry [36] proved the Mordell–Lang Conjecture for subvarieties of abelian varieties. Let A be an abelian variety defined over F and suppose Γ is a subgroup of $A(\overline{F})$. The division closure of Γ is the subgroup

$$\{P \in A(\overline{F}) : \text{there exists an integer } n \geq 1 \text{ with } nP \in \Gamma\}$$

of $A(\overline{F})$. For example, the division closure of the trivial subgroup $\Gamma = \{0\}$ is the subgroup A_{tors} of all points of finite order of $A(\overline{F})$. The following theorem holds for all base fields of characteristic 0.

Theorem 1.2 (Mordell–Lang conjecture, Faltings, Hindry). *Let A be an abelian variety defined over F and let Γ be the division closure of a finitely generated subgroup of $A(\overline{F})$. If V is an irreducible closed subvariety of A, then the Zariski closure of $V(\overline{F}) \cap \Gamma$ in V is a finite union of translates of algebraic subgroups of A.*

The special case when $\Gamma = A_{\mathrm{tors}}$ is called the Manin–Mumford Conjecture and was proved by Raynaud [53].

More recently, Lawrence and Venkatesh [41] gave yet another proof of the Mordell Conjecture. It was inspired by Faltings's original approach and the method of Chabauty–Kim. We refer to the survey [6] on these developments.

In this survey we concentrate mainly on the case of curves and comment on possible extensions to the higher dimensional case.

1.2. Some remarks on effectivity

Despite the variety of approaches to the Mordell Conjecture, no *effective* proof is known. For example, if the curve C is presented explicitly as the vanishing locus of homogeneous polynomial equations with rational coefficients, say, then in full generality we know no algorithm that produces the finite list of rational points of C. The question of effectivity is already open in genus 2, for example, for the family $Y^2 = X^5 + t$ parametrized by t. Proving an effective version of the Mordell Conjecture is among the most important outstanding problems in diophantine geometry.

Although no general algorithm that determines the set of rational points is currently known, it is sometimes possible to determine the set of rational points. For example, we refer to the Chabauty–Coleman method [13,15] which provides a clean upper bound for the number of rational points subject to a hypothesis on the Mordell–Weil rank of the Jacobian of C. In several applications, this bound equals a lower bound for the number of rational points coming from a list of known rational points. Moreover, aspects of Kim's generalization of the Chabauty method were used by Balakrishnan, Dogra, Müller, Tuitman, and Vonk [5] to compute all rational points of the split Cartan modular curve of level 13 which appears in relation to Serre's uniformity question. A different approach motivated by work of Demjanenko and the theory of unlikely intersections was developed in a program by Checcoli, Veneziano, Viada [14]. Here too a condition on the rank of the curve's Jacobian is required for the method to apply. An remarkable aspect to this approach is that the authors obtain an explicit upper bound for the height of a rational point.

1.3. The number of rational points: conjectures and results

Given C and F as in Theorem 1.1, which invariants of C need to appear in an upper bound for $\#C(F)$?

Example 1.3. (i) Consider the hyperelliptic curve C presented by

$$y^2 = (x-1)\cdots(x-2022).$$

Its genus equals $(2022-2)/2 = 1010$. Then C contains the rational points $(1,0),\ldots,(2022,0)$. Together with the two points at infinity, we obtain at least 2024 rational points. This example easily generalizes to higher genus. For any $g \geq 2$ and square-free $f \in \mathbb{Q}[X]$ of degree $2g+2$, the equation $y^2 = f(x)$ determines a hyperelliptic curve C of genus g. If f splits into (pairwise distinct) linear factors over $\mathbb{Q}$, then $\#C(\mathbb{Q}) \geq 2g+4$. So any upper bound for $\#C(\mathbb{Q})$ must depend on the genus.

This lower bound is far from the truth. Stoll discovered a genus 2 curve defined over $\mathbb{Q}$ with 642 rational points in a family of such curves constructed by Elkies. Mestre showed that for all $g \geq 2$ there is a smooth curve of genus g defined over $\mathbb{Q}$ with at least $8g+16$ rational points.

(ii) Let us now fix the curve C and let the number field F vary. We take C as the genus 2 hyperelliptic curve presented by $y^2 = x^5 + 1$. Consider an integer $n \geq 0$ and the points $\{(m, \pm(m^5+1)^{1/2}) : m \in \{0,\ldots,n\}\}$. So $C(F)$ has at least $2n+2+1$ elements where $F = \mathbb{Q}((m^5+1)^{1/2})_{m\in\{1,\ldots,n\}}$. Any upper bound $C(F)$, even for C fixed, must depend on F.

Gabriel Dill pointed out that the number of F-points grows at least logarithmically in the degree $[F:\mathbb{Q}]$ in this case. Indeed, $[F:\mathbb{Q}] \leq 2^n$, so $\#C(F) \geq 2n \geq 2(\log[F:\mathbb{Q}])/\log 2$.

Let us consider the modular curve $X_0(37)$ which has genus 2 and is defined over $\mathbb{Q}$. Let p be one of the infinitely many prime numbers for which the Legendre symbol satisfies $(-p/37) = 1$; so 37 splits in the quadratic field $K = \mathbb{Q}(\sqrt{-p})$. Let F denote the Hilbert Class Field of K. There is an elliptic curve E defined over F with complex multiplication by the ring of integers of K. Moreover, E admits an isogeny of degree 37 to an elliptic curve defined over F. Thus $X_0(37)$ has an F-rational point. The Galois group $\mathrm{Gal}(F/K)$ acts on the F-rational points of $X_0(37)$. It is also known to act transitively on the moduli of elliptic curves with the same endomorphism ring as E. Thus $\#X_0(37)(F)$ is no less than $[F:K]$ which equals the class number of K by Class Field Theory. So $\#X_0(37)(F) \geq [F:K] = [F:\mathbb{Q}]/2$. By the Landau–

Siegel Theorem, $[F : \mathbb{Q}] \to \infty$ as $p \to \infty$. In particular, any upper bound for $\#X_0(37)(F)$ must grow at least linearly in $[F : \mathbb{Q}]$.

The Uniformity Conjecture by Caporaso–Harris–Mazur [11] predicts that the genus and base field of a curve are the only invariants required for a general upper bound.

Conjecture 1.4 (Caporaso–Harris–Mazur). *Let $g \geq 2$ be an integer and F a number field. There exists $c(g, F) \geq 1$ such that if C is a smooth curve of genus g defined over F, then $\#C(F) \leq c(g, F)$.*

Caporaso, Harris, and Mazur showed that the Uniformity Conjecture follows from the Weak Lang Conjecture which is an extension of the Mordell Conjecture to higher dimension. It states that if V is a smooth projective variety defined over F of positive dimension and general type, then $V(F)$ is not Zariski dense in V. Pacelli [50] showed that $\#C(F)$ is bounded from above in function of g and $[F : \mathbb{Q}]$ under the Weak Lang Conjecture after Abramovich [1] treated the case of quadratic and cubic extensions earlier. A refined version of the Weak Lang Conjecture implies, again by work of Caporaso–Harris–Mazur, that $\#C(F)$ can be bounded from above in function of the genus, if we omit finitely many F-isomorphism classes of C, see also [12] for a correction. Rémond [56] has evidence towards this stronger version of the Uniformity Conjecture. Alpoge [2] showed that, on average, a smooth curve of genus 2 defined over $\mathbb{Q}$ has a uniformly bounded number of rational points.

Mazur [46] posed the following question, which is a weaker version of the Uniformity Conjecture. We let $\mathrm{Jac}(C)$ denote the Jacobian of a smooth curve C defined over a field. Then $\mathrm{Jac}(C)$ is a principally polarized abelian variety whose dimension equals the genus of C. If the base field is a number field F, then $\mathrm{Jac}(C)(F)$ is a finitely generated abelian group by the Mordell–Weil Theorem.

Question 1.5 (Mazur [46, p. 223]). Let $g \geq 2$ and r be integers and let F be a number field. There exists $c(g, r, F) \geq 1$ such that if C is a smooth curve of genus g defined over F such that the rank of the Mordell–Weil group satisfies $\mathrm{rk}\,\mathrm{Jac}(C)(F) \leq r$, then $\#C(F) \leq c(g, r, F)$.

Let us review some work on upper bounds for $\#C(F)$. Parshin [59] showed how to extract an upper bound for the number of rational points from Faltings's theorem. In his original paper, Vojta [62] gave a blueprint on how to bound from above the number of rational points for a general C. This bound was refined by Bombieri [8] and de Diego [19]. However, these works did not provide an answer to Mazur's question.

To formulate de Diego's results we need some additional notation. We also require the Weil height on projective space and the Néron–Tate (or canonical) height, they are both defined in Section 2. Let S be an irreducible, smooth, quasiprojective variety defined over a number field F and presented with an immersion $S \subseteq \mathbb{P}^n$ defined over F. De Diego's Theorem holds for a smooth family of curves parametrized by the base S. Indeed, let $\mathscr{C} \to S$ be a smooth morphism such that each fiber is a smooth curve of genus $g \geq 2$. We write $\mathscr{C}_s$ for the fiber of $\mathscr{C} \to S$ above $s \in S(\overline{F})$. This is a smooth curve defined over $\overline{F}$. Let $\mathcal{K}_s$ denote the canonical class on $\mathscr{C}_s$, we identify it with a divisor class modulo linear equivalence of degree $2g-2$. If $P \in \mathscr{C}_s(\overline{F})$, then $(2g-2)[P] - \mathcal{K}_s$ is well defined as a divisor class of degree 0. So it represents a point in $\mathrm{Jac}(\mathscr{C}_s)$. In this way we obtain a morphism

$$j_s : \mathscr{C}_s \to \mathrm{Jac}(\mathscr{C}_s) \quad \text{given by } P \mapsto \big((2g-2)[P] - \mathcal{K}_s\big)/\sim .$$

Let θ_s denote the theta divisor on $\mathrm{Jac}(\mathscr{C}_s)$ and $\hat{h}_s = \hat{h}_{\mathscr{C}_s,\theta_s}$ the canonical height on $\mathrm{Jac}(\mathscr{C}_s)$ attached to this divisor.

Theorem 1.6 (de Diego [19]). *There exists $c(\mathscr{C}) > 1$ such that if F'/F is a finite extension and $s \in S(F')$, then*

$$\#\big\{P \in \mathscr{C}_s(F') : \hat{h}_s\big(j_s(P)\big) \geq c(\mathscr{C})\big(1 + h(s)\big)\big\} \leq \frac{55}{2} \cdot 7^{\mathrm{rk}\,\mathrm{Jac}(C)(F')}.$$

Roughly speaking, this theorem tells us that the number of points of $\mathscr{C}_s$ of sufficiently large canonical height is bounded as in Mazur's question. We will often call these points *large points*. It is striking that the constant 7 is admissible for all genera; a fact that already appeared in Bombieri's work [8]. For smooth curves of genus 2 defined over $\mathbb{Q}$ with a marked Weierstrass point, Alpoge [2] improved 7 to 1.872.

Observe that

$$\big\{P \in \mathscr{C}_s(F') : \hat{h}_s\big(j_s(P)\big) < c(\mathscr{C})\big(1 + h(s)\big)\big\} \tag{1.1}$$

is finite by the Northcott property for height functions which we will review in Section 2. To obtain a positive answer to Mazur's question we need, roughly speaking, to get a similar bound as in Theorem 1.6 for the cardinality of (1.1). There are quantitative versions of Northcott's Theorem. Estimating the cardinality (1.1) with these does, however, introduce a dependence on $h(s)$.

Work of David–Philippon [17] and Rémond [54] further clarified the other value $c(\mathscr{C})$. Indeed, David and Philippon proved a lower bound for the canonical height that, when combined with Rémond's explicit version of the Vojta inequality, yields the next theorem. To state their result, we momentarily shift our focus from families of smooth curves and their Jacobians to a curve immersed in an abelian variety.

Theorem 1.7 (Rémond [17, p. 643]). *Let A be a g-dimensional principally polarized abelian variety defined over F. Let Γ be the division closure of a finitely generated subgroup of $A(\overline{F})$ of rank r and let $C \subsetneq A$ be a curve that is not smooth of genus 1. Then $C(\overline{F}) \cap \Gamma$ is finite of cardinality at most*

$$\big(2^{34} h_0(A) \deg C\big)^{g^{20}(r+1)}.$$

Here $\deg C$ is the degree of C with respect to the principal polarization. Moreover, $h_0(A)$ is a height of the abelian variety A whose definition involves classical theta functions and the degree $[F : \mathbb{Q}]$. Roughly speaking, $h_0(A)$ encodes a bound for the coefficients needed to reconstruct the abelian variety A. Mazur's question does not allow for a dependence on $h_0(A)$. The hypothesis that C is not smooth of genus 1 is natural and cannot be dropped in general. It is equivalent to stating that C is not a translate of an algebraic subgroup of A.

David and Philippon's contribution to Theorem 1.7 was their lower bound for the canonical height, see [17, THÉORÈME 1.4]. Rémond [54] made Vojta's approach (and an inequality of Mumford) completely explicit. David–Philippon and Rémond have a result for subvarieties of A of any dimension. In other words, they provide an explicit version of the Mordell–Lang Conjecture.

David and Philippon's approach to Mazur's question and its higher-dimensional counterparts is via a strong quantitative version of the Bogomolov Conjecture on points of small height. A suitable version is Conjecture 1.5 [18] where the lower bound for the canonical height grows linearly in the Faltings height. We refer to [18, THÉORÈME 1.11] regarding the connection to rational points and more generally the Mordell–Lang Conjecture.

David and Philippon were able to strengthen their height lower bound when A is a power of an elliptic curve. This provided more evidence towards a positive answer for Mazur's question. Here is a version of their result for curves; their general result holds for subvarieties of a power of an elliptic curve.

Theorem 1.8 (David and Philippon [18, THÉORÈME 1.13]). *Let E be an elliptic curve defined over F and let $g \geq 2$ be an integer. Suppose Γ is the division closure of a finitely generated subgroup of $E^g(\overline{F})$ of rank r. If $C \subseteq E^g$ is a curve that is not smooth of genus 1, then $\#C(\overline{F}) \cap \Gamma \leq \deg(C)^{7g^{18}(1+r)}$.*

Thanks to a specialization argument, David and Philippon extended the above result to include the case where F is an arbitrary field of characteristic 0. David, Nakamaye, and Philippon [16] then proved the existence of a $(g-2)$-dimensional family of curves of genus g for which Mazur's question has a positive answer.

We now very briefly turn to some cardinality estimates using the Chabauty–Coleman method, which is based on p-adic analysis. It can produce finiteness of $C(F)$ with a clean cardinality estimate subject to a restriction on the rank of the Mordell–Weil group.

Theorem 1.9 (Coleman [15]). *Suppose C is a smooth curve of genus $g \geq 2$ defined over $\mathbb{Q}$ with $\operatorname{rk} \operatorname{Jac}(C)(\mathbb{Q}) \leq g-1$. If $p > 2g$ is a prime number where C has good reduction $\tilde{C}$, then $\#C(\mathbb{Q}) \leq 2g - 2 + \#\tilde{C}(\mathbb{F}_p)$.*

In combination with the Hasse–Weil bound $\#\tilde{C}(\mathbb{F}_p) \leq p + 1 + 2g\sqrt{p}$, the estimate above yields a bound for $\#C(\mathbb{Q})$ in terms of g and p alone. Observe that a dependence in the arithmetic of C appears through the prime p. Stoll was able to remove this dependence for hyperelliptic curves at the cost of a stronger restriction on the rank of the Mordell–Weil group.

Theorem 1.10 (Stoll [58]). *Let $g \geq 2$ and $d \geq 1$ be integers. There exists $c(g,d) > 0$ with the following property. Suppose C is a smooth hyperelliptic curve of genus g defined over F with $[F : \mathbb{Q}] \leq d$. If $\operatorname{rk} \operatorname{Jac}(C)(\mathbb{Q}) \leq g-3$, then $\#C(F) \leq c(g,d)$.*

Later, Katz, Rabinoff, and Zureick-Brown dropped the hyperellipticity condition.

Theorem 1.11 (Katz, Rabinoff, and Zureick-Brown [38]). *Let $g \geq 2$ and $d \geq 1$ be integers. There exists $c(g,d) > 0$ with the following property. Suppose C is a smooth curve of genus g defined over F with $[F:\mathbb{Q}] \leq d$. If $\operatorname{rk}\operatorname{Jac}(C)(\mathbb{Q}) \leq g-3$, then $\#C(F) \leq c(g,d)$.*

After this detour to the Chabauty–Coleman method, we return to Vojta's method. Vesselin Dimitrov, Ziyang Gao, and the author have recently proved a lower bound for the canonical height that can be used as a replacement for the lower bounds by David and Philippon [17, 18] in the context of Mordell's Conjecture. We recall this height inequality in Section 4.2 below. Indeed, it led to a positive answer to a strengthening of Mazur's question. The following result is new already in genus 2.

Theorem 1.12 (Dimitrov, Gao, and Habegger [24, THEOREM 1.1]). *Let $g \geq 2$ and $d \geq 1$ be integers, there exist $c'(g,d) > 1$ and $c(g,d) > 1$ with the following property. Suppose C is a smooth curve of genus g defined over a number field F such that $[F:\mathbb{Q}] \leq d$. Then*

$$\#C(F) \leq c'(g,d) \cdot c(g,d)^{\operatorname{rk}\operatorname{Jac}(C)(F)}.$$

Regarding the Caporaso–Harris–Mazur Uniformity Conjecture, we ask

Question 1.13. Can the cardinality $\#C(F)$ be bounded from above by a function that is polynomial in $[F:\mathbb{Q}]$ and g?

No one currently knows an algorithm that computes the rank of the Mordell–Weil group $\operatorname{Jac}(C)(F)$. However, upper bounds for this rank follow, for example, from the Ooe–Top Theorem [49]. We discuss this in more depth in Section 6.

Our results also cover points on C that lie in the division closure of a finitely generated subgroup. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The Jacobian $\operatorname{Jac}(C)$ of a smooth curve C of genus g defined over $\overline{\mathbb{Q}}$ corresponds to a $\overline{\mathbb{Q}}$-point of the coarse moduli space $\mathbb{A}_g$ of g-dimensional principally polarized abelian varieties. We let $[\operatorname{Jac}(C)]$ denote the point of $\mathbb{A}_g(\overline{\mathbb{Q}})$ corresponding to $\operatorname{Jac}(C)$ with its canonical principal polarization.

For example, if $g = 1$ then $\mathbb{A}_g = \mathbb{A}^1$ is the affine line. If E is an elliptic curve defined over $\overline{\mathbb{Q}}$, then $[E]$ is the j-invariant of E.

In general, $\mathbb{A}_g$ is a quasiprojective variety of dimension $g(g+1)/2$ defined over $\mathbb{Q}$. We may fix an immersion $\iota : \mathbb{A}_g \hookrightarrow \mathbb{P}^n$ into projective space. Then the absolute logarithmic Weil height h, see Section 2 for a definition, pulls back to a function $h \circ \iota : \mathbb{A}_g(\overline{\mathbb{Q}}) \to [0,\infty)$.

If C is a smooth curve of genus $g \geq 1$ defined over $\overline{\mathbb{Q}}$ and if $P_0 \in C(\overline{\mathbb{Q}})$, then a point $P \in C(\overline{\mathbb{Q}})$ defines a divisor $[P] - [P_0]$ of degree 0. One obtains a closed immersion

$$C \hookrightarrow \operatorname{Jac}(C) \quad \text{from } P \mapsto \big([P]-[P_0]\big)/\sim$$

where $\sim$ again denotes linear equivalence, induces a closed immersion. We will write $C - P_0$ for the image of C in $\operatorname{Jac}(C)$.

Theorem 1.14 (Dimitrov, Gao, and Habegger [24, THEOREM 1.2]). *Let $g \geq 2$ be an integer. There exist $c(g,\iota) > 1$, $c'(g,\iota) > 0$, and $c''(g,\iota) > 0$ that depend on g and the immersion ι with the following property. Suppose C is a smooth curve of genus g defined over $\overline{\mathbb{Q}}$ and let*

$P_0 \in C(\overline{\mathbb{Q}})$. *Let* Γ *be the division closure of a finitely generated subgroup of* $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$ *of rank* r. *If*

$$h\big(\iota\big([\mathrm{Jac}(C)]\big)\big) \geq c''(g,\iota) \quad \textit{then } \#(C-P_0)(\overline{\mathbb{Q}}) \cap \Gamma \leq c'(g,\iota)c(g,\iota)^r.$$

In particular, we may take $\Gamma = \mathrm{Jac}(C)_{\mathrm{tors}}$ and $r = 0$. Thus the theorem yields a uniform bound for the number of torsion points that lie on $C - P_0$ if the height of $\iota([\mathrm{Jac}(C)])$ is sufficiently large.

Suppose that C is defined over a number field F. Then $[\mathrm{Jac}(C)]$ is an F-rational point of the moduli space $\mathbb{A}_g$. If we impose also $h(\iota([\mathrm{Jac}(C)])) < c''(g,\iota)$, then $[\mathrm{Jac}(C)]$ lies in a finite set by the Northcott property. Thus $\mathrm{Jac}(C)$ is in one of at most finitely many $\overline{\mathbb{Q}}$-isomorphism classes and so is C by the Torelli Theorem.

Raynaud proved the following result which is the Manin–Mumford Conjecture for curves.

Theorem 1.15 (Raynaud [52]). *Let* C *be smooth curve defined over* $\mathbb{C}$ *of genus at least* 2. *Then* $(C - P_0) \cap \mathrm{Jac}(C)_{\mathrm{tors}}$ *is finite.*

Theorem 1.14 gives evidence towards the Uniform Manin–Mumford Conjecture which states that $(C - P_0) \cap \mathrm{Jac}(C)_{\mathrm{tors}}$ is bounded from above in terms of the genus g only for any smooth curve C of genus $g \geq 2$ defined over any field in characteristic 0.

Using a different approach involving equidistribution and motivated by dynamical systems, DeMarco, Krieger, and Ye [20] had made substantial progress towards the Uniform Manin–Mumford Conjecture. They proved it for smooth curves of genus 2 defined over $\mathbb{C}$ that are double covers of an elliptic curve when the base point P_0 is a Weierstrass point.

In a preprint, Kühne [39] complemented the method in [24] using ideas from equidistribution to prove the Uniform Manin–Mumford Conjecture.

Theorem 1.16 (Kühne [39]). *Let* $g \geq 2$ *be an integer. There exist* $c(g) > 1$ *and* $c'(g) > 1$ *that depend on* g *with the following property. Suppose* C *is a smooth curve of genus* g *defined over* $\mathbb{C}$ *and let* $P_0 \in C(\mathbb{C})$. *Let* Γ *be the division closure of a finitely generated subgroup of* $\mathrm{Jac}(C)(\mathbb{C})$ *of rank* r. *Then* $\#(C - P_0)(\mathbb{C}) \cap \Gamma \leq c'(g)c(g)^r$.

In contrast to Theorem 1.14, Kühne is able to handle curves C defined over $\overline{\mathbb{Q}}$ for which $[\mathrm{Jac}(C)]$ has height less than $c''(g,\iota)$. Once uniformity is established for all curves over $\overline{\mathbb{Q}}$, Kühne is able to pass to the base field $\mathbb{C}$ using a specialization argument laid out by Dimitrov, Gao, and the author [22] which relies on a result of Masser [43]. Kühne thus answers an older question of Mazur, see the top of page 234 [45], and obtains the full Mordell–Lang variant for curves.

DeMarco and Mavraki's [21] work on a relative version of the Bogomolov conjecture, see [72] and [22], motivated Kühne [39,40] to extend the reach of Arakelovian equidistribution methods of Szpiro–Ullmo–Zhang [60] and Yuan [65] to families of abelian varieties over a quasiprojective base. For algebraic curves, this settles the uniform Bogomolov and the uniform Manin–Mumford conjectures.

Yuan [66] recently gave another proof of Theorem 1.16. His method also runs via a uniform Bogomolov theorem and thus contains aspects related to height lower bounds. However, Yuan's approach relies on arithmetic bigness, rather than on equidistribution. It is independent of the approaches mentioned above and uses a new theory of adelic line bundles over quasiprojective varieties developed by Yuan and Zhang [67] which generalizes Zhang's theory [70] in the projective case. They derive a height inequality for a polarized dynamical system, see Theorem 1.3.2 and Section 6 [67], that extends our own bound. One aspect of Yuan's method is that it works for global fields in any characteristic.

We come to some questions regarding the base constant $c(g)$ in the estimates above. In the context of Mordell's Conjecture, Bombieri observed that the number of large points is bounded by a multiple of $7^{\operatorname{rk} \operatorname{Jac}(C)(F)}$.

Question 1.17. Can the base 7 in the estimate for the number of large points as in Theorem 1.6 be replaced by a function in g that tends to 1 for $g \to \infty$?

Alpoge [2] used the Kabatiansky–Levenshtein estimates on spherical codes to improve on the constant 7 in genus 2. It is quite possible that Alpoge's approach will shed light on this last question.

Concerning the constant $c(g)$ in Theorem 1.16, we pose the following two questions which also cover the moderate, i.e., nonlarge, points. They were inspired by questions of Helfgott.

Question 1.18. Can we choose the $c(g)$ in Theorem 1.16 such that there exists $B \geq 1$ with $c(g) \leq B$ for all integers $g \geq 2$?

Question 1.19. Can we choose the $c(g)$ in Theorem 1.16 with $\lim_{g\to\infty} c(g) = 1$?

Recently, Gao, Ge, and Kühne [32] completed the proof of the Uniform Mordell–Lang Conjecture for a subvariety V of a polarized abelian variety A of any dimension. Uniformity here amounts to bounding the number of irreducible components of the Zariski closure in Theorem 1.2 from above by $c'(\dim A, \deg V)c(\dim A, \deg V)^r$. Their result holds over all base fields in characteristic 0.

We refer to the comprehensive survey by Gao [31] that gives an overview of these recent developments and how they are interlinked.

Here is a brief overview of this survey. In Section 2 we recall some fundamental properties of two height functions: the Weil and Néron–Tate heights. They play a central role in the proof of Theorem 1.12. Then in Section 3 we describe Vojta's approach to the Mordell Conjecture. Later we return to the two height functions and describe their interactions on a family of abelian varieties. This is done in Section 4. Here we also describe the Betti map, an important analytic tool. In Section 5 we sketch how all this fits together in the proof of Theorem 1.12. In the final section we discuss an estimate for the number of rational points on a hyperelliptic curve that does not make reference to Jacobians.

2. HEIGHTS

Height functions are at the heart of Vojta's proof of the Mordell Conjecture and subsequent results such as Theorem 1.12. We will review two flavors of heights. The first one is the absolute logarithmic Weil height which is defined on algebraic points of the projective space. One can also use it to define a class of height functions on a projective variety equipped with an invertible sheaf. The second height function is the canonical or Néron–Tate height on an abelian variety, also equipped with an invertible sheaf. The latter is compatible with the group structure on the abelian variety.

2.1. The absolute logarithmic Weil height

We review here briefly the main properties of the Weil height. For a thorough treatment, we refer to [9, CHAPTERS 1 AND 2] or [37, PART B].

We begin by defining the height of a rational point on projective space $\mathbb{P}^n$.

Definition 2.1. Let $P \in \mathbb{P}^n(\mathbb{Q})$. There exist projective coordinates $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ of $P = [x_0 : \cdots : x_n]$ with $\gcd(x_0, \dots, x_n) = 1$. Then we set

$$h(P) = \log \max\{|x_0|, \dots, |x_n|\}.$$

The vector $(x_0, \dots, x_n)$ is uniquely determined up to a sign, and so $h(P)$ is well defined. For example, $h([2 : 4 : 6]) = h([1 : 2 : 3]) = h([1/3 : 2/3 : 1]) = \log 3$.

The following theorem is a straightforward consequence of the definition of the Weil height.

Theorem 2.2 (Northcott property). *The set* $\{P \in \mathbb{P}^n(\mathbb{Q}) : h(P) \le B\}$ *is finite for all* B.

Defining the height of an algebraic point in $\mathbb{P}^n(\overline{\mathbb{Q}})$ requires some basic algebraic number theory. Indeed, let K be a number field. We let M_K denote the set of absolute values $|\cdot| : K \to [0, \infty)$ that extend either the standard absolute value on $\mathbb{Q}$ or a p-adic absolute value for some prime p. Then M_K is called the set of places of K. For each $v \in M_K$, one sets $d_v = [K_v : \mathbb{Q}_w]$ where K_v is a completion of K with respect to v and $\mathbb{Q}_w$ is the completion of $\mathbb{Q}$ in K_v with respect to $w = v|_{\mathbb{Q}}$.

Definition 2.3. Let $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and let K be a number field such that $P = [x_0 : \cdots : x_n]$ where $(x_0, \dots, x_n) \in K^{n+1} \setminus \{0\}$. The absolute logarithmic Weil height, or just Weil height, is

$$h(P) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{|x_0|_v, \dots, |x_n|_v\}. \tag{2.1}$$

The normalization constants d_v are chosen such that the product formula

$$\prod_{v \in M_K} |x|_v^{d_v} = 1$$

holds for all $x \in K \setminus \{0\}$. This guarantees that the right-hand side of (2.1) is independent of the choice of projective coordinates of P. In particular, we may assume that some projective coordinate of P equals 1. Thus $h(P) \ge 0$ for all $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Moreover, $h(P)$ is independent of the field K containing the projective coordinates.

For applications to diophantine geometry, it is useful to have a height function defined on algebraic points of an irreducible projective variety V defined over $\overline{\mathbb{Q}}$. But without additional data there is no reasonable way to define a height on $V(\overline{\mathbb{Q}})$.

However, if V is a subvariety of the projective space $\mathbb{P}^n$, then we may restrict the Weil height $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$ to a function $V(\overline{\mathbb{Q}}) \to \mathbb{R}$. Slightly more generally, if $V \to \mathbb{P}^n$ is an immersion, then we may pull back the Weil height to $V(\overline{\mathbb{Q}})$.

Recall that an immersion $V \to \mathbb{P}^n$ is induced by a tuple of $(n+1)$ global sections of a very ample invertible sheaf on V. Conversely, given a very ample invertible sheaf $\mathscr{L}$ on V, we can fix a basis of the vector space of global sections of $\mathscr{L}$ and obtain an immersion $\iota_{\mathscr{L}} : V \to \mathbb{P}^n$. So we obtain a function $h \circ \iota_{\mathscr{L}} : V(\overline{\mathbb{Q}}) \to [0, \infty)$. There is a wrinkle here, this function depends not only on $(V, \mathscr{L})$ but also on the basis of the vector space of global sections. A different basis will lead to a function $V(\overline{\mathbb{Q}}) \to [0, \infty)$ that differs from $h \circ \iota_{\mathscr{L}}$ by a bounded function on $V(\overline{\mathbb{Q}})$. We define $h_{V,\mathscr{L}}$ to be the equivalence class of functions $V(\overline{\mathbb{Q}}) \to \mathbb{R}$ modulo bounded functions that contains $h \circ \iota_{\mathscr{L}}$.

If $\mathscr{L}$ is an ample invertible sheaf on V, then there exists an integer $n \geq 1$ such that $\mathscr{L}^{\otimes n}$ is very ample. We then define $h_{V,\mathscr{L}} = \frac{1}{n} h_{V,\mathscr{L}^{\otimes n}}$; this is again only defined up to a bounded function on $V(\overline{\mathbb{Q}})$. The equivalence class does not depend on the choice of n.

Finally, an arbitrary invertible sheaf $\mathscr{L}$ in the Picard group $\operatorname{Pic}(V)$ of V is of the form $\mathscr{F} \otimes \mathscr{M}^{\otimes(-1)}$ with $\mathscr{F}$ and $\mathscr{M}$ ample on V. The difference $h_{V,\mathscr{F}} - h_{V,\mathscr{M}}$ is well defined up to a bounded function on $V(\overline{\mathbb{Q}})$. It does not depend on the pair $\mathscr{F}, \mathscr{M}$ with difference $\mathscr{L}$, and we denote it by $h_{V,\mathscr{L}}$. It is called the Weil height attached to $(V, \mathscr{L})$.

Theorem 2.4. *Let us keep the notation above. In particular, V is an irreducible projective variety defined over $\overline{\mathbb{Q}}$.*

(i) *The association $\mathscr{L} \mapsto h_{V,\mathscr{L}}$ is a group homomorphism with target the group of real-valued maps $V(\overline{\mathbb{Q}}) \to \mathbb{R}$ modulo bounded functions.*

(ii) *For V equal to projective space and $\mathscr{L}$ the hyperplane bundle $\mathcal{O}(1)$, the Weil height from Definition 2.3 represents $h_{\mathbb{P}^n,\mathcal{O}(1)}$.*

(iii) *Suppose W is a further irreducible projective variety defined over $\overline{\mathbb{Q}}$ and $f : W \to V$ is a morphism. For all $\mathscr{L} \in \operatorname{Pic}(V)$ we have $h_{V,\mathscr{L}} \circ f = h_{W,f^*\mathscr{L}}$. As usual, this equality is understood as an equality of equivalence classes of functions.*

(iv) *Suppose $\mathscr{L} \in \operatorname{Pic}(V)$ admits a nonzero global section s. Then $h_{V,\mathscr{L}}$ is bounded from below on the complement of the vanishing locus of s. In particular, $h_{V,\mathscr{L}}$ is bounded from below on a Zariski open and dense subset of V.*

Suppose that V is defined over a number field $F \subseteq \overline{\mathbb{Q}}$ and $\mathscr{L} \in \operatorname{Pic}(V)$ is ample. Then the Northcott property holds for points of bounded degree, i.e.,

$$\left\{P \in V(\overline{F}) : h'_{V,\mathscr{L}}(P) \leq B \text{ and } \left[F(P) : F\right] \leq D\right\}$$

is finite where $h'_{V,\mathscr{L}}$ denotes any representative of $h_{V,\mathscr{L}}$.

Remark 2.5.

Let V be an irreducible projective variety defined over $\overline{\mathbb{Q}}$. We conclude this section by discussing a powerful tool to translate geometric information, here on intersection numbers, into an inequality of heights. The basic question is the following. Given invertible sheaves $\mathscr{F}$ and $\mathscr{M}$ on V, under what conditions can one bound $h_{V,\mathscr{M}}$ from above in terms of $h_{V,\mathscr{F}}$?

(i) We first consider the special case $V = \mathbb{P}^n$. As $\mathrm{Pic}(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}$, any Weil height is some integral multiple of $h_{\mathbb{P}^n,\mathcal{O}(1)}$. So $h_{V,\mathscr{F}}$ and $h_{V,\mathscr{M}}$ are $\mathbb{Z}$-linearly dependent.

(ii) Let us again suppose that V is general and that $\mathscr{F}$ is ample. Then there exists an integer $k \geq 1$ such that $\mathscr{F}^{\otimes k} \otimes \mathscr{M}^{\otimes(-1)}$ is ample. So for some positive integer $l \geq 1$ the power $\mathscr{F}^{\otimes kl} \otimes \mathscr{M}^{\otimes(-l)}$ is very ample. In particular, it admits a global section that does not vanish at a prescribed point of $V(\overline{\mathbb{Q}})$. Theorem 2.4, parts (i) and (iv), imply

$$klh_{V,\mathscr{F}} - lh_{V,\mathscr{M}} = h_{V,\mathscr{F}^{\otimes kl} \otimes \mathscr{M}^{\otimes(-l)}} \geq 0;$$

this must be parsed as an inequality between functions on $V(\overline{\mathbb{Q}})$ defined up to addition of a bounded function. We conclude

$$h_{V,\mathscr{M}} \leq kh_{V,\mathscr{F}}. \tag{2.2}$$

(iii) For some applications, such as Theorem 1.12, the ampleness hypothesis on $\mathscr{F}$ in (i) is not flexible enough. Moreover, we would like some way to estimate the factor k in (2.2) from above. We now describe a criterion of Siu that provides a solution to these two issues.

An invertible sheaf $\mathscr{L} \in \mathrm{Pic}(V)$ is called big if

$$\liminf_{k\to\infty} \frac{\dim H^0(V, \mathscr{L}^{\otimes k})}{k^{\dim V}} > 0;$$

here $H^0(V, \mathscr{L})$ denotes the vector space of global sections of $\mathscr{L}$.

If $\mathscr{L}$ is a big invertible sheaf, then $\mathscr{L}^{\otimes k}$ has a nonzero global section for some $k \geq 1$. Then using (i) and (iv) of Theorem 2.4 we see that $h_{V,\mathscr{L}} = \frac{1}{k}h_{V,\mathscr{L}^{\otimes k}}$ is bounded from below on a Zariski open and dense subset of V.

For example, if $\mathscr{L} = \mathscr{F} \otimes \mathscr{M}^{\otimes(-1)}$ is big, then, again by Theorem 2.4(i), we find $h_{V,\mathscr{F}} \geq h_{V,\mathscr{M}}$ on a Zariski open and dense subset of V.

We now come Siu's Criterion; it ensures that $\mathscr{F} \otimes \mathscr{M}^{\otimes(-1)}$ is big. An invertible sheaf $\mathscr{L} \in \mathrm{Pic}(V)$ is called nef, or numerically effective, if $(\mathscr{L} \cdot [C]) \geq 0$ for all irreducible curves $C \subseteq V$.

Siu's Criterion requires that $\mathscr{F}$ and $\mathscr{M}$ are both nef and that the intersection numbers on V satisfy $(\mathscr{F}^{\cdot \dim V}) > (\dim V)(\mathscr{F}^{\cdot(\dim V-1)} \cdot \mathscr{M})$. With these hypotheses $\mathscr{F} \otimes \mathscr{M}^{\otimes(-1)}$ is big; see **[42, THEOREM 2.2.15]**.

Say $\mathscr{F}$ and $\mathscr{M}$ are nef and $(\mathscr{F}^{\cdot \dim V}) > 0$. Let k and l be positive integers with

$$(\dim V)\frac{(\mathscr{F}^{\cdot(\dim V-1)} \cdot \mathscr{M})}{(\mathscr{F}^{\cdot \dim V})} < \frac{k}{l}$$

then $\mathscr{F}^{\otimes k} \otimes \mathcal{M}^{\otimes(-l)}$ is big. So

$$h_{V,\mathcal{M}}|_U \leq \frac{k}{l} h_{V,\mathscr{F}}|_U$$

holds on some Zariski open and dense $U \subseteq V$.

This allows us to compare the heights $h_{V,\mathcal{M}}$ and $h_{V,\mathscr{F}}$ if we have information on the intersection numbers, at least on a rather large subset of $V(\overline{\mathbb{Q}})$.

Yuan [65] proved an arithmetic version of this criterion in his work on equidistribution. The author [34] used Siu's Criterion to study unlikely intersections in abelian varieties.

2.2. The canonical height on an abelian variety

Let $F \subseteq \overline{\mathbb{Q}}$ be a number field and A an abelian variety defined over F. If $\mathcal{L}$ is an invertible sheaf on A, then we have the Weil height $h_{A,\mathcal{L}}$ from Section 2.1. Recall that $h_{A,\mathcal{L}}$ is only defined up to addition of a bounded function on $A(\overline{\mathbb{Q}})$. For abelian varieties, there is a canonical choice of function in the equivalence class $h_{A,\mathcal{L}}$ called the canonical or Néron–Tate height. A general reference for this section is [9, CHAPTER 9].

For an integer $n \in \mathbb{Z}$, let $[n]$ denote the multiplication-by-n endomorphism of A. Then $\mathcal{L}$ is called even or symmetric if there is an isomorphism $[-1]^*\mathcal{L} \cong \mathcal{L}$. It is called odd or antisymmetric if $[-1]^*\mathcal{L} \cong \mathcal{L}^{\otimes(-1)}$. If $\mathcal{L}$ is any ample invertible sheaf on A, then $\mathcal{L} \otimes [-1]^*\mathcal{L}$ is ample and even. So any abelian variety admits an even, ample invertible sheaf.

Suppose that $\mathcal{L}$ is even. Then $[2]^*\mathcal{L} \cong \mathcal{L}^{\otimes 4}$ is a consequence of the Theorem of the Cube. So Theorem 2.4 implies $h_{A,\mathcal{L}} \circ [2] = 4h_{A,\mathcal{L}}$ as classes and by iteration $h_{A,\mathcal{L}} \circ [2^k] = 4^k h_{A,\mathcal{L}}$ for all $k \geq 1$. We fix a representative $h'_{A,\mathcal{L}}$ of $h_{A,\mathcal{L}}$ and find $h'_{A,\mathcal{L}} \circ [2^k] = 4^k h'_{A,\mathcal{L}} + O_k(1)$ on $A(\overline{\mathbb{Q}})$. Tate's Limit Argument is used to show convergence in the following definition.

Definition 2.6. Let $\mathcal{L}$ be an even invertible sheaf on A and let $P \in A(\overline{\mathbb{Q}})$. Then the limit

$$\hat{h}_{A,\mathcal{L}}(P) = \lim_{k\to\infty} \frac{h'_{A,\mathcal{L}}(P)}{4^k} \tag{2.3}$$

exists and is independent of the choice of representative $h'_{A,\mathcal{L}}$ of $h_{A,\mathcal{L}}$. The real-valued function $P \mapsto \hat{h}_{A,\mathcal{L}}(P)$ is called the canonical or Néron–Tate height (on A attached to $\mathcal{L}$).

If $\mathcal{L}$ is even, then (2.3) immediately implies $\hat{h}_{A,\mathcal{L}}([2](P)) = 4\hat{h}_{A,\mathcal{L}}(P)$ for all $P \in A(\overline{\mathbb{Q}})$. If P has finite order, then $[2^m](P) = [2^n](P)$ for distinct integers $0 \leq m < n$ by the Pigeonhole Principle. Thus $\hat{h}_{A,\mathcal{L}}(P) = 0$.

There is nothing special about [2]. Indeed, one can replace [2] by $[m]$ in (2.3) for any integer $m \geq 2$; one then needs to replace 4^k in the denominator by m^{2k}.

What happens if $\mathcal{L}$ is an odd invertible sheaf? In this case, $[2]^*\mathcal{L} \cong \mathcal{L}^{\otimes 2}$. Then a similar limit (2.3) exists, but now we need to divide by 2^k.

The set of odd invertible sheaves is a divisible subgroup of $\mathrm{Pic}(A)$. From this, one can show that, after possibly extending the base field F, any invertible sheaf $\mathcal{L}$ on A decomposes as $\mathcal{L}_+ \otimes \mathcal{L}_-$ with $\mathcal{L}_+$ even and $\mathcal{L}_-$ odd. One then defines $\hat{h}_{A,\mathcal{L}} = \hat{h}_{A,\mathcal{L}_+} + \hat{h}_{A,\mathcal{L}_-}$; the decomposition of $\mathcal{L}$ is not quite unique, but this ambiguity does not affect $\hat{h}_{A,\mathcal{L}}$.

For our purposes, we often restrict to even invertible sheaves.

Let us collect the some important facts about the Néron–Tate height.

Theorem 2.7. *Let us keep the notation above. In particular, A is an abelian variety defined over a number field $F \subseteq \overline{\mathbb{Q}}$.*

(i) *Then association $\mathcal{L} \mapsto \hat{h}_{A,\mathcal{L}}$ is a group homomorphism from* $\operatorname{Pic}(V)$ *to the additive group of real-valued maps $A(\overline{\mathbb{Q}}) \to \mathbb{R}$.*

Suppose $\mathcal{L}$ is an invertible sheaf on A.

(ii) *The Néron–Tate height $\hat{h}_{A,\mathcal{L}}$ represents the Weil height $h_{A,\mathcal{L}}$.*

(iii) *If $\mathcal{L}$ is even, then the parallelogram equality*

$$\hat{h}_{A,\mathcal{L}}(P+Q) + \hat{h}_{A,\mathcal{L}}(P-Q) = 2\hat{h}_{A,\mathcal{L}}(P) + 2\hat{h}_{A,\mathcal{L}}(Q)$$

holds for all $P, Q \in A(\overline{\mathbb{Q}})$.

(iv) *If $\mathcal{L}$ is even and ample, then $\hat{h}_{A,\mathcal{L}}$ takes nonnegative values and vanishes precisely on A_{tors}.*

(v) *If $\mathcal{L}$ is even and ample, then $\hat{h}_{A,\mathcal{L}}$ induces a well-defined map $A(\overline{\mathbb{Q}}) \otimes \mathbb{R} \to$* $[0, \infty)$. *It is the square of a norm $\|\cdot\|$ on the $\mathbb{R}$-vector space $A(\overline{\mathbb{Q}}) \otimes \mathbb{R}$ and satisfies the parallelogram equality.*

The norm $\|\cdot\|$ allows us to do geometry in the $\mathbb{R}$-vector space $A(\overline{\mathbb{Q}}) \otimes \mathbb{R}$ (which is infinite dimensional if $\dim A \geq 1$). Indeed, for $z, w \in A(\overline{\mathbb{Q}}) \otimes \mathbb{R}$, we define

$$\langle P, Q \rangle = \frac{1}{2}\big(\|P+Q\|^2 - \|P\|^2 - \|Q\|^2\big).$$

Then $\langle \cdot, \cdot \rangle$ is a positive definite, symmetric, bilinear form.

By abuse of notation, we also write $\|P\|$ and $\langle P, Q \rangle$ for $P, Q \in A(\overline{\mathbb{Q}})$. In this notation we have $\langle P, P \rangle = \hat{h}_{A,\mathcal{L}}(P)$.

The Mordell–Weil Theorem implies that $A(F) \otimes \mathbb{R}$ is finite dimensional. We will see that $\|\cdot\|$ is a suitable norm to do Euclidean geometry in $A(F) \otimes \mathbb{R}$.

3. VOJTA'S APPROACH TO THE MORDELL CONJECTURE

Recall that the Mordell Conjecture was proved first by Faltings. In this section we briefly describe Vojta's approach to the Mordell Conjecture [62]. At the core is the deep Vojta inequality which we state here for a curve in an abelian variety.

Let A be an abelian variety defined over a number field $F \subseteq \overline{\mathbb{Q}}$. Let $\mathcal{L}$ be an ample and even invertible sheaf on A. We write $\|\cdot\| = \hat{h}_{A,\mathcal{L}}^{1/2}$ for the norm on $A(\overline{\mathbb{Q}}) \otimes \mathbb{R}$ defined in Theorem 2.7.

Theorem 3.1 (Vojta's inequality). *Let $C \subseteq A$ be a curve that is defined over F and that is not a translate of an algebraic subgroup of A. There are $c_1 > 1$, $c_2 > 1$, and $c_3 > 0$ with the following property. If $P, Q \in C(\overline{\mathbb{Q}})$ satisfy*

$$\langle P, Q\rangle \geq \left(1 - \frac{1}{c_1}\right)\|P\|\|Q\|$$

and

$$\|Q\| \geq c_2\|P\|,$$

then $\|P\| \leq c_3$.

We refer also to Rémond's work [55] for a completely explicit version of Vojta's inequality.

The values c_1, c_2, c_3 depend on the curve C. One remarkable aspect is that Vojta's inequality is a statement about pairs of $\overline{\mathbb{Q}}$-points of the curve C. So all three values c_1, c_2, c_3 are "absolute," i.e., we can take them as independent of the base field F of A and C. Both c_1 and c_2 are of "geometric nature." They depend only on the degree of C with respect to $\mathcal{L}$ and other discrete data attached to A and C. In contrast, c_3 is of "arithmetic nature." Roughly speaking, it depends on suitable heights of coefficients that define the curve C in some projective embedding.

Let us now sketch a proof of Mordell's Conjecture using the Vojta inequality and the classical Mordell–Weil Theorem.

Suppose C has genus $g \geq 2$. Without loss of generality, $C(F) \neq \emptyset$. So we fix a base point $P_0 \in C(F)$, then $P \mapsto P - P_0$ induces an immersion $C \to \operatorname{Jac}(C)$. So we may assume that C is a curve inside the g-dimensional $A = \operatorname{Jac}(C)$. Note that C is not a translate of an algebraic subgroup of its Jacobian since $g \geq 2$.

We observe that $C(F) = C(\overline{\mathbb{Q}}) \cap \operatorname{Jac}(C)(F)$.

By the Northcott property, stated below Theorem 2.4, combined with Theorem 2.7(ii) we find that the "ball"

$$\left\{P \in C(F) : \|P\|^2 \leq B\right\} \tag{3.1}$$

is finite for all B.

We split the set of points $C(F)$ into two subsets:

$$\left\{P \in C(F) : \|P\|^2 > c_3\right\} \quad \textbf{(large points),}$$
$$\left\{P \in C(F) : \|P\|^2 \leq c_3\right\} \quad \textbf{(moderate points).}$$

By the finiteness statement around (3.1), it suffices to show that there are at most finitely many large points.

For any $z \in \operatorname{Jac}(C)(F) \otimes \mathbb{R}$, we define the truncated cone

$$T(z) = \left\{w \in \operatorname{Jac}(C)(F) \otimes \mathbb{R} : \langle z, w\rangle \geq (1 - 1/c_1)\|z\|\|w\| \text{ and } \|w\|^2 > c_3\right\}$$
$$\subseteq \operatorname{Jac}(C)(F) \otimes \mathbb{R}.$$

By the Mordell–Weil Theorem, $\operatorname{Jac}(C)(F) \otimes \mathbb{R}$ is a finite-dimensional $\mathbb{R}$-vector space. So the unit sphere with respect to the norm $\|\cdot\|$ coming from the Néron–Tate height is compact. Therefore, $\{w \in \mathrm{Jac}(C)(F) \otimes \mathbb{R} : \|w\|^2 > c_3\}$ is covered by a finite union $T(z_1) \cup \cdots \cup T(z_N)$. Using a sphere packing argument, one can arrange that N is bounded from above by $c' \cdot c^{\mathrm{rk}\,\mathrm{Jac}(C)(F)}$ where $c' > 0$ and $c > 1$ depend only on c_1. This observation will be important for deriving uniform bounds for $\#C(F)$.

Any large point in $C(F)$ has image in some $T(z_j)$ from above. After possibly adjusting N, one can arrange that each z_j is the image of a point $P_j \in C(F)$ with $\|P_j\|^2 > c_3$ for all $j \in \{1, \ldots, N\}$. If $Q \in C(F)$ has image in $T(z_j)$, then Vojta's inequality implies $\hat{h}_{\mathfrak{L}}(Q)^{1/2} = \|Q\| \leq c_2\|P_j\|$. But then $Q \in C(F)$ lies in a finite ball as in (3.1). So the number of possible Q that come to lie in a single $T(z_j)$ is finite. Thus $C(F)$ is finite.

The constants c_1, c_2, and c_3 in Vojta's inequality can be made effective in terms of A and C. Yet, the proof as a whole is ineffective. Indeed, the height bound for Q depends on the hypothetical point P_j. However, there is no guarantee that P_j exists and if it does not, there is no known way to know for sure.

Using Mumford's Gap Principle, one can show that the number of large points $C(F)$ that come to lie in a single $T(z_j)$ is bounded from above by $c' \cdot c^{\mathrm{rk}\,\mathrm{Jac}(C)(F)}$, after possibly increasing the constants. Now we need to introduce dependency on c_2. But the base c will remain geometric in nature, it depends on the genus of g. But it does not depend on c_3 or other arithmetic properties of C that encode the heights of coefficients defining the said curve. Finally, as observed by Bombieri, 7 is admissible for c for any genus. Indeed, he showed that 4 is admissible for c_1.

Recall that Vojta's inequality with the same values of c_1, c_2, c_3 applies to points in $C(F')$ for all finite extensions F'/F. The upshot is that the number of large points of $C(F')$ is bounded by

$$c' \cdot c^{\mathrm{rk}\,\mathrm{Jac}(C)(F')}$$

where c, c' depend on C, but not on F'.

The dichotomy between large and moderate points was already visible in Vojta's work. But its origin is older and already appears in modified form in work of Thue, Siegel, Mahler, and Roth on diophantine approximation.

Rémond's explicit Théorème 2.1 [54] gives a recipe how to bound the total number of rational points using a bound for the number of moderate points.

With our eyes set on Mazur's question, we aim to obtain good bounds for the number of moderate points. In the coming two sections we explain our general approach to the proof of Theorem 1.12.

4. COMPARING WEIL AND NÉRON–TATE HEIGHTS

The interplay between the Weil and Néron–Tate heights on a family of abelian varieties leads to powerful results including Silverman's Specialization Theorem [57] and more recent work by Masser and Zannier towards the relative Manin–Mumford Conjecture [44]. This interaction also plays a central role in the proof of Theorem 1.12 that resolved Mazur's question.

Having worked with a fixed abelian variety in Sections 2.2 and 3, we now shift gears and work in a family of abelian varieties.

Example 4.1. Let $Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. For $\lambda \in Y(2)(\mathbb{C})$, we have an elliptic curve $\mathcal{E}_\lambda \subseteq \mathbb{P}^2$ determined by

$$y^2z = x(x-z)(x-\lambda z)$$

where the origin is $[0:1:0]$. The total space $\mathcal{E}$ is a surface presented with a closed immersion $\mathcal{E} \hookrightarrow \mathbb{P}^2 \times Y(2)$. It is called the Legendre family of elliptic curves and is an abelian scheme over $Y(2)$. So we can add two complex points of $\mathcal{E}$ if they are in the same fiber above $Y(2)$. More precisely, there is an addition morphism $\mathcal{E} \times_S \mathcal{E} \to \mathcal{E}$ over S, as well as an inversion morphism $\mathcal{E} \to \mathcal{E}$ over S. Finally, the zero section of $\mathcal{E}$ is given by $\lambda \mapsto ([0:1:0], \lambda)$.

Consider a geometrically irreducible smooth quasiprojective variety S defined over a number field $F \subseteq \overline{\mathbb{Q}}$. Let $\pi : \mathcal{A} \to S$ be an abelian scheme over S. So each fiber $\mathcal{A}_s = \pi^{-1}(s)$, where $s \in S(\overline{\mathbb{Q}})$, is an abelian variety. We have an addition morphism on the fibered square $\mathcal{A} \times_S \mathcal{A} \to \mathcal{A}$ and an inversion morphism $\mathcal{A} \to \mathcal{A}$; both are relative over S. Addition induces a multiplication-by-n morphism $[n] : \mathcal{A} \to \mathcal{A}$ over S for all $n \in \mathbb{Z}$.

For simplicity, we assume that $\mathcal{A}$ is presented with an immersion $\mathcal{A} \hookrightarrow \mathbb{P}^n \times S$ over S, much as in Example 4.1 above. Let $\mathcal{L}$ be the restriction of the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{P}^n \times S = \mathbb{P}^n_S$ to $\mathcal{A}$. We also assume that $\mathcal{L}$ is even, that is $[-1]^*\mathcal{L} \cong \mathcal{L}$. This allows us to define a fiberwise Néron–Tate height on $\mathcal{A}(\overline{\mathbb{Q}})$ which we abbreviate by $\hat{h}_\mathcal{A}$.

Let $s \in S(\overline{\mathbb{Q}})$. Then $\mathcal{A}_s$ is an abelian variety in $\mathbb{P}^n$. We have two functions, $\hat{h}_\mathcal{A}|_{\mathcal{A}_s(\overline{\mathbb{Q}})}$ and $h|_{\mathcal{A}_s(\overline{\mathbb{Q}})}$; the latter is the restriction of the Weil height on $\mathbb{P}^n$. By Theorem 2.7(ii), their difference is bounded in absolute value in function of s.

In the example of the Legendre family, the point $[\lambda : 0 : 1] \in \mathcal{E}_\lambda$ is of order 2 for all λ. So its Néron–Tate height vanishes, but its Weil height equals $h([\lambda : 1])$ and is thus unbounded as λ varies.

We would like to understand the difference between Néron–Tate and Weil heights on $\mathcal{A}$ as the base point $s \in S(\overline{\mathbb{Q}})$ varies. As suggested by the Legendre case, the key is the Weil height on the base S. To keep things concrete, we will assume that S comes with an immersion $S \hookrightarrow \mathbb{P}^m$. We identify S with a Zariski locally closed subset of $\mathbb{P}^m$. So S need not be projective, but its Zariski closure $\overline{S}$ in $\mathbb{P}^m$ is. We write h_S for $h|_{\overline{S}(\overline{\mathbb{Q}})} : \overline{S}(\overline{\mathbb{Q}}) \to [0, \infty)$ where h is the Weil height on $\mathbb{P}^m(\overline{\mathbb{Q}})$. In the language of Section 2.1, h_S represents the Weil height attached to $(\overline{S}, \mathcal{O}(1)|_{\overline{S}})$.

The difference between Weil and Néron–Tate heights on the total space $\mathcal{A}(\overline{\mathbb{Q}})$ was clarified in work of Zimmer **[73]** in the elliptic setting and Manin–Zarhin **[69]** and Silverman–Tate **[57]** in the more general setting. In our case the latter result amounts to

$$\hat{h}_\mathcal{A}|_{\mathcal{A}_s(\overline{\mathbb{Q}})} = h|_{\mathcal{A}_s(\overline{\mathbb{Q}})} + O\big(\max\{1, h_S(s)\}\big) \tag{4.1}$$

for all $s \in S(\overline{\mathbb{Q}})$.

We introduce a final player, a geometrically-irreducible subvariety V of $\mathcal{A}$ defined over F.

Theorem 4.2 (Silverman [57]). *Suppose S and $V \subseteq \mathcal{A}$ are curves such that V dominates S. Then*

$$\lim_{\substack{P \in V(\overline{\mathbb{Q}}) \\ h(\pi(P)) \to \infty}} \frac{\hat{h}_{\mathcal{A}}(P)}{h(\pi(P))} \tag{4.2}$$

exists. Suppose, in addition, that the geometric generic fiber of $\mathcal{A} \to S$ has trivial trace over $\overline{\mathbb{Q}}$. Then the limit vanishes if and only if V is an irreducible component of $\ker[N]$ *for some $N \geq 1$. Otherwise the limit is positive.*

Silverman computed the limit in terms of the Néron–Tate height of V restricted to the generic fiber $\mathcal{A} \to S$.

The "if direction" is straightforward: in this case all P in question are of finite order and their Néron–Tate height vanishes; see Theorem 2.7(iv). The "only if" direction is deeper and has many applications: Silverman's Specialization Theorem, Theorem C in [57], as well as applications to unlikely intersections by Masser and Zannier, see [44] and [68] for an overview and more results.

What happens if V has dimension > 1 and S remains a curve? In this case the limit (4.2) does not make sense. Indeed, for fixed $s \in S(\overline{\mathbb{Q}})$, the set of $P \in V(\overline{\mathbb{Q}})$ that map to s has positive dimension and thus unbounded Néron–Tate height.

Motivated by Theorem 4.2, the author showed the next theorem. It may serve as a higher-dimensional substitute for Silverman's Theorem 4.2. For an irreducible subvariety of V of $\mathcal{A}$ that dominates S, we write $V_{\overline{\eta}}$ for the geometric generic fiber of $\pi|_V : V \to S$. This is a possibly reducible subvariety of the geometric generic fiber $\mathcal{A}_{\overline{\eta}}$ of $\mathcal{A} \to S$.

Theorem 4.3 ([35]). *Suppose $S = Y(2)$ and let $\mathcal{A} = \mathcal{E}^{[g]}$ be the g-fold fibered power of the Legendre family of elliptic curves. Suppose $V \subseteq \mathcal{E}^{[g]}$ dominates $Y(2)$ and*

$$V_{\overline{\eta}} \text{ is not a finite union of irreducible components of algebraic subgroups of } \mathcal{A}_{\overline{\eta}}. \tag{4.3}$$

Then there exist $c(V) > 0$ and a Zariski open and dense subset $U \subseteq V$ with

$$h_{Y(2)}\big(\pi(P)\big) \leq c(V) \max\big\{1, \hat{h}_{\mathcal{A}}(P)\big\} \quad \text{for all } P \in U(\overline{\mathbb{Q}}). \tag{4.4}$$

Say (4.3) holds. If $P \in U(\overline{\mathbb{Q}})$ has finite order as a point in its respective fiber, we find $h_{Y(2)}(\pi(p)) \leq c(V)$ and the total Weil height of P is bounded from above by (4.1). This simple observation led to the resolution of several "special points" problems [35] in the spirit of the André–Oort Conjecture. For example, torsion points $P \in V(\overline{\mathbb{Q}})$ that lie in a fiber with complex multiplication are not Zariski dense in V. The proof of Theorem 4.3 makes use of Siu's Criterion, see Remark 2.5(iii), and an investigation of monodromy in $\mathcal{E}^{[g]}$.

The Zariski open U cannot in general be taken to equal V. But there is a natural description of this set in geometric terms through unlikely intersections.

The hypothesis (4.3) is necessary and essentially rules out that V itself is a family of abelian subvarieties.

Gao and the author [33] then generalized Theorem 4.3 to an abelian scheme when the base is again a smooth curve S defined over $\overline{\mathbb{Q}}$. Here more care is needed in connection with the hypothesis (4.3). Indeed, if $\mathcal{A} = A \times S$ is a constant abelian scheme, where A is an abelian variety, then (4.4) cannot hold generically for $V = Y \times S$. Roughly speaking, the condition in [33] that replaces (4.3) also needs to take into account a possible constant part of $\mathcal{A}_{\overline{\eta}}$. If $\mathcal{A}_{\overline{\eta}}$ has no constant part, i.e., if its $\overline{\mathbb{Q}}(\overline{\eta})/\overline{\mathbb{Q}}$-trace is 0, then (4.3) suffices for S a curve. The case of a higher-dimensional base requires even more care, as we will see.

There were two applications of the height bound in [33].

First, and in the same paper, we proved new cases of the geometric Bogomolov Conjecture for an abelian variety defined over the function field of the curve S. This approach relied on Silverman's Theorem 4.2. It was used earlier in [35] to give a new proof of the Geometric Bogomolov Conjecture in a power of an elliptic curve. The number field case of the Bogomolov Conjecture was proved by Ullmo [61] and Zhang [71] in the 1990s. Progress in the function field case was later made by Cinkir, Faber, Moriwaki, Gubler, and Yamaki. For the state of the Geometric Bogomolov Conjecture as of 2017, we refer to a survey of Yamaki [64]. Gubler's strategy works in arbitrary characteristic and was expanded on by Yamaki. In joint work [10] with Cantat, Gao, and Xie, the author later established the Geometric Bogomolov Conjecture in characteristic 0 by bypassing the height inequality (4.4). Very recently, Xie and Yuan [63] announced a proof of the Geometric Bogomolov Conjecture in arbitrary characteristic. Their approach builds on the work of Gubler and Yamaki.

Second, and in later joint work with Dimitrov and Gao [23], we established uniformity for the number of rational points in the spirit of Mazur's question for curves parametrized by the 1-dimensional base S.

As we shall see, the proof of Theorem 1.12 requires a height comparison result like (4.4) for abelian schemes over a base S of any dimension. But now the correct condition to impose on V is more sophisticated and cannot be easily read off of the geometric generic fiber as in (4.3). The condition relies on the Betti map, which we introduce in the next section.

4.1. Degenerate subvarieties and the Betti map

In this section, S is a smooth irreducible quasiprojective variety over $\mathbb{C}$. Let $\pi : \mathcal{A} \to S$ again be an abelian scheme over S of relative dimension $g \geq 1$.

For each $s \in S(\mathbb{C})$, the fiber $\mathcal{A}_s(\mathbb{C})$ is a complex torus of dimension g. Forgetting the complex structure, each g-dimensional complex torus is diffeomorphic to $(\mathbb{R}/\mathbb{Z})^{2g}$ as a real Lie group. By Ehresmann's Theorem, this diffeomorphism extends locally in the analytic topology on the base. That is, there is a contractible open neighborhood U of s in $S(\mathbb{C})$ and a diffeomorphism $\mathcal{A}_U = \pi^{-1}(U) \to (\mathbb{R}/\mathbb{Z})^{2g} \times U$ over U. Fiberwise this diffeomorphism can be arranged to be a group isomorphism above each point of U. Thus we can locally trivialize the abelian scheme at the cost of sacrificing the complex-analytic structure.

The trivialization is not entirely unique as we can let a matrix in $\mathrm{GL}_{2g}(\mathbb{Z})$ act in the natural way on the real torus $(\mathbb{R}/\mathbb{Z})^{2g}$. But since U is connected, this is the only ambiguity. It is harmless for what follows.

The Betti map β_U attached to U is the composition of the trivialization followed by the projection

$$\begin{tikzcd} \mathcal{A}_U \arrow[r] \arrow[rr, bend left, "\beta_U"] & (\mathbb{R}/\mathbb{Z})^{2g} \times U \arrow[r] & (\mathbb{R}/\mathbb{Z})^{2g}. \end{tikzcd}$$

This map has appeared implicitly in diophantine geometry in work of Masser and Zannier [44]. We also refer to more recent work of André, Corvaja, and Zannier [3] for a systematic study of the Betti map.

We list some of the most important properties:

(i) For all $s \in U$, the restriction $\beta_U|_{\mathcal{A}_s(\mathbb{C})} : \mathcal{A}_s(\mathbb{C}) \to (\mathbb{R}/\mathbb{Z})^{2g}$ is a diffeomorphism of real Lie groups. In particular, $P \in \mathcal{A}_U$ has finite order in its respective fiber if and only if $\beta_U(P) \in (\mathbb{Q}/\mathbb{Z})^{2g}$.

(ii) For all $P \in U$ the fiber $\beta_U^{-1}(\beta_U(P))$ is a complex-analytic subset of $\mathcal{A}_U$.

Definition 4.4. An irreducible closed subvariety $V \subseteq \mathcal{A}$ that dominates S is called degenerate if for all U and β_U as above and all smooth points P of $V_U = \pi|_V^{-1}(U)$ the differential of $\mathrm{d}_P(\beta_U|_{V_U})$ satisfies

$$\operatorname{rk} \mathrm{d}_P(\beta_U|_{V_U}) < 2 \dim V. \tag{4.5}$$

It has become customary to call V degenerate if it is not nondegenerate.

For all smooth points P of V_U, the left-hand side of (4.5) is at most the right-hand side, which equals the real dimension of V_U. It is also at most $2g$, the real dimension of a fiber of $\mathcal{A} \to S$. Moreover, if the maximal rank of $\mathrm{d}\beta_U$ on V_U is attained at P then the maximal rank is attained also in a neighborhood of P in V_U. Being nondegenerate is a local property.

Let us consider some examples.

Example 4.5. (i) If S is a point, then $\mathcal{A}$ is an abelian variety and an arbitrary subvariety $V \subseteq \mathcal{A}$ is nondegenerate because β_S is a diffeomorphism.

(ii) Suppose $\dim V > g$. Then $\operatorname{rk} \mathrm{d}_P(\beta_U|_{V_U}) \le 2g < 2 \dim V$ for all smooth P and so V is degenerate. In particular, $\mathcal{A}$ is a degenerate subvariety of $\mathcal{A}$ if $\dim S \ge 1$.

(iii) Suppose $\mathcal{A} = A \times S$ is a constant abelian scheme with A an abelian variety. If $Y \subseteq A$ is a closed irreducible subvariety and if $\dim S \ge 1$, then $Y \times S$ is degenerate. Indeed, the rank is at most $2 \dim Y < 2 \dim Y \times S$.

(iv) Suppose V is an irreducible component of $\ker[N]$ for some integer $N \ge 1$. Any point in $V(\mathbb{C})$ has order dividing N (and, in fact, equal to N). So the image of $\beta_U|_{V_U}$ is finite and hence V is degenerate if $\dim S \ge 1$.

(v) Suppose V is the image of a section $S \to \mathcal{A}$. If the geometric generic fiber of $\mathcal{A} \to S$ has trivial trace, then $(\beta_U)|_{V_U}$ is constant if and only if V is an irreducible component of $\ker[N]$ for some $N \ge 1$. This is Manin's Theorem of the Kernel, we refer to Bertrand's article [7] for the history of this theorem.

(vi) Suppose $\mathcal{A} = \mathcal{E}^{[g]}$ and V are as in Theorem 4.3. One step in the proof of this theorem consisted in verifying that V, subject to hypothesis (4.3), is nondegenerate. Crucial input came from the monodromy action of the fundamental group of the base $Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ on the first homology of a fiber $\mathcal{A}_s$ with s in general position. In this case the monodromy action is unipotent at the cusps 0 and 1 of $Y(2)$. This enabled the author to use a result of Kronecker from diophantine approximation. Already Masser and Zannier [44] used the monodromy action in their earlier work for V a curve.

(vii) If S is a curve, then the monodromy action of the fundamental group of $S(\mathbb{C})$ on the homology of fibers of $\mathcal{A} \to S$ is locally quasiunipotent. But if S is projective, then there are no cusps. So exploiting monodromy in this setting required a different approach. In [33] Gao and the author used o-minimal geometry and the Pila–Wilkie Counting Theorem [51]. A related case was solved by Cantat, Gao, and Xie in collaboration with the author [10]; we used dynamical methods.

(viii) Finally, we consider the case of an abelian scheme $\mathcal{A}$ over a base S of arbitrary dimension. This setting was studied recently in work of André–Corvaja–Zannier [3]. Moreover, the work of Gao on the Ax–Schanuel Theorem [30] for the universal family of abelian varieties led him to formulate a geometric condition [29] that guarantees nondegeneracy. It proves crucial in the application to Mazur's question and we will return to this point. Gao's result also relied on o-minimal geometry and the Pila–Wilkie Theorem.

4.2. Comparing the Weil and Néron–Tate heights on a subvariety

We now come to the generalization of Theorem 4.3 to nondegenerate subvarieties. We retain the notation introduced in Section 4.1. So S is a smooth irreducible quasiprojective variety defined over $\overline{\mathbb{Q}}$ equipped with an immersion in $\mathbb{P}^m$. We have a height h_S on $\overline{S}(\overline{\mathbb{Q}})$. Moreover, $\pi : \mathcal{A} \to S$ is an abelian scheme over S presented with an immersion $\mathcal{A} \to \mathbb{P}^n \times S$ over S. Finally, $\mathcal{L}$ is as in Section 4 and $\hat{h}_{\mathcal{A}}$ is the fiberwise Néron–Tate height on $\mathcal{A}(\overline{\mathbb{Q}})$.

We assume that $\mathcal{A}$ carries symplectic level-ℓ structure for some fixed $\ell \geq 3$ and that $\mathcal{L}$ induces a principal polarization. For the proof of Theorem 1.12, it suffices to have the following height bound under these conditions. We also refer to [24, THEOREM B.1] for a version that relaxes some of the conditions.

Theorem 4.6 ([24, THEOREM 1.6]). *Let V be a nondegenerate irreducible subvariety of $\mathcal{A}$ that dominates S. There exist $c(V) > 0$, $c'(V) \geq 0$, and a Zariski open and dense subset $U \subseteq V$ with*

$$h_S\big(\pi(P)\big) \leq c(V)\hat{h}_{\mathcal{A}}(P) + c'(V) \quad \textit{for all } P \in U(\overline{\mathbb{Q}}).$$

We refer to Yuan and Zhang's Theorem 6.2.2 [67] for a height inequality in the dynamical setting.

Here are just a few words on the proof of Theorem 4.6. Siu's Criterion, see Remark 2.5, is used to compare the Weil height of $\pi(P)$ with a Weil height of P. The nondegeneracy hypothesis is used to extract a volume estimate. The upshot is a lower bound for the top self-intersection number in Siu's Criterion. The predecessor of Theorem 4.6 in the earlier works [33, 35] was proved by counting torsion points using the Geometry of Number; volumes played an important role here as well. Passing from the Weil to the Néron–Tate height introduces an additional dependency on the height of $\pi(P)$, see (4.1). However, this contribution can be eliminated by using Masser's "ruthless strategy of killing Zimmer constants" [68, APPENDIX C]. This task is done by repeated iteration of the duplication morphism [2] which has the effect of truncating Tate's Limit Process after finitely many steps. Our ambient group scheme $\mathcal{A}$ is quasiprojective but in general not projective. So a suitable compactification is required that admits some compatibility with the duplication morphism.

The positive constant $c(V)$ in Theorem 4.6 ultimately comes from the application of Siu's Criterion. As such it can expressed in geometric terms.

5. APPLICATION TO MODERATE POINTS ON CURVES

In this section we sketch the main lines of the proof of Theorem 1.12. It will be enough to bound the number of moderate points, see Section 3.

5.1. The Faltings–Zhang morphism

Smooth curves of genus $g \geq 2$ defined over $\overline{\mathbb{Q}}$ are classified by the $\overline{\mathbb{Q}}$-points of a quasiprojective variety, the coarse moduli space. For us it is convenient to work with symplectic level-ℓ structure on the Jacobian for some fixed integer $\ell \geq 3$. With this extra data, we obtain a fine moduli space $\mathbb{M}_g$, together with a universal family $\mathfrak{C}_g \to \mathbb{M}_g$. Fibers of this family are smooth curves of genus g with the said level structure on the Jacobian. Then $\mathbb{M}_g$ carries the structure of a smooth quasiprojective variety of dimension $3g-3$ defined over a cyclotomic field. For convenience, we replace $\mathbb{M}_g$ by an irreducible component by choosing a complex root of unity of order ℓ and consider it as defined over $\overline{\mathbb{Q}}$.

The Torelli morphism $\tau : \mathbb{M}_g \to \mathbb{A}_g$ takes a smooth curve to its Jacobian with the level structure; here $\mathbb{A}_g$ denotes the fine moduli space of g-dimensional abelian varieties with a principal polarization and symplectic level-ℓ structure.

Let $M \geq 0$ be an integer and consider $M+1$ points $P_0, \ldots, P_M \in \mathfrak{C}_g(\mathbb{C})$ in the same fiber C of $\mathfrak{C}_g \to \mathbb{M}_g$. The differences $[P_1]-[P_0], \ldots, [P_M]-[P_0]$ are divisors of degree 0 on C. We obtain M complex points in the Jacobian of C and so M complex points of $\mathfrak{A}_g$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathfrak{C}_g^{[M+1]} \arrow[r, "\mathscr{D}"] \arrow[dr] & \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} \mathbb{M}_g \arrow[d] \\
 & \mathbb{M}_g
\end{tikzcd}$$

of morphisms of schemes; here the exponent $[M]$ denotes taking the Mth fibered power over the base. The morphism $\mathscr{D}$ is called the Faltings–Zhang morphism; see [26, LEMMA 4.1] and [71, LEMMA 3.1] for important applications to diophantine geometry of variants of this morphism. The morphism $\mathscr{D}$ is proper.

A modified version of this construction is also useful. Say $S \to \mathbb{M}_g$ is a quasifinite morphism with S an irreducible quasiprojective variety defined over $\overline{\mathbb{Q}}$. We obtain a proper morphism $\mathscr{D} : \mathfrak{C}_g^{[M+1]} \times_{\mathbb{M}_g} S \to \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S$, again called Faltings–Zhang morphism.

Gao, using his Ax–Schanuel Theorem for the universal family $\mathfrak{A}_g$ [30] and a characterization [28] of bialgebraic subvarieties of $\mathfrak{A}_g$, obtained

Theorem 5.1 (Gao [29]). *Let $S \to \mathbb{M}_g$ be as above, i.e., a quasifinite morphism from an irreducible quasiprojective variety S defined over $\overline{\mathbb{Q}}$ and $g \geq 2$. If $M \geq \dim \mathbb{M}_g + 1 = 3g - 2$, then $\mathscr{D}(\mathfrak{C}_g^{[M+1]} \times_{\mathbb{M}_g} S)$ is a nondegenerate subvariety of $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S$.*

Mok, Pila, and Tsimerman [47] earlier proved an Ax–Schanuel Theorem for Shimura varieties. Gao's result [30] is a "mixed" version in the abelian setting. We refer to the survey [4] on recent developments in functional transcendence.

The hypothesis $g \geq 2$ is crucial. The definition of the Faltings–Zhang morphism makes sense for $g = 1$. But it will be surjective and the image is degenerate expect in the (for our purposes uninteresting) case $\dim S = 0$.

We consider here for simplicity only the case $S = \mathbb{M}_g$.

Using basic dimension theory, we see $\dim \mathscr{D}(\mathfrak{C}_g^{[M+1]}) \leq M + 1 + \dim \mathbb{M}_g$. The image lies in the fibered power $\mathfrak{A}_g^{[M]}$ where the relative dimension is Mg. A necessary condition for $\mathscr{D}(\mathfrak{C}_g^{[M+1]})$ to be nondegenerate is $\dim \mathscr{D}(\mathfrak{C}_g^{[M+1]}) \leq Mg$, see Example 4.5(ii). This inequality follows from

$$M + 3g - 2 = M + 1 + \dim \mathbb{M}_g \leq Mg. \tag{5.1}$$

If $M \leq 3$, the numerical condition (5.1) is not satisfied for any $g \geq 2$. For this reason, we cannot hope to work with the image of $\mathfrak{C}_g \times_{\mathbb{M}_g} \mathfrak{C}_g$ in $\mathfrak{A}_g$ by taking differences. Moreover, there seems to be no reasonable way to work with a single copy of $\mathfrak{C}_g$, where the relations between dimensions would be even worse. The numerical condition (5.1) is satisfied for all $M \geq 4$ and all $g \geq 2$. Gao's Theorem implies that $M \geq 3g - 2$ is sufficient to guarantee nondegeneracy.

We can thus apply Theorem 4.6 to the image $\mathscr{D}(\mathfrak{C}_g^{[M+1]})$ of the Faltings–Zhang morphism in $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} \mathbb{M}_g$. Let $M = 3g - 2$, then

$$h_{\mathbb{M}_g}(s) \leq c(g)\big(\hat{h}_{\mathfrak{A}_g}(P_1 - P_0) + \cdots + \hat{h}_{\mathfrak{A}_g}(P_M - P_0)\big) + c'(g) \tag{5.2}$$

for all $(P_0, \ldots, P_M) \in U(\overline{\mathbb{Q}})$ above $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ where U is a Zariski open and dense subset of $\mathscr{D}(\mathfrak{C}_g^{[M+1]})$. The constants $c(g) > 0$ and $c'(g) \geq 0$ depend on the various choices made regarding projective immersions of $\mathbb{M}_g$ and $\mathfrak{A}_g$. Ultimately, they depend only on g once these choices have been made.

The Zariski open U cannot be replaced by $\mathscr{D}(\mathfrak{C}_g^{[M+1]})$. Indeed, the right-hand side of (5.2) vanishes on the diagonal $P_0 = P_1 = \cdots = P_M$ whereas the left-hand side is unbounded as s varies.

Let us shift back to using $\|\cdot\|$ to denote the square root of the Néron–Tate height, see Section 2.2. Let us assume that

$$h_{\mathbb{M}_g}(s) \geq 2c'(g). \tag{5.3}$$

As $2M = 6g - 4$ we find

$$h_{\mathbb{M}_g}(s) \leq c(g)(6g-4) \max_{1 \leq j \leq M} \|P_j - P_0\|^2 \quad \text{for all } (P_0, \ldots, P_M) \in U(\overline{\mathbb{Q}}). \tag{5.4}$$

Morally, (5.4) states that among a $(3g-1)$-tuple of points on a curve of genus g in general position, there must be a pair that repels one another with respect to the norm $\|\cdot\|$. The squared distance of such a pair is larger than a positive multiple, depending only on g, of the modular height $h_{\mathbb{M}_g}(s)$; this is the key to bounding the number of moderate points from Section 3.

As stated at the end of Section 4.2, the value $c(g)$ can be expressed in terms of geometry properties of the image of $\mathfrak{C}_g^{[M+1]}$ under the Faltings–Zhang morphism.

Question 5.2. What is an admissible value for $c(g)$?

5.2. Bounding the number of moderate points—a sketch

Recall that, by the discussion at the end of Section 3, we need to bound the number of moderate points.

We retain the notation of Sections 3 and 5. The curve C from Section 3 can be equipped with suitable level structure over a field F'/F with $[F' : F]$ bounded in terms of g. The rank of $\mathrm{Jac}(C)(F')$ may be dangerously larger than the rank of $\mathrm{Jac}(C)(F)$. But recall that we are interested in bounding $\#C(F)$ from above, so only the group $\mathrm{Jac}(C)(F)$ will be relevant. Moreover, c_1, c_2, and c_3 from a suitable version of Vojta's inequality are unaffected by extending F. The effect is that we may identify C with a fiber of $\mathfrak{C}_g$ above some point $s \in \mathbb{M}_g(F')$. For simplicity, we assume $F = F'$ for this proof sketch.

We require some additional information on $c_3(C)$. It turns out that we can take $c_3 = c_4(g)\max\{1, h_{\mathbb{M}_g}(s)\}$ where $c_4(g) > 0$ depends on g. This follows Rémond's work **[55]** on the Vojta inequality. A similar dependency is apparent in de Diego's result, Theorem 1.6.

Suppose now that (5.3) holds, so $h_{\mathbb{M}_g}(s)$ is sufficiently large in terms of g. We fix an auxiliary base point $P' \in C(F)$. We must bound from above the number of points in

$$B(R) = \left\{P \in C(F) : \left\|P - P'\right\|^2 \leq R^2\right\} \quad \text{with } R = \left(c_4(g) h_{\mathbb{M}_g}(s)\right)^{1/2}$$

where $\|\cdot\|$ denotes the square root of the Néron–Tate height on the fiber of $\mathfrak{A}_g \to \mathbb{A}_g$ associated to the Jacobian of C.

Recall $M = 3g - 2$ and suppose $P_0, \ldots, P_M \in C(F)$. If the tuple $(P_0, \ldots, P_M)$ is in general position, i.e., $(P_1 - P_0, \ldots, P_M - P_0)$ lies in $U(\overline{\mathbb{Q}})$ from (5.4), then there is i with

$$P_i \notin B(P_0, r) = \left\{P \in C(F) : \|P - P_0\|^2 \leq r^2\right\} \quad \text{with } r = \left(c_5(g) h_{\mathbb{M}_g}(s)\right)^{1/2}.$$

If we had a guarantee that such $(M+1)$-tuples of pairwise distinct points are always in general position, then $\#B(P_0, r) < M = 3g-2$. By sphere packing, we can cover the image of $B(R)$ in $\mathrm{Jac}(C)(F) \otimes \mathbb{R}$ by at most $(1+2R/r)^{\mathrm{rk}\,\mathrm{Jac}(C)(F)}$ closed balls in $\mathrm{Jac}(C)(F)$ of radius r. One can even arrange for the ball centers to arise as points of $C(F)$. The modular height $h_{\mathbb{M}_g}(s)$ cancels out in the quotient

$$\frac{R}{r} = \left(\frac{c_4(g)}{c_5(g)}\right)^{1/2}.$$

This would complete the proof of Theorem 1.12 except that there is no reason to believe that $(P_1 - P_0, \ldots, P_M - P_0) \in U(\overline{\mathbb{Q}})$ (even if the P_j are pairwise distinct). Treating points with image in the complement of U requires induction on the dimension. Here we rely on the freedom to replace $\mathbb{M}_g$ by a subvariety in that Gao's Theorem 5.1.

Let us briefly explain the resulting induction step. Observe that the dimension of this exceptional set is at most $\dim \mathscr{D}(\mathfrak{C}_g^{[M+1]}) - 1 \leq M + \dim \mathbb{M}_g$. There are two cases for $(P_0, \ldots, P_M)$ with image in the exceptional set $(\mathscr{D}(\mathfrak{C}_g^{[M+1]}) \setminus U)(\overline{\mathbb{Q}})$ on which we do not have the height inequality. For the case study, recall that $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ denotes the point below all the P_j and $\tau(s) \in \mathbb{A}_g(\overline{\mathbb{Q}})$ is its image under the Torelli morphism τ.

First, assume that the fiber of $\mathscr{D}(\mathfrak{C}_g^{[M+1]}) \setminus U \to \mathbb{A}_g$ above $\tau(s)$ has dimension at most M. This fiber contains $(P_1 - P_0, \ldots, P_M - P_0)$. This case is solved using a zero estimate motivated by the following simple lemma.

Lemma 5.3. *Suppose C is an irreducible curve defined over $\mathbb{C}$ and W a proper Zariski closed subset of C^M. If $\Sigma \subseteq C(\mathbb{C})$ with $\Sigma^M \subseteq W(\mathbb{C})$, then Σ is finite.*

This statement can be quantified if C is presented as a curve in some projective space. Using Bézout's Theorem, one can show that $\#\Sigma$ is bounded from above in terms of the degrees of C and W. In our application, both degrees will be uniformly bounded as all varieties arise in algebraic families. This ultimately leads to the desired uniformity estimates.

The second case is if the fiber of $\mathscr{D}(\mathfrak{C}_g^{[M+1]}) \setminus U \to \mathbb{A}_g$ above $\tau(s)$ has dimension at least $M+1$. For dimension reasons, s lies in a proper subvariety S of $\mathbb{M}_g$. Here we apply induction on the dimension and replace $\mathbb{M}_g$ by its subvariety S.

This completes the proof sketch.

Kühne [39] combined ideas from equidistribution with the approach laid out in [24] to get a suitable uniform estimate for $\#B(P_0, r)$ without the restriction (5.3) on $h_{\mathbb{M}_g}(s)$. Yuan's Theorem 1.1 [66] does so as well, but he follows a different approach. He obtains a more general estimate that works also in the function field setting and allows for a larger R.

6. HYPERELLIPTIC CURVES

A hyperelliptic curve is a smooth curve of genus at least 2 that admits a degree 2 morphism to the projective line. Hyperelliptic curves have particularly simple planar models. Indeed, if the base field is a number field F, then a hyperelliptic curve of genus g can be represented by a hyperelliptic equation

$$Y^2 = f(X) \quad \text{with } f \in F[X] \text{ monic and square-free of degree } 2g+1 \text{ or } 2g+2.$$

In this section we determine consequences of Theorem 1.12 for hyperelliptic curves. Our aim is to leave the world of curves and Jacobians and to present a bound for the number of rational solutions of $Y^2 = f(X)$ that can be expressed in terms of f. We refer to Section 6 of [23] for a similar example in a 1-parameter family of hyperelliptic curves.

To keep technicalities to a minimum, we assume that our base field is $F = \mathbb{Q}$ and that $f \in \mathbb{Z}[X]$ is monic of degree $d = 2g+1$ and factors into linear factors in $\mathbb{Q}[X]$. The curve represented by the hyperelliptic equation has a marked Weierstrass point "at infinity." These assumptions can be loosened with some extra effort. For example, if f does not factor in $\mathbb{Q}[X]$, then the class number of the splitting field will play a part.

Say, $f = X^d + f_{d-1}X^{d-1} + \cdots + f_0$. By the assumption above, $f = (X-\alpha_1)\cdots(X-\alpha_d)$ with $\alpha_1,\ldots,\alpha_d \in \mathbb{Q}$ which are necessarily integers. The discriminant of f is

$$\Delta_f = \prod_{1\le i<j\le d} (\alpha_j - \alpha_i)^2 \in \mathbb{Z}\setminus\{0\}.$$

The Mordell Conjecture applied to the hyperelliptic curve represented by $Y^2 = f(X)$ states

$$\#\big\{(x,y) \in \mathbb{Q}^2 : y^2 = f(x)\big\} < \infty.$$

We have the following estimate for the cardinality. Below $\omega(n)$ denotes the number of distinct prime divisors of $n \in \mathbb{Z}\setminus\{0\}$.

Theorem 6.1. *Let $g \ge 2$. There exist $c(g) > 1$ and $c'(g) > 0$ with the following property. Suppose $f \in \mathbb{Z}[X]$ is monic of degree $2g+1$, square-free, and factors into linear factors in $\mathbb{Q}[X]$. Then*

$$\#\big\{(x,y) \in \mathbb{Q}^2 : y^2 = f(x)\big\} \le c'(g)c(g)^{\omega(\Delta_f)}. \tag{6.1}$$

Proof. The hyperelliptic equation represents a curve C defined over $\mathbb{Q}$ of genus g.

If p is a prime number with $p \nmid \Delta_f$, then the α_i are pairwise distinct modulo p. If p is also odd, then the equation $Y^2 = f(X)$ reduced modulo p defines a hyperelliptic curve over $\mathbb{F}_p$. So C has good reduction at all primes that do not divide $2\Delta_f$. Thus the Jacobian $\mathrm{Jac}(C)$ has good reduction at the same primes.

We may embed C into its Jacobian $\mathrm{Jac}(C)$ by sending the marked Weierstrass point to 0. Each root α_i of f corresponds to a rational point in $C(\mathbb{Q})$ and it is sent to a point of order 2 in $\mathrm{Jac}(C)$. Moreover, these points generate the 2-torsion in $\mathrm{Jac}(C)_{\mathrm{tors}}$. In particular, all points of order 2 in $\mathrm{Jac}(C)_{\mathrm{tors}}$ are rational.

Next we bound the rank of $\mathrm{Jac}(C)(\mathbb{Q})$ from above. Indeed, we could use the work of Ooe–Top [49] or [37, THEOREM C.1.9]. The latter applied to $\mathrm{Jac}(C)$, $k = \mathbb{Q}$, $m = 2$, and S the prime divisors of $2\Delta_f$ yields $\mathrm{rk}\,\mathrm{Jac}(C)(\mathbb{Q}) \le 2g\#S \le 2g\omega(2\Delta_f) \le 2g + 2g\omega(\Delta_f)$. Here we use that $\mathbb{Q}$ has a trivial class group; in a more general setup, the class group of the splitting field of f will enter at this point. The estimate (6.1) follows from Theorem 1.12 in the case $F = \mathbb{Q}$ with adjusted constants. ■

It is tempting to average (6.1) over the f bounded in a suitable way, e.g., by bounding the maximal modulus of the roots by a parameter X. As pointed out to the author by Christian Elsholtz and Martin Widmer, this average will be unbounded as $X \to \infty$.

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REFERENCES

[1] D. Abramovich, Uniformité des points rationnels des courbes algébriques sur les extensions quadratiques et cubiques. *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), no. 6, 755–758.

[2] L. Alpoge, The average number of rational points on genus two curves is bounded. 2018, arXiv:1804.05859.

[3] Y. André, P. Corvaja, and U. Zannier, The Betti map associated to a section of an abelian scheme (with an appendix by Z. Gao). *Invent. Math.* **222** (2020), 161–202.

[4] B. Bakker and J. Tsimerman, Lectures on the Ax–Schanuel conjecture. In *Arithmetic geometry of logarithmic pairs and hyperbolicity of moduli spaces*, edited by M. H. Nicole, pp. 1–68, CRM Short Courses, Springer, 2020.

[5] J. Balakrishnan, N. Dogra, J. S. Müller, J. Tuitman, and J. Vonk, Explicit Chabauty–Kim for the split Cartan modular curve of level 13. *Ann. of Math. (2)* **189** (2019), no. 3, 885–944.

[6] J. S. Balakrishnan, A. J. Best, F. Bianchi, B. Lawrence, J. S. Müller, N. Triantafillou, and J. Vonk, Two recent p-adic approaches towards the (effective) Mordell Conjecture. In *Arithmetic L-functions and differential geometric methods*, edited by P. Charollois, G. Freixas i Montplet, V. Maillot, pp. 31–74, Progr. Math. 338, Birkhäuser, 2021.

[7] D. Bertrand, Manin's theorem of the kernel: a remark on a paper of C-L. Chai, preprint, 2008, https://webusers.imj-prg.fr/~daniel.bertrand/Recherche/rpdf/Manin_Chai.pdf.

[8] E. Bombieri, The Mordell Conjecture revisited. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **17** (1990), no. 4, 615–640.

[9] E. Bombieri and W. Gubler, *Heights in diophantine geometry*. Cambridge University Press, 2006.

[10] S. Cantat, Z. Gao, P. Habegger, and J. Xie, The geometric Bogomolov conjecture. *Duke Math. J.* **170** (2021), no. 2, 247–277.

[11] L. Caporaso, J. Harris, and B. Mazur, Uniformity of rational points. *J. Amer. Math. Soc.* **10** (1997), no. 1, 1–35.

[12] L. Caporaso, J. Harris, and B. Mazur, Corrections to uniformity of rational points. 2021, arXiv:2012.14461.

[13] C. Chabauty, Sur les points rationnels des courbes algébriques de genre supérieur à l'unité. *C. R. Acad. Sci. Paris* **212** (1941), 882–885.

[14] S. Checcoli, F. Veneziano, and E. Viada, The explicit Mordell Conjecture for families of curves. *Forum Math. Sigma* **7** (2019), e31, 62 pp.

[15] R. F. Coleman, Effective Chabauty. *Duke Math. J.* **52** (1985), no. 3, 765–770.

[16] S. David, M. Nakamaye, and P. Philippon, Bornes uniformes pour le nombre de points rationnels de certaines courbes. In *Diophantine geometry*, pp. 143–164, CRM Series 4, Ed. Norm, Pisa, 2007.

[17] S. David and P. Philippon, Minorations des hauteurs normalisées des sous-variétés de variétés abeliennes. II. *Comment. Math. Helv.* **77** (2002), no. 4, 639–700.

[18] S. David and P. Philippon, Minorations des hauteurs normalisées des sous-variétés des puissances des courbes elliptiques. *Int. Math. Res. Pap.* **3** (2007), rpm006, 113 pp.

[19] T. de Diego, Points rationnels sur les familles de courbes de genre au moins 2. *J. Number Theory* **67** (1997), no. 1, 85–114.

[20] L. DeMarco, H. Krieger, and H. Ye, Uniform Manin–Mumford for a family of genus 2 curves. *Ann. of Math.* **191** (2020), 949–1001.

[21] L. DeMarco and N. M. Mavraki, Variation of canonical height and equidistribution. *Amer. J. Math.* **142** (2020), no. 2, 443–473.

[22] V. Dimitrov, Z. Gao, and P. Habegger, A consequence of the relative Bogomolov conjecture. *J. Number Theory*, **230** (2022), 146–160.

[23] V. Dimitrov, Z. Gao, and P. Habegger, Uniform bound for the number of rational points on a pencil of curves. *Int. Math. Res. Not. IMRN* **2** (2021), 1138–1159.

[24] V. Dimitrov, Z. Gao, and P. Habegger, Uniformity in Mordell–Lang for curves. *Ann. of Math. (2)* **194** (2021), no. 1, 237–298.

[25] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.* **73** (1983), 349–366.

[26] G. Faltings, Diophantine approximation on abelian varieties. *Ann. of Math. (2)* **133** (1991), no. 3, 549–576.

[27] G. Faltings, The general case of S. Lang's conjecture. In *Barsotti symposium in algebraic geometry (Abano Terme, 1991)*, pp. 175–182, Perspect. Math. 15, Academic Press, San Diego, CA, 1994.

[28] Z. Gao, A special point problem of André–Pink–Zannier in the universal family of Abelian varieties. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **17** (2017), no. 1, 231–266.

[29] Z. Gao, Generic rank of Betti map and unlikely intersections. *Compos. Math.* **156** (2020), no. 12, 2469–2509.

[30] Z. Gao, Mixed Ax–Schanuel for the universal abelian varieties and some applications. *Compos. Math.* **156** (2020), no. 11, 2263–2297.

[31] Z. Gao, Recent developments of the uniform Mordell–Lang conjecture. 2021, arXiv:2104.03431.

[32] Z. Gao, T. Ge, and L. Kühne, The uniform Mordell–Lang conjecture. 2021, arXiv:2105.15085.

[33] Z. Gao and P. Habegger, Heights in families of Abelian varieties and the geometric Bogomolov conjecture. *Ann. of Math.* **189** (2019), no. 2, 527–604.

[34] P. Habegger, Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension. *Invent. Math.* **176** (2009), no. 2, 405–447.

[35] P. Habegger, Special points on fibered powers of elliptic surfaces. *J. Reine Angew. Math.* **685** (2013), 143–179.

[36] M. Hindry, Autour d’une conjecture de Serge Lang. *Invent. Math.* **94** (1988), no. 3, 575–603.

[37] M. Hindry and J. H. Silverman, *Diophantine geometry: an introduction*. Springer, 2000.

[38] E. Katz, J. Rabinoff, and D. Zureick-Brown, Uniform bounds for the number of rational points on curves of small Mordell–Weil rank. *Duke Math. J.* **165** (2016), no. 16, 3189–3240.

[39] L. Kühne, Equidistribution in families of Abelian varieties and uniformity. 2021, arXiv:2101.10272.

[40] L. Kühne, The relative Bogomolov conjecture for fibered products of elliptic surfaces. 2021, arXiv:2103.06203.

[41] B. Lawrence and A. Venkatesh, Diophantine problems and p-adic period mappings. *Invent. Math.* **221** (2020), no. 3, 893–999.

[42] R. Lazarsfeld, *Positivity in algebraic geometry. I*. Ergeb. Math. Grenzgeb. (3). Ser. Mod. Surveys Math. [Res. Math. Relat. Areas (3)] 48, Springer, Berlin, 2004.

[43] D. Masser, Specializations of finitely generated subgroups of abelian varieties. *Trans. Amer. Math. Soc.* **311** (1989), no. 1, 413–424.

[44] D. Masser and U. Zannier, Torsion points on families of squares of elliptic curves. *Math. Ann.* **352** (2012), no. 2, 453–484.

[45] B. Mazur, Arithmetic on curves. *Bull. Amer. Math. Soc.* **14** (1986), no. 2, 207–259.

[46] B. Mazur, Abelian varieties and the Mordell–Lang conjecture. In *Model theory, algebra, and geometry*, pp. 199–227, Math. Sci. Res. Inst. Publ. 39, Cambridge University Press, Cambridge, 2000.

[47] N. Mok, J. Pila, and J. Tsimerman, Ax–Schanuel for Shimura varieties. *Ann. of Math. (2)* **189** (2019), no. 3, 945–978.

[48] L. J. Mordell, On the rational solutions of the indeterminate equations of the third and fourth degrees. *Proc. Camb. Philos. Soc.* **21** (1922), 179–192.

[49] T. Ooe and J. Top, On the Mordell–Weil rank of an abelian variety over a number field. *J. Pure Appl. Algebra* **58** (1989), no. 3, 261–265.

[50] P. L. Pacelli, Uniform boundedness for rational points. *Duke Math. J.* **88** (1997), no. 1, 77–102.

[51] J. Pila and A. J. Wilkie, The rational points of a definable set. *Duke Math. J.* **133** (2006), no. 3, 591–616.

[52] M. Raynaud, Courbes sur une variété abélienne et points de torsion. *Invent. Math.* **71** (1983), no. 1, 207–233.

[53] M. Raynaud, Sous-variétés d'une variété abélienne et points de torsion. In *Arithmetic and geometry, Vol. I*, pp. 327–352, Progr. Math. 35, Birkhäuser Boston, Boston, MA, 1983.

[54] G. Rémond, Décompte dans une conjecture de Lang. *Invent. Math.* **142** (2000), no. 3, 513–545.

[55] G. Rémond, Inégalité de Vojta en dimension supérieure. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **29** (2000), no. 1, 101–151.

[56] G. Rémond, Borne générique pour le problème de Mordell–Lang. *Manuscripta Math.* **118** (2005), no. 1, 85–97.

[57] J. H. Silverman, Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.* **342** (1983), 197–211.

[58] M. Stoll, Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell–Weil rank. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 3, 923–956.

[59] L. Szpiro, Un peu d'effectivité. *Astérisque* **127** (1985), 275–287.

[60] L. Szpiro, E. Ullmo, and S. Zhang, Équirépartition des petits points. *Invent. Math.* **127** (1997), no. 2, 337–347.

[61] E. Ullmo, Positivité et discrètion des points algébriques des courbes. *Ann. of Math. (2)* **147** (1998), no. 1, 167–179.

[62] P. Vojta, Siegel's theorem in the compact case. *Ann. of Math. (2)* **133** (1991), no. 3, 509–548.

[63] Y. Xie and X. Yuan, Geometric Bogomolov conjecture in arbitrary characteristics. 2021, arXiv:2108.09722.

[64] K. Yamaki, Survey on the geometric Bogomolov conjecture. In *Actes de la Conférence "Non-Archimedean analytic geometry: theory and practice"*, pp. 137–193, Publ. Math. Besançon Algèbre Théorie Nr. 2017/1, Presses Univ. Franche-Comté, Besançon, 2017.

[65] X. Yuan, Big line bundles over arithmetic varieties. *Invent. Math.* **173** (2008), no. 3, 603–649.

[66] X. Yuan, Arithmetic bigness and a uniform Bogomolov-type result. 2021, arXiv:2108.05625.

[67] X. Yuan and S. Zhang, Adelic line bundles over quasi-projective varieties. 2021, arXiv:2105.13587.

[68] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*. Ann. of Math. Stud. 181, Princeton University Press, Princeton, NJ, 2012.

[69] J. G. Zarhin and J. I. Manin, Height on families of abelian varieties. *Mat. Sb. (N.S.)* **89** (1972), no. 131, 171–181, 349.

[70] S. Zhang, Small points and adelic metrics. *J. Algebraic Geom.* **4** (1995), no. 2, 281–300.

[71] S. Zhang, Equidistribution of small points on abelian varieties. *Ann. of Math. (2)* **147** (1998), no. 1, 159–165.

[72] S. Zhang, Small points and Arakelov theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, Doc. Math., pp. 217–225, 1998. Extra Vol. II.

[73] H. Zimmer, On the difference of the Weil height and the Néron–Tate height. *Math. Z.* **147** (1976), no. 1, 35–51.

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