THETA LIFTING AND LANGLANDS FUNCTORIALITY

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ABSTRACT

We review various aspects of theta lifting and its role in studying Langlands functoriality. In particular, we discuss realizations of the Jacquet–Langlands correspondence and the Shimura–Waldspurger correspondence in terms of theta lifting and their arithmetic applications.

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1. INTRODUCTION

Langlands functoriality is a principle relating two different kinds of automorphic forms and plays a pivotal role in number theory. Before Langlands formulated this principle in [42], this phenomenon was already observed in the following classical example discovered by Eichler [15] and developed by Shimizu [48]. Consider the space

$$S_k\big(\Gamma_0(N)\big)$$

of elliptic cusp forms of weight k and level N, where k and N are positive integers and $\Gamma_0(N)$ is the congruence subgroup given by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}.$$

This space consists of holomorphic functions f on the upper half-plane $\mathfrak{H}$ which satisfy

$$f(\gamma z) = j(\gamma, z)^k f(z)$$

for all $\gamma \in \Gamma_0(N)$ and $z \in \mathfrak{H}$ and which vanish at all cusps. Here $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ by linear fractional transformations and $j\big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big), z\big) = cz + d$ is the factor of automorphy. It is also equipped with the action of Hecke operators T_n for all positive integers n, which is a central tool in the arithmetic study of automorphic forms. On the other hand, to every indefinite quaternion division algebra B over $\mathbb{Q}$, we may associate a space

$$S_k(\Gamma_B)$$

of modular forms, where Γ_B is the group of norm-one elements in B. Namely, this space is defined similarly by replacing $\Gamma_0(N)$ by Γ_B (which can be regarded as a subgroup of $(B \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong \mathrm{GL}_2(\mathbb{R})$) and is equipped with the action of Hecke operators T_n^B. Now assume that N is the product of an even number of distinct primes and B is ramified precisely at the primes dividing N. Then by the works of Eichler and Shimizu, the trace of T_n^B on $S_k(\Gamma_B)$ coincides with the trace of T_n on the new part of $S_k(\Gamma_0(N))$ for all n prime to N.

This remarkable relation was thoroughly studied by Jacquet–Langlands [37] in the framework of automorphic representations. Let F be a number field with adèle ring $\mathbb{A}$. Let B be a quaternion division algebra over F. Then Jacquet–Langlands proved that for any irreducible automorphic representation $\pi^B \cong \otimes_v \pi_v^B$ of $B^\times(\mathbb{A})$, there exists a unique irreducible automorphic representation $\pi \cong \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A})$ such that

$$\pi_v \cong \pi_v^B$$

for almost all places v of F. Moreover, they described the image of this map $\pi^B \mapsto \pi$ precisely.

The Jacquet–Langlands correspondence gives a basic example of Langlands functoriality. To explain this, let G be a connected reductive group over F. Let ${}^L G$ be the L-group of G, which was introduced by Langlands and which should govern automorphic representations of $G(\mathbb{A})$. Explicitly, ${}^L G$ is defined as a semiproduct $\hat{G} \rtimes \Gamma_F$, where $\hat{G}$ is the complex dual group of G, $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ is the absolute Galois group of F, and the action of

Γ_F on $\hat{G}$ is inherited from the action of Γ_F on the root datum of G. To motivate it, let us admit for the moment the existence of the hypothetical Langlands group $\mathcal{L}_F$ over F, which is equipped with a surjection $\mathcal{L}_F \to \Gamma_F$. Then it is conjectured that irreducible automorphic representations of $G(\mathbb{A})$ are classified in terms of certain L-homomorphisms $\mathcal{L}_F \to {}^L G$, i.e., homomorphisms commuting with the projections to Γ_F. (Strictly speaking, we consider here packets of tempered automorphic representations.) Now suppose that we have another connected reductive quasisplit group G' over F and an L-homomorphism

$$r : {}^L G \to {}^L G'.$$

Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which should correspond to an L-homomorphism

$$\phi : \mathcal{L}_F \to {}^L G.$$

Then Langlands functoriality predicts the existence of an irreducible automorphic representation π' of $G'(\mathbb{A})$ which should correspond to the L-homomorphism

$$r \circ \phi : \mathcal{L}_F \to {}^L G'.$$

This conjectural relation between π and π' can be formulated without assuming the existence of $\mathcal{L}_F$ as follows. Recall that for almost all places v of F, the local component π_v of π at v is unramified, so that it determines and is determined by a $\hat{G}$-conjugacy class $c(\pi_v)$ in ${}^L G$ via the Satake isomorphism. Then π' should satisfy

$$c(\pi'_v) = r\big(c(\pi_v)\big)$$

for almost all v. Note that the Jacquet–Langlands correspondence mentioned above is the special case when $G = B^\times$, $G' = \mathrm{GL}_2$ (so that ${}^L G = {}^L G' = \mathrm{GL}_2(\mathbb{C}) \times \Gamma_F$), and r is the identity map.

Although Langlands functoriality is out of reach in general, it led to substantial developments in the theory of automorphic forms. For example, the trace formula was developed by Arthur to study automorphic representations, culminating in his book [1] which establishes the case when G is a symplectic group or a quasisplit special orthogonal group, G' is a general linear group, and r is the standard embedding. There are also other methods to attack Langlands functoriality, such as the converse theorem [12, 13], the automorphic descent [24], and the theta lifting. In this report, we will discuss various aspects of the theta lifting, which can be viewed as an explicit realization in the case when (G, G') is a certain pair of classical groups.

2. THETA LIFTING

In this section, we recall the notion of the theta lifting, with emphasis on the realization of the Jacquet–Langlands correspondence. We also review some of its applications to explicit formulas for automorphic periods in terms of special values of L-functions.

2.1. Basic definitions and properties

Let F be a number field with adèle ring $\mathbb{A} = \mathbb{A}_F$. Let W be a symplectic space over F equipped with a nondegenerate bilinear alternating form $(\cdot,\cdot)_W$ and let $\mathrm{Sp}(W)$ denote the symplectic group of W. Similarly, let V be a quadratic space over F equipped with a nondegenerate bilinear symmetric form $(\cdot,\cdot)_V$ and let $\mathrm{O}(V)$ denote the orthogonal group of V. Then the pair

$$\big(\mathrm{Sp}(W), \mathrm{O}(V)\big)$$

is an example of a reductive dual pair introduced by Howe [30]. Namely, if we consider the symplectic space $\mathbb{W} = W \otimes_F V$ equipped with the form $(\cdot,\cdot)_W \otimes (\cdot,\cdot)_V$ and the natural homomorphism

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \to \mathrm{Sp}(\mathbb{W}),$$

then $\mathrm{Sp}(W)$ and $\mathrm{O}(V)$ are mutual commutants in $\mathrm{Sp}(\mathbb{W})$.

Roughly speaking, the theta lifting is an integral transform with kernel given by a particular automorphic form on $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$ restricted to $\mathrm{Sp}(W)(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A})$. To be precise, we need to consider the metaplectic group $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$, which is a nontrivial topological central extension

$$1 \to \{\pm 1\} \to \mathrm{Mp}(\mathbb{W})(\mathbb{A}) \to \mathrm{Sp}(\mathbb{W})(\mathbb{A}) \to 1.$$

(Here we have abused notation since $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$ is not the group of $\mathbb{A}$-valued points of an algebraic group over F.) This extension splits over $\mathrm{Sp}(\mathbb{W})(F)$ canonically, so that we may speak of automorphic forms on $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$. We are interested in a particular representation ω of $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$ (depending on a choice of a nontrivial additive character of $\mathbb{A}/F$), called the Weil representation [61], which is a representation theoretic incarnation of theta functions. This representation has an automorphic realization, i.e., there is an $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$-equivariant map $\varphi \mapsto \theta_\varphi$ from ω to the space of automorphic forms on $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$. On the other hand, there exists a dotted arrow making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Mp}(W)(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A}) & \dashrightarrow & \mathrm{Mp}(\mathbb{W})(\mathbb{A}) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(W)(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A}) & \longrightarrow & \mathrm{Sp}(\mathbb{W})(\mathbb{A}) \end{array}$$

(Note that it descends to a homomorphism from the bottom left corner if and only if $\dim V$ is even.) Thus we may regard θ_φ as an automorphic form on $\mathrm{Mp}(W)(\mathbb{A}) \times \mathrm{O}(V)(\mathbb{A})$ by restriction and associate to an automorphic form f on $\mathrm{Mp}(W)(\mathbb{A})$ an automorphic form $\theta_\varphi(f)$ on $\mathrm{O}(V)(\mathbb{A})$ by setting

$$\theta_\varphi(f)(h) = \int_{\mathrm{Sp}(W)(F)\backslash \mathrm{Mp}(W)(\mathbb{A})} \theta_\varphi(g,h)\overline{f(g)}\,dg,$$

provided the integral converges, e.g., if f is cuspidal.

For any irreducible cuspidal automorphic representation π of $\mathrm{Mp}(W)(\mathbb{A})$, we define the theta lift $\theta(\pi)$ of π as the automorphic representation of $\mathrm{O}(V)(\mathbb{A})$ spanned by $\theta_\varphi(f)$ for all $\varphi \in \omega$ and $f \in \pi$. We only consider the case when π descends (resp. does not descend) to a representation of $\mathrm{Sp}(W)(\mathbb{A})$ if $\dim V$ is even (resp. odd); otherwise $\theta(\pi)$ is always zero. To describe $\theta(\pi)$, we need to introduce the local analog of the theta lifting. First, note that the map $(\varphi, f) \mapsto \theta_\varphi(f)$ defines an element in

$$\mathrm{Hom}_{\mathrm{Mp}(W)(\mathbb{A})\times \mathrm{O}(V)(\mathbb{A})}\big(\omega \otimes \bar{\pi}, \theta(\pi)\big) \cong \mathrm{Hom}_{\mathrm{Mp}(W)(\mathbb{A})\times \mathrm{O}(V)(\mathbb{A})}\big(\omega, \pi \otimes \theta(\pi)\big)$$

since π is unitary. Recall that ω can be regarded as the restricted tensor product of the local Weil representations ω_v of $\mathrm{Mp}(\mathbb{W})(F_v)$ via the surjection $\prod'_v \mathrm{Mp}(\mathbb{W})(F_v) \to \mathrm{Mp}(\mathbb{W})(\mathbb{A})$, where $\mathrm{Mp}(\mathbb{W})(F_v)$ is the metaplectic cover of $\mathrm{Sp}(\mathbb{W})(F_v)$. Similarly, π can be decomposed as $\pi \cong \otimes_v \pi_v$, where π_v is an irreducible representation of $\mathrm{Mp}(W)(F_v)$. We define the local theta lift $\theta(\pi_v)$ of π_v as an irreducible representation of $\mathrm{O}(V)(F_v)$ such that

$$\mathrm{Hom}_{\mathrm{Mp}(W)(F_v)\times \mathrm{O}(V)(F_v)}\big(\omega_v, \pi_v \otimes \theta(\pi_v)\big) \neq 0,$$

which is unique (if it exists) by the Howe duality [23, 31, 55]. (When such a representation does not exist, we interpret $\theta(\pi_v)$ as zero.) Now assume that $\theta(\pi)$ is nonzero and cuspidal. Then it follows from the Howe duality that $\theta(\pi)$ is irreducible and can be decomposed as

$$\theta(\pi) \cong \otimes_v \theta(\pi_v).$$

Remark 2.1. We may extend the Weil representation and define the theta lifting for the pair $(\mathrm{GSp}(W), \mathrm{GO}(V))$, where $\mathrm{GSp}(W)$ and $\mathrm{GO}(V)$ are the similitude groups of W and V, respectively.

2.2. Explicit realization of the Jacquet–Langlands correspondence

From now on, we mainly consider the case when

$$\dim W = 2, \quad \dim V = 4,$$

and the discriminant of V is trivial. Then we may identify W with the space F^2, equipped with the form $((x_1, x_2), (y_1, y_2))_W = x_1 y_2 - x_2 y_1$, so that

$$\mathrm{GSp}(W) = \mathrm{GL}_2.$$

We may also identify V with a quaternion algebra B over F equipped with the form $(x, y)_V = \mathrm{Tr}_{B/F}(xy^*)$, where $\mathrm{Tr}_{B/F}$ is the reduced trace and $*$ is the main involution, so that

$$\mathrm{GO}(V)^0 = (B^\times \times B^\times)/F^\times.$$

Here $\mathrm{GO}(V)^0$ is the identity component of $\mathrm{GO}(V)$ and $B^\times \times B^\times$ acts on V by left and right multiplication.

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. We regard the theta lift $\theta(\pi)$ of π (restricted to $\mathrm{GO}(V)^0(\mathbb{A})$) as an automorphic representation of $B^\times(\mathbb{A}) \times B^\times(\mathbb{A})$. Then Shimizu [49] proved that

$$\theta(\pi) = \pi^B \otimes \pi^B,$$

where π^B is the Jacquet–Langlands transfer of π to $B^\times(\mathbb{A})$. (When π does not transfer to $B^\times(\mathbb{A})$, we interpret π^B as zero.)

Remark 2.2. In [36], we gave the following variant of the above realization. Let B, B_1, B_2 be three quaternion division algebras over F such that $B = B_1 \cdot B_2$ in the Brauer group. We consider a 1-dimensional Hermitian space W over B and a 2-dimensional skew-Hermitian space V over B such that

$$\mathrm{GU}(W) = B^\times, \quad \mathrm{GU}(V)^0 = (B_1^\times \times B_2^\times)/F^\times,$$

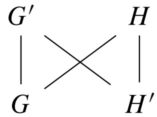
where $\mathrm{GU}(W)$ and $\mathrm{GU}(V)$ are the unitary similitude groups of W and V, respectively. Let π^B be an irreducible automorphic representation of $B^\times(\mathbb{A})$ such that its Jacquet–Langlands transfer to $\mathrm{GL}_2(\mathbb{A})$ is cuspidal. Then we have

$$\theta(\pi^B) = \pi^{B_1} \otimes \pi^{B_2},$$

where π^{B_1} and π^{B_2} are the Jacquet–Langlands transfers of π^B to $B_1^\times(\mathbb{A})$ and $B_2^\times(\mathbb{A})$, respectively. We believe that this realization is useful to study integral period relations.

2.3. Seesaw identities

One of the advantages of the theta lifting is that it produces various period relations in a simple way, which was observed by Kudla [39]. Suppose that we have two reductive dual pairs (G, H) and (G', H') in the same symplectic group such that $G \subset G'$ and $H \supset H'$. This can be illustrated by the following picture, called a seesaw diagram:



Let f and f' be automorphic forms on $G(\mathbb{A})$ and $H'(\mathbb{A})$, respectively. Then the theta lifting produces automorphic forms $\theta_\varphi(f)$ and $\theta_\varphi(f')$ on $H(\mathbb{A})$ and $G'(\mathbb{A})$, respectively, and the so-called seesaw identity

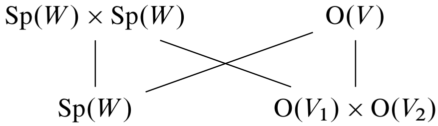
$$\begin{aligned}\langle \theta_\varphi(f)|_{H'(\mathbb{A})}, f' \rangle &= \int_{H'(F)\backslash H'(\mathbb{A})} \theta_\varphi(f)(h')\overline{f'(h')}\, dh' \\ &= \int_{G(F)\backslash G(\mathbb{A})} \int_{H'(F)\backslash H'(\mathbb{A})} \theta_\varphi(g,h')\overline{f(g) f'(h')}\, dg\, dh' \\ &= \int_{G(F)\backslash G(\mathbb{A})} \overline{f(g)}\theta_\varphi(f')(g)\, dg = \overline{\langle f, \theta_\varphi(f')|_{G(\mathbb{A})}\rangle},\end{aligned}$$

provided the double integral converges absolutely. Here $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product.

As an example of this identity, we recall Waldspurger's formula for torus periods. We keep the setup of the previous subsection. Fix a quadratic extension E of F which embeds into B and write $B = E \oplus Ej$ with a trace zero element j in B. Let $V = V_1 \oplus V_2$ be the corresponding decomposition of quadratic spaces, so that

$$\mathrm{GO}(V_1)^0 = \mathrm{GO}(V_2)^0 = E^\times.$$

Then the identification $W \otimes_F V = (W \otimes_F V_1) \oplus (W \otimes_F V_2)$ gives rise to the following seesaw diagram:



Let π^B be an irreducible automorphic representation of $B^\times(\mathbb{A})$ such that its Jacquet–Langlands transfer π to $\mathrm{GL}_2(\mathbb{A})$ is cuspidal. Let χ be an automorphic character of $\mathbb{A}_E^\times$. Assume that the product of the central character of π^B and the restriction of χ to $\mathbb{A}_F^\times$ is trivial and consider the torus period

$$P(f, \chi) = \int_{E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times} f(h)\chi(h)\,dh$$

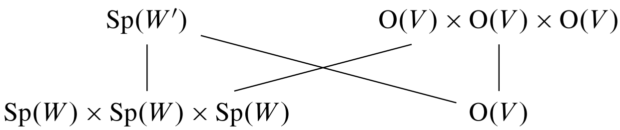
for a decomposable vector $f \in \pi^B$. Then using the above seesaw diagram, Waldspurger [54] proved that

$$\left|P(f, \chi)\right|^2 = \frac{1}{4} \frac{\zeta(2) L(1/2, \pi_E \times \chi)}{L(1, \pi, \mathrm{Ad}) L(1, \mu_{E/F})} \prod_v \alpha_v(f_v, \chi_v),$$

where

- $\zeta(s)$ is the completed Dedekind zeta function of F,
- $L(s, \pi_E \times \chi)$ is the standard L-function of the base change π_E of π to $\mathrm{GL}_2(\mathbb{A}_E)$ twisted by χ,
- $L(s, \pi, \mathrm{Ad})$ is the adjoint L-function of π,
- $L(s, \mu_{E/F})$ is the Hecke L-function of the quadratic automorphic character $\mu_{E/F}$ of $\mathbb{A}_F^\times$ associated to E/F by class field theory,
- $\alpha_v(f_v, \chi_v)$ is a certain normalized local integral of matrix coefficients.

As another example, we consider the 6-dimensional symplectic space $W' = W^3$ over F. Then the identification $W' \otimes_F V = (W \otimes_F V)^3$ gives rise to the following seesaw diagram:



Let $\pi_1^B, \pi_2^B, \pi_3^B$ be irreducible automorphic representations of $B^\times(\mathbb{A})$ such that their Jacquet–Langlands transfers π_1, π_2, π_3 to $\mathrm{GL}_2(\mathbb{A})$ are cuspidal. Assume that the product of the central characters of $\pi_1^B, \pi_2^B, \pi_3^B$ is trivial and consider the trilinear period

$$P(f_1, f_2, f_3) = \int_{B^\times \mathbb{A}^\times \backslash B^\times(\mathbb{A})} f_1(h) f_2(h) f_3(h)\,dh$$

for decomposable vectors $f_1 \in \pi_1^B$, $f_2 \in \pi_2^B$, $f_3 \in \pi_3^B$. Then following the work of Harris–Kudla [27] and using the above seesaw diagram, we proved in [32] that

$$\left|P(f_1, f_2, f_3)\right|^2 = \frac{1}{8} \frac{\zeta(2)^2 L(1/2, \pi_1 \times \pi_2 \times \pi_3)}{L(1, \pi_1, \mathrm{Ad}) L(1, \pi_2, \mathrm{Ad}) L(1, \pi_3, \mathrm{Ad})} \prod_v \alpha_v(f_{1,v}, f_{2,v}, f_{3,v}),$$

where

- $L(s, \pi_1 \times \pi_2 \times \pi_3)$ is the triple product L-function of π_1, π_2, π_3,
- $\alpha_v(f_{1,v}, f_{2,v}, f_{3,v})$ is a certain normalized local integral of matrix coefficients.

Remark 2.3. The above two formulas are special cases of the Gross–Prasad conjecture [25,26] and its refinement [33]. This conjecture (for special orthogonal groups) was extended to all classical groups by Gan–Gross–Prasad [17], and after the breakthrough of Zhang [63,64], the global conjecture for unitary groups has been proved in a series of works [8–11,62] using the relative trace formula. We should also mention the stunning work of Waldspurger [57–60], which led to the proof of the local Gan–Gross–Prasad conjecture for Bessel models [5–7,45] and Fourier–Jacobi models [2,19] in the p-adic case, where the theta lifting is used to deduce the latter from the former.

3. THE SHIMURA–WALDSPURGER CORRESPONDENCE

In this section, we review some applications of the theta lifting to automorphic forms on metaplectic groups.

3.1. Modular forms of half-integral weight

The theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}$$

is a modular form of weight $1/2$ and its significance is well known. Thus it is natural to study modular forms of half-integral weight, but Hecke [28, P. 152] realized the difficulty in developing the arithmetic theory; the Hecke operator T_n is zero unless n is a square. In 1973, Shimura [50] revolutionized the theory of modular forms of half-integral weight by relating them to modular forms of integral weight, i.e., he constructed a modular form of weight $2k$ from a cusp form of weight $k + 1/2$ by using the converse theorem, where k is a positive integer. Soon after the discovery of this correspondence, Niwa [46] and Shintani [51] gave an alternative construction using the theta lifting. This was further investigated by Waldspurger [53, 56] in the framework of automorphic representations. Namely, he established a correspondence between automorphic representations of $\mathrm{Mp}_2(\mathbb{A})$ (where $\mathrm{Mp}_2(\mathbb{A})$ is the metaplectic cover of $\mathrm{SL}_2(\mathbb{A})$) and those of $\mathrm{PGL}_2(\mathbb{A})$, which can be viewed as an example of Langlands functoriality.

3.2. Global correspondence

Now we discuss a generalization of the Shimura–Waldspurger correspondence to metaplectic groups of higher rank. Let F be a number field with adèle ring $\mathbb{A}$. We denote by Sp_{2n} the symplectic group of rank n over F and by $\mathrm{Mp}_{2n}(\mathbb{A})$ the metaplectic cover of $\mathrm{Sp}_{2n}(\mathbb{A})$. Recall that this cover splits over $\mathrm{Sp}_{2n}(F)$ canonically, so that we may speak of the unitary representation of $\mathrm{Mp}_{2n}(\mathbb{A})$ on the Hilbert space

$$L^2\big(\mathrm{Sp}_{2n}(F)\backslash \mathrm{Mp}_{2n}(\mathbb{A})\big)$$

given by right translation. Since we are interested in genuine automorphic representations of $\mathrm{Mp}_{2n}(\mathbb{A})$, i.e., those which do not descend to representations of $\mathrm{Sp}_{2n}(\mathbb{A})$, we only consider its subspace

$$L^2(\mathrm{Mp}_{2n})$$

on which the central subgroup $\{\pm 1\}$ acts by the nontrivial character. Write

$$L^2(\mathrm{Mp}_{2n}) = L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n}) \oplus L^2_{\mathrm{cont}}(\mathrm{Mp}_{2n})$$

for the decomposition into the discrete part and the continuous part. Then the theory of Eisenstein series gives an explicit description of $L^2_{\mathrm{cont}}(\mathrm{Mp}_{2n})$ in terms of automorphic discrete spectra of proper Levi subgroups of Mp_{2n}, i.e., $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{Mp}_{2n_0}$ with $n_1 + \cdots + n_k + n_0 = n$ and $n_0 < n$. Thus the problem is to describe the irreducible decomposition of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$.

To attack this problem, it is better to divide it into two parts as follows:

(1) Describe the decomposition of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ into near equivalence classes. Here we say that two irreducible genuine representations $\pi \cong \otimes_v \pi_v$ and $\pi' \cong \otimes_v \pi'_v$ of $\mathrm{Mp}_{2n}(\mathbb{A})$ are nearly equivalent if π_v and π'_v are equivalent for almost all places v of F. (In particular, if π and π' are equivalent, then they are nearly equivalent.) Note that π_v is unramified for almost all v, so that it determines and is determined by a semisimple conjugacy class $c_{\psi_v}(\pi_v)$ in $\mathrm{Sp}_{2n}(\mathbb{C})$ (depending on a choice of a nontrivial additive character ψ_v of F_v) via the Satake isomorphism. In other words, the near equivalence classes of irreducible genuine representations of $\mathrm{Mp}_{2n}(\mathbb{A})$ can be parametrized by families of semisimple conjugacy classes

$$\{c_v\}_v$$

in $\mathrm{Sp}_{2n}(\mathbb{C})$, where we identify two families if they are equal for almost all v. Thus we want to describe the families $\{c_v\}_v$ which correspond to the near equivalence classes in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$.

(2) Describe the irreducible decomposition of each near equivalence class. Namely, for any near equivalence class C in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ and any irreducible genuine representation π of $\mathrm{Mp}_{2n}(\mathbb{A})$, we want to give an explicit formula for the multiplicity of π in C in terms of the classification of representations.

In [20], we solved (1) completely and (2) partially; we described the families $\{c_v\}_v$ as above in terms of automorphic representations of general linear groups, and admitting that Arthur's endoscopic classification [1] can be extended to nonsplit odd special orthogonal groups, we established the multiplicity formula for the tempered part of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$.

Now we state the first result precisely.

Theorem 3.1 ([20]). *Fix a nontrivial additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}/F$. Then we have a decomposition*

$$L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n}) = \bigoplus_{\phi} L^2_{\phi}(\mathrm{Mp}_{2n}),$$

where ϕ runs over elliptic A-parameters for Mp_{2n}. Here an elliptic A-parameter for Mp_{2n} is defined to be a formal finite direct sum

$$\bigoplus_i \phi_i \otimes S_{d_i}$$

(which is a substitute for a hypothetical L-homomorphism $\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$), where

- *ϕ_i is an irreducible self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ (which is hypothetically identified with an n_i-dimensional irreducible representation of $\mathcal{L}_F$),*
- *S_{d_i} is the d_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$,*
- *if d_i is odd, then ϕ_i is symplectic, i.e., the exterior square L-function $L(s, \phi_i, \wedge^2)$ has a pole at $s = 1$ (and hence n_i is even),*
- *if d_i is even, then ϕ_i is orthogonal, i.e., the symmetric square L-function $L(s, \phi_i, \mathrm{Sym}^2)$ has a pole at $s = 1$,*
- *if $i \neq j$, then $(\phi_i, d_i) \neq (\phi_j, d_j)$,*
- *$\sum_i n_i d_i = 2n$.*

Also, $L^2_{\phi}(\mathrm{Mp}_{2n})$ is defined as the near equivalence class in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ which corresponds to the family of semisimple conjugacy classes

$$\{c_v(\phi_v)\}_v$$

in $\mathrm{Sp}_{2n}(\mathbb{C})$ given as follows (so that any irreducible summand π of $L^2_{\phi}(\mathrm{Mp}_{2n})$ satisfies $c_{\psi_v}(\pi_v) = c_v(\phi_v)$ for almost all v). Suppose that v is finite and $\phi_{i,v}$ is unramified for all i. Let $c_v(\phi_{i,v})$ be the semisimple conjugacy class in $\mathrm{GL}_{n_i}(\mathbb{C})$ which corresponds to $\phi_{i,v}$ and put

$$Q_v(d) = \begin{pmatrix} q_v^{(d-1)/2} & & & \\ & q_v^{(d-3)/2} & & \\ & & \ddots & \\ & & & q_v^{-(d-1)/2} \end{pmatrix}$$

for any positive integer d, where q_v is the cardinality of the residue field of F_v. We regard $c_v(\phi_{i,v}) \otimes Q_v(d_i)$ as a semisimple conjugacy class in $\mathrm{Sp}_{n_i d_i}(\mathbb{C})$. *Then we set*

$$c_v(\phi_v) = \bigoplus_i c_v(\phi_{i,v}) \otimes Q_v(d_i).$$

To state the second result precisely, we need to introduce more notation. For each place v of F, let $\mathcal{W}_{F_v}$ be the Weil group of F_v and put

$$\mathcal{L}_{F_v} = \begin{cases} \mathcal{W}_{F_v} & \text{if } v \text{ is infinite,} \\ \mathcal{W}_{F_v} \times \mathrm{SL}_2(\mathbb{C}) & \text{if } v \text{ is finite.} \end{cases}$$

Let $\phi = \bigoplus_i \phi_i \otimes S_{d_i}$ be an elliptic A-parameter for Mp_{2n}. We regard its local component $\phi_v = \bigoplus_i \phi_{i,v} \otimes S_{d_i}$ at v as a local A-parameter

$$\phi_v : \mathcal{L}_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$$

via the local Langlands correspondence. Note that

$$c_v(\phi_v) = \phi_v \left(\mathrm{Fr}_v, \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right)$$

for almost all v, where Fr_v is a Frobenius element at v. We denote by $\mathcal{S}_{\phi_v}$ the component group of the centralizer of ϕ_v in $\mathrm{Sp}_{2n}(\mathbb{C})$, which is an elementary abelian 2-group, and by

$$\mathcal{S}_\phi = \bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i$$

the global component group of ϕ, which is formally defined as an elementary abelian 2-group with a basis $\{a_i\}_i$ indexed by $\{\phi_i \otimes S_{d_i}\}_i$. Then we have a natural homomorphism $\mathcal{S}_\phi \to \mathcal{S}_{\phi_v}$ for all v. We also consider the compact abelian group $\mathcal{S}_{\phi,\mathbb{A}} = \prod_v \mathcal{S}_{\phi_v}$ and the diagonal map

$$\Delta : \mathcal{S}_\phi \to \mathcal{S}_{\phi,\mathbb{A}}.$$

Theorem 3.2 ([20]). *Assume that ϕ is tempered, i.e., $d_i = 1$ for all i. Then we have a decomposition*

$$L^2_\phi(\mathrm{Mp}_{2n}) \cong \bigoplus_\eta m_\eta \pi_\eta,$$

where $\eta = \otimes_v \eta_v$ runs over continuous characters of $\mathcal{S}_{\phi,\mathbb{A}}$. Here π_η is defined as the restricted tensor product of representations π_{η_v} in the local L-packets

$$\Pi_{\phi_v}\big(\mathrm{Mp}_{2n}(F_v)\big)$$

associated to ϕ_v (depending on ψ_v), which consist of irreducible genuine representations of $\mathrm{Mp}_{2n}(F_v)$ *indexed by characters of $\mathcal{S}_{\phi_v}$. Also, if we define a character ε_ϕ of $\mathcal{S}_\phi$ by*

$$\varepsilon_\phi(a_i) = \varepsilon(1/2, \phi_i),$$

where $\varepsilon(s, \phi_i)$ is the standard ε-function of ϕ_i, then m_η is given by

$$m_\eta = \begin{cases} 1 & \text{if } \eta \circ \Delta = \varepsilon_\phi, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.3. In fact, we gave another proof of the result of Waldspurger for Mp_2 [53, 56], noting that an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ is symplectic if and only if its central character is trivial.

Remark 3.4. If we denote by SO_{2n+1} the split odd special orthogonal group of rank n over F and by $L^2_{\mathrm{disc}}(\mathrm{SO}_{2n+1})$ the discrete part of $L^2(\mathrm{SO}_{2n+1}(F)\backslash \mathrm{SO}_{2n+1}(\mathbb{A}))$, then the decomposition of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$ is similar to that of $L^2_{\mathrm{disc}}(\mathrm{SO}_{2n+1})$ given by Arthur [1], except that the condition $\eta \circ \Delta = \varepsilon_\phi$ in the former has to be replaced by $\eta \circ \Delta = 1$ in the latter.

In the proof of his result for $n = 1$, Waldspurger used the theta lifting between Mp_2 and (inner forms of) $\mathrm{PGL}_2 \cong \mathrm{SO}_3$. Thus, in general, it would be natural to use the theta lifting between Mp_{2n} and (inner forms of) SO_{2n+1}, and then transfer Arthur's endoscopic classification from SO_{2n+1} to Mp_{2n}. However, there is a serious obstacle in this approach. Indeed, if π is an irreducible genuine cuspidal automorphic representation of $\mathrm{Mp}_{2n}(\mathbb{A})$ and its standard L-function $L(s,\pi)$ vanishes at $s = 1/2$, then the theta lift of π to $\mathrm{SO}_{2n+1}(\mathbb{A})$ is zero. When $n = 1$, Waldspurger proved that the twisted standard L-function $L(s,\pi,\chi)$ does not vanish at $s = 1/2$ for some quadratic automorphic character χ of $\mathbb{A}^\times$ and could use the twisted theta lifting to establish the desired correspondence. But for general n, the existence of such a character χ is considered extremely difficult to prove.

To circumvent this difficulty, we used the theta lifting in the so-called stable range studied by Li [44]. More precisely, for any irreducible genuine representation π of $\mathrm{Mp}_{2n}(\mathbb{A})$, we consider its (abstract) theta lift

$$\theta^{\mathrm{abs}}(\pi) = \otimes_v \theta(\pi_v)$$

to $\mathrm{SO}_{2r+1}(\mathbb{A})$ with $r \gg 2n$. Then it follows from the result of Li that if π occurs in $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$, then $\theta^{\mathrm{abs}}(\pi)$ occurs in $L^2_{\mathrm{disc}}(\mathrm{SO}_{2r+1})$. Combining this with the analytic theory of standard L-functions, we may deduce Theorem 3.1 from Arthur's endoscopic classification for SO_{2r+1}. Moreover, if π is an irreducible summand of the tempered part of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$, then we proved that

$$m\big(\theta^{\mathrm{abs}}(\pi)\big) = m(\pi),$$

where $m(\cdot)$ denotes the multiplicity in the automorphic discrete spectrum. (We expect that this equality holds for any irreducible summand π of $L^2_{\mathrm{disc}}(\mathrm{Mp}_{2n})$.) Using this and describing the local theta lifting between Mp_{2n} and SO_{2r+1} explicitly, we may deduce Theorem 3.2.

Remark 3.5. When $n = 2$ and ϕ is nontempered, we proved a similar decomposition of $L^2_\phi(\mathrm{Mp}_4)$ in [21]. Note that π_η is not necessarily irreducible and ε_ϕ has to be modified in this case.

3.3. Local correspondence

There is a local analog of the above correspondence, called the local Shimura correspondence. For simplicity, we only consider the p-adic case and write F for a finite extension of $\mathbb{Q}_p$. We are interested in the set

$$\mathrm{Irr}\,\mathrm{Mp}_{2n}(F)$$

of equivalence classes of irreducible genuine representations of the metaplectic group $\mathrm{Mp}_{2n}(F)$. Recall that there are precisely two $(2n+1)$-dimensional quadratic spaces V^+ and V^- over F with trivial discriminant (up to isometry). Let $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$ denote the special orthogonal groups of V^+ and V^-, respectively. Then the local Shimura correspondence, which was established by Gan–Savin [22] in the p-adic case, says that there is a bijection (depending on a choice of a nontrivial additive character ψ of F)

$$\theta : \mathrm{Irr}\,\mathrm{Mp}_{2n}(F) \to \mathrm{Irr}\,\mathrm{SO}(V^+) \sqcup \mathrm{Irr}\,\mathrm{SO}(V^-)$$

given by the local theta lifting. Namely, for any irreducible genuine representation π of $\mathrm{Mp}_{2n}(F)$, $\theta(\pi)$ is defined as the unique irreducible representation of $\mathrm{SO}(V^\varepsilon)$ with the unique sign $\varepsilon = \pm$ such that

$$\mathrm{Hom}_{\mathrm{Mp}_{2n}(F)\times\mathrm{SO}(V^\varepsilon)}\big(\omega^\varepsilon, \pi \otimes \theta(\pi)\big) \neq 0,$$

where ω^ε is the Weil representation of $\mathrm{Mp}_{2n}(F) \times \mathrm{SO}(V^\varepsilon)$ (depending on ψ). Moreover, they proved various natural properties:

- θ preserves the square-integrability,
- θ preserves the temperedness,
- θ is compatible with the theory of R-groups,
- θ is compatible with the Langlands classification,

and used θ to transfer the local Langlands correspondence from $\mathrm{SO}(V^\varepsilon)$ to $\mathrm{Mp}_{2n}(F)$. (In particular, this defines the local L-packets in the statement of Theorem 3.2.)

Remark 3.6. The local theta lifting has also been described for other reductive dual pairs in terms of the local Langlands correspondence. See [3, 4, 19] for recent progress.

As in Section 2.3, the local theta lifting can produce various relations between local analogs of periods. For example, we consider an irreducible genuine square-integrable representation π of $\mathrm{Mp}_{2n}(F)$ and its formal degree $d(\pi)$. Recall that $d(\pi)$ is defined as the positive real number for which the Schur orthogonality relation

$$\int_{\mathrm{Mp}_{2n}(F)} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{d(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}$$

holds for all $v_1, \dots, v_4 \in \pi$, where $\langle \cdot, \cdot \rangle$ is an invariant Hermitian inner product on π. Note that $d(\pi)$ depends on the choice of a Haar measure dg on $\mathrm{Mp}_{2n}(F)$, but we take the measure determined by a Chevalley basis of the Lie algebra of $\mathrm{Mp}_{2n}(F)$ and a fixed nontrivial additive character ψ of F. The above relation suggests that $d(\pi)$ is a generalization of the dimension of an irreducible representation of a compact group, so that it is a fundamental invariant of a representation. Since a matrix coefficient $g \mapsto \langle \pi(g)v_1, v_2\rangle$ is a local analog of an automorphic form, we may also interpret $d(\pi)^{-1}$ as a local period. Then we proved in [18] that

$$d\big(\theta(\pi)\big) = d(\pi),$$

which is a local analog of the Rallis inner product formula [47], by using the doubling seesaw diagram:

$$\begin{array}{ccc} \mathrm{Mp}_{4n} & & \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1} \\ \Big| & \times & \Big| \\ \mathrm{Mp}_{2n} \times \mathrm{Mp}_{2n} & & \mathrm{SO}_{2n+1} \end{array}$$

Remark 3.7. Recall from Section 2.3 that some automorphic periods can be expressed in terms of special values of L-functions. Similarly, formal degrees should be expressed in terms of arithmetic invariants as follows. Let G be a connected reductive group over F. For simplicity, we assume that G is a pure inner form of a quasisplit group and the center of G is anisotropic. Let π be an irreducible square-integrable representation of $G(F)$. Let $d(\pi)$ denote the formal degree of π with respect to the Haar measure on $G(F)$ determined by a Chevalley basis of the Lie algebra of the split form of G and a fixed nontrivial additive character ψ of F. Then the formal degree conjecture [29] says that

$$d(\pi) = \frac{\dim \eta}{|\mathcal{S}_\phi|} \big| \gamma(0, \mathrm{Ad} \circ \phi, \psi) \big|,$$

where

- $\phi : \mathcal{L}_F \to {}^L G$ is the L-parameter (conjecturally) associated to π,
- $\mathcal{S}_\phi$ is the component group of the centralizer of ϕ in $\hat{G}$,
- η is the irreducible representation of $\mathcal{S}_\phi$ (conjecturally) associated to π,
- Ad is the adjoint representation of ${}^L G$ on its Lie algebra,
- $\gamma(s, \mathrm{Ad} \circ \phi, \psi)$ is the local γ-factor given by

$$\gamma(s, \mathrm{Ad} \circ \phi, \psi) = \varepsilon(s, \mathrm{Ad} \circ \phi, \psi) \frac{L(1-s, \mathrm{Ad} \circ \phi)}{L(s, \mathrm{Ad} \circ \phi)}.$$

In [34], we proved this conjecture for (inner forms of) SO_{2n+1} and its analog for Mp_{2n} by using the main identity of Lapid–Mao [43] and the above relation between formal degrees.

4. GEOMETRIC REALIZATION OF THE JACQUET–LANGLANDS CORRESPONDENCE

Let π be an irreducible automorphic representation of $G(\mathbb{A})$ and suppose that π is cohomological, so that π occurs in the cohomology $H^*(X, \mathbb{C})$, where G is a connected reductive group and X is a locally symmetric space for G. Then it is natural to ask whether functorial transfers of π can be realized geometrically. In this section, we discuss the simplest example, i.e., the Jacquet–Langlands correspondence for GL_2 and its inner forms.

Let F be a totally real number field. Let $\mathbb{A}$ and $\mathbb{A}_f$ denote the rings of adèles and finite adèles of F, respectively. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that π_v is the discrete series of weight 2 for all infinite places v of F. For simplicity, we assume that the central character of π is trivial, the level of π is square-free, and the Hecke eigenvalues of π lie in $\mathbb{Q}$. Let B be a quaternion division algebra over F. For each place v of F, put $B_v = B \otimes_F F_v$. Let $\mathcal{V}_B$ be the set of infinite places v of F such that B_v is split. Assume that $\mathcal{V}_B \neq \emptyset$ and put $d = |\mathcal{V}_B|$. We denote by X_B the Shimura variety for $B^\times$ (with respect to some neat open compact subgroup K_f of $B^\times(\mathbb{A}_f)$), which is a d-dimensional smooth projective variety over the reflex field F', so that

- F' is the number field contained in $\mathbb{C}$ such that
$$\mathrm{Gal}(\bar{\mathbb{Q}}/F') = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid \sigma\mathcal{V}_B = \mathcal{V}_B\},$$
where $\bar{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$,
- the $\mathbb{C}$-valued points of X_B are given by
$$X_B(\mathbb{C}) = B^\times \backslash (\mathfrak{H}^\pm)^d \times B^\times(\mathbb{A}_f)/K_f,$$
where $\mathfrak{H}^\pm$ is the union of the upper and lower half-planes.

Now assume that the Jacquet–Langlands transfer π^B of π to $B^\times(\mathbb{A})$ exists, which is the case if and only if π_v is a discrete series for all v at which B is ramified, and that K_f is chosen appropriately so that $\dim(\pi_f^B)^{K_f} = 1$. Here π_f^B is the finite component of π^B and $(\pi_f^B)^{K_f}$ is the space of K_f-fixed vectors in π_f^B. Then it follows from Matsushima's formula that π_B occurs in the cohomology $H^*(X_B, \mathbb{C})$. More precisely, we consider the rational cohomology $H^*(X_B, \mathbb{Q})$ and its π-isotypic component

$$H^*(X_B, \mathbb{Q})_\pi = \{\alpha \in H^*(X_B, \mathbb{Q}) \mid T_v\alpha = \chi_{\pi_v}(T_v)\alpha \text{ for all } T_v \in \mathcal{H}_v \text{ and almost all } v\},$$

where $\mathcal{H}_v = \mathbb{Q}[K_v \backslash B_v^\times / K_v]$ is the Hecke algebra with respect to the standard maximal compact subgroup K_v of $B_v^\times \cong \mathrm{GL}_2(F_v)$ and $\chi_{\pi_v} : \mathcal{H}_v \to \mathbb{Q}$ is the character by which $\mathcal{H}_v$ acts on $\pi_v^{K_v}$. Then $H^*(X_B, \mathbb{Q})_\pi$ is concentrated in the middle degree d and $H^d(X_B, \mathbb{Q})_\pi$ is a 2^d-dimensional vector space over $\mathbb{Q}$. Moreover, for any prime ℓ, the ℓ-adic representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F')$ on

$$H^d(X_B, \mathbb{Q})_\pi \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^d_{\text{ét}}(X_B \times_{F'} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)_\pi$$

is given by the so-called tensor induction of $\rho_{\pi,\ell}$, where $\rho_{\pi,\ell}$ is the 2-dimensional ℓ-adic representation associated to π. We remark that it depends only on $\rho_{\pi,\ell}$ and $\mathcal{V}_B$.

Suppose that we have two quaternion algebras B_1 and B_2 as above such that

$$\mathcal{V}_{B_1} = \mathcal{V}_{B_2}.$$

Write $X_1 = X_{B_1}$ and $X_2 = X_{B_2}$ for the corresponding Shimura varieties, which are of the same dimension $d = |\mathcal{V}_{B_1}| = |\mathcal{V}_{B_2}|$ over the same reflex field F'. Then we have

(1) an (abstract) isomorphism
$$H^d(X_1, \mathbb{C})_\pi \cong H^d(X_2, \mathbb{C})_\pi$$
which preserves the Hodge decomposition,

(2) an (abstract) isomorphism

$$H^d(X_1, \mathbb{Q}_\ell)_\pi \cong H^d(X_2, \mathbb{Q}_\ell)_\pi$$

of ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/F')$ for all ℓ.

Conjecturally, these isomorphisms are obtained from a single isomorphism

$$H^d(X_1, \mathbb{Q})_\pi \cong H^d(X_2, \mathbb{Q})_\pi$$

given as follows. By (2) and the Künneth formula, the space of $\mathrm{Gal}(\bar{\mathbb{Q}}/F')$-fixed vectors in

$$H^{2d}(X_1 \times X_2, \mathbb{Q}_\ell(d))_{\pi \otimes \pi}$$

is nonzero. Hence the Tate conjecture predicts the existence of an algebraic cycle $Z \in \mathrm{CH}^d(X_1 \times X_2)$ which realizes (1) and (2). Namely, let p_1 and p_2 be the two projections

$$\begin{array}{ccc} & X_1 \times X_2 & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X_1 & & X_2 \end{array}$$

and consider the following map:

$$\begin{array}{ccc} H^d(X_1, \mathbb{Q}) & \xrightarrow{p_1^*} & H^d(X_1 \times X_2, \mathbb{Q}) \\ & & \downarrow{\cup [Z]} \\ & & H^{3d}\big(X_1 \times X_2, \mathbb{Q}(d)\big) \xrightarrow{p_{2*}} H^d(X_2, \mathbb{Q}). \end{array}$$

Then it induces an isomorphism

$$\iota_Z : H^d(X_1, \mathbb{Q})_\pi \to H^d(X_2, \mathbb{Q})_\pi$$

such that

- $\iota_Z \otimes \mathrm{id}_{\mathbb{C}}$ preserves the Hodge decomposition,
- $\iota_Z \otimes \mathrm{id}_{\mathbb{Q}_\ell}$ is $\mathrm{Gal}(\bar{\mathbb{Q}}/F')$-equivariant for all ℓ.

When $d = 1$, the existence of Z in fact follows from the result of Faltings [16]. But for general d, this remains an open problem. On the other hand, noting that the Hodge conjecture reduces it to finding a Hodge cycle on $X_1 \times X_2$, i.e., an element in

$$H^{2d}(X_1 \times X_2, \mathbb{Q}) \cap H^{d,d}(X_1 \times X_2),$$

we gave the following evidence.

Theorem 4.1 ([35]). *Assume that B_1 and B_2 are ramified at some infinite place v of F. Then there exists a Hodge cycle ξ on $X_1 \times X_2$ which induces an isomorphism*

$$\iota_\xi : H^d(X_1, \mathbb{Q})_\pi \to H^d(X_2, \mathbb{Q})_\pi$$

such that

- $\iota_\xi \otimes \mathrm{id}_{\mathbb{C}}$ *preserves the Hodge decomposition,*
- $\iota_\xi \otimes \mathrm{id}_{\mathbb{Q}_\ell}$ *is* $\mathrm{Gal}(\bar{\mathbb{Q}}/F')$*-equivariant for all* ℓ.

Our proof proceeds as follows. First, we choose an ambient variety X equipped with an embedding $j : X_1 \times X_2 \hookrightarrow X$. Then we construct a class $\Xi \in H^{d,d}(X)$ such that the $(\pi \otimes \pi)$-isotypic component $(j^*\Xi)_{\pi\otimes\pi}$ of the pullback $j^*\Xi \in H^{d,d}(X_1 \times X_2)$ is nonzero. Finally, we modify Ξ in such a way that Ξ lies in $H^{2d}(X, \mathbb{Q})$ and $\xi = (j^*\Xi)_{\pi\otimes\pi}$ is the desired Hodge cycle.

More precisely, fix a totally imaginary quadratic extension E of F which embeds into B_1 and B_2. For $i = 1, 2$, let $\mathbf{V}_i = B_i$ be the 2-dimensional Hermitian space over E such that

$$\mathrm{GU}(\mathbf{V}_i) = (B_i^\times \times E^\times)/F^\times.$$

Then we may replace X_i by the Shimura variety for $\mathrm{GU}(\mathbf{V}_i)$. Consider the 4-dimensional Hermitian space $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$ over E and put $\mathbf{G} = \mathrm{GU}(\mathbf{V})$. Note that if we write $v_1, \dots, v_d$ (resp. $v_{d+1}, \dots, v_{[F:\mathbb{Q}]}$) for the infinite places of F at which B_1 and B_2 are split (resp. ramified), then we have

$$\mathbf{G}(F_{v_i}) = \begin{cases} \mathrm{GU}(2,2) & \text{if } i \le d, \\ \mathrm{GU}(4) & \text{if } i > d. \end{cases}$$

Put $\mathbf{G}_\infty = \mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ and let $\mathfrak{g}_\infty$ denote the complexified Lie algebra of $\mathbf{G}_\infty$. Let $\mathbf{K}_\infty$ be the standard maximal connected compact modulo center subgroup of $\mathbf{G}_\infty$. Let X be the Shimura variety for $\mathbf{G}$ (with respect to some neat open compact subgroup $\mathbf{K}_f$ of $\mathbf{G}(\mathbb{A}_f)$), which is equipped with the embedding $j : X_1 \times X_2 \hookrightarrow X$ induced by the natural embedding

$$\mathrm{G}\big(\mathrm{U}(\mathbf{V}_1) \times \mathrm{U}(\mathbf{V}_2)\big) \hookrightarrow \mathrm{GU}(\mathbf{V}),$$

where the left-hand side is the subgroup of $\mathrm{GU}(\mathbf{V}_1) \times \mathrm{GU}(\mathbf{V}_2)$ which consists of elements with the same similitude factor. Then Matsushima's formula says that

$$H^*(X, \mathbb{C}) \cong \bigoplus_\sigma m(\sigma) H^*(\mathfrak{g}_\infty, \mathbf{K}_\infty; \sigma_\infty) \otimes \sigma_f^{\mathbf{K}_f},$$

where

- σ runs over irreducible unitary representations of $\mathbf{G}(\mathbb{A})$,
- σ_∞ and σ_f are the infinite and finite components of σ, respectively,
- $m(\sigma)$ is the multiplicity of σ in the automorphic discrete spectrum of $\mathbf{G}$,
- $H^*(\mathfrak{g}_\infty, \mathbf{K}_\infty; \sigma_\infty)$ is the relative Lie algebra cohomology,
- $\sigma_f^{\mathbf{K}_f}$ is the space of $\mathbf{K}_f$-fixed vectors in σ_f.

Hence, to construct a class Ξ as above, we need to find an irreducible automorphic representation σ of $\mathbf{G}(\mathbb{A})$ which satisfies the following properties:

(a) To achieve the condition $\Xi \in H^{d,d}(X)$, we require that

$$H^{d,d}(\mathfrak{g}_\infty, \mathbf{K}_\infty; \sigma_\infty) \neq 0.$$

If this is the case, then it follows from the result of Vogan–Zuckerman [52] that σ_{v_i} (restricted to $\mathrm{U}(\mathbf{V})(F_{v_i})$) is equal to

$$\begin{cases} \mathbf{1} \text{ or } A_{\mathfrak{q}} & \text{if } i \leq d, \\ \mathbf{1} & \text{if } i > d. \end{cases}$$

Here $\mathbf{1}$ denotes the trivial representation and $A_{\mathfrak{q}}$ is the cohomological representation of $\mathrm{U}(2,2)$ associated to the θ-stable parabolic subalgebra $\mathfrak{q}$ with Levi component $\mathfrak{u}(1,1) \oplus \mathfrak{u}(1,1)$. We further require that $\sigma_{v_i} = A_{\mathfrak{q}}$ if $i \leq d$ in order not to make σ 1-dimensional.

(b) To achieve the condition $(j^*\Xi)_{\pi \otimes \pi} \neq 0$, we require the nonvanishing of the automorphic period

$$\sigma \otimes \overline{(\pi^{B_1} \otimes \pi^{B_2})} \to \mathbb{C}.$$

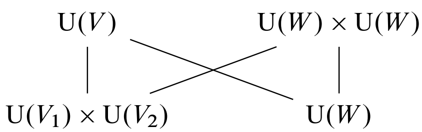
For (a), we use the following variant of the theta lifting from SL_2 to $\mathrm{SO}(4,2) \sim \mathrm{U}(2,2)$ or $\mathrm{SO}(6) \sim \mathrm{U}(4)$, where $\sim$ denotes an isogeny. Let B be the quaternion algebra over F such that $B = B_1 \cdot B_2$ in the Brauer group, so that B is split at all infinite places of F. We may regard $V = \wedge^2 \mathbf{V}$ as a 3-dimensional skew-Hermitian space over B such that

$$\mathrm{GU}(V)^0 / F^\times \cong \mathrm{GU}(\mathbf{V}) / E^\times.$$

Let $W = B$ be the 1-dimensional Hermitian space over B such that

$$\mathrm{GU}(W) = B^\times.$$

Then any σ as in (a) with trivial central character is a theta lift of an irreducible cuspidal automorphic representation τ of $\mathrm{GU}(W)(\mathbb{A})$ such that τ_v is the discrete series of weight 3 for all infinite places v of F. For (b), we can easily find the corresponding τ by using the following seesaw diagram:



Here $V = V_1 \oplus V_2$ is a decomposition into 1- and 2-dimensional skew-Hermitian spaces over B such that

$$\mathrm{GU}(V_1)^0 = E^\times, \quad \mathrm{GU}(V_2)^0 = (B_1^\times \times B_2^\times)/F^\times,$$

and

$$\mathrm{G}\big(\mathrm{U}(V_1) \times \mathrm{U}(V_2)\big)^0 / F^\times \cong \mathrm{G}\big(\mathrm{U}(\mathbf{V}_1) \times \mathrm{U}(\mathbf{V}_2)\big) / E^\times.$$

For simplicity, we further assume that the Hecke eigenvalues of σ lie in $\mathbb{Q}$. Thus we obtain a class $\Xi \in H^{d,d}(X)$ such that $\xi = (j^*\Xi)_{\pi \otimes \pi}$ induces an isomorphism

$$H^d(X_1, \mathbb{C})_\pi \cong H^d(X_2, \mathbb{C})_\pi.$$

To be precise, we need to use the theta lifting valued in cohomology developed by Kudla–Millson [40, 41]. On the other hand, we can determine the near equivalence class of σ and prove that

$$H^{2d}(X,\mathbb{C})_\sigma \subset H^{d,d}(X).$$

Hence we can modify Ξ in such a way that Ξ lies in $H^{2d}(X,\mathbb{Q})_\sigma$, so that it is a Hodge cycle. Finally, it follows from the result of Kisin–Shin–Zhu [38] that

$$H^{2d}(X,\mathbb{Q}_\ell)_\sigma \cong \mathbb{Q}_\ell(-d)^m$$

for some positive integer m, from which Theorem 4.1 follows immediately.

Remark 4.2. It is desirable to upgrade ξ to an absolute Hodge cycle in the sense of Deligne [14], but this remains an open problem.

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