EULER SYSTEMS AND THE BLOCH–KATO CONJECTURE FOR AUTOMORPHIC GALOIS REPRESENTATIONS

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ABSTRACT

We survey recent progress on the Bloch–Kato conjecture, relating special values of L-functions to cohomology of Galois representations, via the machinery of Euler systems. This includes new techniques for the construction of Euler systems, via the étale cohomology of Shimura varieties, and new methods for proving explicit reciprocity laws, relating Euler systems to critical values of L-functions. These techniques have recently been used to prove the Bloch–Kato conjecture for critical values of the degree 4 L-function of GSp_4, and we survey ongoing work aiming to apply this result to the Birch–Swinnerton-Dyer conjecture for modular abelian surfaces, and to generalise it to a range of other automorphic L-functions.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 11G40; Secondary 11F67, 11F80, 11G18

KEYWORDS

Bloch–Kato conjecture, Euler system, Selmer group

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 3, pp. 1918–1939
DOI 10.4171/ICM2022/50

Published by EMS Press

1. WHAT IS THE BLOCH–KATO CONJECTURE?

The Bloch–Kato conjecture, formulated in [11], relates the cohomology of global Galois representations to the special values of L-functions. We briefly recall a weak form of the conjecture, which will suffice for this survey. Let $L/\mathbf{Q}_p$ be a finite extension, let K be a number field, and let V be a representation of $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ on a finite-dimensional L-vector space. We suppose V is unramified outside finitely many primes and de Rham at the primes above p. Then we may attach to V the following two objects:

- Its *L-function*, which is the formal Euler product
$$L(V, s) = \prod_v P_v\big(V, \mathbf{N}(v)^{-s}\big)^{-1},$$
where v varies over (finite) primes of K, and $P_v(V, X) \in L[X]$ is a local Euler factor depending on the restriction of V to a decomposition group at v. It is conjectured that, for any choice of isomorphism $\overline{L} \cong \mathbf{C}$, this product converges for $\Re(s) \gg 0$ and and has meromorphic continuation to all of $\mathbf{C}$.
- Its *Selmer group* $H^1_{\mathrm{f}}(K, V)$, a certain (finite-dimensional) subspace of the Galois cohomology group $H^1(K, V)$ determined by local conditions at each prime, defined in [11].

The *weak Bloch–Kato conjecture* asserts that
$$\mathrm{ord}_{s=1} L\big(V^*, s\big) = \dim H^1_{\mathrm{f}}(K, V) - \dim H^0(K, V).$$
The full conjecture as formulated in [11] also determines the leading term of $L(V^*, s)$ at $s = 1$ up to a p-adic unit, in terms of the cohomology of an integral lattice $T \subset V$.

This conjecture includes as special cases a wide variety of well-known results and conjectures. For example, when V is the 1-dimensional trivial representation, the weak conjecture states that $\zeta_K(s)$ has a simple pole at $s = 1$; and the strong conjecture (for all p at once) is equivalent to the analytic class number formula, relating the residue at this pole to the class group and unit group of K. If $V = T_p(E) \otimes \mathbf{Q}_p$, where E is an elliptic curve over K and $T_p(E)$ is its Tate module, then $L(V^*, s)$ is the Hasse–Weil L-function $L(E/K, s)$, and we recover the Birch and Swinnerton-Dyer conjecture for E over K.

Critical values

The L-function $L(V^*, s)$ is expected to satisfy a functional equation relating $L(V, s)$ and $L(V^*, 1-s)$, after multiplying by a suitable product of Γ-functions $L_\infty(V^*, s)$ (determined by the Hodge–Tate weights of V at p and the action of complex conjugation). These Γ-factors may have poles at $s = 1$, forcing $L(V^*, 1)$ to vanish.

Following [49], we say V is *r-critical*, for some $r \geqslant 0$, if $L_\infty(V^*, 1-s)$ has a pole of order r at $s = 0$, and $L_\infty(V, s)$ is holomorphic there. In particular, V is 0-critical if $L(V^*, 1)$ is a critical value in the sense of Deligne [18]. The most interesting cases of the Bloch–Kato conjecture are when V is 0-critical, and it is these which our main theorems below will address; but 1-critical Galois representations will also play a crucial auxiliary role in our strategy.

Iwasawa theory

The Bloch–Kato conjecture is closely related to the *Iwasawa main conjecture*, in which the finite-dimensional Selmer group $H^1_{\mathrm{f}}(K, V)$ is replaced by a finitely-generated module over an Iwasawa algebra. This connection with Iwasawa theory, together with the proof of the Iwasawa main conjecture in this context by Mazur and Wiles, plays an important role in Huber and Kings' proof **[33]** of the Bloch–Kato conjecture for 1-dimensional representations of $\Gamma_{\mathbf{Q}}$.

2. WHAT IS AN EULER SYSTEM?

For K a number field and V a Γ_K-representation as in Section 1, we have the notion of an *Euler system* for V, defined as follows. Let S be a finite set of places of K containing all infinite places, all primes above p and all primes at which V is ramified.

We define $\mathcal{R}$ to be the collection of integral ideals of K of the form $\mathfrak{m} = \mathfrak{a} \cdot \mathfrak{b}$, where $\mathfrak{a}$ is a square-free product of primes of K not in S, and $\mathfrak{b}$ divides p^∞. For each $\mathfrak{m} \in \mathcal{R}$, let $c[\mathfrak{m}]$ be the ray class field modulo $\mathfrak{m}$. Then an Euler system for (T, S) is a collection of classes

$$\mathbf{c} = \{c[\mathfrak{m}] \in H^1(K[\mathfrak{m}], T) : \mathfrak{m} \in \mathcal{R}\},$$

satisfying the *norm-compatibility relation*

$$\mathrm{cores}^{K[\mathfrak{m}\mathfrak{q}]}_{K[\mathfrak{m}]}\big(c[\mathfrak{m}\mathfrak{q}]\big) = \begin{cases} P_{\mathfrak{q}}(V^*(1), \sigma_{\mathfrak{q}}^{-1}) \cdot c[\mathfrak{m}] & \text{if } \mathfrak{q} \notin S, \\ c[\mathfrak{m}] & \text{if } \mathfrak{q} \mid p, \end{cases}$$

where cores denotes the Galois corestriction (or norm) map, and $\sigma_{\mathfrak{q}}$ is the image of $\mathrm{Frob}_{\mathfrak{q}}$ in $\mathrm{Gal}(K[\mathfrak{m}]/K)$. By an *Euler system for V*, we mean an Euler system for some (T, S). (These general definitions are due to Kato, Perrin-Riou, and Rubin, building on earlier work of Kolyvagin; the standard reference is **[56]**.)

The crucial application of Euler systems is the following: if an Euler system exists for V whose image in $H^1(K, V)$ is non-zero (and V satisfies some auxiliary technical hypotheses), then we obtain a bound for the so-called *relaxed* Selmer group[1]

$$H^1_{\mathrm{rel}}(K, V) := \ker\Big(H^1(K, V) \to \prod_{v \nmid p} H^1_{\mathrm{f}}(K_v, V)\Big).$$

The relaxed Selmer group differs from the Bloch–Kato Selmer group in that we impose no local conditions at p. More generally, under additional assumptions on V and $\mathbf{c}$, we can obtain finer statements taking into account local conditions at p, and hence control the dimension of the Bloch–Kato Selmer group itself.

Euler systems are hence extremely powerful tools for bounding Selmer groups, as long as we can understand whether the image of $\mathbf{c}$ in $H^1(K, V)$ is non-vanishing. In order to

1 See **[49]** for this formulation. Theorem 2.2.3 of **[56]** is an equivalent result, but expressed in terms of a Selmer group for $V^*(1)$, which is related to that of V by Poitou–Tate duality.

use an Euler system to prove new cases of the Bloch–Kato conjecture, one needs to establish a so-called *explicit reciprocity law*, which is a criterion for the non-vanishing of the Euler system in terms of the value $L(V^*, 1)$.

Challenges. In order to use Euler system theory to approach the Bloch–Kato conjecture, and other related problems such as the Iwasawa main conjecture, there are two major challenges to be overcome:

(1) Can we construct "natural" examples of Euler systems (satisfying appropriate local conditions), for interesting global Galois representations V?

(2) Can we prove reciprocity laws relating the images of these Euler systems in $H^1(K, V)$ to the values of L-functions?

This was carried out by Kato [35] for the Galois representations arising from modular forms; but Kato's approach to proving explicit reciprocity laws has turned out to be difficult to generalise. More recently, in a series of works with various co-authors beginning with [40] (building on earlier work of Bertolini–Darmon–Rotger [6]), we developed a general strategy for overcoming these challenges, for Galois representations arising from automorphic forms for a range of reductive groups. We will describe this strategy in the remainder of this article.

Variants. A related concept is that of an *anticyclotomic Euler system*, in which K is a CM field, and we replace the ray-class fields $c[\mathfrak{m}]$ with *ring* class fields associated to ideals of the real subfield K^+. These arise naturally when V is *conjugate self-dual*, i.e. $V^c = V^*(1)$ where c denotes complex conjugation. The most familiar example is Kolyvagin's Euler system of Heegner points [39]; for more recent examples, see, e.g. [12, 15, 25]. Many of the techniques explained here for constructing and studying (full) Euler systems are also applicable to anticyclotomic Euler systems, and we shall discuss examples of both below.

A rather more distant cousin is the concept of a *bipartite Euler system*, which arises naturally in the context of level-raising congruences; cf. [31] for a general account, and [43] for a dramatic recent application to the Bloch–Kato conjecture. These require a rather different set of techniques, and we shall not discuss them further here.

The 1-critical condition. We conjectured in [49] that, in order to construct Euler systems for V by geometric means (i.e. as the images of motivic cohomology classes), we need to impose a condition on V: it needs to be *1-critical*.

However, our intended applications involve the Bloch–Kato conjecture for critical values of L-functions; so we need to construct Euler systems for representations that are 0-critical, rather than 1-critical. So we shall construct Euler systems for these representations in two stages: firstly, we shall construct Euler systems for auxiliary 1-critical representations V, using motivic cohomology; secondly, we shall "p-adically deform" our Euler systems, in order to pass from these 1-critical V to others which are 0-critical. This will be discussed in Section 4 below.

3. EULER SYSTEMS FOR SHIMURA VARIETIES

Shimura varieties. Let $(G, \mathcal{X})$ be a Shimura datum, with reflex field E. For a level $K \subset G(\mathbf{A}_\mathrm{f})$, we write $Y_G(K)$ for the Shimura variety $\mathrm{Sh}_K(G, \mathcal{X})$. Our first goal will be to define Euler systems, either full or anticyclotomic, for Galois representations appearing in the étale cohomology of $Y_G(K)$. We shall attempt to give a systematic general treatment, but the reader should bear the following examples in mind:

(1) $G = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, as in [40];

(2) $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$ for F real quadratic, as in [26, 41];

(3) $G = \mathrm{GSp}_4$, as in [47];

(4) $G = \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, as in [32];

(5) $G = \mathrm{GU}(2, 1)$, as in [48];

(6) $G = \mathrm{U}(2n-1, 1)$ for $n \geqslant 1$, as in [25].

Each of these groups is naturally equipped with a Shimura datum $(G, \mathcal{X})$. In examples (1)–(4), the reflex field E is $\mathbf{Q}$; in (5) and (6), it is the imaginary quadratic field used to define the unitary group. (One can also retrospectively interpret Kato's construction [35] in these terms, taking $G = \mathrm{GL}_2$; and similarly Kolyvagin's anticyclotomic Euler system [39], which is in effect the $n = 1$ case of example (6).)

Étale cohomology. If π is a cuspidal automorphic representation which contributes to $H^*_{\text{ét}}(Y_G(K)_{\bar{E}}, V_\lambda)$ for some level K, where V_λ is the étale local system associated to the representation of G of highest weight λ, then we say π is *cohomological in weight* λ. It is conjectured that if this holds, then there exists a p-adic representation ρ_π of Γ_E, for each prime p and embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, whose local Euler factors are determined by the Satake parameters of π at finite places, and whose Hodge–Tate weights are determined by λ.

For all of the above groups, the existence of such a ρ_π is known, and, moreover, if π is of "general type" (i.e. not a functorial lift from a smaller group), then the π_f-eigenspace in étale cohomology is concentrated in degree $d = \dim(\mathcal{X})$ and isomorphic to $\pi_\mathrm{f} \otimes \rho_\pi$. So we can find projection maps $H^d(Y_G(K)_{\bar{E}}, V_\lambda) \twoheadrightarrow \rho_\pi$, for a suitable choice of K. Via the Hochschild–Serre spectral sequence

$$H^i\big(E, H^j_{\text{ét}}\big(Y_G(K)_{\bar{E}}, V_\lambda\big)\big) \Rightarrow H^{i+j}_{\text{ét}}\big(Y_G(K)_E, V_\lambda\big),$$

we can thus obtain classes in the Galois cohomology of ρ_π as the images of classes in the π_f-eigenspace of $H^{d+1}_{\text{ét}}(Y_G(K)_E, V_\lambda)$. (For simplicity, we shall sketch the construction below assuming $\lambda = 0$, and refer to the original papers for the case of general coefficients.)

Motivic cohomology. In order to construct classes in $H^{d+1}_{\text{ét}}(Y_G(K)_E, V_\lambda)$, we shall use two other, related cohomology theories:

- *Motivic cohomology* (see [3]), which takes values in $\mathbf{Q}$-vector spaces (or $\mathbf{Z}$-lattices in them), and is closely related to algebraic K-theory and Chow groups;

- *Deligne–Beilinson* (or *absolute Hodge*) cohomology (see [34]), which takes values in $\mathbf{R}$-vector spaces, and has a relatively straightforward presentation in terms of pairs (ω, σ), where ω is an algebraic differential form, and σ a real-analytic antiderivative of $\mathrm{Re}(\omega)$.

There is no direct relation between Deligne–Beilinson cohomology and p-adic étale cohomology – one would not expect to compare vector spaces over $\mathbf{R}$ and over $\mathbf{Q}_p$ – but both of these cohomology theories have natural maps ("realisation maps") from motivic cohomology. So we shall use the following strategy, whose roots go back to [3]: we will write down elements of motivic cohomology whose images in Deligne–Belinson cohomology are related to values of L-functions; and we will consider the images of the same motivic cohomology classes in étale cohomology.

Pushforward maps. If $(H, \mathcal{Y}) \hookrightarrow (G, \mathcal{X})$ is the inclusion of a sub-Shimura datum (with the same reflex field E), then we obtain finite morphisms of algebraic varieties over E,

$$Y_H(K \cap H)_E \to Y_G(K)_E,$$

where E is the reflex field of $(H, \mathcal{Y})$. More generally, for each $g \in G(\mathbf{A}_{\mathrm{f}})$ we have a map

$$\iota_g : Y_H(gKg^{-1} \cap H)_E \to Y_G(gKg^{-1})_E \xrightarrow{g} Y_G(K),$$

where the latter arrow is translation by g. So we have associated *pushforward maps* in all of our cohomology theories, namely

$$\iota_{g,\star} : H^j_{\mathrm{mot}}\big(Y_H(K \cap H)_E, \mathbf{Z}(t)\big) \to H^{j+2c}_{\mathrm{mot}}\big(Y_G(K)_E, \mathbf{Z}(t+c)\big)$$

for $j, r \in \mathbf{Z}$, where $c = \dim \mathcal{X} - \dim \mathcal{Y}$ (and similarly for étale cohomology with $\mathbf{Z}_p$ coefficients, or Deligne–Beilinson cohomology with $\mathbf{R}$ coefficients, compatibly with the realisation maps relating the theories).

We shall define motivic cohomology classes for $Y_G(K)$ using the maps $\iota_{g,\star}$. The compatibility of these classes with realisation functors allows us to compute the images of such classes in Deligne–Beilinson cohomology: the projection of such a class to the π_{f}-eigenspace will be computed using integrals over $Y_H(K \cap H)(\mathbf{C})$, involving the pullbacks of differential forms associated to cusp forms in the dual automorphic representation $\pi^\vee$.

Cycle classes and Siegel units. As an input to the above construction, we need a supply of "interesting" classes in $H^j_{\mathrm{mot}}(Y_H(K \cap H), \mathbf{Z}(r))$ for some k, r which are in the image of motivic cohomology.

One possibility is to start with the *identity class* $1 \in H^0_{\mathrm{mot}}(Y_H(K \cap H), \mathbf{Z}(0))$. The image of this class under $\iota_{g,\star}$ is the *cycle class* associated to the image of ι_g, a so-called "special cycle". This case is by no means trivial: indeed, these special cycles are the input used to define anticyclotomic Euler systems, such as Heegner points.

More subtly, one can obtain motivic cohomology classes from *units* in the coordinate ring of Y_H, using the relation

$$H^1_{\mathrm{mot}}\big(Y, \mathbf{Z}(1)\big) = \mathcal{O}(Y)^\times$$

for any variety Y. If Y is a modular curve (i.e. a Shimura variety for GL_2), then we have a canonical family of units: if $Y_1(N)$ is the Shimura variety of level $\{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}$, then we have the *Siegel unit*

$$z_N \in \mathcal{O}\big(Y_1(N)\big)^{\times},$$

denoted ${}_c g_{0,1/N}$ in the notation of [35] (where c is an auxiliary integer coprime to the level). Crucially, we have an explicit formula for the image of this class in Deligne–Beilinson cohomology; it is given by

$$\big(\mathrm{dlog}\, z_N, \log |z_N|\big) = \big(E_2, E_0^{\mathrm{an}}(0)\big) \tag{3.1}$$

where E_2 is a weight 2 Eisenstein series, and $E_0^{\mathrm{an}}(s)$ is a family of real-analytic Eisenstein series depending on a parameter $s \in \mathbf{C}$. (See also [38] for analogues with coefficients, related to Eisenstein series of higher weights.)

Rankin–Eisenstein classes and Rankin–Selberg integrals. We shall consider the following general setting: we consider a Shimura datum $(H, \mathcal{Y})$ equipped with an embedding $\iota : (H, \mathcal{Y}) \to (G, \mathcal{X})$, and also with a family of maps

$$\psi = (\psi_1, \dots, \psi_t) : (H, \mathcal{Y}) \to (\mathrm{GL}_2, \mathbb{H})^t,$$

where $\mathbb{H}$ is the standard GL_2 Shimura datum and $t \geqslant 1$. We then have a collection of classes

$$z_N^H = \psi_1^*(z_N) \cup \dots \cup \psi_t^*(z_N) \in H^t_{\mathrm{mot}}\big(Y_{\mathcal{H}}\big(K_{H,1}(N)\big), \mathbf{Z}(t)\big),$$

for some level $K_{H,1}(N)$, which we call *Eisenstein classes* for H.

Remark 3.1. One might hope for a broader range of "Eisenstein classes" in motivic cohomology, associated to Eisenstein series on other groups which are not just copies of GL_2's. However, this question seems to be very difficult; see [21] for some results in this direction for symplectic groups. If we could construct motivic classes associated to Eisenstein series for the Siegel parabolic of GSp_{2n} (rather than the Klingen parabolic as in [21]), or for the analogous parabolic subgroup in the unitary group $\mathrm{U}(n, n)$, then it would open the way towards a far wider range of Euler system constructions. ■

By a *motivic Rankin–Eisenstein class* for $(G, \mathcal{X})$ (with trivial coefficients), we shall mean a class of the form

$$\iota_{g,\star}\big({}_c z_N^H\big) \in H^{2c+t}_{\mathrm{mot}}\big(Y_G(K)_E, \mathbf{Z}(c+t)\big),$$

for some N and some g and level K. If we choose our data $(H, \mathcal{Y})$ such that $2c + t = 1 + d$, then these classes land in the cohomological degree we want. The twist $c + t$ is then equal to $\frac{d+1+t}{2}$; hence, using the Hochschild–Serre spectral sequence, we can project the étale realisations of these classes into the groups $H^1(E, V_\pi)$, where

$$V_\pi = \rho_\pi\left(\frac{d+1+t}{2}\right).$$

Choosing the data

To define a Rankin–Eisenstein class, we need to choose the group H, and the maps $\iota : H \to G$ and $\psi : H \to (\mathrm{GL}_2)^t$. To guide us in choosing these, we shall use "Rankin–Selberg-type" integral formulas for L-functions of automorphic representations. There are a wide range of such formulas, relating automorphic L-functions to integrals of the form

$$\int_{H(\mathbf{Q})Z_G(\mathbf{A})\backslash H(\mathbf{A})} \iota^*(\phi)\psi^*\big(E^{\mathrm{an}}(s_1) \times \cdots \times E^{\mathrm{an}}(s_n)\big)\mathrm{d}h, \tag{3.2}$$

where $E^{\mathrm{an}}(s_i)$ are real-analytic GL_2 Eisenstein series, and ϕ is a cuspform in the space of π. We call these *period integrals*. Typically, one expects such an integral to evaluate to a product of one or more copies of the L-function of π, evaluated at some linear combination of the parameters s_i. For instance, the Rankin–Selberg integral for $\mathrm{GL}_2 \times \mathrm{GL}_2$ is of this form, as is Novodvorsky's formula for the L-functions of GSp_4 and $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

Using the explicit formula (3.1) relating Siegel units to Eisenstein series, one can often show that the Deligne–Beilinson realisations of Rankin–Eisenstein classes also lead to integrals of the form (3.2), for suitably chosen ϕ and s_i. When this applies, we can use it to relate our motivic Rankin–Eisenstein classes to special values of L-functions[2] (as was carried out in Beilinson's original paper [3] for the L-functions of GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$; see, e.g. [36, 42] for more recent examples).

This gives one a guide to constructing "interesting" Rankin–Eisenstein classes for a given $(G, \mathcal{X})$: one first searches for a Rankin–Selberg integral describing the relevant L-function, and then attempts to breathe motivic life into this real-analytic formula, interpreting it as the Deligne–Beilinson realisation of a motivic Rankin–Eisenstein class. One should hence interpret Rankin–Eisenstein classes as "motivic avatars" of Rankin–Selberg integral formulae.

In the anti-cyclotomic ($t = 0$) case, the period integral (3.2) is more mysterious; but there are still a number of results and conjectures predicting that these period integrals should be related to values of L-functions. For instance, the Gan–Gross–Prasad conjecture [22] gives such a relation in the important cases $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n) \times \mathrm{SO}(n+1)$ and $U(n) \hookrightarrow U(n) \times U(n+1)$. This conjecture has recently been proved in the unitary case [9], although the orthogonal case is still open. We refer to Sakellaridis–Venkatesh [58] for conjectural generalisations to other pairs (G, H).

2 More precisely, we obtain a relation to the first derivative $L'(V_\pi^*, 1)$, with V_π being 1-critical. Unfortunately, this computation does not give us any information about the étale class in $H^1(E, V_\pi)$, since the motivic class might be in the kernel of the étale realisation map. (This is the fundamental obstruction to proving the Bloch–Kato conjecture for 1-critical Galois representations.)

Example 3.2. In our examples (1)–(6) above, we choose H and t as follows:

	G	H	t
(1)	$\mathrm{GL}_2 \times \mathrm{GL}_2$	GL_2	1
(2)	$\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$	GL_2	1
(3)	GSp_4	$\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$	2
(4)	$\mathrm{GSp}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$	1
(5)	$GU(2,1)$	$\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{Res}_{E/\mathbf{Q}} \mathrm{GL}_1$	1
(6)	$U(2n-1,1)$	$U(n-1,1) \times U(n,0)$	0

The integral formulae for L-functions underlying examples (1) and (2) are, respectively, the classical Rankin–Selberg integral and Asai's integral formula for quadratic Hilbert modular forms. Cases (3) and (4) are related to Novodvorsky's integral formula for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-functions (with an additional Eisenstein series on the GL_2 factor in the former case); and case (5) to an integral studied by Gelbart and Piatetski-Shapiro in [23]. Example (6) is related to conjectures of Getz–Wambach [24] on Friedberg–Jacquet periods for automorphic representations of unitary groups. ■

Rankin–Eisenstein classes and norm relations. In order to build Euler systems (either full or anticyclotomic) from Rankin–Eisenstein classes, we need the following conditions to hold:

- ("Open orbit" condition) The group H has an open orbit on the product
$$(G/B_G) \times (\mathbf{P}^1)^t,$$
where B_G is a Borel subgroup of G, and H acts on G/B_G via ι, and on $(\mathbf{P}^1)^t$ via ψ.
- ("Small stabiliser" condition) For a point u in the open orbit, let S_u be the subgroup of H which fixes u and acts trivially on the fibre at u of the tautological $(\mathbf{G}_m)^t$-bundle over $(\mathbf{P}^1)^t$. Then we require that the image of S_u has small image in the maximal torus quotient of H.

The role of the "small stabiliser" condition is to allow us to construct classes over field extensions. Since the connected components of the Shimura variety Y_G are defined over abelian extensions of E, and the Galois action on the component group is described by class field theory, we can modify the Rankin–Eisenstein classes to define elements in $H^{d+1}_{\text{ét}}(Y_G(K)_F, \mathbf{Z}_p(\frac{d+1+t}{2}))$ for a fixed level K and varying abelian extensions F/E. The class of abelian extensions that arise will depend on the image of S_u in the maximal torus quotient; in the examples (1)–(5) above, since $S_u = \{1\}$ and the splitting field of the Galois action is the full maximal abelian extension of E, so we obtain classes over all ray class fields of E. On the other hand, in example (6) we obtain only the anticyclotomic extension (as one would expect, since $t = 0$ in this case).

The “open orbit” condition allows us to prove a so-called *vertical norm relation*, showing that after applying Hida’s ordinary projector, the Rankin–Eisenstein classes form norm-compatible families over the tower $E[p^\infty]/E$, with uniformly bounded denominators relative to the étale cohomology with $\mathbf{Z}_p$-coefficients. This machinery is worked out in considerable generality in [44,46]; the arguments also simultaneously show that the Rankin–Eisenstein classes interpolate in Hida-type p-adic families (in which the weight λ of π varies).

A much more subtle problem is that of *horizontal norm relations*, comparing classes over $E[\mathfrak{m}]$ and $E[\mathfrak{m}\mathfrak{q}]$ for auxiliary primes $\mathfrak{q} \nmid \mathfrak{m}$, with the Euler factors $P_{\mathfrak{q}}$ appearing in the comparison. The strategy developed in [47] and refined in [48] is to use multiplicity-one results in smooth representation theory to reduce the norm relation to a purely local calculation with zeta-integrals, which can then be computed explicitly to give the Euler factor. These multiplicity-one results are themselves closely bound up with the open-orbit condition; see [57].

Remark 3.3. The open-orbit condition, together with the assumption that $2c + t = 1 + d$, amount to stating that the diagonal map $(\iota, \psi) : (H, \mathcal{Y}) \hookrightarrow (G, \mathcal{X}) \times (\mathrm{GL}_2, \mathbb{H})^t$ is a *special pair* of Shimura data in the sense of [59, DEFINITION 3.1]. We can thus interpret the “small stabiliser” condition, at least for $t = 0$, as a criterion for when the special cycles studied in [59] extend to norm-compatible families over field extensions. ■

4. DEFORMATION TO CRITICAL VALUES

Critical values. The above methods allow us to define Euler systems for the automorphic Galois representations $V_\pi = \rho_\pi(\frac{d+1+t}{2})$, where π is cohomological in weight 0; and there are generalisations to representations which are cohomological for a certain range of nonzero weights λ, determined by branching laws for the restriction of algebraic representations from $\tilde{G} = G \times (\mathrm{GL}_2)^t$ to H. Let us write Σ_1 for the set of weights λ which are accessible by these methods, for some specific choice of H and ψ; this is a convex polyhedron in the weight lattice of G, cut out by finitely many linear inequalities. In the examples (1)–(5), one checks that for any π whose weight lies in Σ_1, the representation V_π is 1-critical, consistently with the conjectures of [49].

However, our goal is to prove the Bloch–Kato conjecture for critical L-values; so we are interested in those λ such that, for π of weight λ, the representation V is 0-critical, so $L(\pi^\vee, \frac{1-t}{2})$ is a critical value. The set of such λ is a finite disjoint union of polyhedral regions; and we let Σ_0 be one of these regions, chosen to be adjacent to Σ_1. In order to approach the Bloch–Kato conjecture, we need to find a way of “deforming” our Euler systems from Σ_1 to Σ_0.

Example 4.1. For $G = \mathrm{GL}_2 \times \mathrm{GL}_2$, the Galois representations associated to cohomological representations of G have the form $(\rho_f \otimes \rho_g)(n)$, where f, g are modular forms (of some weights $k + 2, \ell + 2$ with $k, \ell \geqslant 0$) and n is an arbitrary integer. If we set $j = k + \ell + 1 - n$, then the set Σ_1 is given by the inequalities

$$\{0 \leqslant j \leqslant \min(k, \ell)\},$$

and there are two candidates for the set Σ_0, namely

$$\{\ell + 1 \leqslant j \leqslant k\} \quad \text{and} \quad \{k + 1 \leqslant j \leqslant \ell\}. \qquad \blacksquare$$

Remark 4.2. A slightly different, but related, numerology applies for anticyclotomic Euler systems. In these cases, the relevant L-value is always critical, but it lies at the centre of the functional equation, so it may be forced to vanish for sign reasons. Since the local root numbers at the infinite places depend on λ, we have some ranges of weights where the root number is $+1$ (where we expect interesting central L-values) and others where it is -1 (where we expect anticyclotomic Euler systems). These play the roles of the 0-critical and 1-critical regions in the case of full Euler systems. ■

The Bertolini–Darmon–Prasanna strategy. Although the "0-critical" and "1-critical" weight ranges are disjoint, we can relate them together p-adically, using a strategy introduced by Bertolini, Darmon and Prasanna in [5].

The weights λ of cohomological representations can naturally be seen as points of a p-adic analytic space $\mathcal{W}$ (parametrising characters $T(\mathbf{Z}_p) \to \mathbf{C}_p^\times$, where T is a maximal torus in G). This space is isomorphic to a finite union of n-dimensional open discs, where n is the rank of G. Crucially, both Σ_0 and Σ_1 are Zariski-dense in $\mathcal{W}$.

Hida theory shows that there is a finite flat covering $\mathcal{E} \to \mathcal{W}$, the *ordinary eigenvariety* of G, whose points above a dominant integral weight λ ("classical points") biject with automorphic representations π of G which are cohomological of weight λ and p-ordinary.

We thus have two separate families of objects, indexed by different sets of classical points on $\mathcal{E}$:

- at points whose weights lie in Σ_0, we have the critical values of the complex L-function;
- at points whose weights lie in Σ_1, we have Euler systems arising from motivic cohomology.

Our first goal will be to "analytically continue" the Euler system classes from Σ_1 into Σ_0.

This is not all that we require, however, since we also need a relation between the resulting Euler system for each 0-critical V and the L-value $L(V^*, 1)$. Relations of this kind are known as *explicit reciprocity laws*, and they are the crown jewels of Euler system theory. Following a strategy initiated in [5] and further developed in [37], in order to prove explicit reciprocity laws, we shall use a second kind of p-adic deformation: besides deforming Euler system classes from Σ_1 to Σ_0, we shall also deform L-values from Σ_0 into Σ_1. The strategy consists of the following steps:

(i) We shall construct a function on the eigenvariety – an "analytic p-adic L-function" – whose values in Σ_0 are critical L-values (modified by appropriate periods and Euler factors).

(ii) Using the Perrin-Riou regulator map of p-adic Hodge theory, we construct a second analytic function on the eigenvariety – a "motivic p-adic L-function" – whose value at some cohomological π measures the non-triviality of Euler system classes for π locally at p.
Note that the motivic p-adic L-function has no *a priori* reason to be related to complex L-values; however, its values in Σ_1 are by definition related to the Euler system classes (which arise from motivic cohomology, hence the terminology).

(iii) We shall prove a "p-adic regulator formula", showing that the values of the analytic p-adic L-function in at points in Σ_1 are related to the localisations of the Euler system classes at p.

(iv) Using the regulator formula of step (iii), we can deduce that the motivic and analytic p-adic L-function coincide at points in Σ_1. Since weights lying in Σ_1 are Zariski-dense in $\mathcal{E}$, this implies the two p-adic L-functions coincide in Σ_0 as well. Since the values of the analytic p-adic L-function in Σ_0 are complex L-values, we obtain the sought-for explicit reciprocity law.

At the time of writing, this strategy has only been fully carried out for the examples (1) and (3) in our list, and partially for (4). However, the remaining cases are being treated in ongoing work of members of our research groups; and we expect the strategy to extend to many other Euler systems (both full and anticyclotomic) besides these.

5. CONSTRUCTING p-ADIC L-FUNCTIONS

Coherent cohomology

To construct the analytic p-adic L-function, we shall use the integral formula (3.2). Previously, for weights in Σ_1, we interpreted this integral as a cup-product in Deligne–Beilinson cohomology. We shall now give a different cohomological interpretation of the same formula, for weights in the range Σ_0. Following a strategy introduced by Harris [28,29], we can choose the cusp-form ϕ, and the Eisenstein series, to be harmonic differential forms (with controlled growth at the boundary) representing Dolbeault cohomology classes valued in automorphic vector bundles. These can then be interpreted algebraically, via the comparison between Dolbeault cohomology and Zariski sheaf cohomology. The upshot is that $L(\pi^\vee, \frac{1-t}{2})$ can be related to a cup-product in the cohomology of coherent sheaves on a smooth toroidal compactification $\mathrm{Sh}_K(H, \mathcal{Y})^{\mathrm{tor}}_{\Sigma}$ of $\mathrm{Sh}_K(H, \mathcal{Y})$.

Interpolation

In order to construct a p-adic L-function, we need to show that the cohomology classes appearing in our formula for the L-function interpolate in Hida-type p-adic families, and that the cup-product of these families makes sense.

Until recently, there was a fundamental limitation in the available techniques: we could only interpolate cohomology classes corresponding to holomorphic automorphic forms (i.e. degree 0 coherent cohomology), or (via Serre duality) those in the top-degree cohomology, which correspond to anti-holomorphic forms. This is an obstacle for our intended applications, since the integral formulas relevant for Euler systems always involve coherent cohomology in degrees close to the middle of the possible range. (More precisely, the relevant degree is $\frac{d+t-1}{2}$, where t is the number of Eisenstein series present, which is typically $0, 1, 2$.) So unless d is rather small, using holomorphic or anti-holomorphic classes will not work.

A slightly wider range of "product type" examples arises when $(G, \mathcal{X})$ is a product of two Shimura data $(G_1, \mathcal{X}_1) \times (G_2, \mathcal{X}_2)$ of approximately equal dimension, with $\dim(\mathcal{X}_1) - \dim(\mathcal{X}_2) = t - 1$; then we can build a class in the correct degree as the product of an anti-holomorphic form on $\mathcal{X}_1$ and a holomorphic one on $\mathcal{X}_2$, and the resulting cup-products can often be understood as Petersson-type scalar products in Hida theory. For instance, the Rankin–Selberg integral formula can be analysed in this way [30]. However, for $G = \mathrm{GSp}_4$ (with $t = 2$ and $d = 3$), we need to work with a class in coherent H^2, and these are not seen by orthodox Hida theory.

Higher Hida theory. A beautiful solution to this problem is provided by the "higher Hida theory" developed in [55]. Pilloni's work shows that degree 1 coherent cohomology for the GSp_4 Shimura variety interpolates in a "partial" Hida family, with one weight fixed and the other varying p-adically.

A key ingredient in this work is to consider a certain stratification of the mod p fibre of the GSp_4 Shimura variety Y_G (for some level structure unramified at p). This space parametrises abelian surfaces A with a principal polarisation and some prime-to-p level structure. There is an open subspace Y_G^{ord}, whose complement has codimension 1, where A is ordinary; and a slightly larger open set, with complement of codimension 2, where the p-rank of A (the dimension of the multiplicative part of $A[p]$) is $\geqslant 1$. This stratification can be extended to a toroidal compactification X_G of Y_G; and Pilloni's approach to studying H^1 in p-adic families is based on restricting to the tube of this p-rank $\geqslant 1$ locus in the p-adic completion $\mathfrak{X}_G$ of X_G. (In contrast, orthodox Hida theory for GSp_4 involves working over the ordinary locus; this is very effective for studying H^0 but disastrous for studying H^1, since the ordinary locus is affine in the minimal compactification, so its cuspidal cohomology vanishes in positive degrees.)

In [45] we carried out a (slightly delicate) comparison of stratifications, showing that we can find an embedding ι_g of an H-Shimura variety, for a carefully chosen g, so that the preimage of the p-rank $\geqslant 1$ locus in $\mathfrak{X}_G$ is the ordinary locus in $\mathfrak{X}_H$, that is, the image of $\mathfrak{X}_H$ "avoids" the locus where the p-rank is exactly 1. Using this, we constructed pushforward maps from the orthodox (H^0) Hida theory for H to Pilloni's H^1 theory for G, interpolating the usual coherent-cohomology pushforward maps for varying weights. This is the tool we need to construct analytic p-adic L-functions for GSp_4 and for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

At present higher Hida theory, in the above sense, is only available for a few specific groups, although these include many of the ones relevant for Euler systems: besides GSp_4, the group $\mathrm{GU}(2,1)$ is treated in [53], and Hilbert modular groups in [27] (in both cases assuming G is locally split at p). In the GSp_4 and $\mathrm{GU}(2,1)$ cases the results are also slightly weaker than one might ideally hope, since we only obtain families in which one component of the weight is fixed and the others vary (so the resulting p-adic L-functions have one variable fewer than one would expect). However, we expect that these restrictions will be lifted in future work.

Remark 5.1. A related theory, *higher Coleman theory*, has been developed by Boxer and Pilloni in [13]. This theory also serves to interpolate higher-degree cohomology in families, with all components of the weight varying; and the theory applies to any Shimura variety of abelian type. However, unlike the higher Hida theory of [55], this theory only applies to cohomology classes satisfying an "overconvergence" condition. This rules out the 2-parameter GL_2 Eisenstein family which plays a prominent role in the constructions of [45], as this Eisenstein series is not overconvergent. It may be possible to work around this problem by combining the higher Coleman theory of [13] with the theory of families of nearly-overconvergent modular forms for GL_2 introduced by Andreatta–Iovita [1]; but the technical obstacles in carrying this out would be formidable. ■

6. P-ADIC REGULATORS

We now turn to step (iii) of the BDP strategy: relating values of the analytic p-adic L-function in the range Σ_1 to the localisations at p of the Euler system classes.

Syntomic cohomology. For all but finitely many primes, the Shimura variety has a smooth integral model over $\mathbf{Z}_p$, and the motivic Rankin–Eisenstein classes can be lifted to the cohomology of this integral model. This allows us to study them via another cohomology theory, Besser's *rigid syntomic cohomology* [7]. This is a cohomology theory for smooth $\mathbf{Z}_p$-schemes $\mathcal{Y}$, which has two vital properties:

- Via works of Fontaine–Messing and Nizioł, one can define a comparison map relating syntomic cohomology of $\mathcal{Y}$ to étale cohomology of its generic fibre Y; and this is compatible with motivic cohomology, in the sense that we have a commutative diagram (see [8]):

$$\begin{array}{ccc} H^*_{\mathrm{mot}}(\mathcal{Y}, n) & \longrightarrow & H^*_{\mathrm{mot}}(Y, n) \\ \downarrow & & \downarrow r_{\text{ét}} \\ H^*_{\mathrm{syn}}(\mathcal{Y}, n) & \xrightarrow{\mathrm{FM}} & H^*_{\text{ét}}(Y, n) \end{array}$$

  where the map $r_{\text{ét}}$ is the étale realisation map.

  The Fontaine–Messing–Niziol map induces the Bloch–Kato exponential map on Galois cohomology; so, for a class in $H^*_{\text{ét}}(Y, n)$ in the image of motivic cohomology of $\mathcal{Y}$, one can express its Bloch–Kato logarithm via cup-products in syntomic cohomology ("syntomic regulators").

- Rigid syntomic cohomology and its variant, fp-cohomology, were defined by Besser as a generalisation of Coleman's theory of p-adic integration. It is computed by an explicit complex of sheaves which is a p-adic analogue of the real-analytic Deligne–Beilinson complex: sections of this complex are pairs (ω, σ), where ω is an algebraic differential form, and σ is an overconvergent rigid–analytic differential form such that $d\sigma = (1 - \varphi)\omega$, where φ is a local lift of the Frobenius of the special fibre.

In a series of works, beginning with the breakthrough [16] by Darmon–Rotger (see also [6,10,38]), rigid syntomic cohomology has been systematically exploited to compute the Bloch–Kato logarithms of Rankin–Eisenstein classes when G is a product of copies of GL_2, in terms of Petersson products of (non-classical) p-adic modular forms. These can then be interpreted as values of p-adic L-functions in a "1-critical" region Σ_1. All of these p-adic L-functions are "product type" settings in the sense explained above, involving coherent cohomology in either top or bottom degree.

Remark 6.1. A key role in these constructions is played by an explicit formula for the image of the Siegel unit in the syntomic cohomology of the ordinary locus of the modular curve, which is the p-adic counterpart of equation (3.1): it is represented by the pair

$$\left(\operatorname{dlog} z_N, (1-\varphi)\log_p z_N\right) = \left(E_2, E_0^{(p)}\right)$$

where $E_0^{(p)}$ is a p-adic Eisenstein series of weight 0.

We can thus understand these syntomic regulator formulae as p-adic counterparts of the integral formula (3.2), with the integral understood via Coleman's p-adic integration theory, and the real-analytic Eisenstein class replaced by a p-adic one. ■

The GSp_4 regulator formula. The approach to computing regulators of étale classes via syntomic cohomology generalises to Euler systems for other Shimura varieties, such as GSp_4: one can always express the image of the Euler system class under the Bloch–Kato logarithm, paired against a suitable de Rham cohomology class (lying in the $\pi^\vee$-eigenspace), as a cup product in syntomic cohomology.

However, syntomic cohomology of the whole Shimura variety is not well-suited to explicit computations, since there is generally no global lift of the Frobenius of the special fibre. The first major problem is hence to express the pairing in terms of the syntomic cohomology of certain open subschemes of the Shimura variety which do possess an explicit Frobenius lift. This requires some results on the Hecke eigenspaces appearing in the rigid cohomology of Newton strata of the special fibre, which are the $\ell = p$ counterparts of the vanishing theorems proved by Caraiani–Scholze [14] for ℓ-adic cohomology for $\ell \neq p$.

The second major problem is to establish a link between rigid syntomic cohomology and coherent cohomology, for varieties admitting a Frobenius lifting. We succeeded in proving such a relation via a new a spectral sequence (the so-called *Poznań spectral sequence*) which is a syntomic analogue of the Hodge–de Rham spectral sequence: its E_1 page is given by the mapping fibre of $1 - \varphi$ on coherent cohomology, and its abutment is rigid syntomic cohomology. In the case of the ordinary locus of the modular curve, where all coherent cohomology in positive degrees vanishes, this reduces to the description of a syntomic class as a pair of global sections (ω, σ) as described above.

Thanks to this new spectral sequence, we were able to express the syntomic regulator of our Rankin–Eisenstein class for GSp_4 as a pairing in coherent cohomology, which we could identify as a specialisation of the pairing in higher Hida theory defining the p-adic L-function. We can hence identify the logarithm of the GSp_4 Euler system class with a non-critical value of a p-adic L-function. This is the first example of a p-adic regulator formula where the p-adic L-function is *not* of product type. We expect this strategy to be applicable to all the other Euler systems mentioned in Section 3 above. Cases (2) and (6) are currently work in progress by Giada Grossi, and by Andrew Graham and Waqar Shah, respectively; and case (5) is being explored by some members of our research groups.

7. DEFORMATION TO CRITICAL VALUES

We can now proceed to the final step of the Bertolini–Darmon–Prasanna strategy: deforming from Σ_1 to Σ_0.

First, we must show that Euler systems interpolate over the eigenvariety. The étale cohomology eigenspaces attached to cohomological, p-ordinary automorphic representations are known to interpolate in families, giving rise to sheaves of Galois representations over $\mathcal{E}$. With this in hand, the machinery of [44, 46] then shows that the Euler system classes themselves interpolate, giving families of Euler systems taking values in these sheaves.

A generalisation of Coleman and Perrin-Riou's theory of "big logarithm" maps (cf. [37]) also allows us to define a *motivic p-adic L-function* $\mathcal{L}^{\mathrm{mot}}$ associated to the bottom class in our family of Euler systems. Perrin-Riou's local reciprocity formula implies that $\mathcal{L}^{\mathrm{mot}}$ has an interpolation property both in Σ_0 and in Σ_1. For classical points π whose weights lie in Σ_1, the value of $\mathcal{L}^{\mathrm{mot}}$ interpolates the Bloch–Kato logarithm of the geometrically-defined Euler system class for V_π. Much more subtly, if we evaluate $\mathcal{L}^{\mathrm{mot}}$ at points π whose weights lie in Σ_0, it computes the image under the *dual-exponential* map of the bottom class in the Euler system for V_π which we have just defined using analytic continuation.

We would like to make the following argument: "the regulator formula shows that $\mathcal{L}^{\mathrm{mot}}$ and the analytic p-adic L-function $\mathcal{L}$ agree at points in Σ_1, and these are Zariski-dense; so $\mathcal{L} = \mathcal{L}^{\mathrm{mot}}$ everywhere". This is essentially how we proved an explicit reciprocity law for $\mathrm{GL}_2 \times \mathrm{GL}_2$ in [37]. Unfortunately, there are two subtle technical hitches which occur in making this argument precise for GSp_4.

The first is that $\mathcal{L}$ and $\mathcal{L}^{\mathrm{mot}}$ take values in different line bundles over the eigenvariety $\mathcal{E}$ (one interpolating coherent cohomology, and the other $\mathbf{D}_{\mathrm{cris}}$ of a certain subquotient of étale cohomology). At each classical point of $\mathcal{E}$, we have a canonical isomorphism between the fibres of these two line bundles; but it is far from obvious *a priori* that these "pointwise" isomorphisms at classical points interpolate into an isomorphism of line bundles. For the

GL_2 ordinary eigenvariety, we do have such an isomorphism, the p-adic Eichler–Shimura isomorphism of Ohta [54] (extended to non-ordinary families in [2]). However, the case of higher-dimensional Shimura varieties such as GSp_4 is more difficult: one expects several Eichler–Shimura isomorphisms, each capturing coherent cohomology in a different degree, and at present only the case of H^0 is available in the literature [19]. For the problem at hand, it is the coherent H^1 (and dually H^2) which is relevant.

The second is that, while Σ_1 is indeed Zariski-dense in the eigenvariety, the function $\mathcal{L}$ is only defined on a lower-dimensional "slice" of the eigenvariety (on which the GSp_4 form has weight (r_1, r_2), with r_1 varying and r_2 fixed), and the intersection of each individual slice with Σ_1 is not Zariski-dense in the slice.

In [50], we circumvented these problems in a somewhat indirect way, by appealing to a second, independent construction of an analytic p-adic L-function, defined using Shalika models for GL_4 [20]. As written this construction shares with [45] the shortcoming of requiring r_2 to be fixed, but the methods of [44] can be applied in order to extend this construction by varying r_2 as well. Using this we were able to

The lack of an Eichler–Shimura isomorphism in families – or, more precisely, of an isomorphism between the sheaves in which $\mathcal{L}^{\mathrm{mot}}$ and the GL_4 p-adic L-function take values – can be dealt with via the so-called "leading term argument". This proceeds as follows. There is clearly a *meromorphic* isomorphism between these sheaves which maps one p-adic L-function to the other (since both are clearly non-zero).[3] If this meromorphic isomorphism degenerates to zero at some "bad" 0-critical π, then the bottom class in our Euler system for π lies in the kernel of the Perrin-Riou regulator. However, this would also apply to all the classes $c[m]$ in this Euler system, for all values of m. So we obtain an Euler system satisfying a very strong local condition at p; and a result of Mazur–Rubin [52] shows that this condition is so strong that it forces the entire Euler system to be zero. Hence we can replace all of these classes by their derivatives in the weight direction, which amounts to renormalising the Eichler–Shimura map to reduce its order of vanishing by 1.

Iterating this process, we eventually obtain a non-trivial Euler system for π; and if $L(\pi^\vee, \frac{1-t}{2}) \neq 0$, the bottom class of this Euler system is non-zero. We can now deduce the vanishing of $H^1_{\mathrm{f}}(\mathbf{Q}, V_\pi)$, where $V_\pi = \rho_\pi(\frac{d+1+t}{2})$, as predicted by the Bloch–Kato conjecture.

Non-regular weights

The above strategy can also be used to study automorphic forms which are not cohomological (so π does not contribute to étale cohomology), as long as π contributes to *coherent* cohomology in the correct degree. For instance, this applies to weight 1 modular forms, which is crucial in several works such as [17] which use Euler systems to study the Birch–Swinnerton-Dyer conjecture for Artin twists of elliptic curves. It also applies to

3 This argument can be used to *construct* an Eichler–Shimura isomorphism in families for GSp_4, which interpolates the classical H^1 comparison isomorphism at almost all classical points – see [51].

paramodular Siegel modular forms for GSp_4 of parallel weight 2, such as those corresponding to paramodular abelian surfaces.

In this situation, if π is ordinary at p, it follows from the results of [13] that it defines a point on the eigenvariety $\mathcal{E}$. However, in contrast to the case of cohomological weights, it is not clear if the eigenvariety is smooth, or étale over weight space, at π; results of Bellaiche–Dimitrov show that this can fail even for GL_2 [4].

If A is a paramodular abelian surface over $\mathbf{Q}$ which is ordinary at p, and has analytic rank 0, then we can use the above approach to prove the finiteness of $A(\mathbf{Q})$ (as predicted by the Birch–Swinnerton-Dyer conjecture), and of the p-part of the Tate–Shafarevich group, under the assumption that the GSp_4 eigenvariety be smooth at the point corresponding to A. This is work in progress.

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