POINTS ON SHIMURA VARIETIES MODULO PRIMES

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ABSTRACT

We survey recent developments on the Langlands–Rapoport conjecture for Shimura varieties modulo primes and an analogous conjecture for Igusa varieties. We discuss resulting implications on the automorphic decomposition of the Hasse–Weil zeta functions and ℓ-adic cohomology of Shimura varieties, along with further applications to the Langlands correspondence and related problems.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 11G18; Secondary 11F72, 11R39

KEYWORDS

Shimura varieties, Igusa varieties, trace formula, cohomology

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 3, pp. 1966–1988
DOI 10.4171/ICM2022/62

1. INTRODUCTION

Shimura varieties have been vital to number theory for their intrinsic beauty and wide-ranging applications. They are simultaneously locally symmetric spaces and quasi-projective varieties over number fields, serving as a geometric bridge between automorphic forms and arithmetic. This feature has been particularly fruitful in the Langlands program.

This paper concentrates on the problem of understanding the Hasse–Weil zeta functions and ℓ-adic cohomology of Shimura varieties following the approach due to Langlands, Kottwitz, Rapoport, and others. As such, we are naturally led to study integral models and special fibers of Shimura varieties at each prime (Section 2), as epitomized by the Langlands–Rapoport (LR) conjecture (Section 3). Below is a partial summary of this article:

$$\begin{array}{ccccccc} \mathrm{LR}_1(\mathrm{Sh}) & \Longleftarrow & \boxed{\mathrm{LR}_0(\mathrm{Sh})} & \Longrightarrow & \mathrm{TF}(\mathrm{Sh}) & \Longrightarrow & \mathrm{app}(\mathrm{Sh}) \\ & & \Big\Downarrow & & & & \Updownarrow \\ & & \mathrm{LR}_0(\mathrm{Ig}) & \Longrightarrow & \mathrm{TF}(\mathrm{Ig}) & \Longrightarrow & \mathrm{app}(\mathrm{Ig}). \end{array}$$

Central to this paper is Theorem 3.2, asserting that $\mathrm{LR}_0(\mathrm{Sh})$, a version of the LR conjecture, is true for Shimura varieties of abelian type with good reduction. This is a strengthening of another version $\mathrm{LR}_1(\mathrm{Sh})$ which was previously verified by Kisin. Even though $\mathrm{LR}_0(\mathrm{Sh})$ is weaker than the original LR conjecture (still wide open), it opens doors for most applications. Indeed, the diagram shows how $\mathrm{LR}_0(\mathrm{Sh})$ implies a (stabilized) trace formula for cohomology of Shimura varieties, designated as TF(Sh), which in turn leads to interesting applications (Section 4). In Sections 5–6, we survey related problems and directions in the bad reduction case. Finally in Section 7, we review a parallel story for Igusa varieties, where $\mathrm{LR}_0(\mathrm{Sh})$ provides a key ingredient for proving the analogous assertion $\mathrm{LR}_0(\mathrm{Ig})$ for Igusa varieties. The dotted vertical arrow suggests that interactions occur between certain applications to Shimura and Igusa varieties, e.g., through Mantovan's formula.

Conventions

Unless otherwise stated, cohomology means the ℓ-adic étale cohomology with $\overline{\mathbb{Q}}_l$-coefficients. For an inverse limit of varieties $X = (X_i)$ over a field k, we write $H(X, \overline{\mathbb{Q}}_l)$ for $\varinjlim_i H(X_i \times_k \bar{k}, \overline{\mathbb{Q}}_l)$, with $\bar{k}$ a separable closure of k; likewise for cohomology with compact support. We adhere to cohomology with constant coefficients for simplicity, though the discussed results are valid more generally. In the LR conjecture, we omit $Z_G(\mathbb{Q}_p)$-equivariance to keep the statements simple. If Γ is a topological group, $\mathcal{H}(\Gamma)$ is the Hecke algebra of locally constant compactly supported functions on Γ. We write $\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ for the ring of finite adèles, and $\mathbb{A}^{\infty,p}$ for the analogous ring without the p-component. Put $\breve{\mathbb{Z}}_p := W(\overline{\mathbb{F}}_p)$ for the Witt ring of $\overline{\mathbb{F}}_p$, and $\breve{\mathbb{Q}}_p := \breve{\mathbb{Z}}_p[1/p]$. Denote by σ the Frobenius operator on $\breve{\mathbb{Q}}_p$ or a finite unramified extension of $\mathbb{Q}_p$. When we have cohomology spaces $H^i(X)$ (supported on finitely many i's) with a group action, denote by $[H(X)] = \sum_{i \geq 0} (-1)^i H^i(X)$ the alternating sum viewed in a suitable Grothendieck group of representations. For an algebraic group G over $\mathbb{Q}$ and a field k over $\mathbb{Q}$, write $G_k := G \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\, k$. We quietly fix field embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}}_v \hookrightarrow \mathbb{C}$ at each place v of $\mathbb{Q}$, and identify the residue field of $\overline{\mathbb{Q}}_p$ with $\overline{\mathbb{F}}_p$.

2. SHIMURA VARIETIES WITH GOOD REDUCTION

Let G be a connected reductive group over $\mathbb{Q}$, and X a $G(\mathbb{R})$-conjugacy class of $\mathbb{R}$-group morphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$. We say that (G, X) is a Shimura datum if it satisfies axioms (2.1.1.1)–(2.1.1.3) of **[10]**. Each (G, X) determines a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$ over $\mathbb{C}$, whose field of definition is a number field $E = E(G, X) \subset \mathbb{C}$. There is an obvious notion of morphisms between Shimura data.

Thanks to Shimura, Deligne, Borovoi, and Milne, we have a $G(\mathbb{A}^{\infty})$-scheme Sh over E (in the sense of **[10, 2.7.1]**, cf. **[31, 1.5.1]**), which is a projective limit of quasiprojective varieties over E with a $G(\mathbb{A}^{\infty})$-action. If (G, X) is a Siegel datum, i.e., $G = \mathrm{GSp}_{2n}$ and X is realized by the Siegel half-spaces of genus n for some $n \in \mathbb{Z}_{\geq 1}$, then we obtain (a projective limit of) Siegel modular varieties as output. There is a hierarchy of Shimura data:

$$\text{(PEL type)} \subset \text{(Hodge type)} \subset \text{(abelian type)} \subset \text{(all)}.$$

Roughly speaking, Shimura varieties coming from PEL-type data are realized as moduli spaces of abelian varieties with polarizations (P), endomorphisms (E), and level (L) structures.[1] This case includes modular curves and, more generally, Siegel modular varieties. A Shimura datum of Hodge type embeds in a Siegel datum by definition, and the corresponding Shimura varieties embed in Siegel modular varieties. Abelian-type data are generalized from those of Hodge type to cover the case when the Dynkin diagram of $G_{\overline{\mathbb{Q}}}$ consists of only types A, B, C, and D, with a small exception in the type D case, cf. **[10, §2.3]**.

Now we turn to integral models of Shimura varieties in the good reduction case. A starting point is an *unramified Shimura datum* $(G, X, p, \mathcal{G})$, where (G, X) is a Shimura datum, p is a prime, and $\mathcal{G}$ is a reductive model of G over $\mathbb{Z}_p$. The existence of $\mathcal{G}$ is equivalent to the condition that $G_{\mathbb{Q}_p}$ is an unramified group (i.e., quasisplit over $\mathbb{Q}_p$ and split over an unramified extension of $\mathbb{Q}_p$). Now we put $K_p := \mathcal{G}(\mathbb{Z}_p)$ and consider the $G(\mathbb{A}^{\infty,p})$-scheme Sh_{K_p} over E, which is similar to Sh as above but has a fixed level K_p at p (while the level subgroup away from p varies). Kisin **[28]** ($p > 2$) and Kim–Madapusi Pera **[27]** ($p = 2$) proved the following fundamental result.

Theorem 2.1. *If (G, X) is of abelian type, then there exists a canonical integral model $\mathscr{S}_{K_p}$, which is an $\mathcal{O}_{E,(p)}$-scheme with a $G(\mathbb{A}^{\infty,p})$-action, such that the generic fiber of $\mathscr{S}_{K_p}$ is $G(\mathbb{A}^{\infty,p})$-equivariantly isomorphic to* Sh_{K_p}.

Here "canonical" means that $\mathscr{S}_{K_p}$ is formally smooth over $\mathcal{O}_{E,(p)}$ and satisfies the extension property of **[28, (2.3.7)]**, which characterizes $\mathscr{S}_{K_p}$ uniquely up to a unique isomorphism. The proof of the theorem reduces to the Hodge-type case and utilizes the

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1 A caveat is that such a moduli space is in general a finite disjoint union of Shimura varieties due to a possible failure of the Hasse principle for G. See **[34, §8]** for details.

known canonical integral models in the Siegel case. Kisin constructs $\mathscr{S}_{K_p}$ by normalizing the closure of Sh_{K_p} in an ambient Siegel modular variety. The key point is to show formal smoothness of $\mathscr{S}_{K_p}$ over $\mathcal{O}_{E,(p)}$ by deformation theory and integral p-adic Hodge theory. The existence of canonical integral models is completely open beyond the abelian-type case.

3. THE LANGLANDS–RAPOPORT CONJECTURE

Given an unramified Shimura datum, the Langlands–Rapoport (LR) conjecture consists of two parts: (i) the existence of canonical integral models and (ii) a group-theoretic description of $\overline{\mathbb{F}}_p$-points of such integral models. We already addressed (i) in Section 2, which is a prerequisite for discussing (ii) in this section. There is an instructive analogy between (ii) and a description of $\mathbb{C}$-points [47, §16]. See Section 6.1 below for the case of bad reduction. We recommend the introduction of [31] for a more detailed survey of the content in this section.

3.1. Galois gerbs

Let k be a perfect field with an algebraic closure $\bar{k}$. A *Galois gerb* over k consists of a pair $(G, \mathfrak{G})$, where G is a connected linear algebraic group over $\bar{k}$, and $\mathfrak{G}$ is a topological group extension (with discrete topology on $G(\bar{k})$ and profinite topology on $\mathrm{Gal}(\bar{k}/k)$),

$$1 \to G(\bar{k}) \xrightarrow{i} \mathfrak{G} \xrightarrow{\pi} \mathrm{Gal}(\bar{k}/k) \to 1, \tag{3.1}$$

such that (i) for every $g \in \mathfrak{G}$, the conjugation by g on $G(\bar{k})$ is induced by a $\bar{k}$-group isomorphism $\pi(g)^* G \xrightarrow{\sim} G$, and (ii) there exists a finite extension K/k in $\bar{k}$ such that π admits a continuous section over $\mathrm{Gal}(\bar{k}/K)$. If G is a torus, then (ii) determines a model of G over k. We often refer to $(G, \mathfrak{G})$ as $\mathfrak{G}$ and write $\mathfrak{G}^{\Delta}$ for G.

There is a natural notion of morphisms between Galois gerbs over k. Passing to projective limits, we define pro-Galois gerbs $(G, \mathfrak{G})$ over k, which still fit in (3.1) but with G a pro-algebraic group over $\bar{k}$. When $\mathfrak{G}$ is a (pro-)Galois gerb over $\mathbb{Q}$, we can localize it at each place v of $\mathbb{Q}$ to obtain a (pro-)Galois gerb over $\mathbb{Q}_v$, to be denoted by $\mathfrak{G}(v)$.

The most basic example is the neutral Galois gerb $\mathfrak{G}_G$ which arises when G is already defined over k. By definition, $\mathfrak{G}_G := G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$ as a semidirect product with the natural action of $\mathrm{Gal}(\bar{k}/k)$ on $G(k)$.

We introduce a (pro-)Galois gerb $\mathfrak{G}_v$ over $\mathbb{Q}_v$ at each v. Take $\mathfrak{G}_\infty$ to be the real Weil group (in particular, $\mathfrak{G}_\infty^{\Delta} = \mathbb{G}_{m,\mathbb{C}}$); the definition of $\mathfrak{G}_p$ is involved but intended to encode isocrystals. For $v \neq p, \infty$, put $\mathfrak{G}_v := \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$, namely the trivial neutral Galois gerb.

Central to the LR conjecture is a quasimotivic pro-Galois gerb $\mathfrak{Q}$ over $\mathbb{Q}$ whose algebraic part $\mathfrak{Q}^{\Delta}$ is a pro-torus. The gerb $\mathfrak{Q}$ comes equipped with morphisms $\zeta_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$, and the datum $(\mathfrak{Q}, \{\zeta_v\})$ is uniquely characterized up to a suitable equivalence. A quasimotivic gerb (more precisely, its quotient called a pseudomotivic gerb) is devised as a substitute for the Galois gerb which should arise via Tannaka duality from the category of motives over $\overline{\mathbb{F}}_p$. The morphisms ζ_v should come from the fiber functors on the latter category coming from cohomology and polarization structures. See Langlands–Rapoport **[41, §§3–4]** (complemented by **[56, §8]**) and **[58, B2.7, B2.8]** for further information.

For each torus T over $\mathbb{Q}$ and each cocharacter $\mu : \mathbb{G}_m \to T$ (defined over a finite extension of $\mathbb{Q}$), there is a recipe **[29, (3.1.10)]** to define a morphism

$$\Psi_{T,\mu} : \mathfrak{Q} \to \mathfrak{G}_T. \tag{3.2}$$

As a special case, if (T, h) is a toral Shimura datum, then we obtain Ψ_{T,μ_h} with $\mu_h : \mathbb{G}_m \to T$ coming from h. In terms of the heuristics for $\mathfrak{Q}$, the construction of Ψ_{T,μ_h} mirrors the operation of taking the mod p fiber of a CM abelian variety in characteristic 0.

3.2. Versions of the LR conjecture

Let $(G, X, p, \mathcal{G})$ be an unramified Shimura datum. Write $\mathfrak{p}$ for the prime of E over p, determined by the field embeddings in Section 1, with residue field $k(\mathfrak{p})$. A canonical integral model $\mathscr{S}_{K_p}$ over $\mathcal{O}_{E_\mathfrak{p}}$ is available in the abelian-type case (Theorem 2.1) and conjectured to exist in general. For the moment, we assume (G, X) to be of Hodge type. Then we can take the partition

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p) = \coprod_{\mathscr{I}\in\mathbb{I}} S(\mathscr{I}) \tag{3.3}$$

according to the set $\mathbb{I}$ of isogeny classes, and then parametrize the set $S(\mathscr{I})$ consisting of points in each isogeny class $\mathscr{I}$ relative to a "base point" of choice in $\mathscr{I}$. This was obtained by Kisin **[29, §1.4]**, where a subtlety in the notion of p-power isogenies was handled by a result on the connected components of affine Deligne–Lusztig varieties **[9]**. Each $S(\mathscr{I})$ is $\Phi^{\mathbb{Z}} \times G(\mathbb{A}^{\infty,p})$-stable, where Φ acts as the geometric Frobenius over $k(\mathfrak{p})$, and

$$S(\mathscr{I}) \cong \varprojlim_{K^p\subset G(\mathbb{A}^{\infty,p})} I_{\mathscr{I}}(\mathbb{Q})\backslash\big(X_p(\mathscr{I}) \times X^p(\mathscr{I})/K^p\big), \tag{3.4}$$

as a right $\Phi^{\mathbb{Z}} \times G(\mathbb{A}^{\infty,p})$-set, where $X_p(\mathscr{I})$ and $X^p(\mathscr{I})$ account for p-power and prime-to-p isogenies (from a base point). The quotient by $I_{\mathscr{I}}(\mathbb{Q})$ takes care of redundant counting up to self-isogenies. Since (G, X) is of Hodge type, (3.4) simplifies as $I_{\mathscr{I}}(\mathbb{Q})\backslash(X_p(\mathscr{I}) \times X^p(\mathscr{I}))$.

We return to general unramified Shimura data. Following **[29, (3.3.6)]**, one defines *admissible morphisms* as morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ satisfying certain conditions to ensure that ϕ contributes to $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$. In analogy with isogeny classes above, ϕ gives rise to a right $G(\mathbb{A}^{\infty,p})$-torsor $X^p(\phi)$ and a nonempty affine Deligne–Lusztig variety $X_p(\phi)$ with a $\Phi^{\mathbb{Z}}$-action, where Φ is the Frobenius operator. Write I_ϕ for the $\mathbb{Q}$-group of automorphisms of ϕ. Then $I_\phi(\mathbb{A}^\infty)$ naturally acts on $X_p(\phi) \times X^p(\phi)$. (This is analogous to the self-isogeny of an abelian variety over $\overline{\mathbb{F}}_p$ acting on its étale cohomology away from p and crystalline cohomology at p.) Now let I_ϕ^{ad} denote the $\mathbb{Q}$-group of inner automorphisms of I_ϕ. Each $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}^\infty)$ can be used to twist the natural action of $I(\mathbb{Q})$ on $X_p(\phi) \times X^p(\phi)$:

$$I(\mathbb{Q}) \subset I\big(\mathbb{A}^\infty\big) \xrightarrow{\tau} I\big(\mathbb{A}^\infty\big) \circlearrowright X_p(\phi) \times X^p(\phi).$$

Taking the left quotient by this action (denoted $\backslash_\tau$ below), we define a $\Phi^{\mathbb{Z}} \times G(\mathbb{A}^{\infty,p})$-set

$$S_\tau(\phi) := \varprojlim_{K^p\subset G(\mathbb{A}^{\infty,p})} I_\phi(\mathbb{Q})\backslash_\tau\big(X_p(\phi) \times X^p(\phi)/K^p\big). \tag{3.5}$$

We just write $S(\phi)$ if τ is trivial. The isomorphism class of $S_\tau(\phi)$ depends only on $[\tau] \in \mathcal{H}(\phi) := I_\phi^{\mathrm{ad}}(\mathbb{Q})\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}^\infty)/I_\phi(\mathbb{A}^\infty)$ represented by τ. (The right quotient is taken with respect to the multiplication through the natural map $I_\phi \to I_\phi^{\mathrm{ad}}$.) If ϕ, ϕ' are $G(\overline{\mathbb{Q}})$-conjugate, then $S_\tau(\phi) \cong S_\tau(\phi')$ and canonically $\mathcal{H}(\phi) \cong \mathcal{H}(\phi')$. Denoting by $\mathbb{J}$ the set of $G(\overline{\mathbb{Q}})$-conjugacy classes of admissible morphisms, we write $\mathcal{H}(\mathcal{J})$ and $S_\tau(\mathcal{J})$ respectively for $\mathcal{H}(\phi)$ and $S_\tau(\phi)$, when $\mathcal{J}$ is the $G(\overline{\mathbb{Q}})$-conjugacy class of ϕ.

It is convenient to name a "rationality" condition on the adelic element $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}^\infty)$ that is technical but useful. For each maximal torus T of I_ϕ over $\mathbb{Q}$, we have the maps

$$I_\phi^{\mathrm{ad}}(\mathbb{A}^\infty) \xrightarrow{\partial} H^1(\mathbb{A}^\infty, Z_{I_\phi}) \to H^1(\mathbb{A}^\infty, T),$$

where ∂ is the connecting homomorphism, and the second map is induced by $Z_{I_\phi} \subset T$. We say that τ is *tori-rational* if the image of τ in $H^1(\mathbb{A}^\infty, T)$ lies in the subset of the image of $H^1(\mathbb{Q}, T) \to H^1(\mathbb{A}^\infty, T)$ which maps trivially into the abelianized cohomology of G, for every T. This condition depends only on $[\tau] \in \mathcal{H}(\phi)$.

We are ready to state versions of the LR conjecture in increasing order of strength. (To be precise, the conjecture requires extra compatibility conditions on $\tau(\mathcal{J})$ under cohomological twistings of $\mathcal{J}$ in (LR_1) and (LR_0), but we avoid mentioning them explicitly in this exposition. See **[31, §§2.6–2.7]**, where these conditions correspond to $\underline{\tau} \in \Gamma(\mathcal{H})_1$ and $\underline{\tau} \in \Gamma(\mathcal{H})_0$, respectively. With this correction, the Langlands–Rapoport-τ conjecture therein is exactly (LR_0) below.)

Conjecture 3.1. *The following assertions hold true:*

(LR_1) *There exists a $\Phi^{\mathbb{Z}} \times G(\mathbb{A}^{\infty,p})$-equivariant bijection*

$$\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p) \cong \coprod_{\mathcal{J} \in \mathbb{J}} S_{\tau(\mathcal{J})}(\mathcal{J}),$$

for some family of elements $\{\tau(\mathcal{J}) \in \mathcal{H}(\mathcal{J})\}_{\mathcal{J} \in \mathbb{J}}$.

(LR_0) *The conclusion of* (LR_1) *holds with $\tau(\mathcal{J})$ tori-rational for every $\mathcal{J} \in \mathbb{J}$.*

(LR) *The conclusion of* (LR_1) *holds with $\tau(\mathcal{J})$ trivial for every $\mathcal{J} \in \mathbb{J}$.*

Statement (LR) is nothing but the original LR conjecture. In the Hodge-type case (to which the abelian-type case can be reduced in practice), a natural approach in view of (3.3) is to establish a bijection $\mathcal{I} \in \mathbb{I} \leftrightarrow \mathcal{J} \in \mathbb{J}$ such that there exists a $\Phi^{\mathbb{Z}} \times G(\mathbb{A}^{\infty,p})$-equivariant bijection $S(\mathcal{I}) \cong S_{\tau(\mathcal{J})}(\mathcal{J})$ with constraints on $\tau(\mathcal{J})$ as in the conjecture.

It is known (**[31, §3]**, cf. **[42, 46]**) that (LR) implies (3.7) below, which is the gateway to applications, but (LR) remains to be completely open even for Siegel modular varieties (of genus ≥ 2). Milne proved that the original LR conjecture follows from the Hodge conjecture for CM abelian varieties (see **[48, p. 4]** and the references therein), but the latter conjecture is also wide open.

On a positive note, Kisin **[29]** made a major breakthrough to prove (LR_1) for all unramified Shimura data $(G, X, p, \mathcal{G})$ of abelian type. Unfortunately, (LR_1) by itself is not strong enough for the next steps. This motivated us to formulate and prove the strengthening

(LR_0) in [31], which suffices for the trace formula (Section 3.3) and applications (Section 4) below.

Theorem 3.2. *For every unramified Shimura datum* $(G, X, p, \mathcal{G})$ *of abelian type, Conjecture* (LR_0) *holds true.*

Let us sketch some ideas of proof when (G, X) is of Hodge type. The reduction to this case is nontrivial and convoluted, cf. [31, §6]. Already in [29], Kisin proved a refinement of (LR_1) in order to propagate (LR_1) through Deligne's formalism of connected Shimura varieties. With that said, we focus on the Hodge-type setting for simplicity.

The proof consists of two parts: (i) constructing a bijection $\mathscr{I} \in \mathbb{I} \leftrightarrow \mathscr{J} \in \mathbb{J}$ and (ii) showing that $S(\mathscr{I}) \cong S_{\tau(\mathscr{J})}(\mathscr{J})$ with some control over $\tau(\mathscr{J})$. A crucial idea is to use special point data, namely toral Shimura data (T, h_T) with embeddings into (G, X), to probe both sides of the bijection. Such data can be mapped into $\mathbb{I}$ by taking mod p of the corresponding special points on Sh_{K_p}, and to $\mathbb{J}$ by composing (3.2) with the induced embedding $\mathfrak{G}_T \hookrightarrow \mathfrak{G}_G$. The map to $\mathbb{I}$ is onto by Kisin [29], generalizing Honda's result on CM lifting of an abelian variety over $\overline{\mathbb{F}}_p$ up to isogeny. The surjectivity onto $\mathbb{J}$ is due to Langlands–Rapoport [41]:

$$\begin{array}{ccccc} \{(T,h_T) \hookrightarrow (G,X)\} & \xrightarrow{\text{Kisin}} \!\!\!\!\rightarrow & \mathbb{I} = \{\text{isog. classes}\} & \longrightarrow & \{(\gamma_0,\gamma,\delta)\}/\sim \\ \text{special points data} & & \Big\updownarrow & & \text{Kottwitz triples} \\ & \searrow^{\text{Langlands–Rapoport}} & \mathbb{J} = \{\text{conj. classes}\} & \nearrow & \end{array} \qquad (3.6)$$

From each of $\mathbb{I}$ and $\mathbb{J}$, Kisin [29] constructed Kottwitz triples consisting of certain conjugacy classes on G up to an equivalence, and showed that the outer diagram above commutes. This determines a bijection $\mathbb{I} \cong \mathbb{J}$ up to a finite ambiguity since the maps to Kottwitz triples have finite fibers. However, $S(\mathscr{I})$ and $S(\mathscr{J})$ need not be isomorphic through this bijection, since Kottwitz triples forget part of their structures. The possible deviation is recorded by $\tau(\mathscr{J})$, which is a priori under little control. This is still enough for deducing (LR_1).

To prove (LR_0), various refinements and improvements are made on both (i) and (ii) of the argument. Since Shimura varieties from toral Shimura data have canonical base points, a special point datum not only determines $\mathscr{I} \in \mathbb{I}$ but also a distinguished point on $S(\mathscr{I})$. Similarly, we have a base point on $S(\mathscr{J})$ as well if $\mathscr{J}$ comes from the same special point datum. The two points on $S(\mathscr{I})$ and $S(\mathscr{J})$ are difficult to relate, but they are shown to be compatible on the level of the maximal abelian quotient G^{ab}, based on integral p-adic Hodge theory of crystalline lattices in G-valued Galois representations, among other things. (A relevant technical issue is that the $\mathbb{Q}_p$-embedding $T \hookrightarrow G$ does not extend to a $\mathbb{Z}_p$-map from the Néron model of T to $\mathcal{G}$ over $\mathbb{Z}_p$, but this is fine if $G, \mathcal{G}$ are replaced with $G^{\mathrm{ab}}, \mathcal{G}^{\mathrm{ab}}$.) Further arguments (sketched in [31, §0.5]) amplify this compatibility to (LR_0).

3.3. From the LR conjecture to a stabilized trace formula

Here we return to a general unramified Shimura datum $(G, X, p, \mathcal{G})$ (possibly not of abelian type). Set $r \in \mathbb{Z}$ to be the inertia degree of $\mathfrak{p}$ over p. As indicated above, (LR_0)

is designed as a substitute for (LR) to imply the following formula predicted by **[32, 41]**. The implication is shown in **[31, §3]** (refer to the latter for undefined notation):

$$\mathrm{tr}\big(f^{\infty,p} \times \Phi_{\mathfrak{p}}^{j} \mid \big[H_c(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_l)\big]\big) = \sum_{\mathfrak{c}} c(\mathfrak{c}) O_{\gamma(\mathfrak{c})}(f^{\infty,p}) TO_{\delta(\mathfrak{c})}(\phi^{(j)}), \quad j \gg 1. \tag{3.7}$$

Here $\phi^{(j)}$ is an explicit function in the unramified Hecke algebra of $G(\mathbb{Q}_{p^{jr}})$ (with respect to $\mathcal{G}(\mathbb{Z}_{p^{jr}})$), and the sum runs over certain group-theoretic data $\mathfrak{c}$ (called Kottwitz parameters) fibered over the set of stable conjugacy classes in $G(\mathbb{Q})$ which are elliptic in $G(\mathbb{R})$. Here $\mathfrak{c}$ determines an explicit constant $c(\mathfrak{c}) \in \mathbb{Q}$, $\gamma(\mathfrak{c}) \in G(\mathbb{A}^{\infty,p})$ up to conjugacy, and $\delta(\mathfrak{c}) \in G(\mathbb{Q}_{p^{jr}})$ up to σ-conjugacy. In particular, the orbital integral $O_{\gamma(\mathfrak{c})}(f^{\infty,p})$ on $G(\mathbb{A}^{\infty,p})$ and the σ-twisted orbital integral $TO_{\delta(\mathfrak{c})}(\phi^{(j)})$ on $G(\mathbb{Q}_{p^{jr}})$ are well defined. Stabilizing the right-hand side, we arrive at the following, which is a rough version of **[31, THM. 3 AND 4]**.

Theorem 3.3. *Assume that* (LR$_0$) *is true. For every $f^{\infty,p} \in \mathcal{H}(G(\mathbb{A}^{\infty,p}))$, there exists a constant j_0 such that for every $j \geq j_0$, a formula of the following form holds:*

$$\mathrm{tr}\big(f^{\infty,p} \times \Phi^{j} \mid \big[H_c(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_l)\big]\big) = \sum_{e \in \mathcal{E}_{\mathrm{ell}}(G)} \mathrm{ST}_{\mathrm{ell}}^{\mathfrak{e}}\big(f^{e,\infty,p} f_p^{e,(j)} f_\infty^{\mathfrak{e}}\big), \tag{3.8}$$

where $\mathcal{E}_{\mathrm{ell}}(G)$ is the set of elliptic endoscopic data for G up to isomorphism, and $\mathrm{ST}_{\mathrm{ell}}^{\mathfrak{e}}$ is the stable elliptic distribution associated with the endoscopic datum $\mathfrak{e}$.

In light of Theorem 3.2, the conclusion of the theorem is unconditionally true for (G, X) of abelian type. We can easily allow a nonconstant coefficient as done in **[31]**.

The proof of (3.7) from (LR$_0$) is mostly close to the deduction from (LR) (cf. **[46]**), and starts from the fixed-point formula for (improper) varieties over finite fields due to Fujiwara and Varshavsky **[13,70]**; this explains the condition on j. To compute the cohomology of the generic fiber via that of the special fiber, we apply Lan–Stroh's result **[39]**. Tori-rationality in (LR$_0$) is the main point to ensure that the fixed-point counting is not affected by the presence of $\tau(\mathcal{J})$ even if $\tau(\mathcal{J})$ is nontrivial. The stabilization from (3.7) to Theorem 3.3 follows the argument in **[32]** with small improvements to work without technical hypotheses. We note that $f^{e,\infty,p}$ is the Langlands–Shelstad transfer of $f^{\infty,p}$ whereas $f_p^{e,(j)}$ and $f_\infty^{\mathfrak{e}}$ are constructed differently. (See **[31, §8.2]**.) As usual in endoscopy, auxiliary z-extensions are chosen if the derived subgroup of G is not simply connected, and the right-hand side of (3.8) should be interpreted appropriately.

Remark 3.4. Sometimes it is possible to obtain (3.8) bypassing any version of the LR conjecture. When (G, X) is of PEL type A or C, this is done by Kottwitz **[34]**; for Hodge-type data, this is worked out by Lee **[42]**. It is unclear how their methods interact with connected components of Shimura varieties, so their results do not easily extend to the abelian-type setup. In contrast, the formalism of the LR conjecture is well suited to such extensions.

Remark 3.5. If the adjoint quotient G/Z_G is isotropic over $\mathbb{Q}$ or, equivalently, if Sh is not proper over E (at each fixed level), it is desirable to prove the analogue of (3.8) for the intersection cohomology of the Satake–Baily–Borel compactification; see **[51, §§4–5]** for what new problems need to be solved. This has been carried out for certain unitary and orthogonal Shimura varieties, as well as Siegel modular varieties, in [50,52,73].

4. APPLICATIONS

4.1. The Hasse–Weil zeta functions and ℓ-adic cohomology

As pioneered by Eichler, Shimura, Deligne, Kuga, Sato, and Ihara, a central problem on Shimura varieties is to compute their ζ-functions and ℓ-adic cohomology. The goals are (i) to express the ζ-function as a quotient of products of automorphic L-functions (thereby deduce a meromorphic continuation and a functional equation when the L-functions are sufficiently understood), cf. [6, CONJ. 5.2], and (ii) to decompose the ℓ-adic cohomology according to automorphic representations and identify the Galois action on each piece. To this end, Langlands and Kottwitz developed a robust method in a series of papers in the 1970–1980s (from [40] to [32]). At the heart is a comparison between the Arthur–Selberg trace formula and a conjectural trace formula for the Hecke–Frobenius action on the cohomology at good primes p, where the latter should come from a fixed-point formula for the special fiber of Shimura varieties modulo p.

When G/Z_G is anisotropic over $\mathbb{Q}$ (equivalently, when Sh is an inverse limit of projective varieties), Theorem 3.3 should be sufficient for the goals (i) and (ii) (up to semisimplifying the Galois action), by following the outline in [32, §§8–10]. We say "should" for two reasons. Firstly, we do not have enough knowledge about automorphic representations in general (e.g., endoscopic classification, cf. [32, §8]). Thus complete details have not been worked out apart from low-rank examples, some special cases such as [33], or under simplifying hypotheses. Secondly, we typically need a positive answer to the following problem to proceed.[2] The reason is that $\mathrm{ST}^{\mathfrak{e}}$ should admit a relatively clean spectral expansion in terms of the discrete automorphic spectrum of endoscopic groups for G, but the spectral interpretation of $\mathrm{ST}^{\mathfrak{e}}_{\mathrm{ell}}$ is expected to be quite complicated in general.

Problem 4.1. Assume that G/Z_G is $\mathbb{Q}$-anisotropic. In (3.8), prove that

$$\mathrm{ST}^{\mathfrak{e}}_{\mathrm{ell}}\big(f^{\mathfrak{e},\infty,p} f_p^{\mathfrak{e},(j)} f_\infty^{\mathfrak{e}}\big) = \mathrm{ST}^{\mathfrak{e}}\big(f^{\mathfrak{e},\infty,p} f_p^{\mathfrak{e},(j)} f_\infty^{\mathfrak{e}}\big), \quad \forall \mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G),$$

where $\mathrm{ST}^{\mathfrak{e}}$ stands for the stable distribution as defined in [52, §5.4].

Although this problem is open, there are quite a few examples where it is known either by the nature of G or under a simplifying hypothesis on the test function. This provides a starting point for the Langlands correspondence (Section 4.2 below).

Now we remove the assumption on G/Z_G. In fact, the argument outlined in [32, §§8–10] is given in this generality, conditional on an affirmative answer to the following.

2 A shortcut getting around Problem 4.1 is possible when G has "no endoscopy," e.g., if G is a form of GL_2 or a certain unitary similitude group as in [33].

Problem 4.2. Prove a formula of the form

$$\operatorname{tr}\big(f^{\infty,p} \times \Phi^j \mid [\mathrm{IH}(\overline{\mathrm{Sh}}, \overline{\mathbb{Q}}_l)]\big) = \sum_{e \in \mathcal{E}_{\mathrm{ell}}(G)} \mathrm{ST}^e\big(f^{e,\infty,p} f_p^{e,(j)} f_\infty^e\big), \tag{4.1}$$

where $\mathrm{IH}(\overline{\mathrm{Sh}}, \overline{\mathbb{Q}}_l)$ is the intersection cohomology of the Satake–Baily–Borel compactification of Sh (see [51, 3.4–3.5], for instance).

To obtain (4.1) from Theorem 3.3, one has to match the nonelliptic terms in ST^e with the contribution to $[\mathrm{IH}(\overline{\mathrm{Sh}}, \overline{\mathbb{Q}}_l)]$ from the boundaries. As a special case, if G/Z_G is $\mathbb{Q}$-anisotropic, then $\mathrm{IH}^i(\overline{\mathrm{Sh}}, \overline{\mathbb{Q}}_l) = H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_l)$ for each $i \geq 0$, and the identity of Problem 4.1 should hold since there are no boundaries. In this sense, Problem 4.2 generalizes Problem 4.1. Problem 4.2 has been solved for Siegel modular varieties and certain unitary/orthogonal Shimura varieties by Morel and Zhu [50,52,73].

4.2. The global Langlands correspondence

The computation of ℓ-adic cohomology in Section 4.1 often leads to new instances of the global Langlands correspondence satisfying a local–global compatibility in the direction from automorphic representations to Galois representations, roughly stated as follows. Refer to Buzzard–Gee [7] for the definition of L-algebraicity and a full discussion of the conjecture, including a variant conjecture for C-algebraic representations.

Conjecture 4.3. *Let F be a number field, and $\pi = \otimes'_v \pi_v$ an L-algebraic cuspidal automorphic representation of $G(\mathbb{A}_F)$. Then for each prime ℓ and each isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there exists a continuous representation $\rho_{\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \to {}^L G(\overline{\mathbb{Q}}_l)$ such that the restriction of $\rho_{\ell,\iota}$ to $\mathrm{Gal}(\overline{F}_v/F_v)$ is isomorphic to the unramified Langlands parameter of π_v at almost all finite places v of F (where π_v is unramified).*

The relevance of Shimura varieties to the conjecture is as follows. A Shimura datum (G, X) determines a representation $r_X : {}^L G \to \mathrm{GL}(V)$ (up to isomorphism). Then one expects that the Galois representation $r_X \circ \rho_{\ell,\iota}$ is realized in the ℓ-adic cohomology of the associated Shimura varieties (more precisely, the π^∞-isotypic part thereof), with several caveats including normalization issues (e.g., C-algebraic vs L-algebraic), Arthur packets, and endoscopic problems. These caveats often present much difficulty, and even if they are ignored, it is generally a subtle group-theoretic problem to recover $\rho_{\ell,\iota}$ from $r \circ \rho_{\ell,\iota}$ for a set of representations r of ${}^L G$. (Over global function fields, V. Lafforgue [38] solved the analogous problem in a revolutionary way via generalized pseudocharacters.)

The most fundamental case of Conjecture 4.3 is when $G = \mathrm{GL}_n$. When F is a totally real or CM field and π satisfies a suitable self-duality condition, then the conjecture is proven in a series of papers making use of PEL-type Shimura varieties arising from a unitary similitude group by Clozel, Kottwitz, and others. (See [67] for a discussion and references.) The duality condition allows π to "descend" to an automorphic representation on the unitary similitude group as first observed by Clozel. The self-duality condition was later removed independently by Harris–Lan–Taylor–Thorne and Scholze [20,61], by exquisite p-adic congruences which are beyond the scope of this article.

The above results for GL_n imply new cases of Conjecture 4.3 (or a weaker form) for quasisplit unitary, symplectic, or special orthogonal groups G over a totally real or CM field via twisted endoscopy by Arthur and Mok [1,49]. However, the conjecture for symplectic or orthogonal *similitude* groups does not follow easily. To get a feel for the difference, note that the dual groups of $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$ are $\mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}$, which are embeddable in $\mathrm{GL}_{2n+1}, \mathrm{GL}_{2n}$. In contrast, the dual groups of GSp_{2n} and GSO_{2n} are GSpin_{2n+1} and GSpin_{2n}, whose faithful representations have dimensions at least 2^n (achieved by the spin representation). In [35,37], Conjecture 4.3 is verified for GSp_{2n} and a (possibly outer) form of GSO_{2n} over a totally real field F under a simplifying hypothesis on π. The basic input comes from Shimura varieties of abelian type associated with a form of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_{2n}$, resp. $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSO}_{2n}$, where r_X is essentially the spin representation, resp. a half-spin representation. Both problems in Section 4.1 have positive answers in that setup.

5. SHIMURA VARIETIES WITH BAD REDUCTION, PART I

Let (G, X) be a Shimura datum. At each prime p such that (G, X) can be promoted to an unramified Shimura datum $(G, X, p, \mathcal{G})$, we discuss three methods to study the cohomology $H_c(\mathrm{Sh}, \overline{\mathbb{Q}}_l)$ as a $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}_\mathfrak{p}/E_\mathfrak{p})$-module at each prime $\mathfrak{p}$ of E over p.[3] The "bad reduction" in the section title means that the level subgroup $K'_p \subset K_p = \mathcal{G}(\mathbb{Z}_p)$ at p is allowed to be arbitrarily small, in which case integral models typically have bad reduction mod p. The complicated geometry may be understood better through stratifications.

There are several stratifications of interest on Shimura varieties (cf. [22]) but the most relevant to us is the Newton stratification. In the Hodge-type case, this yields a partition of the mod p Shimura variety into finitely many locally closed subsets, which can be equipped with the reduced subscheme structure, according to the isogeny class of p-divisible groups with additional structure. The unique closed stratum is called the *basic* stratum and corresponds to the p-divisible group that is "most supersingular" under the given constraint.

The first method is a p-adic uniformization of Shimura varieties as pioneered by Čerednik and Drinfeld, and further developed by Rapoport–Zink, Fargues, Kim, and Howard–Pappas [11,23,26,57]. Let $\mathrm{Sh}^{\mathrm{basic}}_{K^pK'_p}$ denote the basic locus in the rigid analytification of $\mathrm{Sh}_{K^pK'_p}$ over $E_\mathfrak{p}$, defined to be the preimage of the basic stratum under the specialization map towards the special fiber. The fundamental result asserts that $\mathrm{Sh}^{\mathrm{basic}}_{K^pK'_p}$ is uniformized by the Rapoport–Zink space with level K'_p arising from the corresponding basic isogeny class. A prominent application is to prove new cases of the Kottwitz conjecture on the cohomology of basic Rapoport–Zink spaces and their generalizations [11,19,25]. Hansen's work [19] points to a synergy between the global method here and Fargues–Scholze's purely local geometric construction of the local Langlands correspondence [12].

3 Sometimes these methods extend beyond the stated hypotheses. For example, [18] also works for Kisin–Pappas models (Section 6), and [26] proves a uniformization result also in the nonbasic case. However, we will not try to present the methods in their maximally general settings.

Next we discuss the Harris–Taylor method [21, CHAPS. IV–V] based on a product structure, namely coverings of Newton strata by the products of Igusa varieties and Rapoport–Zink spaces. The outcome is known as Mantovan's formula [44] (generalizing [21, CHAP. IV]), which expresses the cohomology of Newton strata in terms of that of Igusa varieties and Rapoport–Zink spaces. In the basic case, this is closely related to the p-adic uniformization. Hamacher–Kim [18] extended Mantovan's formula and the product structure to Hodge-type Shimura varieties.

To go further, it is desirable to understand the cohomology of Igusa varieties – we address this problem in Section 7 below. Granting this, and putting different Newton strata together, we have a formula relating $[H_c(\mathrm{Sh}, \overline{\mathbb{Q}}_l)]$ to the cohomology of Rapoport–Zink spaces. Then our knowledge about $[H_c(\mathrm{Sh}, \overline{\mathbb{Q}}_l)]$ tells us something nontrivial about the cohomology of Rapoport–Zink spaces, and vice versa. This observation turned out to be useful for proving local–global compatibility, i.e., identifying the local Galois action for the Galois representations in Conjecture 4.3 at ramified primes (see [21, CHAP. VII], [65]) and also for understanding the cohomology of basic/nonbasic Rapoport–Zink spaces [2–4, 66].

Last but not least, there is Scholze's extension of the Langlands–Kottwitz approach from the hyperspecial level at p to arbitrarily small level subgroup at p. (See Section 6.2 below for another generalization.) One seeks for the following analogue of (3.7), where $\tau \in W_{E_{\mathfrak{p}}}$ is a Weil group element with positive valuation, $h \in \mathcal{H}(G(\mathbb{Q}_p))$ has support contained in $\mathcal{G}(\mathbb{Z}_p)$, and the sum is over the same set of c:

$$\operatorname{tr}\big(f^{\infty,p} \times h \times \tau \;\big|\; \big[H_c(\mathrm{Sh}, \overline{\mathbb{Q}}_l)\big]\big) = \sum_{\mathrm{c}} c(\mathrm{c}) O_{\gamma(\mathrm{c})}(f^{\infty,p}) TO_{\delta(\mathrm{c})}(\phi_{\tau,h}). \tag{5.1}$$

This has been verified by Scholze [59] for PEL-type data and by Youcis [71] for abelian-type data. As an application, Scholze gave a new proof and characterization of the local Langlands correspondence for GL_n over p-adic fields [60] via a base-change transfer of $\phi_{\tau,h}$. A generalization of the latter to other groups was conjectured in [62] and partially proved for unitary groups by Bertoloni Meli and Youcis [5].

In the proof of (5.1), one can push-forward from arbitrarily small level K'_p down to hyperspecial level K_p at the expense of complicating the coefficient sheaf. Applying the fixed-point formula to this, one can exploit knowledge of the fixed-points (coming from results on the LR conjecture). The main problem is to identify the local terms, which are shown to be encoded by a locally constant compactly supported function $\phi_{\tau,h}$ at p constructed from deformation spaces of p-divisible groups with additional structures.

6. SHIMURA VARIETIES WITH BAD REDUCTION, PART II

Let (G, X) be a Shimura datum, and p a prime. In this section, we survey generalizations of Sections 2–5 in the setup where $G_{\mathbb{Q}_p}$ is allowed to be ramified (thus there may be no unramified Shimura datum of the form $(G, X, p, \mathcal{G})$). We recommend the articles [14, 53, 56] for introductions to the contents of this section.

6.1. The LR conjecture in the parahoric case

From now until the end of Section 6, assume that (G, X) is of abelian type. We fix $K_p \subset G(\mathbb{Q}_p)$ a parahoric subgroup, and $\mathfrak{p}$ a place of E over p. In this setting, Kisin–Pappas **[30]** constructed an integral model $\mathscr{S}_{K_p}$ over $\mathcal{O}_{E_\mathfrak{p}}$ under a mild hypothesis, which are canonical in the sense of **[54]**.

With the integral model as above, we can state versions of the Langlands–Rapoport conjecture analogous to Conjecture 3.1, cf. **[56, §9]**.[4] One can extend the notion of isogeny classes on $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ and admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ to the parahoric setup, following **[30]** and **[56]**, respectively. Thus we can consider the set $\mathbb{I}$ of isogeny classes and the set $\mathbb{J}$ of conjugacy classes of admissible morphisms. The set $S(\mathscr{I})$ of $\overline{\mathbb{F}}_p$-points in each isogeny class $\mathscr{I} \in \mathbb{I}$ is still described by (3.4), with $X_p(\mathscr{I})$ a suitable affine Deligne–Lusztig variety at the parahoric level K_p. Analogously we define $S_\tau(\phi)$ and $S_\tau(\mathscr{J})$ for each admissible ϕ and $\mathscr{J} \in \mathbb{J}$, with $X_p(\phi)$ in (3.5) also adapted to the parahoric level K_p.

Conjecture 6.1. *With the above definitions, there exists a $G(\mathbb{A}^{\infty,p}) \times \Phi^{\mathbb{Z}}$-equivariant bijection $\mathscr{S}_{K_p}(\overline{\mathbb{F}}_p) \cong \coprod_{\mathscr{J}\in\mathbb{J}} S_{\tau(\mathscr{J})}(\mathscr{J})$ such that the exact analogue of* (LR$_1$)*, resp.* (LR$_0$) *and* (LR)*, holds true.*

Van Hoften and Zhou **[68,72]** proved the following theorem.

Theorem 6.2. *Statement* (LR$_1$) *of Conjecture 6.1 is true if $G_{\mathbb{Q}_p}$ is quasisplit, under a mild technical hypothesis.*

The stronger statement (LR$_0$) is expected to be within reach under the same hypothesis, by extending the argument from **[31]** to the parahoric setting.

To prove Theorem 6.2, the essential case is when (G, X) is of Hodge type. Zhou proves the very special parahoric case of the conjecture. Van Hoften deduces the case of general parahoric subgroup K'_p, contained in a very special parahoric K_p, by studying the localization maps from Shimura varieties of level K'_p and K_p, to their respective moduli spaces of local Shtukas. The maps are roughly given by assigning to each abelian variety the associated p-divisible group in terms of the moduli problems. Via the forgetful maps from level K'_p down to level K_p, one can form a commutative square diagram. The central claim is that the diagram is Cartesian, from which the LR conjecture at level K'_p can be deduced from the known case at level K_p. The proof of the claim eventually rests on understanding the irreducible components of Kottwitz–Rapoport strata in the situation of the diagram.

6.2. Semisimple zeta functions and Haines–Kottwitz's test function conjecture

At primes of bad reduction, it is useful to compute the semisimple local zeta factor at p instead of the (true) local factor of the Hasse–Weil zeta function (of Shimura varieties) as the former is more amenable to computation. The latter can be recovered from the former in the cases where the weight-monodromy conjecture is known **[55]**.

4 One can remove the assumption in **[56]** that G^{der} is simply connected, by adopting Kisin's formulation in **[29]** via strictly monoidal categories.

Just like the local zeta factor at p can be computed in terms of the trace of powers of Frobenius on the Frobenius-invariant subspace of the cohomology with compact support, the semisimple local factor can be described in terms of the trace of powers of Frobenius on the derived Frobenius-invariants; such a trace is called the semisimple trace and will be denoted by $\mathrm{tr}^{\mathrm{ss}}$ (see **[55, §2]**, **[15, §3.1]**). Thus a key is to establish the following generalization of (3.7), which recovers (3.7) if K_p is hyperspecial, due to Haines and Kottwitz **[14, §6.1]**. The summation is over the same set as in (3.7).

Conjecture 6.3. *Let $f^{\infty,p} \in \mathcal{H}(G(\mathbb{A}^{\infty,p}))$. For all sufficiently large integers $j \gg 1$, there exist test functions $\phi_{\mathrm{HK}}^{(j)} \in \mathcal{H}(G(\mathbb{Q}_{p^{jr}}))$ such that*

$$\mathrm{tr}^{\mathrm{ss}}\big(f^{\infty,p} \times \Phi_{\mathfrak{p}}^{j} \mid [H_c(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_l)]\big) = \sum_{\mathfrak{c}} c(\mathfrak{c}) O_{\gamma(\mathfrak{c})}(f^{\infty,p}) TO_{\delta(\mathfrak{c})}\big(\phi_{\mathrm{HK}}^{(j)}\big). \tag{6.1}$$

Moreover, $\phi_{\mathrm{HK}}^{(j)}$ may be given by an explicit recipe only in terms of local data at p.

When K_p is a parahoric subgroup of $G(\mathbb{Q}_p)$, one can be more concrete about $\phi_{\mathrm{HK}}^{(j)}$ following **[14, §7]**: $\phi_{\mathrm{HK}}^{(j)}$ should admit a geometric construction via nearby cycles on local models, as well as a representation-theoretic description in terms of the Langlands correspondence. That the two descriptions for $\phi_{\mathrm{HK}}^{(j)}$ coincide is the test function conjecture verified by Haines–Richarz **[16, 17]** under a very mild hypothesis. (See **[14, §8]** for prior and related results.) The proof is based on geometry of mixed-characteristic affine Grassmanians and the geometric Satake equivalence. It remains to combine their theorem with the results in Section 6.1 to obtain new cases of Conjecture 6.3 and its stabilized form, so as to determine the semisimple zeta factor at p. This requires an endoscopic understanding of $\phi_{\mathrm{HK}}^{(j)}$, cf. **[14, §6.2]**; a simple exemplary case is demonstrated in **[14, §6.3]**, where endoscopic problems disappear.

In a related but somewhat different direction (cf. the last two paragraphs in **[14, §8.4]**), the Langlands–Kottwitz–Scholze approach discussed in Section 5 should extend to the current setup despite the absence of hyperspecial subgroups at p, at least when the results of Section 6.1 are available for some parahoric subgroups.

Problem 6.4. Prove the analogue of (5.1) for general Shimura data (G, X) and primes p.

7. IGUSA VARIETIES

Igusa curves were introduced to understand the geometry of modular curves modulo p when the level is divisible by a prime p **[24]**. The construction has been generalized by Harris–Taylor **[21]** and Mantovan **[44]** in the PEL-type case, and most recently to the setup of Kisin–Pappas models for Hodge-type Shimura varieties by Hamacher–Kim **[18]**. Igusa varieties have a variety of applications to p-adic and mod p modular forms, cohomology of Shimura varieties, the Langlands correspondence, and some more. We refer to the introduction of **[36]** for further details and references. In this section, we concentrate on computing the ℓ-adic cohomology of Igusa varieties via an analogue of the LR conjecture.

7.1. The LR conjecture for Igusa varieties

Let $(G, X, p, \mathscr{G})$ be an unramified Shimura datum of Hodge type, with a fixed embedding of (G, X) into a Siegel Shimura datum. Put $K_p := \mathscr{G}(\mathbb{Z}_p)$, a hyperspecial subgroup of $G(\mathbb{Q}_p)$. Let $\mathcal{A}$ denote the abelian scheme over $\mathscr{S}_{K_p}$ pulled back from the universal abelian scheme over the ambient Siegel moduli scheme. Thus we have a p-divisible group $\mathcal{A}[p^\infty]$ over $\mathscr{S}_{K_p}$ with $G_{\mathbb{Q}_p}$-structure in some precise sense.

Now fix a p-divisible group Σ over $\overline{\mathbb{F}}_p$ with $G_{\mathbb{Q}_p}$-structure. By Dieudonné theory, this determines $b \in G(\breve{\mathbb{Q}}_p)$ (up to replacing b with $g^{-1} b_x \sigma(g)$ for $g \in \mathscr{G}(\breve{\mathbb{Z}}_p)$). Write $J_b(\mathbb{Q}_p)$ for the group of self-quasi-isogenies of Σ respecting the $G_{\mathbb{Q}_p}$-structure. As a $\mathbb{Q}_p$-algebraic group, J_b is known to be an inner form of a Levi subgroup of $G_{\mathbb{Q}_p}$. We define Ig_b to be the parameter space (in the category of perfect $\overline{\mathbb{F}}_p$-schemes) of $G_{\mathbb{Q}_p}$-structure-preserving isomorphisms between $\mathcal{A}[p^\infty]$ and (the constant family of) Σ over $\mathscr{S}_{K_p,\overline{\mathbb{F}}_p}$. The scheme Ig_b is nonempty if and only if the image of b in Kottwitz's set $B(G)$ lies in the finite subset $B(G, \mu_X^{-1})$, where μ_X is the Hodge cocharacter of G determined by (G, X) up to conjugacy. (The same set $B(G, \mu_X^{-1})$ labels the Newton strata, cf. Section 5.) There is a natural action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on Ig_b, thus also on its ℓ-adic cohomology.

We may and will replace Σ with an isogenous p-divisible group which is completely slope divisible, since the isomorphism class of Ig_b with the group action is invariant under such a replacement. The advantage of doing so is that Ig_b can be written as the projective limit of finite-type varieties (up to taking perfection, which does not affect cohomology) by trivializing only a finite p-power torsion subgroup and by fixing a level subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$ away from p at a time. With some care, the projective system of varieties can be defined over a common finite field. This enables us to apply the Fujiwara–Varshavsky fixed-point formula to compute the cohomology (with compact support) at each finite level, provided that we understand the structure of $\mathrm{Ig}_b(\overline{\mathbb{F}}_p)$. Thus we are prompted to think about the analogue of the LR conjecture for Igusa varieties.

In analogy with (3.3) and (3.4), keeping the same definition of $\mathbb{I}$ and $\mathbb{J}$, we have the partition $S^{\mathrm{Ig}_b}(\overline{\mathbb{F}}_p) = \coprod_{\mathscr{I} \in \mathbb{I}} S^{\mathrm{Ig}_b}(\mathscr{I})$ according to isogeny classes of abelian varieties, with

$$S^{\mathrm{Ig}_b}(\mathscr{I}) \cong I_{\mathscr{I}}(\mathbb{Q}) \backslash \big(X_p^{\mathrm{Ig}_b}(\mathscr{I}) \times X^p(\mathscr{I}) \big). \tag{7.1}$$

The $G(\mathbb{A}^{\infty,p})$-set $X^p(\mathscr{I})$ is the same as before, but the difference from Section 3 is that $X_p^{\mathrm{Ig}_b}(\mathscr{I})$ is no longer an affine Deligne–Lusztig variety but a right $J_b(\mathbb{Q}_p)$-torsor. Turning to the other side of the LR conjecture, let $\mathscr{J} \in \mathbb{J}$. It is natural to impose the so-called b-admissibility condition on $\mathscr{J}$ at p, which is the group-theoretic analogue of the condition that a p-divisible group is isogenous to Σ (with $G_{\mathbb{Q}_p}$-structure). For b-admissible $\mathscr{J}$, we set

$$S_\tau^{\mathrm{Ig}_b}(\mathscr{J}) := I_\phi(\mathbb{Q}) \backslash_\tau \big(X_p^{\mathrm{Ig}_b}(\mathscr{J}) \times X^p(\mathscr{J}) \big),$$

with the same $X^p(\mathscr{J})$ as in Section 3, and a suitably defined right $J_b(\mathbb{Q}_p)$-torsor $X_p^{\mathrm{Ig}_b}(\mathscr{J})$, where the τ-twisted quotient can be defined again as in Section 3. We are ready to state versions of the LR conjecture for Igusa varieties in parallel with Conjecture 3.1, for unramified Shimura data $(G, X, p, \mathscr{G})$ of Hodge type.

Conjecture 7.1. *There is a bijection of right $J_b(\mathbb{Q}_p)\times G(\mathbb{A}^{\infty,p})$-sets*

$$\mathrm{Ig}_b(\overline{\mathbb{F}}_p)\simeq \coprod_{\substack{\mathcal{J}\in\mathbb{J}\\ b\text{-adm.}}} S^{\mathrm{Ig}_b}_{\tau(\mathcal{J})}(\mathcal{J}),$$

where $\{\tau(\mathcal{J})\}$ over the set of b-admissible $\mathcal{J}$ satisfies the conditions in (LR$_1$)*, resp.* (LR$_0$) *and* (LR).

Mack-Crane proved the following theorem in his thesis **[43]**, where $\mathfrak{c}$, $c(\mathfrak{c})$, and $\gamma(\mathfrak{c})$ are the same as in Section 3, but we impose a b-admissibility condition on the Kottwitz parameter $\mathfrak{c}$ inherited from the similar condition on $\mathcal{J}$, and each $\mathfrak{c}$ gives rise to a conjugacy class of $\delta'(\mathfrak{c})$ in $J_b(\mathbb{Q}_p)$, along which we compute the (untwisted) orbital integral $O_{\delta'(\mathfrak{c})}(\phi'_p)$.

Theorem 7.2. *The* (LR$_0$)*-version of Conjecture 7.1 is true. Moreover, the following analogue of* (3.7) *holds true for $f^{\infty,p}\in\mathcal{H}(G(\mathbb{A}^{\infty,p}))$ and sufficiently many functions $\phi'_p\in\mathcal{H}(J_b(\mathbb{Q}_p))$:*

$$\mathrm{tr}\big(f^{\infty,p}\times\phi'_p \mid \big[H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_l)\big]\big)=\sum_{\mathfrak{c}:\,b\text{-adm.}} c(\mathfrak{c})O_{\gamma(\mathfrak{c})}(f^{\infty,p})O_{\delta'(\mathfrak{c})}(\phi'_p). \tag{7.2}$$

By "sufficiently many," we mean that the traces for such a set of functions are enough to determine $[H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_l)]$ in the Grothendieck group of $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$-representations. (The precise condition has to do with twisting by a high power of Frobenius in the Fujiwara–Varshavsky formula.) The proof of the theorem proceeds by carefully adapting the methods of **[29, 31]** but with significant changes occurring at p, thus often requiring different techniques and arguments.

Formula (7.2) was obtained for some simple PEL-type Shimura varieties in **[21, CHAP. 5]** and **[63]** without formulating and proving the LR conjecture. In contrast, the above theorem represents the first LR-style approach to Igusa varieties, giving it two advantages. Firstly, the new approach makes the similarities between Shimura and Igusa varieties transparent. An important consequence is that the hard-won statement (LR$_0$) for Shimura varieties can be transferred to the Igusa side. (If we had the full (LR) for Shimura varieties, then that would carry over to Igusa varieties, too.) Going from (LR$_0$) for Igusa varieties to (7.2) is mostly the same as for Shimura varieties. Secondly, just like for Shimura varieties, the LR-style approach makes it feasible to extend the theorem to the abelian-type case, cf. Remark 3.4. (This extension has not been worked out, yet.) It should also be possible to go beyond good reduction and work in the setup of Kisin–Pappas models (Section 6). For the sake of proposing a problem, we can be even more general but still stick to (LR$_0$) rather than (LR) as this should suffice for most applications:

Problem 7.3. Construct Igusa varieties modulo p for all Shimura data (G,X) and all primes p. Prove the (LR$_0$)-version of Conjecture 7.1, thereby deduce formula (7.2).

Assuming a positive answer (known in the setting of Theorem 7.2), the next step is to unconditionally stabilize (7.2) into the following form:

$$\mathrm{tr}\big(f^{\infty,p}\times\phi'_p \mid \big[H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_l)\big]\big)=\sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(G)} \mathrm{ST}^{\mathfrak{e}}_{\mathrm{ell}}\big(f^{\mathfrak{e},\infty,p}f^{\mathfrak{e},\prime}_p f^{\mathfrak{e}}_\infty\big). \tag{7.3}$$

The formula is an exact analogue of (3.8). Indeed, $f^{\mathfrak{e},\infty,p}$ and $f^{\mathfrak{e}}_{\infty}$ are constructed in the same way. However, $f^{\mathfrak{e},\prime}_{p}$ is constructed from ϕ'_p via a "nonstandard" transfer of functions – this is the main novelty in the stabilization. Even when $G_{\mathbb{Q}_p}$ is a product of general linear groups so that local endoscopy at p disappears, the transfer goes to $G_{\mathbb{Q}_p}$ from an inner form of a Levi subgroup of $G_{\mathbb{Q}_p}$. The transfer was constructed and studied in [64] in a somewhat ad hoc manner, and later streamlined in [2]. Unfortunately, both papers make a set of technical hypotheses, to be removed in the work in progress with Bertoloni Meli.

7.2. Applications

The stabilization (7.3) is a significant step towards the following:

Problem 7.4. Obtain a decomposition of $[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)]$ according to automorphic representations of G and its endoscopic groups, and describe each piece in the decomposition.

Just from the definition of Igusa varieties, it is not even clear whether the entirety of $[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)]$ can be understood through automorphic representations. For Shimura varieties over $\mathbb{C}$, the connection is made through Matsushima's formula and its generalizations, but there is no analogue for Igusa varieties.

We have a concrete answer for some simple PEL-type Shimura varieties arising from (G, X) such that (i) endoscopy for G disappears over $\mathbb{Q}$ and $\mathbb{Q}_p$, (ii) G is anisotropic modulo center over $\mathbb{Q}$, and (iii) $G_{\mathbb{Q}_p}$ is a product of general linear groups. Recall that J_b is an inner form of a $\mathbb{Q}_p$-rational Levi subgroup, say M_b, of $G_{\mathbb{Q}_p}$. In fact, b determines a particular parabolic subgroup P_b^{op} containing M_b as a Levi component. Write Red^b for the composite morphism on the Grothendieck group of representations

$$\mathrm{Red}^b : \mathrm{Groth}\big(G(\mathbb{Q}_p)\big) \to \mathrm{Groth}\big(M_b(\mathbb{Q}_p)\big) \to \mathrm{Groth}\big(J_b(\mathbb{Q}_p)\big), \tag{7.4}$$

where the first map is the Jacquet module relative to P_b^{op} (up to a character twist), and the second is Badulescu's Langlands–Jacquet map. The answer to Problem 7.4 in this setting is given by [21, THM. V.5.4] and [66, THM. 6.7]:

$$\big[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)\big] = \big[\mathrm{Red}^b H_c(\mathrm{Sh}, \overline{\mathbb{Q}}_l)\big] \quad \text{in } \mathrm{Groth}\big(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)\big). \tag{7.5}$$

Since $H_c(\mathrm{Sh}, \overline{\mathbb{Q}}_l)$ is well understood by Matsushima's formula via relative Lie algebra cohomology, (7.5) is indeed a satisfactory answer for $[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)]$. Through Mantovan's formula, (7.5) sheds light on the cohomology of Rapoport–Zink spaces [3, 66], cf. Section 5.

When $G_{\mathbb{Q}_p}$ is still a product of general linear groups but G exhibits endoscopy over $\mathbb{Q}$, the formula for $[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)]$ is no longer as simple as (7.5). Computing the formula in certain endoscopic cases was crucial in the proof of local–global compatibility in [65], cf. Section 5. (See [67, §6] for an expository account.)

In general, Problem 7.4 seems out of reach. Firstly, just like for Shimura varieties, the lack of endoscopic classification is a major obstacle. Secondly, a new difficulty in the Igusa setup is that the analogue of Problem 4.1 has no conceptual reason to have a positive answer (cf. last paragraph of Section 4.1), and the analogue of Problem 4.2 is even less clear. (The second point is related to the question at the end of this section.) Assuming that both issues go away, a conjectural formula for $[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)]$ has been given in **[2]** under some hypotheses on G, resembling Kottwitz's conjectural formula for $[\mathrm{IH}(\overline{\mathrm{Sh}}, \overline{\mathbb{Q}}_l)]$ in **[32, §10]**. The formula for $[H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)]$ is far more complicated than (7.5) and involves endoscopic versions of Red^b. (In the stable case, which is the simplest, the correct analogue of (7.4) is the Jacquet module followed by a stable transfer between inner forms.) A main observation in **[2]** is that the endoscopic versions of Red^b should interact with the cohomology of Rapoport–Zink spaces (and their generalizations) in an interesting way by a global reason.

For an unconditional result towards Problem 7.4, we managed to compute the $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module $H^0(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)$ for unramified Shimura data $(G, X, p, \mathcal{G})$ of Hodge type in joint work with Kret **[36]**, when b is nonbasic[5] (in every $\mathbb{Q}$-simple factor of the adjoint group of G). In analogy with (7.5), the result may be expressed as

$$H^0(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l) = \mathrm{Red}^b H^0(\mathrm{Sh}, \overline{\mathbb{Q}}_l) \quad \text{as} \quad G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)\text{-modules}. \tag{7.6}$$

Here $H^0(\mathrm{Sh}, \overline{\mathbb{Q}}_l)$ has a well-known description in terms of 1-dimensional automorphic representations of $G(\mathbb{A})$, and Red^b is unequivocally defined for 1-dimensional representations of $G(\mathbb{Q}_p)$ (which are always stable). The proof starts from the results of Section 7.1. The main point is to get around the two essential difficulties mentioned in the last paragraph, by incorporating asymptotic and inductive arguments to extract the H^0-part from a very complicated identity coming from (stabilized) trace formulas. The study of H^0 was motivated by geometric applications to the irreducibility of Igusa varieties and to the discrete Hecke orbit conjecture. The reader is referred to **[36]** for details and further references. Similar geometric results were independently obtained by van Hoften and Xiao **[68, 69]** via a more geometric approach without using automorphic forms or trace formulas.

We conclude this section with an unrefined question. Igusa varieties (at finite level) are almost never proper varieties, so the answer to Problem 7.4 does not determine $H^i_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_l)$ for $i \geq 0$ due to possible cancelations in the Grothendieck group. (For improper Shimura varieties, the intersection cohomology is free from such a cancelation thanks to purity.) Thus we can ask whether there are useful compactifications of Igusa varieties to help us understand the cohomology more precisely. This was undertaken by Mantovan **[45]** in a special case (with a different goal). In general, it is unclear how to proceed even when Shimura varieties are proper. If a strategy is found in that case, it may be possible to deal with improperness of Shimura varieties by virture of Caraiani–Scholze's partial compactification **[8]**.

ACKNOWLEDGMENTS

Special thanks are due to Michael Harris, Robert Kottwitz, and Richard Taylor for introducing the author to Shimura varieties and the Langlands program, and patiently answering his questions over many years. The author is grateful to Mark Kisin, Arno Kret, and Yihang Zhu for sharing their insight through years of recent and ongoing collaboration

5 If b is basic then Igusa varieties are 0-dimensional and the picture is quite different from (7.6), cf. **[36, §1.6]**.

that this paper is largely based on. He is also thankful for exciting mathematical journeys to the other coworkers: Sara Arias-de Reyna, Ana Caraiani, Luis Dieulefait, Matthew Emerton, Jessica Fintzen, Toby Gee, David Geraghty, Wushi Goldring, Julia Gordon, Junehyuk Jung, Tasho Kaletha, Julee Kim, Keerthi Madapusi Pera, Simon Marshall, Alberto Minguez, Vytautas Paškūnas, Peter Sarnak, Peter Scholze, Nicolas Templier, Paul-James White, and Gabor Wiese. Finally, he thanks Alex Youcis for comments to improve the exposition of the paper.

FUNDING

This work was partially supported by NSF grant DMS-1802039/2101688, NSF RTG grant DMS-1646385, and a Miller Professorship.

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