MOTIVIC COHOMOLOGY

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ABSTRACT

We give a survey of the development of motivic cohomology, motivic categories, and some of their recent descendants.

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1. INTRODUCTION

Motivic cohomology arose out of a marriage of Grothendieck's ideas about motives with a circle of conjectures about special values of zeta functions and L-functions. It has since taken on a very active life of its own, spawning a multitude of developments and applications. My intention in this survey is to present some of the history of motivic cohomology and the framework that supports it, its current state, and some thoughts about its future directions. I will say very little about the initial impetus given by the conjectures about zeta functions and L-functions, as there are many others who are much better qualified to tell that story. I will also say next to nothing about the many applications motivic cohomology has seen: I think this would be like writing about the applications of cohomology up to, say, 1950, and would certainly make this already lengthy survey completely unmanageable.

My basic premise is that motivic cohomology is supposed to be universal cohomology for algebro-geometric objects. As "universal" depends on the universe one happens to find oneself in, motivic cohomology is an ever-evolving construct. My plan is to give a path through some of the various universes that have given rise to motivic cohomologies, to describe the resulting motivic cohomologies and put them in a larger, usually categorical, framework. Our path will branch into several directions, reflecting the different aspects of algebraic and arithmetic geometry that have been touched by this theory. We begin with the conjectures of Beilinson and Lichtenbaum about motivic complexes that give rise to the universal Bloch–Ogus cohomology theory on smooth varieties over a field, and the candidate complexes constructed by Bloch and Suslin. We then take up Voevodsky's triangulated category of motives over a field and the embedding of the motivic complexes and motivic cohomology in this framework. The next developments moving further in this direction give us motivic homotopy categories that tell us about "generalized motivic cohomology" for a much wider class of schemes, analogous to the development of the stable homotopy category and generalized cohomology for spaces; this includes a number of candidate theories for motivic cohomology over a general base-scheme. We conclude with three variations on our theme:

- Milnor–Witt motives and Milnor–Witt motivic cohomology, incorporating information about quadratic forms,
- Motives with modulus, relaxing the usual condition of homotopy invariance with respect to the affine line, and
- p-adic, étale motivic cohomology in mixed characteristic $(0, p)$, with its connection to p-adic Hodge theory.

This last example does not yet, as far as I know, have a categorical framework, while one for a motivic cohomology with modulus is still in development.

There is already an extensive literature on the early development of motives and motivic cohomology. It was not my intention here to cover this part in detail, but I include a section on this topic to give a quick overview for the sake of background, and to isolate a few main ideas so the reader could see how they have influenced later developments.

I would like to thank all those who helped me prepare this survey, especially Tom Bachmann, Federico Binda, Dustin Clausen, Thomas Geisser, Wataru Kai, Akhil Mathew, Hiroyasu Miyazaki, Matthew Morrow, and Shuji Saito. In spite of their efforts, I feel certain that a number of errors have crept in, which are, of course, all my responsibility. I hope that the reader will be able to repair them and continue on.

2. BACKGROUND AND HISTORY

2.1. The conjectures of Beilinson and Lichtenbaum

Beilinson pointed out in his 1983 paper "Higher regulators and values of L-functions" [13] that the existence of Gillet's Chern character [53] from algebraic K-theory to an arbitrary Bloch–Ogus cohomology theory [30] with coefficients in a $\mathbb{Q}$-algebra implies that one can form the universal Bloch–Ogus cohomology $H^a_\mu(-,\mathbb{Q}(b))$ with $\mathbb{Q}$-coefficients by decomposing algebraic K-theory into its weight spaces for the Adams operations ψ_k. In terms of the indexing, one has

$$H^a\big(X,\mathbb{Q}(b)\big) := K_{2b-a}(X)^{(b)}$$

where $K_{2b-a}(X)^{(b)} \subset K_{2b-a}(X)_\mathbb{Q}$ is the weight b eigenspace for the Adams operations

$$K_{2b-a}(X)^{(b)} := \big\{x \in K_{2b-a}(X)_\mathbb{Q} \mid \psi_k(x) = k^b \cdot x\big\}.$$

This raised the question of finding the universal integral Bloch–Ogus cohomology theory. Let Sch_k denote the category of separated finite-type k-schemes with full subcategory Sm_k of smooth k-schemes. Beilinson [13] and Lichtenbaum [87] independently conjectured that this universal theory $H^a_\mu(-,\mathbb{Z}(b))$ should arise as the hypercohomology of a complex of sheaves $X \mapsto \Gamma_X(b)$ on Sm_k (for the Zariski or étale topology)

$$H^a_\mu\big(X,\mathbb{Z}(b)\big) := \mathbb{H}^a\big(X,\Gamma_X(b)\big),$$

with the $\Gamma_X(b)$ satisfying a number of axioms. We give Beilinson's list of axioms for motivic complexes in the Zariski topology (axiom iv(p) was added by Milne [90, §2]):

(i) In the derived category of sheaves on X, $\Gamma(0)$ is the constant sheaf $\mathbb{Z}$ on Sm_k, $\Gamma(1) = \mathbb{G}_m[-1]$ and $\Gamma(n) = 0$ for $n < 0$.

(ii) The graded object $\Gamma(*) := [X \mapsto \bigoplus_{n\ge 0}\Gamma_X(n)]$ is a commutative graded ring in the derived category of sheaves on Sm_k.

(iii) The cohomology sheaves $\mathcal{H}^m(\Gamma(n))$ are zero for $m > n$ and for $m \le 0$ if $n > 0$; $\mathcal{H}^n(\Gamma(n))$ is the sheaf of Milnor K-groups $X \mapsto \mathcal{K}^M_{n,X}$.

(iv)(a) Letting $\alpha : \mathrm{Sm}_{k,\text{ét}} \to \mathrm{Sm}_{k,\mathrm{Zar}}$ be the change of topology morphism, the étale sheafification $\Gamma(n)_{\text{ét}} := \alpha^*\Gamma(n)$ of $\Gamma(n)$ satisfies $\Gamma(n)_{\text{ét}}/m \cong \mu_m^{\otimes n}$ for m prime to the characteristic, where μ_m is the étale sheaf of mth roots of unity.

(iv)(b) For m prime to the characteristic, the natural map $\Gamma(n)/m \to R\alpha_*\Gamma(n)_{\text{ét}}/m$ induces an isomorphism $\Gamma(n)/m \to \tau_{\le n}R\alpha_*\Gamma(n)_{\text{ét}}/m$. Integrally,

$\mathbb{Z}(n) \to R\alpha_* \Gamma(n)_{\text{ét}}$ induces an isomorphism $\Gamma(n) \to \tau_{\leq n} R\alpha_* \Gamma(n)_{\text{ét}}$ and

$$R^{n+1}\alpha_* \Gamma(n)_{\text{ét}} = 0.$$

(iv)(p) For k of characteristic $p > 0$, let $W_\nu \Omega^n_{\log}$ denote the ν-truncated logarithmic de Rham–Witt sheaf. The $d\log$ map $d\log : \mathcal{K}^M_n / p^n \to W_\nu \Omega^n_{\log}$ induces via (ii) a map $\Gamma(n)/p^\nu \to W_\nu \Omega^n_{\log}[-n]$, which is an isomorphism.

One then defines motivic cohomology by

$$H^p\big(X, \mathbb{Z}(q)\big) := \mathbb{H}^p\big(X_{\text{Zar}}, \Gamma_X(q)\big).$$

(v) There should also be a spectral sequence starting with integral motivic cohomology and converging to algebraic K-theory, analogous to the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological K-theory. Explicitly, this should be

$$E_2^{p,q} := H^{p-q}\big(X, \mathbb{Z}(-q)\big) \Rightarrow K_{-p-q}(X).$$

This spectral sequence should degenerate rationally, and give an isomorphism

$$H^p\big(X, \mathbb{Q}(q)\big) := H^p\big(X, \mathbb{Z}(q)\big) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_{2q-p}(X)^{(q)}.$$

The vanishing $\mathcal{H}^m(\Gamma(n)) = 0$ for $n > 0$ and $m \leq 0$ is the *integral Beilinson–Soulé vanishing conjecture*. The mod m-part of axiom (iv)(b) is known as the *Beilinson–Lichtenbaum conjecture*; this implies the integral part of (iv)(b) with the exception of the vanishing of $R^{n+1}\alpha_* \Gamma(n)_{\text{ét}}$, which is known as *Hilbert's theorem 90* for the motivic complexes. In weight $n = 1$, with the identity $\Gamma(1) = \mathbb{G}_m[-1]$, this translates into the classical Hilbert theorem 90

$$H^1_{\text{ét}}(\mathcal{O}, \mathbb{G}_m) = 0$$

for $\mathcal{O}$ a local ring, while the mod m part of (iv)(b) follows from the Kummer sequence of étale sheaves

$$1 \to \mu_m \to \mathbb{G}_m \xrightarrow{\times m} \mathbb{G}_m \to 1.$$

In light of axiom (iii), the Merkurjev–Suslin theorem [89, THEOREM 14.1] settled the degree ≥ 2 part of (iv)(b) for $n = 2$ even before the complex $\Gamma(2)$ was defined.

Beilinson [14, §5.10] rephrased and refined these conjectures to a categorical statement, invoking a conjectural category of mixed motivic sheaves, and an embedding of the hypercohomology of the Beilinson–Lichtenbaum complexes into a categorical framework.

In this framework, motivic cohomology should arise via an abelian tensor category of motivic sheaves on Sch_S, $X \mapsto \text{Sh}^{\text{mot}}(X)$, admitting the six functor formalism of Grothendieck, f^*, f_*, $f_!$, $f^!$, $\mathcal{H}om$, $\otimes$, on the derived categories. There should be Tate objects $\mathbb{Z}_X(n) \in \text{Sh}^{\text{mot}}(X)$, and objects $M(X) := p_{X!}p_X^! \mathbb{Z}_S(0)$ in the derived category of $\text{Sh}^{\text{mot}}(S)$, $p_X : X \to S$ the structure morphism, and motivic cohomology should arise as the Hom-groups

$$H^a_\mu\big(X, \mathbb{Z}(b)\big) = \text{Hom}_{D(\text{Sh}^{\text{mot}}(S))}\big(M(X), \mathbb{Z}_S(b)[a]\big).$$

For X smooth over S, this gives the identity

$$H^a_\mu\big(X, \mathbb{Z}(b)\big) = \mathrm{Ext}^a_{\mathrm{Sh}^{\mathrm{mot}}(X)}\big(\mathbb{Z}_X(0), \mathbb{Z}_X(b)\big).$$

This is a very strong statement, with implications that have not been verified to this day. For instance, the vanishing of $\mathrm{Ext}^a_{\mathcal{A}}(-,-)$ for an abelian category $\mathcal{A}$ and for $a < 0$ gives a vanishing $\mathcal{H}^m(\mathbb{Z}_S(n)) = 0$ for $m < 0$. The stronger vanishing posited by axiom (iii) above (with $\mathbb{Q}$-coefficients) is the *Beilinson–Soulé vanishing conjecture,* and even the weak version is only known for weight $n = 1$ (for which the strong version holds).

Beilinson's conjecture on categories of motivic sheaves is still an open problem. However, other than the integral Beilinson–Soulé vanishing conjecture, the axioms do not rely on the existence of an abelian category of motivic sheaves, and can be framed in the setting of a functorial assignment $X \mapsto \mathrm{DM}(X)$ from S-schemes to tensor-triangulated categories. Such a functor has been constructed and the axioms (except for the vanishing conjectures) have been verified. We will discuss this construction in Section 2.4.

2.2. Bloch's higher Chow groups and Suslin homology

The first good definition of motivic cohomology complexes was given by Spencer Bloch, in his landmark 1985 paper "Algebraic cycles and higher Chow groups" [24]. As suggested by the title, the starting point was the classical Chow group $\mathrm{CH}_*(X)$ of algebraic cycles modulo rational equivalence.

For X a finite type k-scheme, recall that the group of dimension d algebraic cycles on X, $\mathrm{Z}_d(X)$, is the free abelian group on the integral closed subschemes Z of X of dimension d over k. The group of cycles modulo rational equivalence, $\mathrm{CH}_d(X)$, has the following presentation. Let $n \mapsto \Delta^n$ be the cosimplicial scheme of *algebraic n-simplices*

$$\Delta^n := \mathrm{Spec}\, \mathbb{Z}[t_0, \ldots, t_n]/\sum_{i=0}^{n} t_i - 1 \cong \mathbb{A}^n_{\mathbb{Z}}.$$

The coface and codegeneracy maps are defined just as for the usual real simplices $\Delta^n_{\mathrm{top}} \subset \mathbb{R}^n$. A *face* of Δ^n is a closed subscheme defined by the vanishing of some of the t_i. Let $z_d(X, n)$ be the subgroup of the $(n + d)$-dimensional algebraic cycles $Z_{n+d}(X \times \Delta^n)$ generated by the integral closed $W \subset X \times \Delta^n$ such that $\dim W \cap X \times F = m + d$ for each m-dimensional face F (or the intersection is empty). For cycles $w \in z_d(X, n)$, the face condition gives a well-defined pullback $(\mathrm{Id}_X \times g)^* : z_d(X, n) \to z_d(X, m)$ for each map $g : \Delta^m \to \Delta^n$ in the cosimplicial structure, forming the simplicial abelian group $n \mapsto z_d(X, n)$ and giving the associated chain complex $z_d(X, *)$, Bloch's cycle complex. The degree 0 and 1 terms of $z_d(X, *)$ give our promised presentation of $\mathrm{CH}_d(X)$,

$$H_0\big(z_d(X, *)\big) = \mathrm{CH}_d(X),$$

and Bloch defines his higher Chow group $\mathrm{CH}_d(X, n)$ as

$$\mathrm{CH}_d(X, n) := H_n\big(z_d(X, *)\big).$$

If X has pure dimension N over k, we index by codimension

$$z^q(X, *) := z_{N-q}(X, *); \quad \mathrm{CH}^q(X, n) := \mathrm{CH}_{N-q}(X, n).$$

With some technical difficulties due to the necessity of invoking moving lemmas to allow for pullback morphisms, the assignment

$$X \mapsto z^q(X, 2q - *)$$

can be modified via isomorphisms in the derived category to a presheaf of cohomological complexes $\mathbb{Z}_{Bl}(q)$ on Sm_k.

Following a long series of works [25, 29, 43, 52, 94, 96, 112, 113, 115–117, 120, 121, 123–125, 127] (see also [56, 102] for detailed discussions of the Bloch–Kato conjecture, the essential point in axiom (iv)(b) and the most difficult of the Beilinson axioms to prove), it has been shown that the complexes $\mathbb{Z}_{\mathrm{Bl}}(q)$ satisfy all the Beilinson–Lichtenbaum–Milne axioms, except for the Beilinson–Soulé vanishing conjecture in axiom (iii).

After Bloch introduced his cycle complexes, Suslin [111] constructed an algebraic version of singular homology. For a k-scheme X, instead of a naive generalization of the singular chain complex of a topological space by taking the free abelian group on the morphisms $\Delta^n_k \to X$, Suslin's insight was to enlarge this to the abelian group of *finite correspondences*.

A subvariety W of a product $Y \times X$ of varieties (with Y smooth) defines an irreducible finite correspondence from Y to X if $p_1 : W \to Y$ is finite and surjective to some irreducible component of Y. The association $y \mapsto p_2(p_1^{-1}(y))$ can be thought of as a multivalued map from Y to X.

The group of finite correspondences $\mathrm{Cor}_k(Y, X)$ is defined as the free abelian group on the irreducible finite correspondences. Given a morphism $f : Y' \to Y$, there is a pullback map $f^* : \mathrm{Cor}_k(Y, X) \to \mathrm{Cor}_k(Y', X)$, compatible with the interpretation as multivalued functions, and making $\mathrm{Cor}_k(-, X)$ into a contravariant functor from smooth varieties over k to abelian groups.

Suslin defines $C_n^{\mathrm{Sus}}(X) := \mathrm{Cor}_k(\Delta^n_k, X)$; the structure of Δ^*_k as smooth cosimplicial scheme makes $n \mapsto C_n^{\mathrm{Sus}}(X)$ a simplicial abelian group. As above, we have the associated complex $C_*^{\mathrm{Sus}}(X)$, the *Suslin complex* of X, whose homology is the *Suslin homology* of X:

$$H_n^{\mathrm{Sus}}(X, \mathbb{Z}) := \pi_n\big(\big|m \mapsto C_m^{\mathrm{Sus}}(X)\big|\big) = H_n\big(C_*^{\mathrm{Sus}}(X)\big).$$

In fact, the monoid of the $\mathbb{N}$-linear combinations of irreducible correspondences $W \subset X \times Y$ is the same as the monoid of morphisms

$$\phi : X \to \coprod_{n \geq 0} \mathrm{Sym}^n Y$$

where $\mathrm{Sym}^n Y$ is the quotient Y^n / Σ_n of Y^n by the symmetric group permuting the factors, with the monoid structure induced by the sum map

$$\mathrm{Sym}^n Y \times \mathrm{Sym}^m Y \to \mathrm{Sym}^{m+n} Y.$$

Suslin's complex and his definition of algebraic homology can thus be thought of as an algebraic incarnation of the theorem of Dold–Thom [34, SATZ 6.4], that identifies the homotopy groups of the infinite symmetric product of a pointed CW complex T with the reduced homology of T. The main result of [112] gives an isomorphism of the mod n Suslin homology, $H_*^{\mathrm{Sus}}(X, \mathbb{Z}/n)$, for X of finite type over $\mathbb{C}$, with the mod n singular homology of $X(\mathbb{C})$, a first major success of the theory.

Let Δ^n_{top} denote the usual n-simplex

$$\Delta^n_{\mathrm{top}} := \Big\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geq 0\Big\},$$

with the inclusion $\Delta^n_{\mathrm{top}} \subset \Delta^n(\mathbb{C})$.

Theorem 2.1 ([112, THEOREM 8.3]). *Let X be separated finite type scheme over $\mathbb{C}$ and let $n \geq 2$ be an integer. Then the map*

$$\mathrm{Hom}\Big(\Delta^*_{\mathbb{C}}, \bigsqcup_{d\geq 0} \mathrm{Sym}^d X\Big) \to \mathrm{Hom}_{\mathrm{top}}\Big(\Delta^*_{\mathrm{top}}, \bigsqcup_{d\geq 0} \mathrm{Sym}^d X(\mathbb{C})\Big)$$

induced by the inclusions $\Delta^m_{\mathrm{top}} \subset \Delta^m(\mathbb{C})$ gives rise to an isomorphism $H_*^{\mathrm{Sus}}(X, \mathbb{Z}/n) \to H_*^{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}/n)$.

There is also a corresponding statement for X over an arbitrary algebraically closed field k of characteristic zero in terms of étale cohomology [112, THEOREM 7.8]; this extends to characteristic $p > 0$ and n prime to p by using alterations.

2.3. Quillen–Lichtenbaum conjectures

Quillen's computation of the higher algebraic K-theory of finite fields and of number rings led to a search for a relation of higher algebraic K-theory with special values of zeta-functions and L-functions. We will not go into this in detail here, but to large part, this was responsible for the Beilinson–Lichtenbaum conjectures on the existence of motivic complexes computing the conjectural motivic cohomology. Going back to K-theory, this suggested that algebraic K-theory with mod-ℓ coefficients should be the same as mod-ℓ étale K-theory (a purely algebraic version of mod-ℓ topological K-theory, see [35]), at least in large enough degrees. This is more precisely stated as the Quillen–Lichtenbaum conjecture

Conjecture 2.2 ([101], [42, CONJECTURE 3.9]). *Let ℓ be a prime and let X be a regular, noetherian scheme with ℓ invertible on X. Suppose X has finite ℓ-étale cohomological dimension* $\mathrm{cd}_\ell(X)$. *Then the canonical map*

$$K_n(X; \mathbb{Z}/\ell^r) \to K_n^{\text{ét}}(X; \mathbb{Z}/\ell^r)$$

is an isomorphism for $n \geq \mathrm{cd}_\ell(X) - 1$ and is injective for $n = \mathrm{cd}_\ell(X) - 2$.

Here $K_n^{\text{ét}}(X; \mathbb{Z}/\ell)$ is the étale K-theory developed by Dwyer and Friedlander [35, 41, 42].

Conjecture 2.2 for a smooth k-scheme is essentially a consequence of the Beilinson–Lichtenbaum axioms (without Beilinson–Soulé vanishing). The Beilinson–Lichtenbaum conjecture (iv)(a,b) says that the comparison map $\Gamma(q)/\ell^r \to R\alpha_*\mu_{\ell^r}^{\otimes q}$ induces an isomorphism on cohomology sheaves up to degree q. Combining the local–global spectral sequence for some $X \in \mathrm{Sm}_k$ with the Atiyah–Hirzebruch spectral sequences from motivic cohomology to K-theory (axiom (v)) and from étale cohomology to étale K-theory, and keeping track of the cohomological bound in the Beilinson–Lichtenbaum conjecture gives the result.

2.4. Voevodsky's category DM and modern motivic cohomology

One can almost realize Beilinson's ideas of a categorical framework for motivic cohomology by working in the setting of triangulated categories, viewed as a replacement for the derived category of Beilinson's conjectured abelian category of motivic sheaves. Once this is accomplished, one could hope that an abelian category of mixed motives could be constructed out of the triangulated category as the heart of a suitable t-structure.

Constructions of a triangulated category of mixed motives over a perfect base-field were given by Hanamura [57–59], Voevodsky [127], and myself [83]. All three categories yield Bloch's higher Chow groups as the categorical motivic cohomology, however, Voevodsky's sheaf-theoretic approach has had the most far-reaching consequences and has been widely adopted as the correct solution. The construction of a motivic t-structure is still an open problem.[1] There are also constructions of triangulated categories of mixed motives by the method of compatible realizations, such as by Huber [64], or Nori's construction of an abelian category of mixed motives, described in [65, PART II]; we will not pursue these directions here. We also refer the reader to Jannsen's survey on mixed motives [68].

Voevodsky's triangulated category of motives over k, $\mathrm{DM}(k)$, is based on the category of *finite correspondences* on Sm_k, a refinement of Grothendieck's composition law for correspondences on smooth projective varieties. Grothendieck had constructed categories of motives for smooth projective varieties, with the morphisms from X to Y given by the group of cycles modulo rational equivalence $\mathrm{CH}_{\dim X}(X \times Y)$. The composition law is given by

$$W' \circ W = p_{XZ*}\big(p_{XY}^*(W) \cdot p_{YZ}^*(W')\big), \tag{2.1}$$

with $W \in \mathrm{CH}_{\dim X}(X \times Y)$ and $W' \in \mathrm{CH}_{\dim Y}(Y \times Z)$; one needs to pass to cycle classes to define $p_{XY}^*(W) \cdot p_{YZ}^*(W')$ and the projection p_{XZ} needs to be proper (that is, Y needs to be proper over k) to define p_{XZ*}.

Voevodsky's key insight was to restrict to finite correspondences, so that all the operations used in the composition law of correspondence classes would be defined on the level of the cycles themselves, without needing to pass to rational equivalence classes, and without needing the varieties involved to be proper. Voevodsky's idea of having a well-defined composition law on a restricted class of correspondences has been modified and applied in a wide range of different contexts, somewhat similar to the use of various flavors of bordism theories in topology.

Let X and Y be in Sm_k. Recall from Section 2.2 the subgroup $\mathrm{Cor}_k(X, Y) \subset Z_{\dim X}(X \times Y)$ generated by the integral $W \subset X \times Y$ that are finite over X and map surjectively to a component of X.

1 Voevodsky showed this is not possible integrally, so the best one can hope for is a t-structure with $\mathbb{Q}$-coefficients.

Lemma 2.3. *Let X, Y, Z be smooth k-varieties and take $\alpha \in \mathrm{Cor}_k(X,Y)$, $\beta \in \mathrm{Cor}_k(Y,Z)$. Then*

(i) *The cycles $p_{YZ}^*(\beta)$ and $p_{XY}^*(\alpha)$ intersect properly on $X \times Y \times Z$, so the intersection product $p_{YZ}^*(\beta) \cdot p_{XY}^*(\alpha)$ exists as a well-defined cycle on $X \times Y \times Z$.*

(ii) *Letting $|\alpha| \subset X \times Y$, and $|\beta| \subset Y \times Z$ denote the support of α and β, respectively, each irreducible component of the intersection $X \times |\beta| \cap |\alpha| \times Z$ is finite over $X \times Z$, and maps surjectively onto some component of X.*

In other words, the formula

$$\beta \circ \alpha = p_{XZ*}(p_{YZ}^*\beta \cdot p_{XY}^*\alpha)$$

makes sense for $\alpha \in \mathrm{Cor}_k(X,Y)$ and $\beta \in \mathrm{Cor}_k(Y,Z)$, and the resulting cycle on $X \times Z$ is in $\mathrm{Cor}_k(X,Z)$. This defines the composition law in Voevodsky's *category of finite correspondences*, Cor_k, with objects as for Sm_k, and morphisms $\mathrm{Hom}_{\mathrm{Cor}_k}(X,Y) = \mathrm{Cor}_k(X,Y)$. Sending a usual morphism $f : X \to Y$ of smooth varieties to its graph defines a faithful functor $[-] : \mathrm{Sm}_k \to \mathrm{Cor}_k$.

Once one has the category Cor_k, the path to $\mathrm{DM}(k)$ is easy to describe. One takes the category of additive presheaves of abelian groups on Cor_k, the category of *presheaves with transfer* $\mathrm{PST}(k)$. Inside $\mathrm{PST}(k)$ is the category $\mathrm{NST}(k)$ of Nisnevich sheaves with transfer, that is, a presheaf that is a Nisnevich sheaf when restricted to $\mathrm{Sm}_k \subset \mathrm{Cor}_k$. Each $X \in \mathrm{Sm}_k$ defines an object $\mathbb{Z}_{\mathrm{tr}}(X) \in \mathrm{NST}(k)$, as the representable (pre)sheaf $Y \mapsto \mathrm{Cor}_k(Y,X)$. Inside the derived category $D(\mathrm{NST}(k))$ is the full subcategory of complexes K whose homology presheaves $\underline{h}_i(K)$ are $\mathbb{A}^1$-*homotopy invariant*: $\underline{h}_i(K)(X) \cong \underline{h}_i(K)(X \times \mathbb{A}^1)$ for all $X \in \mathrm{Sm}_k$. This is the category of effective motives $\mathrm{DM}^{\mathrm{eff}}(k)$. The Suslin complex construction, $\mathcal{P} \mapsto C_*^{\mathrm{Sus}}(\mathcal{P})$, with

$$C_*^{\mathrm{Sus}}(\mathcal{P})(X) := \mathcal{P}(X \times \Delta^*)$$

extends to a functor $RC_*^{\mathrm{Sus}} : D(\mathrm{NST}(k)) \to \mathrm{DM}^{\mathrm{eff}}(k)$, and realizes $\mathrm{DM}^{\mathrm{eff}}(k)$ as the localization of $D(\mathrm{NST}(k))$ with respect to the complexes $\mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \xrightarrow{p_*} \mathbb{Z}_{\mathrm{tr}}(X)$. Via RC_*^{Sus}, $\mathrm{DM}^{\mathrm{eff}}(k)$ inherits a tensor structure $\otimes$ from $D(\mathrm{NST}(k))$. The functor $\mathbb{Z}_{\mathrm{tr}} : \mathrm{Sm}_k \to \mathrm{NST}(k)$ defines the functor $M^{\mathrm{eff}} := RC_*^{\mathrm{Sus}} \circ \mathbb{Z}_{\mathrm{tr}}$,

$$M^{\mathrm{eff}} : \mathrm{Sm}_k \to \mathrm{DM}^{\mathrm{eff}}(k).$$

The *Tate object* $\mathbb{Z}(1) \in \mathrm{DM}^{\mathrm{eff}}(k)$ is the image of the complex $\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k) \xrightarrow{i_{\infty *}} \mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1)$ (with $\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1)$ in degree 2) via RC_*^{Sus}. One forms the triangulated tensor category $\mathrm{DM}(k)$ as the category of $-\otimes \mathbb{Z}(1)$-spectrum objects in $\mathrm{DM}^{\mathrm{eff}}(k)$, inverting the endofunctor $-\otimes \mathbb{Z}(1)$; for $M \in \mathrm{DM}(k)$, one has the Tate twists $M(n) := M \otimes Z(1)^{\otimes n}$ for $n \in \mathbb{Z}$; in particular, we have the Tate objects $\mathbb{Z}(n)$. The functor M^{eff} induces the functor $M : \mathrm{Sm}_k \to \mathrm{DM}(k)$.

Bloch's higher Chow groups, Suslin homology, and the motivic complexes $\mathbb{Z}_{\mathrm{Bl}}(q)$ are represented in $\mathrm{DM}(k)$ via canonical isomorphisms

$$\mathrm{CH}^q(X, 2q-p) = \mathbb{H}^p\big(X_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{Bl}}(q)\big) \cong \mathrm{Hom}_{\mathrm{DM}(k)}\big(M(X), \mathbb{Z}(q)[p]\big),$$
$$H_n^{\mathrm{Sus}}(X, \mathbb{Z}) = H_n\big(C_*^{\mathrm{Sus}}(X)\big) \cong \mathrm{Hom}_{\mathrm{DM}(k)}\big(\mathbb{Z}[n], M(X)\big).$$

In addition, one has the presheaf of complexes $\mathbb{Z}_V(q)$ on Sm_k

$$\mathbb{Z}_V(q)(X) := C_{-*}^{\mathrm{Sus}}\big(\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m)^{\otimes q}[-q]\big)(X),$$

where $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m)$ is the quotient presheaf $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1 \setminus \{0\})/\mathbb{Z}_{\mathrm{tr}}(\{1\})$. The complexes $\mathbb{Z}_V(q)$ and $\mathbb{Z}_{\mathrm{Bl}}(q)$ define isomorphic objects in $\mathrm{DM}^{\mathrm{eff}}(k)$, in particular, are isomorphic in the derived category of Nisnevich sheaves on Sm_k. The details of these constructions and results are carried out in [127] (with a bit of help from [117]).

2.5. Motivic homotopy theory

Although Voevodsky's triangulated category of motives does give motivic cohomology a categorical foundation, this is really a halfway station on the way to a really suitable categorical framework. As analogy, embedding the Beilinson–Lichtenbaum/Bloch–Suslin theory of motivic complexes in $\mathrm{DM}(k)$ is like considering the singular chain or cochain complex of a topological space as an object in the derived category of abelian groups. A much more fruitful framework for singular (co)homology is to be found in the stable homotopy category SH.

A parallel representability for motivic cohomology for schemes over a base-scheme B in a wider category of good cohomology theories is to be found in the *motivic stable homotopy category over* B, $\mathrm{SH}(B)$. This, together with the motivic unstable homotopy category, $\mathcal{H}(B)$, gives the proper setting for the deeper study of motivic cohomology, besides placing this theory on a equal footing with all cohomology theories on algebraic varieties that satisfy a few natural axioms.

Just as the category $\mathrm{DM}(k)$ starts out as a category of presheaves, the category $\mathrm{SH}(B)$ starts out with the category of presheaves of simplicial sets on Sm_B. The construction of the unstable motivic homotopy category $\mathcal{H}(B)$ over a general base-scheme B as a suitable localization of this presheaf category was achieved by Morel–Voevodsky [94] and the stable version $\mathrm{SH}(B)$ was described by Voevodsky in his ICM address [116]. The important six-functor formalism of Grothendieck was sketched out by Voevodsky and realized in detail by Ayoub [5, 6]. A general theory of motivic categories with a six-functor formalism, including $\mathrm{SH}(-)$, was established by Cisinski–Déglise [33], and Hoyois [62] gave a construction on the level of infinity categories for an equivariant version. A new point of view, the approach of *framed correspondences*, also first sketched by Voevodsky [126], is a breakthrough in our understanding of the infinite loop objects in the motivic setting, and concerning our main interest, motivic cohomology, has led to a natural construction of motivic cohomology over a general base-scheme.

In topology, the representation of singular (co)homology via the singular (co)chain complexes is placed in the setting of stable homotopy theory through the construction of the

Eilenberg–MacLane spectra, giving a natural isomorphism for each abelian group A,

$$H^n(X, A) \cong \mathrm{Hom}_{\mathrm{SH}}\big(\Sigma^\infty X_+, \Sigma^n \mathrm{EM}(A)\big),$$

with the Eilenberg–MacLane spectrum $\mathrm{EM}(A)$ being characterized by its stable homotopy groups

$$\pi_n^s\big(\mathrm{EM}(A)\big) = \begin{cases} A & \text{for } n = 0, \\ 0 & \text{else.} \end{cases}$$

The assignment $A \mapsto \mathrm{EM}(A)$ extends to a fully faithful embedding $\mathrm{EM} : D(\mathbf{Ab}) \to \mathrm{SH}$. This realizes the ordinary (co)homology as being represented by the derived category $D(\mathbf{Ab})$ via its Eilenberg–MacLane embedding in SH, which in turn is to be viewed as the category of all cohomology theories on reasonable topological spaces.

The stable homotopy category SH is the stabilization of the unstable pointed homotopy category $\mathcal{H}_\bullet$ with respect to the suspension operator $\Sigma X := S^1 \wedge X$, which becomes an invertible endofunctor on SH. The resulting functor of $\mathcal{H}_\bullet$ to its stabilization is the infinite suspension functor Σ^∞ and gives us the "effective" subcategory $\mathrm{SH}^{\mathrm{eff}} \subset \mathrm{SH}$, as the smallest subcategory containing $\Sigma^\infty(\mathcal{H}_\bullet)$ and closed under homotopy cofibers and small coproducts. This in turn gives a decreasing filtration on SH by the subcategories $\Sigma^n\mathrm{SH}^{\mathrm{eff}}$, $n \in \mathbb{Z}$. This rather abstract looking filtration is simply the filtration by connectivity: E is in $\Sigma^n\mathrm{SH}^{\mathrm{eff}}$ if and only if $\pi_m^s E = 0$ for $m < n$. The layers in this filtration are isomorphic to the category $\mathbf{Ab}$, by the functor $E \mapsto \pi_n E$, and in fact, this filtration is the one given by a natural t-structure on SH with heart $\mathbf{Ab}$; concretely, the 0th truncation $\tau_0 E$ is given by the Eilenberg–MacLane spectrum $\mathrm{EM}(\pi_0(E))$.

A central example is the sphere spectrum $\mathbb{S} := \Sigma^\infty S^0$. Since

$$\pi_0^s\mathbb{S} = \mathrm{colim}_m \pi_m(S^m) = \mathbb{Z},$$

we have $\tau_0\mathbb{S} = \mathrm{EM}(\mathbb{Z})$, establishing the natural relation between homology and homotopy.

In the motivic world, we have a somewhat parallel picture. The pointed unstable category $\mathcal{H}_\bullet(B)$ has a natural 2-parameter family of "spheres." Let S^n denote the constant presheaf with value the pointed n-sphere, and let $\mathbb{G}_m$ denote the representable presheaf $\mathbb{A}^1 \setminus \{0\}$ pointed by 1. Define

$$S^{a,b} := S^{a-b} \wedge \mathbb{G}_m^{\wedge b}$$

for $a \geq b \geq 0$. We consider $\mathbb{P}^1$ as the representable presheaf, pointed by 1; there is a canonical isomorphism $\mathbb{P}^1 \cong S^{2,1}$ in $\mathcal{H}_\bullet(B)$.

In order to achieve the analog of Spanier–Whitehead duality in the motivic setting, one needs to use spectra with respect to $\mathbb{P}^1$-suspension rather than with respect to S^1-suspension. The category $\mathrm{SH}(B)$ is constructed as a homotopy category of $\mathbb{P}^1$-spectra in $\mathcal{H}_\bullet(B)$, so $\mathbb{P}^1$-suspension becomes invertible and our family of spheres extends to a family of invertible suspension endofunctors

$$\Sigma^{a,b} : \mathrm{SH}(B) \to \mathrm{SH}(B), \quad a, b \in \mathbb{Z}.$$

Each $E \in \mathrm{SH}(B)$ gives the bigraded cohomology theory on Sm_B by

$$E^{a,b}(X) := \mathrm{Hom}_{\mathrm{SH}(B)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{a,b} \wedge E).$$

Note that the translation in $\mathrm{SH}(B)$ is given by S^1-suspension, not $\mathbb{P}^1$-suspension.

The effective subcategory $\mathrm{SH}^{\mathrm{eff}}(B)$ is defined as the localizing subcategory (i.e., a triangulated subcategory closed under small coproducts) generated by the $\mathbb{P}^1$-infinite suspension spectra $\Sigma^\infty_{\mathbb{P}^1} \mathcal{X}$ for $\mathcal{X} \in \mathcal{H}_\bullet(B)$. We replace the filtration of SH with respect to S^1-connectivity with the filtration on $\mathrm{SH}(B)$ with respect to $\mathbb{P}^1$-connectivity, via the subcategories $\Sigma^n_{\mathbb{P}^1} \mathrm{SH}^{\mathrm{eff}}(B)$. This is Voevodsky's slice filtration, with associated nth truncation denoted f_n, giving for each $E \in \mathrm{SH}(B)$ the tower

$$\cdots \to f_{n+1}E \to f_n E \to \cdots \to E.$$

One has the layers $s_n E$ of this tower, fitting into a distinguished triangle

$$f_{n+1}E \to f_n E \to s_n E \to f_{n+1}E[1] = \Sigma^{1,0} f_{n+1} E.$$

An important difference from the topological case is that this is a filtration by triangulated subcategories; the $\mathbb{P}^1$-suspension is not the shift in the triangulated structure on $\mathrm{SH}(B)$, and so the slice filtration does not arise from a t-structure.

We concentrate for a while on the case $B = \mathrm{Spec}\, k$, k a perfect field. There is an Eilenberg–MacLane functor

$$\mathrm{EM} : \mathrm{DM}(k) \to \mathrm{SH}(k),$$

giving the motivic cohomology spectrum $\mathrm{EM}(\mathbb{Z}(0)) \in \mathrm{SH}(k)$ representing motivic cohomology as

$$H^p\big(X, \mathbb{Z}(q)\big) = \mathrm{EM}\big(\mathbb{Z}(0)\big)^{p,q}(X).$$

One has the beautiful internal description of motivic cohomology via Voevodsky's isomorphism [122]

$$s_0 \mathbb{S}_k \cong \mathrm{EM}\big(\mathbb{Z}(0)\big); \tag{2.2}$$

see also [85, THEOREM 10.5.1] and the recent paper of Bachmann–Elmanto [9]. In other words, the 0th slice truncation of the motivic sphere spectrum represents motivic cohomology. Röndigs–Østvær [103] show that the homotopy category of $\mathrm{EM}(\mathbb{Z}(0))$-modules in $\mathrm{SH}(k)$ is equivalent to $\mathrm{DM}(k)$ and represents the Eilenberg–MacLane functor as the forgetful functor, right-adjoint to the free $\mathrm{EM}(\mathbb{Z}(0))$ functor

$$\mathrm{EM}\big(\mathbb{Z}(0)\big) \wedge - : \mathrm{SH}(k) \rightleftarrows \mathrm{EM}(\mathbb{Z}(0))\text{-}\mathbf{Mod} \ : \mathrm{EM}$$
$$\mathrm{EM}(\mathbb{Z}(0))\text{-}\mathbf{Mod} = \mathrm{DM}(k)$$

This is the triangulated motivic analog of the classical result, that the heart of the t-structure on SH is $\mathbf{Ab}$.

2.6. Motivic cohomology and the rational motivic stable homotopy category

In classical homotopy theory, the Eilenberg–MacLane functor $\mathrm{EM} : D(\mathbf{Ab}) \to \mathrm{SH}$ has a nice structural property: after $\mathbb{Q}$-localization, the functor $\mathrm{EM}_{\mathbb{Q}} : D(\mathbf{Ab})_{\mathbb{Q}} \to \mathrm{SH}_{\mathbb{Q}}$ is an equivalence. Does the same happen for the motivic Eilenberg–MacLane functor $\mathrm{EM} : \mathrm{DM}(k) \to \mathrm{SH}(k)$? In general, the answer is no, and the reason goes back to Morel's $\mathbb{C}$–$\mathbb{R}$ dichotomy for $\mathrm{SH}(k)$.

We discuss the case of a characteristic zero field k as base. Suppose that k admits a real embedding $\sigma : k \to \mathbb{R}$. The embedding σ induces a *realization functor*

$$\Re^{\sigma}_{\mathbb{R}} : \mathrm{SH}(k) \to \mathrm{SH},$$

which sends the $\mathbb{P}^1$-suspension spectrum $\Sigma^{\infty}_{\mathbb{P}^1} X_+$ of a smooth k-scheme X to the infinite suspension spectrum of the real manifold of real points $X(\mathbb{R})$. For an embedding $\sigma : k \to \mathbb{C}$, one has the realization functor $\Re^{\sigma}_{\mathbb{C}} : \mathrm{SH}(k) \to \mathrm{SH}$, sending $\Sigma^{\infty}_{\mathbb{P}^1} X_+$ to $\Sigma^{\infty} X(\mathbb{C})_+$. If we take $X = \mathbb{P}^1$, the real embedding gives you $\Sigma\mathbb{S}$ and the complex embedding yields $\Sigma^2\mathbb{S}$, since $\mathbb{P}^1(\mathbb{R}) = S^1$, $\mathbb{P}^1(\mathbb{C}) = S^2$. This has the effect that the switch map $\tau : \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1$ induces an automorphism of $\mathbb{S}_k$ that maps to -1 under the real embedding and to $+1$ under the complex embedding. Thus, if we invert 2 and decompose the motivic sphere spectrum into ± 1 eigenfactors with respect to τ, we decompose $\mathrm{SH}(k)[1/2]$ into corresponding summands $\mathrm{SH}(k)_{\pm}$, with all of $\mathrm{SH}(k)_+$ going to zero under the real embedding and all of $\mathrm{SH}(k)_-$ going to zero under the complex one (after inverting 2 in SH).

Alternatively, the minus part is $\mathrm{SH}(k)[1/2, \eta^{-1}]$, where η is the $\mathbb{P}^1$-stabilization of the algebraic Hopf map

$$\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1, \quad \eta(x, y) = [x : y].$$

A motivic spectrum $E \in \mathrm{SH}(k)$ is *orientable* if E has a good theory of Thom classes. For $V \to X$ a vector bundle with 0-section $s_0 : X \to V$, we have the Thom space $\mathrm{Th}(V) := V/(V \setminus s_0(X)) \in \mathcal{H}_{\bullet}(k)$ (defined as the quotient of representable presheaves). An orientation for E consists of giving a class

$$\mathrm{th}(V) \in E^{2r,r}\big(\mathrm{Th}(V)\big)$$

for each rank r vector bundle $V \to X$ over $X \in \mathrm{Sm}_k$, satisfying axioms parallel to the notion of a $\mathbb{C}$-orientation in topology; a choice of Thom classes defines E as an oriented cohomology theory. After inverting 2, all the orientable E live in the plus part; this includes motivic cohomology, as well as algebraic K-theory and algebraic cobordism. These theories E all have the property that η induces zero on E-cohomology.

Theories that live in the minus part will contrariwise invert η (after inverting 2); these include things like Witt theory or cohomology of the sheaf of Witt groups. The real and complex avatars of this are seen by noting that the complex realization of the algebraic Hopf map is the usual Hopf map, which is the 2-torsion element of stable π_1 of the sphere spectrum, while the real realization is the multiplication map $\times 2 : S^1 \to S^1$.

The analog of the fact that $\mathrm{EM}_{\mathbb{Q}} : D(\mathbb{Q}) \to \mathrm{SH}_{\mathbb{Q}}$ is an equivalence is the following result of Cisinski–Déglise

Theorem 2.4 ([33, THEOREM 16.2.13]). *The unit map* $\mathbb{S}_k \to \mathrm{EM}(\mathbb{Z}(0))$ *induces an isomorphism*

$$\mathrm{SH}(k)_{+\mathbb{Q}} \to \mathrm{DM}(k)_{\mathbb{Q}}$$

with inverse the Eilenberg–MacLane functor followed by the plus-projection

$$\mathrm{DM}(k)_{\mathbb{Q}} \to \mathrm{SH}(k)_{\mathbb{Q}} \to \mathrm{SH}(k)_{+\mathbb{Q}}.$$

The rational minus part is also a homotopy category of modules over a suitable cohomology theory, namely Witt sheaf cohomology. For a field F, we have the Witt ring $W(F)$ of virtual non-degenerate quadratic forms, modulo the hyperbolic form. This extends to a sheaf $\mathcal{W}$ on Sm_k, and the functor $X \mapsto H^p_{\mathrm{Nis}}(X, \mathcal{W})$ is represented in $\mathrm{SH}(k)$ by a suitable spectrum $\mathrm{EM}(\mathcal{W})$. We have

Theorem 2.5 ([3, THEOREM 4.2, COROLLARY 4.4]). *The functor* $E \mapsto \mathrm{EM}(\mathcal{W})_{\mathbb{Q}} \wedge E$ *induces a natural isomorphism of* $\mathrm{SH}(k)_{-\mathbb{Q}}$ *with the homotopy category* $\mathrm{EM}(\mathcal{W})_{\mathbb{Q}}$*-modules.*

From this point of view, one can see the $\mathbb{Z}$-graded cohomology theory

$$X \mapsto \bigoplus_{n \geq 0} H^n_{\mathrm{Nis}}(X, \mathcal{W})$$

as the motivic cohomology for the minus part; this theory picks up information about the real points of schemes. To get the complete theory, one also needs to include twists of $\mathcal{W}$ by line bundles, an analog of orientation local systems in the topological setting. We will say more about this in Section 4.

2.7. Slice tower and motivic Atiyah–Hirzebruch spectral sequences

The classical Atiyah–Hirzebruch spectral sequence for a spectrum $E \in \mathrm{SH}$ is the spectral sequence of the Postnikov tower of E, and looks like

$$E_2^{p,q} := H^p(X, \pi_{-q}E) \Rightarrow E^{p+q}(X).$$

This comes by identifying the qth layer in the Postnikov tower with the shifted Eilenberg–MacLane spectrum $\Sigma^q \mathrm{EM}(\pi_q(E))$.

Together with results of Pelaez [99] and Gutierrez–Röndigs–Spitzweck, Voevodsky's isomorphism (2.2) has a structural expression, namely, for any $E \in \mathrm{SH}(k)$, each slice $s_q(E)$ has a canonical structure of an $\mathrm{EM}(\mathbb{Z}(0))$-module. We write corresponding object of $\mathrm{DM}(k)$ as $\pi^{\mathrm{mot}}_q(E)$, satisfying

$$s_q(E) = \Sigma^q_{\mathbb{P}^1} \mathrm{EM}\big(\pi^{\mathrm{mot}}_q(E)\big) = S^{2q,q} \wedge \mathrm{EM}\big(\pi^{\mathrm{mot}}_q(E)\big),$$

This gives the motivic Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q}(n) := H^{p-q}\big(X, \pi^{\mathrm{mot}}_{-q}(E)(n-q)\big) \Rightarrow E^{p+q,n}(X).$$

These slices have been explicitly identified in a number of important cases. The first case was algebraic K-theory, $\mathrm{KGL} \in \mathrm{SH}(k)$. Voevodsky [118, 119] and Levine [85] show

$$s_q(\mathrm{KGL}) = \mathrm{EM}\big(\mathbb{Z}(q)[2q]\big) = \Sigma^q_{\mathbb{P}^1} \mathrm{EM}\big(\mathbb{Z}(0)\big)$$

so

$$\pi_q^{\text{mot}}(\text{KGL}) = \mathbb{Z}(0),$$

corresponding to classical computation for topological K-theory,

$$\pi_q^s KU = \begin{cases} \mathbb{Z} & \text{for } q \text{ even}, \\ 0 & \text{for } q \text{ odd}. \end{cases}$$

Using "algebraic Bott periodicity" for KGL: $\text{KGL}^{a+2n,b+n}(X) = \text{KGL}^{a,b}(X) = K_{2b-a}(X)$, this yields the Atiyah-Hirzebruch spectral sequence of the Beilinson-Lichtenbaum axiom (v),

$$E_2^{p,q} := H^{p-q}\big(X, \mathbb{Z}(-q)\big) \Rightarrow K_{-p-q}(X).$$

There is also a corresponding spectral sequence with $\mathbb{Z}/m$-coefficients.

This Atiyah–Hirzebruch spectral sequence for algebraic K-theory was first constructed for X the spectrum of a field by Bloch and Lichtenbaum [29], by a completely different approach and without recourse to motivic homotopy theory or Voevodsky's slice tower. Their construction was generalized to general $X \in \text{Sm}_k$ by Friedlander–Suslin [43], also without using the categorical machinery. The rough idea is to give a filtration by codimension of support on $X \times \Delta^*$ (with additional conditions), and then identify the layers with a suitable complex of cycles. Another approach, by Grayson [54], relies on the K-theory of exact categories with commuting isomorphisms. For smooth finite-type schemes over a perfect field, all these approaches yield the same spectral sequence (see [85, THEOREM 7.1.1, THEOREM 9.0.3], [44])).

3. MOTIVIC COHOMOLOGY OVER A GENERAL BASE

It is natural to ask if this picture of a good motivic cohomology theory for schemes over a perfect field can be extended to more general base-schemes, not just as an interesting technical question but for a wide range of applications, especially in arithmetic. Over a perfect field, we have a number of different constructions that all lead to the same groups, each of which have their advantages and disadvantages: Bloch's higher Chow groups, the cohomology of a suitable Suslin complex, the morphisms in $\text{DM}(k)$, the cohomology theory represented in $\text{SH}(k)$ by $\text{EM}(\mathbb{Z}(0))$, or by $s_0\mathbb{S}_k$, or by $s_0\text{KGL}$.

One would expect motivic cohomology to be an *absolute* theory, like algebraic K-theory, that is, its value on a given scheme should not depend on the choice of base-scheme. In terms of a spectrum $H\mathbb{Z}_S \in \text{SH}(S)$ that would represent our putative theory, this is the *cartesian condition*: there should be canonical isomorphisms $H\mathbb{Z}_T \cong f^*H\mathbb{Z}_S$ for each morphism of schemes $f : T \to S$.

The identity (2.2) raises the possibility of defining motivic cohomology over a general base-scheme B by this formula. One problem here is that the slice filtration has only a limited functoriality: for $f : C \to B$ a map of schemes, one does not in general have a natural isomorphism $f^* \circ s_0 \cong s_0 \circ f^*$. For the cartesian property to hold for a motivic cohomology defined via the slice filtration, one would want the compatibility of the slices with pullback; this latter is in fact the case for $f : C \to B$ is a morphism of separated, finite type schemes over a field k of characteristic zero (or assuming resolution of singularities for separated, finite type k-schemes), by results of Pelaez [100, COROLLARY 4.3]. This compatibility also holds for arbitrary smooth f, but is not known in general.

Another concrete candidate for the motivic Borel–Moore homology is given by the hypercohomology of a version of Bloch's cycle complex, suitably extended to the setting of finite type schemes over a Dedekind domain. This theory is nearly absolute, as it depends only on a good notion of dimension or codimension, which one would have for say equi-Krull-dimensional schemes. In general, however, this theory lacks a full functoriality under pullback and also lacks a multiplicative structure.

There is a $\mathbb{P}^1$-spectrum $\mathrm{KGL}_S \in \mathrm{SH}(S)$ that represents the so-called homotopy invariant K-theory over an arbitrary base and is cartesian, so one could try $s_0\mathrm{KGL}$ as a representing spectrum. Again, the problem is the functoriality of the slice filtration, but perhaps KGL would be easier to handle than the sphere spectrum in this regard.

3.1. Cisinski–Déglise motivic cohomology

Over an base-scheme S that is noetherian and of finite Krull dimension, Cisinski–Déglise [33, §11] have followed the program of Voevodsky to define a triangulated category of motives $\mathrm{DM}_{\mathrm{CD}}(S)$, with Tate objects $\mathbb{Z}_S(n)$, and with a "motives functor"

$$M : \mathrm{Sm}_S \to \mathrm{DM}_{\mathrm{CD}}(S); X \mapsto M(X) \in \mathrm{DM}_{\mathrm{CD}}(S).$$

This extends Voevodsky's construction of $\mathrm{DM}(k)$ for a perfect field k. The main point is that the notion of a finite correspondence for smooth finite type schemes over a field extends to a corresponding notion over a general base-scheme (see [33, §8]). This gives rise to a theory of motivic cohomology generalizing Voevodsky's definition as

$$H^{p,q}(X, \mathbb{Z}) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{CD}}(S)}\big(M(X), \mathbb{Z}_S(q)[p]\big)$$

for X smooth over S. They show that the assignment $S \mapsto \mathrm{DM}_{\mathrm{CD}}(S)$ defines a functor to the category of triangulated tensor categories, $\mathrm{DM}_{\mathrm{CD}}(-) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathbf{Tr}^{\otimes}$, admitting a six-functor formalism. There are also Tate twists $M \mapsto M(n)$. This gives a definition of motivic cohomology of an general scheme Y by

$$H^{p,q}(Y, \mathbb{Z}) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{CD}}(Y)}\big(\mathbb{Z}_Y(0), \mathbb{Z}_Y(q)[p]\big),$$

which for $Y \in \mathrm{Sm}_S$ agrees with the definition given above.

They construct an adjunction

$$\phi^* : \mathrm{SH}(Y) \rightleftarrows \mathrm{DM}_{\mathrm{CD}}(Y) : \phi_*,$$

with ϕ_* playing the role of the Eilenberg–MacLane functor, giving rise to the spectrum $\mathcal{M}\mathbb{Z}_Y \in \mathrm{SH}(Y)$ representing $H^{*,*}(Y, \mathbb{Z})$ [33, DEFINITION 11.2.17]. They discuss the question of whether $Y \mapsto \mathcal{M}\mathbb{Z}_Y$ is cartesian (see [33, CONJECTURE 11.2.22, PROPOSITION 11.4.7]), without reaching a general resolution.

Cisinski–Déglise have a different approach for representing motivic cohomology with $\mathbb{Q}$-coefficients, much in the same spirit as Beilinson's construction of universal cohomology using algebraic K-theory. Using the spectrum $\mathrm{KGL}_S \in \mathrm{SH}(S)$, which represents homotopy invariant algebraic K-theory, they use the Adams operations to decompose $\mathrm{KGL}_{S\mathbb{Q}}$ into summands, $\mathrm{KGL}_{S\mathbb{Q}} = \bigoplus_i \mathrm{KGL}_S^{(i)}$, with $\mathrm{KGL}_S^{(i)}$ representing the ith graded piece of K-theory for the γ-filtration. This gives them a nice commutative monoid object (i.e., commutative ring spectrum) $H_S^{\mathbb{B}} := \mathrm{KGL}_S^{(0)} \in \mathrm{SH}(S)_{\mathbb{Q}}$, whose module category they call the category of *Beilinson motives over* S. This construction is cartesian, gives a good theory of motivic cohomology with $\mathbb{Q}$-coefficients over a general base-scheme and agrees with $\mathrm{DM}_{\mathrm{CD}}(S)_{\mathbb{Q}}$ for S a uni-branch scheme. See **[33, §14]** for details.

3.2. Spitzweck's motivic cohomology

In **[110]**, Spitzweck constructs a motivic cohomology theory over an arbitrary base-scheme. The Bloch cycle complex gives rise to a general version of Bloch's higher Chow groups for finite type schemes over a Dedekind domain, which has nice localization properties (by **[25]** and **[84]**), but has poor functoriality and lacks a multiplicative structure. On the other hand, using the Bloch–Kato conjectures, established by Voevodsky et al., the ℓ-completed higher Chow groups are recognized as a truncated ℓ-adic étale cohomology, for ℓ prime to all residue characteristics. The theorem of Geisser–Levine **[52]** describes the p-completed higher Chow groups in characteristic $p > 0$ in terms of logarithmic de Rham–Witt sheaves. Finally, there is the good theory with $\mathbb{Q}$-coefficients given by Beilinson motivic cohomology of Cisinski–Déglise, as described above.

Each of these three theories, namely the ℓ-adic étale cohomology, the cohomology of the logarithmic de Rham–Witt sheaves, and the rational Beilinson motivic cohomology, has good functoriality and multiplicative properties. Gluing the ℓ-adic, p-adic, and rational theories together via their respective comparisons with the Bloch cycle complex, Spitzweck constructs a theory with good functoriality and multiplicative properties, and which is described by a presheaf of complexes on smooth schemes over a given Dedekind domain as base-scheme. The corresponding theory agrees with Voevodsky's motivic cohomology for smooth schemes over a perfect field, and is given additively by the hypercohomology of the Bloch complex for smooth schemes over a Dedekind domain (even in mixed characteristic).

Taking the base-scheme to be $\mathrm{Spec}\,\mathbb{Z}$, Spitzweck's construction yields a representing object $M\mathbb{Z}_{\mathbb{Z}}$ in $\mathrm{SH}(\mathbb{Z})$ and one can thus define absolute motivic cohomology for smooth schemes over a given base-scheme S by pulling back $M\mathbb{Z}_{\mathbb{Z}}$ to $M\mathbb{Z}_S \in \mathrm{SH}(S)$. The resulting motivic cohomology agrees with Voevodsky's for smooth schemes of finite type over a perfect base-field, and with the hypercohomology of the Bloch cycle complex for smooth finite type schemes over a Dedekind domain. This gives rise to a triangulated category of motives $\mathrm{DM}_{\mathrm{Sp}}(S)$ over a base-scheme S, defined as the homotopy category of $M\mathbb{Z}_S$-modules, and the functor $S \mapsto \mathrm{DM}_{\mathrm{Sp}}(S)$ inherits a Grothendieck six-functor formalism from that of $S \mapsto \mathrm{SH}(S)$.

3.3. Hoyois' motivic cohomology

Spitzweck's construction gives a solution to the problem of constructing a triangulated category of motives over an arbitrary base, admitting a six-functor formalism and thus yielding a good theory of motivic cohomology. His construction is a bit indirect and it would be nice to have a direct construction of a representing motivic ring spectrum $H\mathbb{Z}_S \in \mathrm{SH}(S)$ for each base-scheme S, still satisfying the cartesian condition.

Hoyois has constructed such a theory of motivic cohomology over an arbitrary base-scheme by using a recent breakthrough in our understanding of the motivic stable homotopy categories $\mathrm{SH}(S)$. This is a new construction of $\mathrm{SH}(S)$ more in line with Voevodsky construction of $\mathrm{DM}(k)$. The basic idea is sketched in notes of Voevodsky [126], which were realized in a series of works by Ananyevskiy, Garkusha, Panin, Neshitov [2,4,45–48](authorship in various combinations). Building on these works, Elmanto, Hoyois, Khan, Sosnilo, and Yakerson [36–38] construct an infinity category of framed correspondences, and use the basic program of Voevodsky's construction of $\mathrm{DM}(k)$ to realize $\mathrm{SH}(S)$ as arising from presheaves of spectra *with framed transfers*, just as objects of $\mathrm{DM}(k)$ arise from presheaves of complexes of sheaves with transfers for finite correspondences. It is not our purpose here to give a detailed discussion of this beautiful topic; we content ourselves with sketching some of the basic principles.

An integral closed subscheme $Z \subset X \times Y$ that defines a finite correspondence from X to Y can be thought of a special type of a *span* via the two projections

$$X \xleftarrow{\;p_1\;} Z \xrightarrow{\;p_2\;} Y$$

For X and Y smooth and finite type over a given base-scheme S, a framed correspondence from X to Y is also a span,

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} Y$$

satisfying certain conditions, together with some additional data (the framing). For simplicity, assume that X is connected. The morphism p is required to be a finite, flat, local complete intersection (lci) morphism, called a finite *syntomic* morphism (the terminology was introduced by Mazur). The lci condition means that p factors as closed immersion $i : Z \to P$ followed by a smooth morphism $f : P \to X$, and the closed subscheme $i(Z)$ of P is locally defined by exactly $\dim_X P - \dim_X Z$ equations forming a regular sequence. The morphism p factored in this way has a relative cotangent complex $\mathbb{L}_p$ admitting a simple description, namely

$$\mathbb{L}_p = \big[\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{P/X}\big];$$

the conditions on i and p say that both $\mathcal{I}_Z/\mathcal{I}_Z^2$ and $i^*\Omega_{P/X}$ are locally free coherent sheaves on Z of rank $\dim_X P - \dim_X Z$ and $\dim_X P$, respectively. For p an lci morphism, the perfect complex $\mathbb{L}_p$ defines a point $\{\mathbb{L}_p\}$ in the space $\mathcal{K}(Z)$ defining the K-theory of Z; in the case of a finite syntomic morphism, the virtual rank of $\{\mathbb{L}_p\}$ is zero.

A *framing* for a syntomic map $p : Z \to X$ is a choice of a path $\gamma : [0,1] \to \mathcal{K}(Z)$ connecting $\{\mathbb{L}_p\}$ with the base-point $0 \in \mathcal{K}(Z)$. For a framing to exist, the class $[\mathbb{L}_p] \in K_0(Z)$ must be zero, but the choice of γ is additional data. The morphism $q : Z \to Y$ is arbitrary.

One has the usual notion of a composition of spans:

$$\left(Y \xleftarrow{p'} Z' \xrightarrow{q'} W\right) \circ \left(X \xleftarrow{p} Z \xrightarrow{q} Y\right) := \left(X \xleftarrow{p\circ p_1} Z \times_Y Z' \xrightarrow{q'\circ p_2} W\right)$$

which preserves the finite syntomic condition. However, one needs a higher categorical structure to take care of associativity constraints. The composition of paths is even trickier, since we are dealing here with actual paths, not paths up to homotopy. In the end, this produces an infinity category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$ of framed correspondences on smooth S-schemes, rather than a category; roughly speaking, the composition is only defined "up to homotopy and coherent higher homotopies."

Via the infinity category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$, we have the infinity category of framed motivic spaces, $\mathbf{H}^{\mathrm{fr}}(S)$, this being the infinity category of $\mathbb{A}^1$-invariant, Nisnevich sheaves of spaces on $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$. There is a stable version, $\mathbf{SH}^{\mathrm{fr}}(S)$, an infinite suspension functor $\Sigma^\infty_{\mathrm{fr}} : \mathbf{H}^{\mathrm{fr}}(S) \to \mathbf{SH}^{\mathrm{fr}}(S)$, and an equivalence of infinity categories $\gamma_* : \mathbf{SH}^{\mathrm{fr}}(S) \to \mathbf{SH}(S)$, where $\mathbf{SH}(S)$ is the infinity category version of the triangulated category $\mathrm{SH}(S)$, that is, the homotopy category of $\mathbf{SH}(S)$ is $\mathrm{SH}(S)$. The equivalence γ_* can be thought of as a version of the construction of infinite loop spaces from Segal's Γ-spaces, with a framed correspondence $X \leftarrow Z \to Y$ of degree n over X being viewed as a generalization of the map $[n]_+ \to [0]_+$ in Γ^{op}.

With this background, we can give a rough idea of Hoyois' construction of the spectrum representing motivic cohomology over S in [63]. He considers spans $X \xleftarrow{p} Z \xrightarrow{q} Y$, $X, Y \in \mathrm{Sm}_S$, with $p : Z \to X$ a finite morphism such that $p_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_X$-module; note that this condition is satisfied if p is a syntomic morphism, but not conversely. These spans form a category $\mathbf{Corr}^{\mathrm{flf}}(\mathrm{Sm}_S)$ under span composition ("flf" stands for "finite, locally free") and forgetting the paths γ defines a morphism of (infinity) categories $\pi_{\mathrm{ad}} : \mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S) \to \mathbf{Corr}^{\mathrm{flf}}(\mathrm{Sm}_S)$.

Given a commutative monoid A, the constant Nisnevich sheaf on Sm_S with value A extends to a functor

$$A_S : \left(\mathbf{Corr}^{\mathrm{flf}}\right)^{\mathrm{op}} \to \mathbf{Ab},$$

where the pullback from Y to X by $X \xleftarrow{p} Z \xrightarrow{q} Y$ is given by multiplication by $\mathrm{rnk}_{\mathcal{O}_X}\mathcal{O}_Z$ if X and Y are connected; one extends to general smooth X and Y by additivity. This gives us the presheaf (of abelian monoids) with framed transfers $A^{\mathrm{fr}}_S := A_S \circ \pi^{\mathrm{op}}_{\mathrm{ad}}$, and the machinery of [36–38] converts this into the motivic spectrum $\gamma_*\Sigma^\infty_{\mathrm{fr}} A^{\mathrm{fr}}_S \in \mathrm{SH}(S)$. Hoyois shows [63, LEMMA 20] that this construction produces a cartesian family, and that taking $A = \mathbb{Z}$ recovers Spitzweck's family $S \mapsto M\mathbb{Z}_S$ [63, THEOREM 21].

This gives us a conceptually simple construction of a motivic Eilenberg–MacLane spectrum, and the corresponding motivic category $\mathrm{DM}_H(S)$, much in the spirit of Voevodsky original construction of $\mathrm{DM}(k)$ and the Röndigs–Østvær theorem identifying $\mathrm{DM}(k)$ with the homotopy category of $\mathrm{EM}(\mathbb{Z}(0))$-modules.

4. MILNOR–WITT MOTIVIC COHOMOLOGY

The classical Chow group $\mathrm{CH}^n(X)$ of codimension n algebraic cycles modulo rational equivalence on a smooth variety X is part of the motivic cohomology of X via the isomorphism $\mathrm{CH}^n(X) = H^{2n}(X, \mathbb{Z}(n))$. Barge and Morel [12] have introduced a refinement of the Chow groups, the *Chow–Witt groups*, that incorporates information about quadratic forms. Their construction has been embedded in a larger theory of *Milnor–Witt motives* and *Milnor–Witt motivic cohomology*, which we describe in this section. The quadratic information given by the Chow–Witt groups, Milnor–Witt motivic cohomology and related theories has proven useful in recent efforts to give quadratic refinements for intersection theory and enumerative geometry; see [10, 11, 21, 61, 76, 77, 86] for some examples. We refer the reader to [8, 31, 39, 92] for details on the theory described in this section.

4.1. Milnor–Witt K-theory and the Chow–Witt groups

A codimension n algebraic cycle $Z := \sum_i n_i Z_i$ can be thought of as the set of its generic points z_i together with the $\mathbb{Z}$-valued function n_i on z_i, from which we can write the group $Z^n(X)$ of codimension n algebraic cycles as

$$Z^n(X) = \bigoplus_{z \in X^{(n)}} \mathbb{Z},$$

where $X^{(n)}$ is the set of points $z \in X$ with closure $Z := \bar{z} \subset X$ of codimension n.

Let $\mathrm{GW}(F)$ denote the Grothendieck–Witt ring of virtual non-degenerate quadratic forms over F and let $W(F) = \mathrm{GW}(F)/(H)$ where H is the hyperbolic form $H(x, y) = x^2 - y^2$ (we assume throughout that the characteristic is $\neq 2$ to avoid technical difficulties); $W(F)$ is the *Witt ring* of anisotropic quadratic forms over F (see [107]).

One can consider a finite set of codimension n points $z_i \in X^{(n)}$, together with a collection of classes $\{q_i \in \mathrm{GW}(k(z_i))\}$; one recovers a $\mathbb{Z}$-valued function on z_i by taking the rank of q_i. This gives the group

$$\tilde{Z}^n(X) := \bigoplus_{z \in X^{(n)}} \mathrm{GW}\big(k(z)\big)$$

with rank homomorphism $\mathrm{rnk} : \tilde{Z}^n(X) \to Z^n(X)$. In contrast with integer-valued functions, an element $q \in \mathrm{GW}(k(z))$ does not always extend to all of $\bar{z}$; there is an obstruction given by a certain boundary map

$$\partial : \mathrm{GW}\big(k(z)\big) \to \bigoplus_{w \in \bar{z} \cap X^{(n+1)}} W\big(k(w)\big).$$

This starts to look more like classical homology, in that one should consider $\tilde{Z}^n(X)$ as a group of chains rather than a group of cycles.

This is not enough, as one needs a quadratic refinement for the classical relation given by rational equivalence. The original construction of Barge–Morel defined this relation, but later developments put their construction in a rather more natural form, which we now describe.

We recall that the Milnor K-theory ring $K_*^M(F) := \bigoplus_{n\geq 0} K_n^M(F)$ of a field F is defined as the quotient of the tensor algebra on the abelian group of units $F^\times$, modulo the Steinberg relation

$$K_*^M(F) := (F^\times)^{\otimes_{\mathbb{Z}} *}/\big(\{a \otimes (1-a) \mid a \in F \setminus \{0,1\}\}\big).$$

The quadratic refinement of $K_*^M(F)$ is the Hopkins–Morel *Milnor–Witt K-theory* of F.

Definition 4.1 (Hopkins–Morel [92, DEFINITION 6.3.1]). Let F be a field. The *Milnor–Witt K-theory of F*, $K_*^{\mathrm{MW}}(F) := \bigoplus_{n\in\mathbb{Z}} K_n^{\mathrm{MW}}(F)$, is the $\mathbb{Z}$-graded associative algebra defined by the following generators and relations.

Generators

(G1) For each $u \in F^\times$, we have the generator $[u]$ of degree 1;

(G2) There is an additional generator η of degree -1.

Relations

(R0) $\eta \cdot [u] = [u] \cdot \eta$;

(R1) $[uv] = [u] + [v] + \eta \cdot [u] \cdot [v]$;

(R2) $[u] \cdot [1-u] = 0$ for $u \in F \setminus \{0,1\}$;

(R3) Let $h = (2 + \eta \cdot [-1])$. Then $\eta \cdot h = 0$.

It follows directly that sending $[u]$ to $\{u\} \in K_1^M(F)$ and sending η to zero defines a surjective graded algebra homomorphism $K_*^{MW}(F) \to K_*^M(F)$ with kernel (η). We write $[u_1, \ldots, u_n]$ for the product $[u_1]\cdots[u_n]$.

Theorem 4.2 (Hopkins–Morel [92, THEOREM 6.4.5]). *Let $I(F) \subset \mathrm{GW}(F)$ be the kernel of the rank homomorphism $\mathrm{GW}(F) \to \mathbb{Z}$, with the nth power ideal $I^n(F) \subset \mathrm{GW}(F)$ for $n > 0$. Define $I^n(F) = W(F)$ for $n \leq 0$. Then for each $n \in \mathbb{Z}$, the surjection $K_n^{\mathrm{MW}}(F) \to K_n^M(F)$ extends to an exact sequence*

$$0 \to I^{n+1}(F) \to K_n^{\mathrm{MW}}(F) \to K_n^M(F) \to 0.$$

For $n = 0$, $K_0^M(F) = \mathbb{Z}$, $K_0^{\mathrm{MW}}(F)$ is isomorphic to $\mathrm{GW}(F)$ and the above sequence is isomorphic to the defining sequence for $I(F)$. For $n < 0$, $K_n^M(F) = 0$ and $K_n^{\mathrm{MW}}(F) \cong$

$W(F)$. Finally, we have, for each $n < 0$, a commutative diagram

$$\begin{array}{ccc} K_n^{\mathrm{MW}}(F) & \xrightarrow{\sim} & W(F) \\ \Big\downarrow{\scriptstyle \times\eta} & & \Big\| \\ K_{n-1}^{\mathrm{MW}}(F) & \xrightarrow{\sim} & W(F) \end{array}$$

and, for $n = 0$, the commutative diagram

$$\begin{array}{ccc} K_0^{\mathrm{MW}}(F) & \xrightarrow{\sim} & \mathrm{GW}(F) \\ \Big\downarrow{\scriptstyle \times\eta} & & \Big\downarrow{\scriptstyle \pi} \\ K_{-1}^{\mathrm{MW}}(F) & \xrightarrow{\sim} & W(F) \end{array}$$

where π is the canonical surjection.

The isomorphism $\mathrm{GW}(F) \xrightarrow{\sim} K_0^{\mathrm{MW}}(F)$ sends $\langle u\rangle$ to $1 + \eta[u]$, where $\langle u\rangle$ is the rank one form $\langle u\rangle(x) := ux^2$; since a quadratic form over F is diagonalizable ($\mathrm{char} F \neq 2$), the isomorphism is completely determined by its value on the forms $\langle u\rangle$. Given a 1-dimensional F-vector space L, we have the $\mathrm{GW}(F)$-module $\mathrm{GW}(F; L)$ of non-degenerate, L-valued quadratic forms $q : V \to L$; each vector space isomorphism $\phi : L \to F$ gives an isomorphism of $\mathrm{GW}(F)$-modules $\mathrm{GW}(F; L) \cong \mathrm{GW}(F)$. Since $K_*^{\mathrm{MW}}(F)$ is a $\mathbb{Z}$-graded $K_0^{\mathrm{MW}}(F) = \mathrm{GW}(F)$-module, we can form the $\mathbb{Z}$-graded $K_*^{\mathrm{MW}}(F)$-module $K_*^{\mathrm{MW}}(F; L) := \mathrm{GW}(F; L) \otimes_{\mathrm{GW}(F)} K_*^{\mathrm{MW}}(F)$.

Given a dvr $\mathcal{O}$ with residue field k, quotient field F, and generator t for the maximal ideal, one has the map

$$\partial_t : K_n^{\mathrm{MW}}(F) \to K_{n-1}^{\mathrm{MW}}(k)$$

determined by the formulas

$$\partial_t\big([t, u_2, \dots, u_n]\big) = [\bar{u}_2, \dots, \bar{u}_n], \quad \partial_t\big([u_1, u_2, \dots, u_n]\big) = 0, \quad \partial_t(\eta \cdot x) = \eta \cdot \partial_t(x)$$

for $u_1, \dots, u_n \in \mathcal{O}^\times$, and $x \in K_{n+1}^{\mathrm{MW}}(F)$, where $\bar{u}_i$ is the image of u_i in $k^\times$. This is similar to the well-known boundary map $\partial : K_n^M(F) \to K_{n-1}^M(k)$, with the difference, that ∂ does not depend on the choice of t while ∂_t does. To get a boundary map that is independent of the choice of parameter t, one needs to include the twisting. This yields the well-defined boundary map

$$\partial : K_n^{\mathrm{MW}}(F; L \otimes_{\mathcal{O}} F) \to K_{n-1}^{\mathrm{MW}}\big(k; L \otimes_{\mathcal{O}} (\mathfrak{m}/\mathfrak{m}^2)^\vee\big)$$

for L a free rank-one $\mathcal{O}$-module, independent of the choice of generator for the maximal ideal $\mathfrak{m}$, where ∂ is defined by choosing a generator t and an $\mathcal{O}$-basis λ for L, and setting

$$\partial(x \otimes \lambda) := \partial_t(x) \otimes \lambda \otimes \partial/\partial t.$$

Definition 4.3. Let X be a smooth finite type k-scheme, and let $\mathcal{L}$ be an invertible sheaf on X. The nth *$\mathcal{L}$-twisted Rost–Schmid complex* for Milnor–Witt K-theory is the complex $\mathrm{RS}^*(X, \mathcal{L}, n)$ with

$$\mathrm{RS}^m(X, \mathcal{L}, n) := \bigoplus_{x \in X^{(m)}} K_{n-m}^{\mathrm{MW}}\Big(k(x); \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \bigwedge^m (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee\Big)$$

and boundary map $\partial^m : \mathrm{RS}^m(X, \mathcal{L}, n) \to \mathrm{RS}^{m+1}(X, \mathcal{L}, n)$ the sum of the maps

$$\partial_{w,x} : K^{\mathrm{MW}}_{n-m}\Big(k(x); \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \bigwedge^m (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee\Big) \to K^{\mathrm{MW}}_{n-m-1}\Big(k(w); \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \bigwedge^{m+1} (\mathfrak{m}_w/\mathfrak{m}_w^2)^\vee\Big)$$

associated to the normalization of the local ring $\mathcal{O}_{\bar{x},w}$ for $w \in \bar{x} \cap X^{(m+1)}$. Here we have cheated a bit in the definition of $\partial_{w,x}$. This is correct if $\mathcal{O}_{\bar{x},w}$ is a dvr, which is the case outside of finitely many points $w \in \bar{x} \cap X^{(m+1)}$; in general, one needs to use a push-forward map in Milnor–Witt K-theory for finite field extensions to define $\partial_{w,x}$.

The twisted Milnor–Witt sheaf $\mathcal{K}^{\mathrm{MW}}_n(\mathcal{L})_X$ is the Nisnevich sheaf on X associated to the presheaf

$$U \mapsto H^0\big(\mathrm{RS}^*(U, \mathcal{L} \otimes \mathcal{O}_U, n)\big).$$

The codimension n twisted Chow–Witt group of X, $\widetilde{\mathrm{CH}}^n(X; \mathcal{L})$, is defined as

$$\widetilde{\mathrm{CH}}^n(X; \mathcal{L}) := H^n\big(\mathrm{RS}^*(X, \mathcal{L}, n)\big)$$

For details, see [93, CHAP. 5] or [31, CHAP. 2].

For Milnor K-theory, one has the Gersten complex $G^*(X, n)$,

$$G^*(X,n) := \bigoplus_{x\in X^{(0)}} K^{\mathrm{MW}}_n\big(k(x)\big) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-m+1}} \bigoplus_{x\in X^{(m)}} K^M_{n-m}\big(k(x)\big) \xrightarrow{\partial^{n-m}} \cdots \xrightarrow{\partial^{n-1}} \bigoplus_{x\in X^{(n)}} K^M_0\big(k(x)\big),$$

with essentially the same definition as the Rost–Schmid complex, without the twisting. This gives us the Milnor K-theory sheaf $\mathcal{K}^M_{n,X} := \ker \partial^0$, and it follows easily from the definitions that $\mathrm{CH}^n(X) = H^n(G^*(X, n))$. The same ideas that lead to the Bloch–Kato formula [78]

$$\mathrm{CH}^n(X) \cong H^n(X_{\mathrm{Nis}}, \mathcal{K}^M_{n,X})$$

give the isomorphism

$$\widetilde{\mathrm{CH}}^n(X; \mathcal{L}) \cong H^n\big(X_{\mathrm{Nis}}, \mathcal{K}^{\mathrm{MW}}_n(\mathcal{L})_X\big)$$

(see the discussion following [31, DEFINITION 3.1] for details). The maps $\mathcal{K}^{\mathrm{MW}}_n \to \mathcal{K}^{\mathrm{M}}_n$ give the map of complexes $\mathrm{RS}^*(X, \mathcal{L}, n) \to G^*(X, n)$ and the corresponding map $\mathrm{rnk}_{X,n} : \widetilde{\mathrm{CH}}^n(X; \mathcal{L}) \to \mathrm{CH}^n(X)$.

The twists by an invertible sheaf are not just a device for defining the Rost–Schmid complexes and the Chow–Witt groups, they play an integral part in the structure of the overall theory. The Chow groups of smooth varieties admit the functorialities of a Borel–Moore homology theory: they have functorial pullback maps $f^* : \mathrm{CH}^n(Y) \to \mathrm{CH}^n(X)$ for each morphism $f : X \to Y$ in Sm_k, and for $f : X \to Y$ a proper morphism of relative dimension d, one has the functorial proper push-forward map $f_* : \mathrm{CH}^n(X) \to \mathrm{CH}^{n-d}(Y)$. The Chow–Witt groups also have a contravariant functoriality; for $f : X \to Y$, and $\mathcal{L}$ an invertible sheaf on Y, one has the functorial pullback

$$f^* : \widetilde{\mathrm{CH}}^n(Y, \mathcal{L}) \to \widetilde{\mathrm{CH}}^n(X, f^*\mathcal{L}).$$

But for the proper push-forward, one needs to include the orientation sheaf, this being the usual relative dualizing sheaf $\omega_f := \omega_{X/k} \otimes f^*\omega_{Y/k}^{-1}$, where $\omega_{X/k} := \det \Omega^1_{X/k}$ is the sheaf of top-dimensional forms. The push-forward takes the form

$$f_* : \widetilde{\mathrm{CH}}^n(X, \omega_f \otimes f^*\mathcal{L}) \to \widetilde{\mathrm{CH}}^{n-d}(Y, \mathcal{L}).$$

This limits the possible twists $\widetilde{\mathrm{CH}}^n(X, \mathcal{M})$ for which a push-forward f_* is even defined; this type of restricted push-forward is typical of so-called SL-oriented theories, such as hermitian K-theory. See [1] for a detailed discussion of SL-oriented theories and [31, CHAP. 3] for the details concerning the push-forward in $\widetilde{\mathrm{CH}}^*$.

4.2. The homotopy t-structure and Morel's theorem

Building on the Bloch–Kato formula, $\mathrm{CH}^n(X) \cong H^n(X_{\mathrm{Nis}}, \mathcal{K}^M_{n,X})$, one can construct a good bigraded cohomology theory $\mathrm{EM}(\mathcal{K}^M_*)^{**}$ by using all the cohomology groups. To get the correct bigrading, one should set

$$\mathrm{EM}(\mathcal{K}^M_*)^{a,b}(X) := H^{a-b}(X_{\mathrm{Nis}}, \mathcal{K}^M_b),$$

giving in particular $\mathrm{EM}(\mathcal{K}^M_*)^{2n,n}(X) = \mathrm{CH}^n(X)$. It was recognized early on that this theory is not the sought-after motivic cohomology, for instance, for $X = \mathrm{Spec}\, F$, F a field, one gets exactly the Milnor K-theory of F, and none of the other parts of the K-theory of F. In spite of this, this theory and the similarly defined theory for Milnor–Witt K-theory have a natural place in the universe of motivic cohomology theories, which we now explain.

The classical stable homotopy category SH is a triangulated category with a natural t-structure measuring connectedness, mentioned in Section 2.5. For SH, the truncations give the terms in the Moore–Postnikov tower

$$\cdots \to \tau_{\geq n+1}E \to \tau_{\geq n}E \to \cdots \to E$$

with $\tau_{\geq n}E \to E$ characterized by inducing an isomorphism on π_m for $m \geq n$ and with $\pi_m\tau_{\geq n}E = 0$ for $m < n$. The heart of SH is the category of spectra E with $\pi_m E = 0$ for $m \neq 0$, which are just the Eilenberg–MacLane spectra $\mathrm{EM}(A)$, A an abelian group. Thus, the heart of SH is **Ab** and the cohomology theory represented by $\tau_0 E$ is

$$\mathrm{EM}(\pi_0 E)^n(X) := H^n(X, \pi_0 E),$$

singular cohomology with coefficients in the abelian group $\pi_0 E$.

We have a parallel t-structure on $\mathrm{SH}(k)$, introduced by Morel [92, §5.2], called the *homotopy t-structure* (and *not* coming from Voevodsky's slice tower discussed in Section 2.5). This is similar to the t-structure on SH, where one takes into account the fact that one has bigraded homotopy sheaves $\pi_{a,b}E$ for $E \in \mathrm{SH}(k)$, rather than a $\mathbb{Z}$-graded family of homotopy groups $\pi_n E$ for $E \in \mathrm{SH}$. The truncation $\tau_{\geq n}E$ is characterized by

$$\pi_{a,b}(\tau_{\geq n}E) = \begin{cases} \pi_{a,b}(E) & \text{if } a-b \geq n, \\ 0 & \text{if } a-b < n \end{cases}$$

Recalling that the sphere $S^{a,b}$ is $S^{a-b} \wedge \mathbb{G}_m^b$, the homotopy t-structure on $\mathrm{SH}(k)$ is measuring S^1-connectedness, instead of the $\mathbb{P}^1$-connectedness measured by Voevodsky's slice tower.

We denote the 0th truncation $\tau_0 E$ for $E \in \mathrm{SH}(k)$ by $\mathrm{EM}(\pi_{-*,-*}E)$; the notation comes from Morel's identification of the heart with his category of *homotopy modules*; for details, see [92, §5.2]. The corresponding cohomology theory satisfies, for $X \in \mathrm{Sm}_k$,

$$\mathrm{EM}(\pi_{-*,-*}E)^{a,b}(X) = H^{a-b}\big(X_{\mathrm{Nis}}, \pi_{-b,-b}(E)\big).$$

Here we have Morel's fundamental theorem [92, THEOREM 6.4.1] computing τ_0 of the sphere spectrum $1_k \in \mathrm{SH}(k)$.

Theorem 4.4 (Morel). *Let k be a perfect field. Then there are canonical isomorphisms of sheaves on* Sm_k

$$\pi_{-n,-n}(1_k) = \mathcal{K}_n^{\mathrm{MW}}.$$

Consequently,

$$\tau_0 1_k = \mathrm{EM}(\mathcal{K}_*^{\mathrm{MW}})$$

and

$$\mathrm{EM}(\mathcal{K}_*^{\mathrm{MW}})^{a,b}(X) = H^{a-b}(X_{\mathrm{Nis}}, \mathcal{K}_{b,X}^{\mathrm{MW}}).$$

Going back in time a bit, we have the theorem of Totaro [115] and Nesterenko–Suslin [96]

$$H^n\big(F, \mathbb{Z}(n)\big) \cong K_n^M(F)$$

for F a field. Combined with the isomorphism

$$s_0 1_k \cong H\mathbb{Z}$$

of [9, 85, 122], we have

Theorem 4.5. *Let k be a perfect field. Then*

$$\tau_0 s_0 1_k = \tau_0 H\mathbb{Z} = \mathrm{EM}(\mathcal{K}_*^M)$$

and

$$\mathrm{EM}(\mathcal{K}_*^M)^{a,b}(X) = H^{a-b}(X_{\mathrm{Nis}}, \mathcal{K}_{b,X}^M)$$

for $X \in \mathrm{Sm}_k$.

Bachmann proves an extension of this result. Recall Voevodsky's slice tower

$$\cdots \to f_{n+1}E \to f_n E \to \cdots \to f_0 E \to \cdots \to E$$

with $s_n E$ the layer given by the distinguished triangle

$$f_{n+1}E \to f_n E \to s_n E \to f_{n+1}E[1].$$

Recall that this is *not* the truncation tower of a t-structure, as the subcategories defined by the layers $s_n := f_n/f_{n+1}$ are triangulated categories, not abelian categories.

Proposition 4.6 ([7, LEMMA 12]). *Let $1_k \to \mathrm{EM}(\mathcal{K}_*^M)$ be the composition $1_k \to \tau_0 1_k = \mathrm{EM}(\mathcal{K}_*^{\mathrm{MW}}) \to \mathrm{EM}(\mathcal{K}_*^M)$, the latter map induced by the surjection $\mathcal{K}_*^{\mathrm{MW}} \to \mathcal{K}_*^M$. Then the induced maps*

$$s_0(1_k) \to s_0\mathrm{EM}(\mathcal{K}_*^M) \leftarrow f_0\mathrm{EM}(\mathcal{K}_*^M) = f_0\tau_0 H\mathbb{Z}$$

are all isomorphisms, so all of these objects are isomorphic to the motivic cohomology spectrum $H\mathbb{Z}$.

The truncation functors for the homotopy t-structure and for the Voevodsky slice tower do not commute. Since 1_k is effective, we have $f_0 1_k = 1_k$ and so $\tau_0 f_0 1_k = \tau_0 1_k = \mathrm{EM}(\mathcal{K}_*^{\mathrm{MW}})$. The truncations in the other order give us something new.

4.3. Milnor–Witt motivic cohomology

Definition 4.7 ([7, NOTATION, P. 1134, JUST BEFORE LEMMA 12]). Let k be a perfect field. Define the *Milnor–Witt motivic cohomology spectrum* $\tilde{H}\mathbb{Z} \in \mathrm{SH}(k)^{\mathrm{eff}}$ by

$$\tilde{H}\mathbb{Z} := f_0(\tau_0 1_k) = f_0\mathrm{EM}(\mathcal{K}_*^{\mathrm{MW}}).$$

The canonical map $\tau_0 1_k \to \tau_0 s_0 1_k = \tau_0 H\mathbb{Z}$ induces the map

$$\tilde{H}\mathbb{Z} = f_0(\tau_0 1_k) \xrightarrow{\Xi} f_0\tau_0 H\mathbb{Z} = H\mathbb{Z}.$$

For $X \in \mathrm{Sm}_k$, the *Milnor–Witt motivic cohomology in bidegree* (a, b) is defined as $\tilde{H}\mathbb{Z}^{a,b}(X)$.

Remarkably, one can compute $\tilde{H}\mathbb{Z}^{a,b}(X)$ in terms of the Milnor–Witt sheaves, at least for some of the indices (a, b); one also recovers the Chow–Witt groups. For $X = \mathrm{Spec}\, F$, the spectrum of a field F, one has a complete computation in terms of the Milnor–Witt K-groups and the usual motivic cohomology $H\mathbb{Z}^{a,b}(X) := H^a(X, \mathbb{Z}(b))$.

Theorem 4.8 (Bachmann). *For $X \in \mathrm{Sm}_k$ and $b \le 0$, there are natural isomorphisms*

$$\tilde{H}\mathbb{Z}^{a,b}(X) \cong H^{a-b}(X_{\mathrm{Nis}}, \mathcal{K}_{b,X}^{\mathrm{MW}}) = \begin{cases} H^{a-b}(X_{\mathrm{Nis}}, \mathcal{W}_X) & \text{for } b < 0, \\ H^{a-b}(X_{\mathrm{Nis}}, \mathcal{GW}_X) & \text{for } b = 0. \end{cases}$$

Here $\mathcal{W}_X$ is the sheaf of Witt groups and $\mathcal{GW}_X$ is the sheaf of Grothendieck–Witt rings.

For $X \in \mathrm{Sm}_k$ and $n \in \mathbb{Z}$, we have

$$\tilde{H}\mathbb{Z}^{2n,n}(X) \cong \widetilde{\mathrm{CH}}^n(X).$$

For F a field, we have isomorphisms

$$\tilde{H}\mathbb{Z}^{a,b}(\mathrm{Spec}\, F) \cong \begin{cases} K_n^{\mathrm{MW}}(F) & \text{for } a = b = n, \\ H\mathbb{Z}^{a,b}(\mathrm{Spec}\, F) & \text{for } a \neq b. \end{cases}$$

This follows from

Theorem 4.9 ([7, THEOREM 17]). *Let $\tilde{\mathcal{H}}\mathbb{Z}^{a,b}$, $\mathcal{H}\mathbb{Z}^{a,b}$ denote the respective homotopy sheaves $\pi_{-a,-b}(\tilde{H}\mathbb{Z})$, $\pi_{-a,-b}(H\mathbb{Z})$. Then for $a \neq b$, the map*

$$\Xi^{a,b} : \tilde{\mathcal{H}}\mathbb{Z}^{a,b} \to \mathcal{H}\mathbb{Z}^{a,b}$$

is an isomorphism. Moreover, we have canonical isomorphisms $\tilde{\mathcal{H}}\mathbb{Z}^{b,b} = \mathcal{K}_b^{\mathrm{MW}}$, $\mathcal{H}\mathbb{Z}^{b,b} = \mathcal{K}_b^M$, *and* $\Xi^{a,b} : \tilde{\mathcal{H}}\mathbb{Z}^{b,b} \to \mathcal{H}\mathbb{Z}^{b,b}$ *is canonical surjection* $\mathcal{K}_b^{\mathrm{MW}} \to \mathcal{K}_b^M$.

To prove Theorem 4.8, one applies this to the local–global spectral sequence

$$E_2^{p,q}(n) := H^p(X_{\mathrm{Nis}}, \tilde{\mathcal{H}}\mathbb{Z}^{q,n}) \Rightarrow \tilde{H}\mathbb{Z}^{p+q,n}(X),$$

noting that $\mathcal{H}\mathbb{Z}^{q,n} = 0$ for $n < 0$. This implies that the Gersten resolution of $\mathcal{H}\mathbb{Z}^{q,n}$ has length $\leq n$ and thus $H^p(X_{\mathrm{Nis}}, \mathcal{H}\mathbb{Z}^{q,n}) = 0$ for $p > n$.

In general, one can approximate $\tilde{H}\mathbb{Z}^{a,b}(X)$ using the local–global sequence. Combined with Theorem 4.9 and the exact sheaf sequence

$$0 \to \mathcal{I}^{n+1} \to \mathcal{K}_n^{\mathrm{MW}} \to \mathcal{K}_n^M \to 0,$$

this tells us that the Milnor–Witt cohomology of X is built out of the usual motivic cohomology combined with information arising from quadratic forms.

4.4. Milnor–Witt motives

Rather than pulling the Milnor–Witt cohomology out of the motivic stable homotopy hat, there is another construction that is embedded in a Voevodsky-type triangulated category built out of a modified category of correspondences. We refer to [8] and [31] for details.

The Chow–Witt groups on a smooth X have been defined using the Rost–Schmid complex; one can also define Chow–Witt cycles with a fixed support using a modified version of the Rost–Schmid complex.

Definition 4.10. Let X be a smooth k-scheme, $\mathcal{L}$ an invertible sheaf on X, and $T \subset X$ a closed subset. The *nth $\mathcal{L}$-twisted Rost–Schmid complex* with supports in T, $\mathrm{RS}_T^*(X, n; \mathcal{L})$, is the subcomplex of $\mathrm{RS}^*(X, \mathcal{L}, n)$ with

$$\mathrm{RS}_T^m(X, \mathcal{L}, n) := \bigoplus_{x \in T \cap X^{(p)}} K_{n-m}^{\mathrm{MW}}\Big(k(x); \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \bigwedge^m (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee\Big) \subset \mathrm{RS}^m(X, \mathcal{L}, n).$$

The usual arguments used to prove Gersten's conjecture yield the following result.

Lemma 4.11. *Let X be a smooth k-scheme, $\mathcal{L}$ an invertible sheaf on X, and $T \subset X$ a closed subset. The cohomology with support $H_T^p(X, \mathcal{K}_n^{\mathrm{MW}}(\mathcal{L})_X)$ is computed as*

$$H_T^p\big(X, \mathcal{K}_n^{\mathrm{MW}}(\mathcal{L})_X\big) = H^p\big(\mathrm{RS}_T^*(X, \mathcal{L}, n)\big).$$

Suppose T has pure codimension n on X. Then $X^{(m)} \cap T = \emptyset$ for $m < n$, $X^{(n)} \cap T$ is the finite set of generic points $T^{(0)}$ of T and $X^{(n+1)} \cap T = T^{(1)}$ is the set of codimension one points of T. This gives us the exact sequence

$$\begin{aligned} 0 \to H_T^n\big(X, \mathcal{K}_n^{\mathrm{MW}}(\mathcal{L})_X\big) &\to \bigoplus_{z \in T^{(0)}} \mathrm{GW}\big(k(z), \det{}^{-1}(\mathfrak{m}_z/\mathfrak{m}_z^2) \otimes \mathcal{L}\big) \\ &\to \bigoplus_{z \in T^{(1)}} W\big(k(z), \det{}^{-1}(\mathfrak{m}_z/\mathfrak{m}_z^2) \otimes \mathcal{L}\big) \end{aligned}$$

which allows us to think of $H^n_T(X, \mathcal{K}^{\mathrm{MW}}_n(\mathcal{L})_X)$ as the group of "Grothendieck–Witt cycles" supported on T, whose definition we hinted at in the beginning of this section. We write this as $\tilde{Z}^n_T(X, \mathcal{L}, n)$, with the warning that this is only defined for T a closed subset of a smooth X of pure codimension n.

Note that the fact that T has pure codimension n implies that there are no relations in $H^n_T(X, \mathcal{K}^{\mathrm{MW}}_n(\mathcal{L})_X)$ coming from $K^{\mathrm{MW}}_1(k(w))$ for w a codimension $n-1$ point of X. For similar reasons, the corresponding group for the Chow groups, $H^n_T(X, \mathcal{K}^M_{n,X})$, is just the subgroup $Z^n_T(X)$ of $Z^n(X)$ freely generated by the irreducible components of T, that is, the group of codimension n cycles on X with support contained in T.

For $T \subset T' \subset X$, two codimension-n closed subsets, we have the evident map $\tilde{Z}^n_T(X, \mathcal{L}, n) \to \tilde{Z}^n_{T'}(X, \mathcal{L}, n)$. The rank map $\mathrm{GW}(-) \to \mathbb{Z}$ gives the homomorphism $\tilde{Z}^n_T(X, \mathcal{L}, n) \to Z^n_T(X)$.

Definition 4.12. For X, Y in Sm_k, let $\mathcal{A}(X, Y)$ be the set of closed subsets $T \subset X \times Y$ such that each component of T is finite over X and maps surjectively onto an irreducible component of X. We make $\mathcal{A}(X, Y)$ a poset by the inclusion of closed subsets.

Note that if Y is irreducible of dimension n, then a closed subset $T \subset X \times Y$ is in $\mathcal{A}(X, Y)$ if and only if T is finite over X and has pure codimension n on $X \times Y$.

Definition 4.13 (Calmès–Fasel [31, §4.1]). Let X, Y be in Sm_k and suppose Y is irreducible of dimension n. Define

$$\widetilde{\mathrm{Corr}}_k(X, Y) = \mathrm{colim}_{T \in \mathcal{A}(X,Y)} \tilde{Z}^n_T(X \times Y, p_2^*\omega_{Y/k}).$$

Extend the definition to general Y by additivity.

Using the functorial properties of pullback, intersection product and proper push-forward for the Chow–Witt groups with support, we have a well-defined composition law

$$\widetilde{\mathrm{Corr}}_k(Y, Z) \times \widetilde{\mathrm{Corr}}_k(X, Y) \to \widetilde{\mathrm{Corr}}_k(X, Z)$$

via the same formula used to define the composition in Cor_k,

$$Z_2 \circ Z_1 := p_{XZ*}\big(p^*_{YZ}(Z_2) \cap p^*_{XY}(Z_1)\big).$$

The twisting by the relative dualizing sheaf in the definition of $\widetilde{\mathrm{Corr}}_k(-,-)$ is exactly what is needed for the push-forward map p_{XZ*} to be defined.

This defines the additive category $\widetilde{\mathrm{Corr}}_k$ with objects Sm_k and morphisms $\widetilde{\mathrm{Corr}}_k(X, Y)$. The rank map gives an additive functor

$$\mathrm{rnk} : \widetilde{\mathrm{Corr}}_k \to \mathrm{Cor}_k.$$

One then follows the program used by Voevodsky to define the abelian category of Nisnevich sheaves with Milnor–Witt transfers, $\mathrm{Sh}^{\mathrm{MWtr}}_{\mathrm{Nis}}(k)$, and then $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k) \subset D(\mathrm{Sh}^{\mathrm{MWtr}}_{\mathrm{Nis}}(k))$ as the full subcategory of complexes with strictly $\mathbb{A}^1$-homotopy invariant cohomology sheaves. One has the localization functor

$$\tilde{L}_{\mathbb{A}^1} : D\big(\mathrm{Sh}^{\mathrm{MWtr}}_{\mathrm{Nis}}(k)\big) \to \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)$$

constructed using the Suslin complex, the representable sheaves $\tilde{\mathbb{Z}}^{\mathrm{tr}}(X)$ for $X \in \mathrm{Sm}_k$, their corresponding motives $\tilde{M}^{\mathrm{eff}}(X) := \tilde{L}_{\mathbb{A}^1}(\tilde{\mathbb{Z}}^{\mathrm{tr}}(X)) \in \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)$ and the Tate motives $\tilde{\mathbb{Z}}(n)$ arising from the reduced motive of $\mathbb{P}^1$. Finally, one constructs $\widetilde{\mathrm{DM}}(k)$ as a category of $\tilde{\mathbb{Z}}(1)$-spectra in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)$ and we have the motive $\tilde{M}(X)$ defined as the suspension spectrum of $\tilde{M}^{\mathrm{eff}}(X)$.

Definition 4.14. For $X \in \mathrm{Sm}_k$, categorical Milnor–Witt cohomology is

$$H^p\big(X, \tilde{\mathbb{Z}}(q)\big) := \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k)}\big(\tilde{M}(X), \tilde{\mathbb{Z}}(q)[p]\big).$$

Theorem 4.15. *There is a natural isomorphism*

$$H^p\big(X, \tilde{\mathbb{Z}}(q)\big) \cong H\tilde{\mathbb{Z}}^{p,q}(X).$$

The proof is very much the same as for motivic cohomology. One shows there is an equivalence of $\widetilde{\mathrm{DM}}(k)$ with the homotopy category of $H\tilde{\mathbb{Z}}$-modules (this is [8, THEOREM 5.2]). This gives an adjunction

$$H\tilde{\mathbb{Z}} \wedge - : \mathrm{SH}(k) \rightleftarrows \widetilde{\mathrm{DM}}(k) : \widetilde{\mathrm{EM}}$$

with $H\tilde{\mathbb{Z}} \wedge -$ the free $H\tilde{\mathbb{Z}}$ module functor and the Eilenberg–MacLane functor $\widetilde{\mathrm{EM}}$ the forgetful functor. This gives $\widetilde{\mathrm{EM}}(\tilde{\mathbb{Z}}(0)) = H\tilde{\mathbb{Z}}$, $\tilde{M}(X) = H\tilde{\mathbb{Z}} \wedge \Sigma^\infty_{\mathbb{P}^1} X_+$, and induces the isomorphism

$$\begin{aligned} H^p\big(X, \tilde{\mathbb{Z}}(q)\big) &= \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k)}\big(\tilde{M}(X), \tilde{\mathbb{Z}}(q)[p]\big) \\ &\cong \mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q} H\tilde{\mathbb{Z}}) = H\tilde{\mathbb{Z}}^{p,q}(X). \end{aligned}$$

5. CHOW GROUPS AND MOTIVIC COHOMOLOGY WITH MODULUS

Up to now, all the version of motivic cohomology we have considered share the $\mathbb{A}^1$-homotopy invariance property, namely, that $H^*(X, \mathbb{Z}(*)) \cong H^*(X \times \mathbb{A}^1, \mathbb{Z}(*))$; essentially by construction, this property is enjoyed by all theories that are represented in the motivic stable homotopy category. Although this is a fundamental property controlling a large collection of cohomology theories, this places a serious restriction in at least two naturally occurring areas.

One is the use of deformation theory. This relies on having useful invariants defined on non-reduced schemes, but a cohomology theory that satisfies $\mathbb{A}^1$-invariance will not distinguish between a scheme and its reduced closed subscheme. The second occurs in ramification theory. An $\mathbb{A}^1$-homotopy invariant theory will not detect Artin–Schreyer covers, and would not give invariants that detect wild ramification.

Fortunately, we have an interesting cohomology theory that is not $\mathbb{A}^1$-homotopy invariant, namely, algebraic K-theory, that we can use as a model for a general theory. Algebraic K-theory does satisfy the $\mathbb{A}^1$-invariance property when restricted to regular schemes, but in general this fails. Besides allowing K-theory to have a role in deformation theory and ramification theory, this lack of $\mathbb{A}^1$-invariance gives rise to interesting invariants of singularities.

5.1. Higher Chow groups with modulus

The theory of Chow groups with modulus attempts to refine the classical theory of the Chow groups to be useful in both of these areas. This is still a theory in the process of development; just as in the early days of motivic cohomology, many approaches are inspired by properties of algebraic K-theory.

The tangent space to the functor $X \mapsto \mathcal{O}_X^\times$ is given by the structure sheaf, $X \mapsto \mathcal{O}_X$, via the isomorphism

$$\mathcal{O}^\times_{X[\varepsilon]/(\varepsilon^2)} \cong \mathcal{O}_X^\times \oplus \varepsilon \cdot \mathcal{O}_X.$$

Via the isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$, this shows that the tangent space at X to the functor $\mathrm{Pic}(-)$ is $H^1(X, \mathcal{O}_X)$.

In [23], Bloch computes the tangent space to K_2 (on local $\mathbb{Q}$-algebras), giving the isomorphism of sheaves on X_{Zar} (for X a $\mathbb{Q}$-scheme)

$$\mathcal{K}_{2,X[\varepsilon]/(\varepsilon^2)} \cong \mathcal{K}_{2,X} \oplus \Omega_X$$

where Ω_X is the sheaf of absolute Kähler differentials. Bloch then uses his formula from [22],

$$H^2(X_{\mathrm{Zar}}, \mathcal{K}_2) \cong \mathrm{CH}^2(X),$$

to justify defining $\mathrm{CH}^2(X[\varepsilon]/(\varepsilon^2))$ as $H^2(X[\varepsilon]/(\varepsilon^2)_{\mathrm{Zar}}, \mathcal{K}_2)$, giving

$$\mathrm{CH}^2\big(X[\varepsilon]/(\varepsilon^2)\big) = \mathrm{CH}^2(X) \oplus H^2(X, \Omega_X).$$

For X a smooth projective surface over $\mathbb{C}$ with $H^2(X, \mathcal{O}_X) \neq 0$, the exact sheaf sequence

$$0 \to \Omega_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega_X \to \Omega_{X/\mathbb{C}} \to 0$$

along the fact that $\Omega_{\mathbb{C}/\mathbb{Q}}$ is a $\mathbb{C}$-vector space of uncountable dimension show that $\Omega_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} H^2(X, \mathcal{O}_X)$ makes a huge contribution to the tangent space $H^2(X, \Omega_X)$ of $\mathrm{CH}^2(-)$ on X. This is reflected in Mumford's result [95], that if $H^2(X, \mathcal{O}_X) \cong H^0(X, \Omega^2_{X/\mathbb{C}})$ is nonzero, then $\mathrm{CH}^2(X)$ is "infinite-dimensional," and gives some evidence for Bloch's conjecture [23, CONJECTURE (0.4)] on 0-cycles on surfaces X with $H^0(X, \Omega^2_{X/\mathbb{C}}) = 0$.

The algebraic cycles have disappeared in this approach to Chow groups of non-reduced schemes. Bloch and Esnault [26] gave the first construction of a cycle-theoretic theory that could say something interesting about higher cycles on the non-reduced scheme $\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$. In a second paper [27], they modified and extended this construction to give a theory of *additive higher Chow groups with modulus* m, for the field k. This was motivated by Bloch's earlier use of K-theory on the affine line, relative to $\{0, 1\}$, to study K_3. Letting 1 tend to 0, they were led to consider the relative K-theory space $K(k[\varepsilon], (\varepsilon^2))$, this being the homotopy fiber of the restriction map $K(k[\varepsilon]) \to K(k[\varepsilon]/\varepsilon^2)$, whose homotopy groups are the relative K-theory groups $K_n(k[\varepsilon], (\varepsilon^2))$. Replacing 2 with $m \geq 2$ gives the relative K-theory groups $K_n(k[\varepsilon], (\varepsilon^m))$. This led to the consideration of a complex of cycles on $\mathrm{Spec}\, k[\varepsilon]$, with an additional condition imposed on the mth order limiting behavior of the cycles; an explicit construction of such a cycle complex with modulus, $z^q(k, *, m)$ was given in [27]. The homology $A\mathrm{CH}^q(k, p, m) := H_p(z^q(k, *, m))$ defines the *additive codimension* *q higher Chow groups with modulus m for* Spec k. Bloch–Esnault recover the computation $ACH^n(k, n-1, 2) \cong \Omega_k^{n-1}$ from [26], and relate the additive analogue of weight two K_3, $ACH^2(k, 2, 2)$, with the additive dilogarithm of Cathelineau [32].

Rülling [104] studied the projective system

$$\cdots \to ACH^n(k, n-1, m+1) \to ACH^n(k, n-1, m) \to \cdots .$$

He showed this is endowed with additional endomorphisms F_n and V_n, and the graded group $\bigoplus_n ACH^n(k, n-1, *+1)_{*\geq 2}$ has the structure of a pro-differential graded algebra. In fact, we have

Theorem 5.1 (Rülling). *Let k be a field of characteristic $\neq 2$. The pro-dga $\bigoplus_n ACH^n(k, n-1, *+1)$, with F_n as Frobenius and V_n as Verschiebung, is isomorphic to the de Rham–Witt complex of Madsen–Hesselholt,*

$$\bigoplus_n ACH^n(k, n-1, *+1) \cong \bigoplus_n W_*\Omega_k^{n-1}.$$

With essentially the same definition as given by Bloch–Esnault, the additive cycle complex and additive Chow groups were extended to arbitrary k-schemes Y by Park [98], replacing $\mathbb{A}^1$ and divisor $m \cdot 0$ with the scheme $Y \times \mathbb{A}^1$ and divisor $m \cdot Y \times 0$. Binda and Saito [20] went one step further, defining complexes $z^q(X, D, *)$ for a pair (X, D) of a finite type separated k-scheme X and a Cartier divisor D, using essentially the same definition as before. The homology is the *higher Chow group with modulus*

$$\mathrm{CH}^q(X, D, p) := H_p\big(z^q(X, D, *)\big).$$

The constructions of Bloch–Esnault, Park, and Binda–Saito all use a cubical model of Bloch's cycle complex. Here one replaces the algebraic n-simplex, $\Delta_k^n = \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$, with the algebraic n-cube

$$\square^n := \big(\mathbb{P}^1 \setminus \{1\}, 0, \infty\big)^n.$$

The notation means that one considers $(\mathbb{P}^1 \setminus \{1\})^n \cong \mathbb{A}^n$ with its "faces" defined by setting some of the coordinates equal to 0 or ∞. The corresponding cycle complex $z^q(X, *)_c$ has degree n component $z^q(X, n)_c$ the codimension q cycles on $X \times \square^n$ that intersect $X \times F$ properly for all faces F of $\square^n$; one also needs to quotient out by the degenerate cycles, these being the ones that come by pullback via projection to a $\square^m$ with $m < n$. The differential is again an alternating sum of restrictions to the maximal faces $t_i = 0$ and $t_i = \infty$.

This complex also computes the motivic cohomology of X, just as Bloch's simplicial cycle complex does. In the Binda–Saito construction, the modulus condition arises by considering the closed box $\overline{\square}^n := (\mathbb{P}^1)^n$. Let $F_n^i \subset (\mathbb{P}^1)^n$ be the divisor defined by $t_i = 1$ and let $F_n = \sum_{i=1}^n F_n^i$. In $(\mathbb{P}^1)^n \times X$ we have two distinguished Cartier divisors, $(\mathbb{P}^1)^n \times D$ and $F_n \times X$. A subvariety $Z \subset (\mathbb{P}^1 \setminus \{1\})^n \times X$ that is in $z^q(X, n)_c$ satisfies the modulus condition if

$$p^*(F_n \times X) \geq p^*\big((\mathbb{P}^1)^n \times D\big)$$

where $p : \bar{Z}^N \to (\mathbb{P}^1)^n \times X$ is the normalization of the closure of Z in $\overline{\square}^n \times X$. Restricting to the subgroup of $Z^q(\square^n \times X)$ generated by codimension q subvarieties $Z \subset \square^n \times X$ that intersect faces properly and satisfy the modulus condition yields the cycle complex with modulus $z^q(X; D, *) \subset z^q(X, *)_c$; the higher Chow groups with modulus is then defined as

$$\mathrm{CH}^q(X; D, p) := H_p\big(z^q(X; D, *)\big).$$

The second construction of Bloch–Esnault, and Park's generalization, are recovered as the special cases $X = \mathbb{A}^1_k$ and $D = m \cdot 0$ in the Bloch–Esnault version and $X = Y \times \mathbb{A}^1$, $D = m \cdot Y \times 0$ in Park's version.

For X a finite type k-scheme, recall the Bloch motivic complex $\mathbb{Z}_{\mathrm{Bl}}(q)^*_X$ defined as the Zariski sheafification of the presheaf $U \mapsto z^q(X, 2q - *)$ (this is already a Nisnevich sheaf). Bloch's cycle complexes satisfy an important localization property: the natural maps to Zariski and Nisnevich hypercohomology

$$H^p\big(z^q(X, 2q - *)\big) \to \mathbb{H}^p\big(X_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{Bl}}(q)^*_X\big) \to \mathbb{H}^p\big(X_{\mathrm{Nis}}, \mathbb{Z}_{\mathrm{Bl}}(q)^*_X\big)$$

are isomorphisms. This fails for the cycle complex with modulus, although the comparison between the Zariski and Nisnevich hypercohomology seems to be still an open question.

Iwasa and Kai consider the Nisnevich sheafification $\mathcal{Z}(q)^*_{(X;D)}$ of the presheaf

$$U \mapsto z^q(U; D \times_X U, 2q - *).$$

We call $\mathbb{H}^p(X_{\mathrm{Nis}}, \mathcal{Z}(q)^*_{(X;D)})$ the *motivic cohomology with modulus* for (X, D). Kai [74] shows that this sheafified version has contravariant functoriality. Iwasa and Kai [67] construct Chern class maps from relative K-theory

$$c_{p,q} : K_{2q-p}(X; D) \to \mathbb{H}^p\big(X_{\mathrm{Nis}}, \mathcal{Z}(q)^*_{X,\mathrm{Nis}}\big).$$

5.2. 0-cycles with modulus and class field theory

There is a classical theory of 0-cycles on a smooth complete curve C with a modulus condition at a finite set of points S, due to Rosenlicht and Serre [109, 111]. The idea is quite simple, instead of relations coming from divisors (zeros minus poles) of an arbitrary rational function f, f is required to have a power series expansion at each point $p \in S$, with leading term 1 and the next nonzero term of the form $ut_p^{n_p}$, with $u(p) \neq 0$, t_p a local coordinate at p and the integer $n_p > 0$ being the "modulus." This is applied to the class field theory of a smooth open curve $U \subset C$ over a finite field [109, THEOREM 4], that identifies the inverse limit of the groups of degree 0 cycle classes on U, with modulus supported in $C \setminus U$, with the kernel of the map $\pi_1^{\text{ét}}(U)^{ab} \to \mathrm{Gal}(\bar{k}/k)$.

In their class field theory for higher-dimensional varieties, Kato and Saito [80] introduce a group of 0-cycles on a k-scheme X with modulus D, defined by

$$\mathrm{CH}_0(X, D) := H^n(X, \mathcal{K}^M_{n,(X,D)})$$

with $\mathcal{K}^M_{n,(X,D)}$ a relative version of the Milnor K-theory sheaf, recalling Kato's isomorphism $H^n(X, \mathcal{K}^M_n) \cong \mathrm{CH}^n(X)$ for X a smooth k-scheme [78]. Kerz and Saito give a different definition of a group of relative 0-cycles $C(X, D)$ on a normal k-scheme X with effective Cartier divisor D such that $X \setminus D$ is smooth. It follows from their comments in [81, DEFINITION 1.6] that $C(X,D) = \mathrm{CH}^n(X;D,0)$ for X of dimension n, and it is easy to see that the Kato–Saito and Kerz–Saito relative 0-cycles agree with the Rosenlicht–Serre groups in the case of curves.

Kerz and Saito consider a smooth finite-type k-scheme U, choose a normal compactification X and define the topological group $C(U) := \lim_D C(X,D)$, as D runs over effective Cartier divisors on X, supported in $X \setminus U$, and with each $C(X,D)$ given the discrete topology. They show that $C(U)$ is independent of the choice of X, and their main result generalizes class field theory for smooth curves over a finite field as described above.

Theorem 5.2 ([81, THEOREM 3.3]). *Let k be a finite field of characteristic $\neq 2$ and let U be a smooth variety over k, Then $C(U)$ is isomorphic as topological group to a dense subgroup of the abelianized étale fundamental group $\pi_1^{\acute{e}t}(U)^{ab}$ and this isomorphism induces an isomorphism of the degree 0 part $C(U)^0$ of $C(U)$ with the kernel $\pi_1^{\acute{e}t}(U)_0^{ab}$ of $\pi_1^{\acute{e}t}(U)^{ab} \to \pi_1^{\acute{e}t}(k)$.*

5.3. Categories of motives with modulus

There has been a great deal of interest in constructing a categorical framework for motivic cohomology with modulus. A central issue is the lack of $\mathbb{A}^1$-homotopy invariance for this theory, which raises the question of what type of homotopy invariance should replace this.

One direction has been the construction of a reasonable replacement for the category of homotopy invariant Nisnevich sheaves with transfers. A non-homotopy invariant version has been developed via the theory of *reciprocity sheaves*, the name coming from the reciprocity laws in class field theory of curves and its relation to the group of 0-cycles with modulus of Rosenlicht–Serre. We will say a bit about reciprocity sheaves later on, in the context of motives for log schemes Section 5.4.

For now, we will look at categories of motives with modulus constructed on the Voevodsky model by introducing a new notion of correspondence and a suitable replacement for $\mathbb{A}^1$-homotopy invariance.

Looking at algebraic K-theory, the closest replacement for $\mathbb{A}^1$-homotopy invariance seems to be the $\mathbb{P}^1$-bundle formula

$$K_n(X \times \mathbb{P}^1) \cong K(X) \cdot [\mathcal{O}_{X \times \mathbb{P}^1}] \oplus K(X) \cdot \big[\mathcal{O}_{X \times \mathbb{P}^1}(-1)\big],$$

valid for a general scheme X. This has led to attempts to create a category of motives with modulus based on a notion of "$\overline{\square}$-invariance."

Here one has the problem that $\mathbb{P}^1$ does not have the structure of an interval, a structure enjoyed by $\mathbb{A}^1$. One considers $\mathbb{A}^1$ together with "endpoints" $0, 1$. Following the general theory of a site with interval, as developed by Morel–Voevodsky [94, CHAP. 2], one needs the multiplication map $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ to allow one to consider $(\mathbb{A}^1, 0, 1)$ as an abstract interval. In the construction of the cycle complex with modulus, one identifies $(\mathbb{A}^1, 0, 1)$ with $(\mathbb{P}^1 \setminus \{1\}, 0, \infty)$, and the corresponding multiplication map $m' : (\mathbb{P}^1 \setminus \{1\}) \times (\mathbb{P}^1 \setminus \{1\}) \to \mathbb{P}^1 \setminus \{1\}$ only extends as a rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$. However, m' becomes a morphism after blowing up the point $(1,1)$, which suggests that one should consider the closure of the graph of m' in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as an allowable correspondence from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$.

With this as starting point, Kahn, Miyazaki, Saito, and Yamazaki [69–71] follow Voevodsky's program, defining a category of modulus correspondences $\underline{\mathrm{M}}\mathrm{Cor}_k$. Objects are pairs $(\bar{M}, M^\infty)$ with $\bar{M}$ a separated finite-type k-scheme and M^∞ an effective Cartier divisor on $\bar{M}$ such that the open complement $M^\circ := \bar{M} \setminus M^\infty$ is smooth. The morphism group $\underline{\mathrm{M}}\mathrm{Cor}_k((\bar{M}, M^\infty), (\bar{N}, N^\infty))$ is the subgroup of $\mathrm{Cor}_k(M^\circ, N^\circ)$ generated by subvarieties Z (finite and surjective over a component of M°) such that

(i) The closure $\bar{Z}$ of Z in $\bar{M} \times \bar{N}$ is proper over $\bar{M}$ (not necessarily finite).

(ii) Let $f : \bar{Z}^N \to \bar{M} \times \bar{N}$ be the normalization of $\bar{Z}$. Then

$$f^* p_1^* M^\infty \geq f^* p_2^* N^\infty.$$

The composition law in Cor_k preserves conditions (i) and (ii), giving the category $\underline{\mathrm{M}}\mathrm{Cor}_k$ with functor $\underline{\mathrm{M}}\mathrm{Cor}_k \to \mathrm{Cor}_k$ sending $(\bar{M}, M^\infty)$ to M° and with $\underline{\mathrm{M}}\mathrm{Cor}_k((\bar{M}, M^\infty), (\bar{N}, N^\infty)) \to \mathrm{Cor}_k(M^\circ, N^\circ)$ the inclusion. The product of pairs makes $\underline{\mathrm{M}}\mathrm{Cor}_k$ a symmetric monoidal category and the functor to Cor_k is symmetric monoidal.

Let $\overline{\square}$ be the object $(\mathbb{P}^1, \{1\})$. As hinted above, the closure of the graph of $m' : (\mathbb{P}^1 \setminus \{1\}) \times (\mathbb{P}^1 \setminus \{1\}) \to \mathbb{P}^1 \setminus \{1\}$ defines a morphism $m : \overline{\square} \times \overline{\square} \to \overline{\square}$ in $\underline{\mathrm{M}}\mathrm{Cor}_k$.

They then consider the abelian category of additive presheaves of abelian groups on $\underline{\mathrm{M}}\mathrm{Cor}_k$, $\underline{\mathrm{M}}\mathrm{PST}_k := \mathrm{PreSh}^{\mathbf{Ab}}(\underline{\mathrm{M}}\mathrm{Cor}_k)$. There is also a version MCor_k of *proper* modulus pairs (X, D), with X a proper k-scheme, as a full subcategory of $\underline{\mathrm{M}}\mathrm{Cor}_k$, with its presheaf category MPST_k.

They define a category of *effective proper motives with modulus*, $\mathrm{MDM}^{\mathrm{eff}}(k)$, by localizing the derived category $D(\mathrm{MPST}_k)$. Roughly speaking, they follow the Voevodsky program, replacing the $\mathbb{A}^1$-localization with $\overline{\square}$ localization. To get the proper Nisnevich localization is a bit technical; we refer the reader to [71, DEFINITION 1.3.9] for details.

There is still quite a bit that is not known. One central problem is how to realize the various constructions of the higher Chow groups with modulus as morphisms in a suitable triangulated category. There is a connection, at least for the modulus version of Suslin homology and the Suslin complex, which we now describe.

One can show that the cubical version of the Suslin complex

$$C_*^{\mathrm{Sus}}(X)_c(Y) := \mathrm{Hom}_{\mathrm{Cor}_k}(Y \times \square^*, X)/\operatorname{degn}$$

is naturally quasi-isomorphic to the simplicial version $C_*^{\mathrm{Sus}}(X)(Y)$, where $/\operatorname{degn}$ means taking the quotient by the image of the pullback maps via the projections $Y \times \square^n \to Y \times \square^{n-1}$. For a modulus pair (X, D), one can similarly form the *naive Suslin complex*

$$C_*^{\mathrm{Sus}}(X, D)(Y, E) := \mathrm{Hom}_{\underline{\mathrm{M}}\mathrm{Cor}_k}\big((Y, E) \otimes \overline{\square}^*, X\big)/\operatorname{degn}.$$

Taking $(Y, E) = (\operatorname{Spec} k, \emptyset)$, we have the complex

$$C_*^{\mathrm{Sus}}(X, D) := C_*^{\mathrm{Sus}}(X, D)(\operatorname{Spec} k, \emptyset).$$

Next, there is a *derived* Suslin complex $RC_*^{\mathrm{Sus}}(X,D)_c(-)$ with a natural map of presheaves

$$C_*^{\mathrm{Sus}}(X,D)_c(-) \to RC_*^{\mathrm{Sus}}(X,D)_c(-).$$

By [71, THEOREM 2], for (X,D) a proper modulus pair, $RC_*^{\mathrm{Sus}}(X,D)_c(-)$ computes the maps in $\mathrm{MDM}^{\mathrm{eff}}(k)$ as

$$H_n\big(RC_*^{\mathrm{Sus}}(X,D)_c(\operatorname{Spec} k,\emptyset)\big) = \operatorname{Hom}_{\mathrm{MDM}^{\mathrm{eff}}(k)}\big((\operatorname{Spec} k,\emptyset), M^{\mathrm{eff}}(X,D)\big).$$

However, one should not expect that the Suslin complex or its derived version should yield a version of the higher Chow groups. If one looks back at the setting of $\mathrm{DM}(k)$, the object that most naturally yields the higher Chow groups for an arbitrary finite type k-scheme X is the motive with compact supports $M^c(X)$. This is defined as $C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{tr}}^c(X))$, where $\mathbb{Z}_{\mathrm{tr}}^c(X)$ is the presheaf with transfers with $\mathbb{Z}_{\mathrm{tr}}^c(X)(Y)$ the free abelian group on integral $W \subset Y \times X$, with $W \to Y$ *quasi-finite* and dominant over a component of $Y \in \mathrm{Sm}_k$. See [127, CHAP. 5, PROPOSITION 4.2.9] for the relation of $M^c(X)$ with Bloch's higher Chow groups.

One can define a similar version with modulus as the object $M^c(X,D)$ associated to the presheaf $\mathbb{Z}_{\mathrm{tr}}^c(X,D)$, with $\mathbb{Z}_{\mathrm{tr}}^c(X,D)(Y,E) \subset Z_{\dim Y}(Y \times X)$ the subgroup generated by closed subvarieties $W \subset (Y \setminus E) \times (X \setminus D)$ that are quasi-finite and dominant over Y, and with the usual modulus condition, that the normalization $\nu : \bar{W}^N \to Y \times X$ of the closure of W in $Y \times X$ satisfies

$$\nu^*(E \times X) \geq \nu^*(Y \times D).$$

There is an analog of Suslin's comparison theorem in the affine case, due to Kai–Miyazaki [75]: They define an equi-dimensional cycle complex with modulus

$$z_d^{\mathrm{equi}}(X,D,*) \subset z_d(X,D,*)$$

which for $d=0$ is the Suslin complex with modulus $C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{tr}}^c(X,D))(\operatorname{Spec} k,\emptyset)$

Theorem 5.3 (Kai–Miyazaki). *Let (X,D) be a modulus pair, with X affine. Then there is a pro-isomorphism*

$$\big\{H_*\big(z_d^{\mathrm{equi}}(X,mD,*)\big)\big\}_m \cong \big\{\mathrm{CH}_d(X,mD,*)\big\}_m.$$

Miyazaki [91] has defined objects $z^{\mathrm{equi}}(X,D,d) \in \underline{\mathrm{M}}\mathrm{NST}_k$, with $\mathbb{Z}_{\mathrm{tr}}^c(X,D) = z^{\mathrm{equi}}(X,D,0)$. The sheaf $z^{\mathrm{equi}}(X,D,r)$ is defined similarly to $\mathbb{Z}_{\mathrm{tr}}^c(X,D)$, with $z^{\mathrm{equi}}(X,D,d)(Y,E)$ the group of cycles on $(Y \setminus E) \times (X \setminus D)$ generated by closed, integral $W \subset (Y \setminus E) \times (X \setminus D)$ that are equi-dimensional of dimension d over $Y \setminus E$, dominate a component of $Y \setminus E$, and with $\nu : \bar{W}^N \to Y \times X$ satisfying the modulus condition

$$\nu^*(E \times X) \geq \nu^*(Y \times D).$$

Moreover, for an arbitrary modulus pair (X,D), one has

$$z_d^{\mathrm{equi}}(X,D,*) = C_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,d)\big)(\operatorname{Spec} k,\emptyset),$$

and there is the canonical map

$$C_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,d)\big) \to RC_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,d)\big).$$

Letting $\mathrm{CH}_q^{\mathrm{equi}}(X,D,p) = H_p(z_q^{\mathrm{equi}}(X,D,*))$, we have the natural map

$$\mathrm{CH}_q^{\mathrm{equi}}(X,D,p) \to \mathrm{CH}_q(X,D,p)$$

which is an isomorphism for X affine, and we have the natural maps for (X,D) a proper modulus pair

$$\begin{aligned}\mathrm{CH}_q^{\mathrm{equi}}(X,D,p) &\to H_p\big(RC_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,q)\big)(\operatorname{Spec} k,\emptyset)\big)\\ &\to \mathrm{Hom}_{\mathrm{MDM}^{\mathrm{eff}}(k)}\big(M(\operatorname{Spec} k,\emptyset)[p], RC_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,q)\big)\big).\end{aligned}$$

For a proper modulus pair, let $M^c(X,D)$ denote the image of $\mathbb{Z}_{\mathrm{tr}}^c(X,D)$ in $\mathrm{MDM}^{\mathrm{eff}}(k)$. One can ask if there are analogs of the theorem of Kahn–Miyazaki–Saito–Yamazaki.

Question 5.4. *For (X,D) a proper modulus pair, are the maps*

$$H_p\big(RC_*^{\mathrm{Sus}}\big(\mathbb{Z}_{tr}^c(X,D)\big)(\operatorname{Spec} k,\emptyset)\big) \to \mathrm{Hom}_{\mathrm{MDM}^{\mathrm{eff}}(k)}\big(M(\operatorname{Spec} k,\emptyset)[p], M^c(X,D)\big)$$

isomorphisms? More generally, are the maps

$$\begin{aligned}&H_p\big(RC_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,q)\big)(\operatorname{Spec} k,\emptyset)\big)\\ &\quad\to \mathrm{Hom}_{\mathrm{MDM}^{\mathrm{eff}}(k)}\big(M(\operatorname{Spec} k,\emptyset)[p], RC_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,q)\big)\big)\end{aligned}$$

isomorphisms?

It is also not clear if the map

$$\mathrm{CH}_q^{\mathrm{equi}}(X,D,p) \to H_p\big(RC_*^{\mathrm{Sus}}\big(z^{\mathrm{equi}}(X,D,q)\big)(\operatorname{Spec} k,\emptyset)\big)$$

should be an isomorphism. Possibly one should also consider the Nisnevich hypercohomology $\mathbb{H}^{-p}(X_{\mathrm{Nis}}, \mathcal{Z}_q^{\mathrm{equi}}(X,D,*))$, with $\mathcal{Z}_q^{\mathrm{equi}}(X,D,*)$ defined by sheafifying $U \mapsto z_q^{\mathrm{equi}}(U, U\cap D, *)$.

For Voevodsky motives, and for X a finite type k-scheme, the motivic Borel-Moore homology is defined by

$$\begin{aligned}H_p^{\mathrm{B.M.}}\big(X,\mathbb{Z}(q)\big) &:= \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k)}\big(\mathbb{Z}(q)[p], M(X)^c\big)\\ &\cong H_{p-2q}\big(z_q^{\mathrm{equi}}(X,*)\big) \cong H_{p-2q}\big(z_q(X,*)\big) = \mathrm{CH}_q(X,p-2q).\end{aligned}$$

This uses the duality $M(X)^c \cong M(X)^\vee(d)[2d]$ for X of dimension d (valid in characteristic zero, or after inverting p in characteristic $p>0$), and the extension of Suslin's quasi-isomorphism $z_q^{\mathrm{equi}}(X,*) \hookrightarrow z_q(X,*)$ to arbitrary X. Moreover, we have $M(X)^c = M(X)$ for X smooth and proper.

However, a corresponding motivic *cohomology* of modulus pairs seems to need a larger category. This is hinted at by the use of the duality (in $\mathrm{DM}(k)$) $M(X)^c \cong M(X)^\vee(d)[2d]$ in the computations described above. This says in particular that each motive $M(X)$ admits a "twisted" dual in $\mathrm{DM}^{\mathrm{eff}}(k)$, more precisely, the usual evaluation and coevaluation maps associated with a dual exist, but as maps with target or source some $\mathbb{Z}(d)[2d]$ rather than the unit. For a general proper modulus pair (X, D), this does not seem to be the case; one seems to need modulus pairs with an anti-effective Cartier divisor. Another way to say the same thing, if one looks for a proper modulus pair (X, D') such that $\mathrm{Hom}_{\mathrm{MDM}^{\mathrm{eff}}(k)}(M(X, D'), \mathbb{Z}(q)[p])$ looks at all like $\mathrm{CH}^q(X, D, 2q - p)$ for some given proper modulus pair (X, D), the defining inequalities in Corr_k suggest that D' could be $-D$. See the section "Perspectives" in [71, INTRODUCTION] for further details in this direction.

5.4. Logarithmic motives and reciprocity sheaves

Grothendieck motives for log schemes have been constructed in [66], where a version for mixed motives has also been constructed using systems of realizations. There the emphasis is on versions of motives for homological or numerical equivalence in the setting of log schemes. In this section we discuss a recent construction of a triangulated category of log motives, by Binda–Park–Østvær [19], that follows the Voevodsky program. We refer the reader to the lectures notes of Ogus [97] for the facts about log schemes.

Recall that a log scheme is a pair $(X, \alpha : \mathcal{M} \to (\mathcal{O}_X, \times))$ consisting of a scheme X and a homomorphism of sheaves of commutative monoids $\alpha : \mathcal{M} \to (\mathcal{O}_X, \times)$ such that $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism; without this last condition, the pair $(X, \alpha : \mathcal{M} \to (\mathcal{O}_X, \times))$ is called a *pre-log structure*. A pre-log structure $\alpha : \mathcal{M} \to (\mathcal{O}_X, \times)$ induces a log structure $\alpha^{\log} : \mathcal{M}^{\log} \to (\mathcal{O}_X, \times)$ by taking $\mathcal{M}^{\log}$ to be the push-out (in the category of sheaves of monoids) in

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \longrightarrow & \mathcal{M} \\ \big\downarrow & & \\ \mathcal{O}_X^\times & & \end{array}$$

Given a modulus pair (X, D), there are a number of (in general distinct) induced log structures on X. For example, one can take the *compactifying* log structure, with $\mathcal{M} := \mathcal{O}_X \cap j_*\mathcal{O}_U^\times$, where $U = X \setminus D$ and $j : U \to X$ is the inclusion. There are other log structures, which in general depend on a choice of decomposition of D as a sum of effective Cartier divisors (for example, the Deligne–Faltings log structure, discussed in [97, III, DEFINITION 1.7.1]).

Replacing the category of smooth k-schemes is the category lSm_k of *fine, saturated, log smooth* and separated log schemes over the log scheme $\mathrm{Spec}\, k$ endowed with the trivial log structure. We refer the reader to [19] for details; one needs these technical conditions to construct the category of finite log correspondences. We call a separated, fine, saturated log scheme an fs log scheme.

We sketch the construction of the category of finite log correspondences, and describe how Binda–Park–Østvær follow Voevodsky's program to define the triangulated category $\mathrm{logDM}^{\mathrm{eff}}(k)$ of effective log motives over k.

For $X \in \mathrm{lSm}_k$, let $\underline{X}$ denote the underlying k-scheme. We let $X^\circ \subset \underline{X}$ denote the maximal open subscheme over which the log structure $\mathcal{M}_X \to \mathcal{O}_X$ is trivial, that is, $\mathcal{M}_{X|U} = \mathcal{O}_U^\times$, and let $\partial X = \underline{X} \setminus X^\circ$.

Definition 5.5. 1. For $X, Y \in \mathrm{lSm}_k$, the group $\mathrm{lCor}_k(X, Y)$ consisting of *finite log correspondences* from X to Y is the free abelian group on integral closed subschemes $\underline{Z} \subset \underline{X} \times \underline{Y}$ such that

(i) $\underline{Z} \to \underline{X}$ is finite and is surjective to a component of $\underline{X}$.

(ii) Let Z^N be the log scheme with underlying scheme the normalization $\nu : \underline{Z}^N \to X \times Y$ of $\underline{Z}$ and log structure $(\nu \circ p_1)^*_{\log}\mathcal{M}_X \to \mathcal{O}_{Z^N}$. Here $\mathcal{M}_X \to \mathcal{O}_X$ is the given log structure on X and $(\nu \circ p_1)^*_{\log}\mathcal{M}_X \to \mathcal{O}_Z$ is the log structure induced by the pre-log structure $(\nu \circ p_1)^{-1}\mathcal{M}_X \to (\nu \circ p_1)^{-1}\mathcal{O}_X \to \mathcal{O}_Z$. Then the map of schemes $p_2 \circ \nu : \underline{Z}^N \to \underline{Y}$ extends to a map of log schemes $Z^N \to Y$.

Remark 5.6. It follows from (i) and (ii) above that, for $\underline{Z} \in \mathrm{lCor}_k(X, Y)$, the restriction of $\underline{Z}$ to a cycle on the open subset $X^\circ \times Y^\circ$ of $\underline{X} \times \underline{Y}$ actually lands in $\mathrm{Cor}_k(X^\circ, Y^\circ)$. Moreover, by [19, LEMMA 2.3.1], if the extension in (ii) exists, it is unique, so there is no need to include this as part of the data. In particular, the restriction map $\mathrm{lCor}_k(X, Y) \to \mathrm{Cor}_k(X^\circ, Y^\circ)$ is injective ([19, LEMMA 2.3.2]).

The condition that there exists a map of log schemes $(Z^N, (p_1 \circ \nu)^*_{\log}\mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ extending $p_2 \circ \nu : \underline{Z}^N \to \underline{Y}$ is analogous to the modulus condition

$$\nu^*(D \times Y) \geq \nu^*(X \times E)$$

for a subvariety $W \subset X \setminus D \times Y \setminus E$ to define a finite correspondence of modulus pairs from (X, D) to (Y, E).

For the composition law, the proof of [19, LEMMA 2.3.3] shows that, given elementary log correspondences $\underline{W} \in \mathrm{lCor}_k(X, Y)$, and $\underline{W}' \in \mathrm{lCor}_k(Y, Z)$, each integral component $\underline{R}$ of $\underline{W} \times \underline{Z} \cap \underline{X} \times \underline{W}'$ is the underlying scheme of a (unique!) elementary log correspondence $\underline{R} \in \mathrm{lCor}_k(X, Z)$. It is then easy to show that there is a unique composition law

$$\circ : \mathrm{lCor}_k(Y, Z) \times \mathrm{lCor}_k(X, Y) \to \mathrm{lCor}_k(X, Z)$$

that is compatible with the composition law in Cor_k via the respective restriction maps.

This defines the additive category of *finite log correspondences* lCor_k with the same objects as for lSm_k, giving the category of *presheaves with log transfers*, lPST_k, defined as the category of additive presheaves of abelian groups on lCor_k. For a log scheme $X \in \mathrm{lSm}_k$, let $\mathbb{Z}_{\mathrm{ltr}}(X)$ denote the representable presheaf

$$\mathbb{Z}_{\mathrm{ltr}}(X)(Y) := \mathrm{lCor}_k(Y, X).$$

The fiber product of log schemes induces a tensor product structure on lPST_k.

The next step is to define the log version of the Nisnevich topology.

A morphism of log schemes $f : (X, \mathcal{M}_X \to \mathcal{O}_X), (Y, \mathcal{M}_Y \to \mathcal{O}_Y)$ is *strict* if the map of log structures $f^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism. An *elementary log Nisnevich square* is a cartesian square in the category of fs log schemes

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & X \end{array} \tag{5.1}$$

where f is strict étale, g is an open immersion, and f induces an isomorphism on reduced schemes $\underline{Y} \setminus \underline{V} \to \underline{X} \setminus \underline{U}$.

A log modification is a generalization of the notion of a log blow-up, which in turn is a morphism of log schemes modeled on the birational morphism of toric varieties given by a subdivision of the fan defining the target. We refer the reader to [19, APPENDIX A] for details. The Grothendieck topology generated by the log modifications and strict Nisnevich elementary squares is called the *dividing Nisnevich topology* on fs log schemes. In a sense, this is a log version of the cdh topology, where all the modifications are taking place in the boundary.

With this topology in hand, we have the subcategory lNST_k of lPST_k of Nisnevich sheaves with log transfers, just as for $\mathrm{NST}_k \subset \mathrm{PST}_k$, by requiring that a presheaf with log transfers be a sheaf for the dividing Nisnevich topology when restricted to lSm_k.

Finally, we need a suitable interval object to define a good notion of homotopy invariance. This is just as for the category $\mathrm{MDM}^{\mathrm{eff}}(k)$, where we consider $\overline{\square}$ as the scheme $\mathbb{P}^1$ with compactifying log structure for $(\mathbb{P}^1, \{1\})$. The product log scheme $\overline{\square}^2$ also has the compactifying log structure for the divisor $1 \times \mathbb{P}^1 + \mathbb{P}^1 \times 1$. However, the closure $\bar{\Gamma}_m$ of the graph of the multiplication map $m : \overline{\square}^2 \to \overline{\square}$ is not a morphism $\tilde{m}$ in lCor_k, as the requirement that the map of $\bar{\Gamma}_m$ to $\overline{\square}^2$ be finite is not satisfied.

Another way to look at this is to note that the projection $\bar{\Gamma}_m \to \overline{\square}^2$ is a cover of $\overline{\square}^2$ in the dividing Nisnevich topology, and becomes an isomorphism after dNis-localization. In a sense, this allows one to consider the sheaf $a_{\mathrm{dNis}}\overline{\square}$ as a version of a cylinder object and allows many of the constructions of Morel–Voevodsky for a site with interval to go through, although there are occasional technical difficulties that arise.

Definition 5.7. The tensor triangulated category of *effective log motives over* k, $\mathrm{logDM}^{\mathrm{eff}}(k)$, is the Verdier localization of the derived category $D(\mathrm{lPST}_k)$ with respect to the localizing subcategory generated by:

(lMV) for an elementary log Nisnevich square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g} & X \end{array}$$

we have the complex

$$\mathbb{Z}_{\mathrm{ltr}}(V) \to \mathbb{Z}_{\mathrm{ltr}}(U) \oplus \mathbb{Z}_{\mathrm{ltr}}(Y) \to \mathbb{Z}_{\mathrm{ltr}}(X).$$

(lM) For a log modification $f : Y \to X$ in lSm_k, we have the complex

$$\mathbb{Z}_{\mathrm{ltr}}(Y) \to \mathbb{Z}_{\mathrm{ltr}}(X).$$

(lCI) For $X \in \mathrm{lSm}_k$, we have the complex

$$\mathbb{Z}_{\mathrm{ltr}}(X \times \underline{\square}) \to \mathbb{Z}_{\mathrm{ltr}}(X).$$

For each fs smooth log scheme $X \in \mathrm{lSm}_k$, the image of $\mathbb{Z}_{\mathrm{ltr}}(X)$ in $\mathrm{logDM}^{\mathrm{eff}}(k)$ is the *effective log motive* $\mathrm{lM}^{\mathrm{eff}}(X)$, giving the functor

$$\mathrm{lM}^{\mathrm{eff}} : \mathrm{lSm}_k \to \mathrm{logDM}^{\mathrm{eff}}(k).$$

The functor $\mathrm{lM}^{\mathrm{eff}}$ shares many of the formal properties of $M^{\mathrm{eff}} : \mathrm{Sm}_k \to \mathrm{DM}^{\mathrm{eff}}(k)$; we refer the reader to the [19, INTRODUCTION] for an overview.

Questions of representing known constructions such as the higher Chow groups with modulus in $\mathrm{logDM}^{\mathrm{eff}}(k)$, or finding direct connections of $\mathrm{logDM}^{\mathrm{eff}}(k)$ with the category $\mathrm{MDM}^{\mathrm{eff}}(k)$ are not discussed in [19]. However, for (X, D) a proper modulus pair, one has the log scheme $l(X, D)$, defined using the Deligne–Faltings log structure on X associated to the ideal sheaf $\mathcal{O}_X(-D)$. In general, this is not saturated. Still, there should be presheaves with log transfers $\mathbb{Z}_{\mathrm{ltr}}(X, D)$ and $\mathbb{Z}^c_{\mathrm{ltr}}(X, D)$ using finite and quasi-finite "log correspondences," with value on $Y \in \mathrm{lSm}_k$ the free abelian group of integral subschemes W of $\underline{Y} \times X$ that admit a map of log schemes $(W^N, (p_1 \circ \nu)^*(\mathcal{M}_Y)) \to l(X, D)$, as in the definition of $\mathrm{lCor}_k(-,-)$. One could also expect to have presheaves $lz(X, D, r)$ similarly defined, and corresponding to the presheaves with modulus transfers $z(X, D, r)$ constructed by Miyazaki. These could be used to give a map

$$H_p\big(z_r^{\mathrm{equi}}(X, D, *)\big) \to \mathrm{Hom}_{\mathrm{logDM}^{\mathrm{eff}}(k)}\big(\mathbb{Z}(0)[p], M\big(lz(X, D, r)\big)\big).$$

We have briefly mentioned *reciprocity sheaves* in our discussion of motives with modulus. There is a nice connection of $\mathrm{logDM}^{\mathrm{eff}}(k)$ with the theory of reciprocity sheaves, so we take the opportunity to say a few words about reciprocity sheaves before we describe the theorem of Shuji Saito, which gives the connection between these two theories.

The notion of a reciprocity sheaf and its relation to motives with modulus goes back to the theorem of Rosenlicht–Serre. In our discussion of reciprocity sheaves, we work over a fixed perfect field k.

Theorem 5.8 (Rosenlicht–Serre [109, III]). *Let k be a perfect field, let C be a smooth complete curve over k, let G be an smooth commutative algebraic group over k, and let $f : C \dashrightarrow G$ be a rational map over k. Let $S \subset C$ be a finite subset such that f is a morphism on $C \setminus S$. Then there is an effective divisor D supported in S such that, for g a rational function on C with $g \equiv 1 \bmod D$, one has*

$$\sum_{P \in C \setminus S} \mathrm{ord}_P(g) \cdot f(P) = 0$$

in G.

In [72], reciprocity functors and reciprocity sheaves are defined. We will just give a sketch. One first defines for F a presheaf with transfers (in the Voevodsky sense), and for a proper modulus pair (X, D) with a section $a \in F(X \setminus D)$, what it means for a to have modulus D. As an example, if $p : C \to X$ is a non-constant morphism of a smooth proper integral curve C over k to X with $p(C)$ not contained in D, and g is a rational function on C such that $g \equiv 1 \bmod p^*(D)$, then one is required to have

$$a\big(p_*\big(\mathrm{div}(g)\big)\big) = 0 \in F(\mathrm{Spec}\, k).$$

Here, for a 0-cycle $\sum_x n_x \cdot x$ on $X \setminus D$, $a(\sum_x n_x \cdot x) = \sum_x n_x \cdot p_{x*} i_x^*(a) \in F(\mathrm{Spec}\, k)$, where for a closed point x of $X \setminus D$, $i_x : x \to X \setminus D$ is the inclusion and $p_x : x \to \mathrm{Spec}\, k$ is the (finite) structure morphism. In general, one imposes a similar condition in $F(S)$ for a "relative curve" on $X \times S$ over some smooth base scheme S.

A presheaf with transfers F is a reciprocity sheaf if for each quasi-affine U and section $a \in F(U)$, there is a proper modulus pair (X, D) with $U = X \setminus D$ such that a has modulus D. Roughly speaking, one should think that each section of F has "bounded ramification," although the "ramification" for F itself may be unbounded.

This definition is not quite accurate, as a slightly different notion of "modulus pair" from what we have defined here is used in [72]. A more elegant definition of reciprocity sheaf is given in [73]. This new notion is a bit more restrictive than the old one, but by [73, THEOREM 2], the two notions agree on for $F \in \underline{\mathrm{M}}\mathrm{NST}_k$.

Using the definition of [73], the reciprocity sheaves define a the full subcategory $\mathbf{RST}_k$ of PST_k, strictly containing the subcategory $\mathbf{HI}_k \subset \mathrm{PST}_k$ of $\mathbb{A}^1$-homotopy invariant presheaves with transfer. There is also the subcategory $\mathbf{RST}_{\mathrm{Nis},k}$ of NST_k, consisting of those reciprocity presheaves that are Nisnevich sheaves.

Some examples of non-homotopy invariant sheaves in $\mathbf{RST}_{\mathrm{Nis},k}$ include the sheaf of n-forms over k, $X \mapsto \Omega^n_{X/k}$, the sheaf of absolute n-forms, $X \mapsto \Omega^n_X$, and for k of positive characteristic, the truncated de Rham–Witt sheaves, $X \mapsto W_m\Omega^n_X$. The representable sheaf of a commutative algebraic group G over k, $X \mapsto G(X)$, is in $\mathbf{RST}_{\mathrm{Nis},k}$, and for some G (e.g. $G = \mathbb{G}_m^n$) this is also $\mathbb{A}^1$-homotopy invariant. This is not the case for unipotent G (e.g. $G = \mathbb{G}_a^n$).

Here is the promised theorem of Saito. For a sheaf $G \in \mathrm{lNST}_k$, we say that G is *strictly $\overline{\square}$-invariant* if for all $X \in \mathrm{lSm}_k$, the map

$$H^*_{d\mathrm{Nis}}(X, G_{|X_{d\mathrm{Nis}}}) \to H^*_{d\mathrm{Nis}}(X \times \overline{\square}, G_{|X \times \overline{\square}_{d\mathrm{Nis}}})$$

induced by the projection $X \times \overline{\square} \to X$ is an isomorphism. Here $_{d\mathrm{Nis}}$ refers to the divided Nisnevich site.

Theorem 5.9 (Saito [105, THEOREM 0.2]). *There exists a fully faithful exact functor*

$$\log : \mathbf{RST}_{\mathrm{Nis},k} \to \mathrm{lNST}_k$$

such that $\log(F)$ *is strictly* $\overline{\square}$*-invariant for every* $F \in \mathbf{RST}_{\mathrm{Nis},k}$*. Moreover, for each* $X \in \mathrm{Sm}_k$*, there is a natural isomorphism*

$$H^i_{\mathrm{Nis}}(X, F_{|X}) \cong \mathrm{Hom}_{\mathrm{logDM}^{\mathrm{eff}}(k)}\big(\mathrm{lM}^{\mathrm{eff}}(X), \log(F)[i]\big).$$

6. p-ADIC ÉTALE MOTIVIC COHOMOLOGY AND p-ADIC HODGE THEORY

We discuss yet another theory of motivic cohomology that is not $\mathbb{A}^1$-homotopy invariant.

Working over a base-field k and for m prime to the characteristic, we have the isomorphism of the étale sheafification $\mathbb{Z}/m(r)_{\text{ét}}$ with the étale sheaf $\mu_m^{\otimes r}$. The étale sheaves $\mathbb{Z}/m(r)_{\text{ét}}$ can be considered as objects in a version of Voevodsky's DM constructed using the étale topology rather than the Nisnevich topology, and their categorical cohomology agrees with the usual étale cohomology [127, CHAP. 5, §3.3]. In particular, the complexes $\mathbb{Z}/m(r)_{\text{ét}}$ have $\mathbb{A}^1$-homotopy invariant étale cohomology.

On the other hand, if k has characteristic $p > 0$, we have the isomorphism [52] of the Nisnevich sheaves on Sm_k,

$$\mathbb{Z}/p^n(r) \cong W_n\Omega^r_{\log}[-r], \tag{6.1}$$

hence of étale sheaves

$$\mathbb{Z}/p^n(r)_{\text{ét}} \cong W_n\Omega^r_{\log}[-r].$$

Here something strange happens: the étale sheaf $\mathbb{Z}/p^n(r)_{\text{ét}}$ is no longer strictly homotopy invariant! In fact, the existence of the Artin–Schreyer étale cover $\mathbb{A}^1 \to \mathbb{A}^1$ of degree p implies that the étale version of $\mathrm{DM}^{\mathrm{eff}}(k)$ with coefficients modulo p^n is zero if k has characteristic $p > 0$ [127, CHAP. 5, PROPOSITION 3.3.3]. Thus $\mathbb{Z}/p^n(r)_{\text{ét}}$ leaves the world of Voevodsky's motives and motivic cohomology.

For $S = \operatorname{Spec}\Lambda$, with Λ a mixed characteristic $(0, p)$ dvr, the complex $\mathbb{Z}/p^n(r)_{\text{ét}}$ on $\mathrm{Sm}_{S,\text{ét}}$ yields an interesting gluing of $\mathbb{Z}/p^n(r)_{\text{ét}} = \mu_{p^n}^{\otimes r}$ over the characteristic zero quotient field of Λ and $\mathbb{Z}/p^n(r)_{\text{ét}} = W_n\Omega^r_{\log}[-r]$ over the characteristic p residue field. The positive characteristic part again says that we have left homotopy invariance behind.

The complexes $\mathbb{Z}/p^n(r)_{\text{ét}}$ have an interesting connection with a certain complex of sheaves arising in p-adic Hodge theory. A version of this complex first appears in the paper [40] of Fontaine–Messing, and plays an important role in the proof of their main result. Its construction was reinterpreted by Kurihara [82], relying on the work of Bloch–Kato [28] and Kato [79], and was generalized by Sato [106]. Geisser [49], following work of Schneider [108], established the connection of the Fontaine–Messing/Kurihara/Sato complex with $\mathbb{Z}/p^n(r)_{\text{ét}}$ in the case of a smooth degeneration, and this connection was partially extended by Zhong [128] to the semi-stable case.

In their recent work on integral p-adic Hodge theory, Bhatt–Morrow–Scholze [18] have introduced a "motivic filtration" on p-adic étale K-theory, relying on a Postnikov tower for topological cyclic homology, and the layers in this tower have been identified with the pro-system $\{\mathbb{Z}/p^n(r)_{\text{ét}}\}_n$ in a work-in-progress [16] by Bhargav Bhatt and Akhil Mathew.

Our goal in this section is to give some details of the story sketched above.

We first discuss the papers of Bloch–Kato, Fontaine–Messing, Kurihara and Sato without reference to all the advances in p-adic Hodge theory that followed these works; we wanted to give the reader just enough background to put the connections with motivic cohomology in context. We will then describe the works of Geisser and Zhong, as well as results of Geisser–Hesselholt that form some of the foundations for the work of Bhatt–Morrow–Scholze. We conclude with a description of the Bhatt–Morrow–Scholze motivic tower and its connection with the p-adic cycle complexes.

We refer the reader to [15] for background on crystalline cohomology.

6.1. A quick overview of some p-adic Hodge theory

We begin with a few comments on the paper of Bloch and Kato [28], which we have already mentioned in our discussion of the Beilinson–Lichtenbaum conjectures. They consider the spectrum S of a complete dvr Λ with generic point $\eta = \operatorname{Spec} K \hookrightarrow S$ and closed point $s = \operatorname{Spec} k \hookrightarrow S$, and a smooth and proper S-scheme $X \to S$ with generic fiber $V := X_\eta$ and special fiber $Y := X_s$. $\bar{V}$, $\bar{Y}$ denote V, Y over the respective algebraic closures $\bar{K}$ and $\bar{k}$ of K and k. Let $\bar{\Lambda}$ be the integral closure of Λ in $\bar{K}$, $\bar{S} := \operatorname{Spec} \bar{\Lambda}$, and $\bar{X} = X \times_S \bar{S}$. Let $G = \operatorname{Gal}(\bar{K}/K)$ and let C denote the completion of $\bar{K}$.

The closure $\bar{Y}$ has its crystalline cohomology $H^*_{\text{crys}}(\bar{Y}/W(\bar{k}))$ with action of Frobenius, giving the p^i-eigenspace

$$H^*_{\text{crys}}\big(\bar{Y}/W(\bar{k})\big)^{(i)} \subset H^*_{\text{crys}}\big(\bar{Y}/W(\bar{k})\big)_{\mathbb{Q}}.$$

We say $\bar{Y}$ is *ordinary* if

$$\dim_{W(\bar{k})_{\mathbb{Q}}} H^m_{\text{crys}}\big(\bar{Y}/W(\bar{k})\big)^{(i)} = \dim_{\bar{k}} H^{m-i}(\bar{Y}, \Omega^i_{\bar{Y}/\bar{k}}).$$

We have the inclusions $\bar{i} : \bar{Y} \to \bar{X}$, $\bar{j} : \bar{V} \to \bar{X}$ and the spectral sequence

$$E_2^{s,t} = H^s_{\text{ét}}\big(\bar{Y}, \bar{i}^* R^t \bar{j}_*(\mathbb{Z}/p^n\mathbb{Z})\big) \Rightarrow H^{s+t}_{\text{ét}}(\bar{V}, \mathbb{Z}/p^n\mathbb{Z}),$$

inducing a descending filtration $F^* H^*_{\text{ét}}(\bar{V}, \mathbb{Q}_p)$ on $H^*_{\text{ét}}(\bar{V}, \mathbb{Q}_p)$ with $F^0 H^q = H^q$ and $F^{q+1} H^q = 0$.

We have the de Rham–Witt sheaf $W\Omega^i$ on $\mathrm{Sm}_{\bar{k}}$ and the sheaf of differential forms $\Omega^i_{-/K}$ on Sm_K.

Theorem 6.1 (Bloch–Kato [28, THEOREM 0.7]). *Suppose that k is perfect and that Y is ordinary. Then there are natural G-equivariant isomorphisms*

$$\text{(i)} \quad \operatorname{gr}^{q-i} H^q_{ét}(\bar{V}, \mathbb{Q}_p) \cong H^q_{\text{crys}}\big(\bar{Y}/W(\bar{k})\big)^{(i)}_{\mathbb{Q}}(-i),$$

$$\text{(ii)} \quad \operatorname{gr}^{q-i} H^q_{ét}(\bar{V}, \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} W(\bar{k}) \cong H^q_{\text{crys}}(\bar{Y}, W\Omega^i)_{\mathbb{Q}}(-i),$$

$$\text{(iii)} \quad \operatorname{gr}^{q-i} H^q_{ét}(\bar{V}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong H^q(V, \Omega^i_{V/K}) \otimes_K C(-i).$$

We will not give any details of the proof here, but do want to mention that what ties these different theories together is the sheaf of Milnor K-groups $\mathcal{K}^M_q$. This maps to étale cohomology by the Galois symbol

$$\theta_{q,m} : \mathcal{K}^M_q/m \to \mathcal{H}^q_{\text{ét}}(\mu_m^{\otimes q})$$

for m prime to the characteristic, to the de Rham–Witt sheaf by the $d\log$ map on Sm_k,

$$d\log^W_{q,p^n} : \mathcal{K}^M_q/p^n \to W_n\Omega^q,$$

and to the sheaf of differential forms, by the $d\log$ map on Sm_S,

$$d\log_{q,p^n} : \mathcal{K}_q^M/p^n \to \Omega^q_{-/S}/p^n.$$

The main structural results that underpin the proof of the Bloch–Kato theorem are two comparison isomorphisms on the sheaf level. For the first, let $W_n\Omega^q_{\log} \subset W_n\Omega^q$ be the étale subsheaf locally generated by the image $d\log(\mathcal{K}_q^M/p^n)$.

Theorem 6.2 ([28, COROLLARY 2.8]). *Let F be a field of characteristic $p > 0$. Then the map $d\log : K_q^M(F)/p^n \to W_n\Omega^q(F)$ defines an isomorphism of $K_q^M(F)/p^n$ with $W_n\Omega^q_{\log}(F)$.*

Note that the composition

$$\mathbb{Z}/p^n(q) \to \tau_{\geq q}\mathbb{Z}/p^n(q) \cong \mathcal{K}_q^M/p^n[-q] \xrightarrow{d\log} W_n\Omega^q_{\log}[-q]$$

is the map that defines the isomorphism (6.1).

The second result is a special case of the Bloch–Kato conjecture.

Theorem 6.3 (Bloch–Kato [28, THEOREM 5.12]). *Let F be a henselian discretely valued field of characteristic 0, with residue field of characteristic $p > 0$. Then the Galois symbol*

$$K_q^M(F)/p^n \to H^q_{\acute{e}t}(F, \mu_{p^n}^{\otimes q})$$

is an isomorphism for all $n \geq 1$.[2]

Bloch and Kato use $\mathcal{K}_q^M$ to relate $i^*R^q j_*\mu_{p^n}^{\otimes q}$ to $\Omega^q_{-/K}/p^n$ and $W_n\Omega^{q-1}_{\log}$ via the respective $d\log$ maps. Relying on the isomorphisms of Theorem 6.2 and Theorem 6.3, these maps from Milnor K-theory tie de Rham cohomology, crystalline cohomology and étale cohomology together, and eventually lead to a proof of Theorem 6.1.

As part of the proof, they define a surjective map

$$\gamma : i^*R^q j_*\mu_{p^n}^{\otimes r} \to W_n\Omega^{q-1}_{\log} \tag{6.2}$$

on $Y_{\text{ét}}$ with the following property: Let $\theta : i^*j_*\mathcal{K}^M_{q,\text{ét}} \to i^*R^q j_*\mu_{p^n}^{\otimes r}$ be the Galois symbol map, let $u_2, \dots, u_q$ be units on X near some point y of Y with restrictions $\bar{u}_1, \dots, \bar{u}_q$ to Y and let π be a parameter in Λ. Then

$$\gamma \circ \theta\big(\{u_1, \dots, u_{q-1}, \pi\}\big) = d\log\big(\{\bar{u}_1, \dots, \bar{u}_{q-1}\}\big).$$

We highlight this because it will be used later on in a gluing construction that defines an object of central interest for this section.

The next paper I want to mention is by Fontaine–Messing [40]. They construct a comparison isomorphism between de Rham cohomology and étale cohomology for a smooth, proper scheme X over the ring of integers $\mathcal{O}_K$ for K a characteristic zero local field (under some additional assumptions). The de Rham cohomology $H^q_{\mathrm{dR}}(V/K)$ has its Hodge

2 In fact, at the beginning of §3 of [28], Bloch and Kato write, “The cohomological symbol defined by Tate [114] gives a map . . . , which one conjectures to be an isomorphism quite generally.”

filtration and via the comparison isomorphism $H^q_{\mathrm{dR}}(V/K) \cong H^q_{\mathrm{crys}}(Y/W(k)) \otimes_{W(k)} K$ $H^q_{\mathrm{dR}}(V/K)$ acquires a Frobenius operator ϕ; call this object $H^q_{\mathrm{crys}}(X)$. Fontaine–Messing construct the p-adic period ring $B_{\mathrm{crys}} \supset K$ with a Galois action, a Frobenius and a filtration, and show there are isomorphisms

$$\mathrm{Fil}^0\big(B_{\mathrm{crys}} \otimes_K H^q_{\mathrm{crys}}(X)\big)^{\phi=\mathrm{Id}} \cong H^q_{\mathrm{\acute{e}t}}(V_{\bar{K}}, \mathbb{Q}_p)$$

and

$$\big(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^q_{\mathrm{\acute{e}t}}(V_{\bar{K}}, \mathbb{Q}_p)\big)^G \cong H^q_{\mathrm{crys}}(X)$$

both compatible with the "remaining" structures.

To set this up, they consider the syntomic topology on $\mathrm{Sch}_{W_n(k)}$, where a cover is a surjective syntomic map (we described syntomic maps in Section 3.3). The crystalline structure sheaf $\mathcal{O}_n^{\mathrm{crys}}$ defines a sheaf for the syntomic topology with a surjection to the usual structure sheaf $\mathcal{O}_n$ on $\mathrm{Sch}_{W_n(k)}$. Letting J_n denote the kernel of $\mathcal{O}_n^{\mathrm{crys}} \to \mathcal{O}_n$, one has the rth divided power $J_n^{[r]}$; this gives us the sheaf $\tilde{S}_n^r$ defined as the kernel of $\phi - p^r : J_n^{[r]} \to \mathcal{O}_n^{\mathrm{crys}}$. Modifying this by taking the image S_n^r of the reduction map $\tilde{S}_{n+r}^r \to \tilde{S}_n^r$ gives the inverse system $\{S_n^r\}_n$ and the cohomology

$$H^*(\bar{X}, S^r_{\mathbb{Q}_p}) := \Big(\varprojlim_n H^*(\bar{X}, S_n^r)\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The ring B^+_{crys} is defined as follows. The characteristic p ring $\mathcal{O}_{\bar{K}}/p$ forms an inverse system via the Frobenius endomorphism; let

$$\mathcal{O}^\flat = \varprojlim_{\mathrm{Frob}} \mathcal{O}_{\bar{K}}/p,$$

a perfect characteristic p ring. We have the ring of truncated Witt vectors $W_n(\mathcal{O}^\flat)$ and a surjection $\pi_n : W_n(\mathcal{O}^\flat) \to \mathcal{O}_{\bar{K}}/p^n$. Let $W_n^{\mathrm{DP}}(\mathcal{O}_{\bar{K}})$ be the divided power envelope of the kernel of π_n, forming the inverse system $\{W_n^{\mathrm{DP}}(\mathcal{O}_{\bar{K}})\}_{n\geq 0}$. Let

$$B^+_{\mathrm{crys}} := K \otimes_{W(k)} \varprojlim_n W_n^{\mathrm{DP}}(\mathcal{O}_{\bar{K}}).$$

The Frobenius on $W_n(\mathcal{O}^\flat)$ induces a Frobenius on B^+_{crys} and the filtration $J_n^{[*]}$ of $W_n^{\mathrm{DP}}(\mathcal{O}_{\bar{K}})$ induces a filtration $\mathrm{Fil}^* B^+_{\mathrm{crys}}$ on B^+_{crys}.

The derived push-forward of the complex $J_n^{[r]} \xrightarrow{\phi - p^r} \mathcal{O}_n^{\mathrm{crys}}$ is an analog of the Deligne complex, as expressed in the following theorem.

Theorem 6.4 (**[40, Corollary to Theorem 1.6, Lemma 3.1]**). *Suppose that X is admissible*[3] *and $\Lambda = W(k)$. Then for $m \leq r < p$ there is an exact sequence*

$$0 \to H^m(\bar{X}, S^r_{\mathbb{Q}_p}) \to \mathrm{Fil}^r\big(B^+_{\mathrm{crys}} \otimes_K H^m_{\mathrm{dR}}(V/K)\big) \xrightarrow{\phi - p^r} H_{\mathrm{dR}}(V/K) \to 0.$$

In other words,

$$H^m(\bar{X}, S^r_{\mathbb{Q}_p}) = \big(\mathrm{Fil}^r\big(B^+_{\mathrm{crys}} \otimes_K H^m_{\mathrm{dR}}(V/K)\big)\big)^{\phi = p^r}.$$

3 See **[40, §2.1]** for the definition of admissible X.

To involve étale cohomology in the picture, Fontaine–Messing introduce the syntomic–étale site on formal Spf$(W(k))$-schemes, where a cover is a map $\{U_n\}_n \to \{V_n\}_n$ such that $U_n \to V_n$ is a syntomic cover for all n and is an étale cover on the rigid analytic generic fibers. This extends to the syntomic–étale site on $\bar{X}$, where an object is $U \to \bar{X}$, quasi-finite and syntomic, with $U_{\bar{K}} \to V_{\bar{K}}$ étale. Letting $\mathcal{Z}$ be the formal completion of $\bar{X}$, we have the diagram of sites

$$\mathcal{Z}_{\text{syn}-\text{ét}} \xrightarrow{i} \bar{X}_{\text{syn}-\text{ét}} \xleftarrow{j} V_{\bar{K},\text{ét}}.$$

Fontaine–Messing prove a patching result, that a sheaf on $\bar{X}_{\text{syn}-\text{ét}}$ is given by a triple $(\mathcal{F}, \mathcal{G}, \alpha)$, with $\mathcal{F}$ a sheaf on $\mathcal{Z}_{\text{syn}-\text{ét}}$, $\mathcal{G}$ a sheaf on $V_{\bar{K},\text{ét}}$, and $\alpha : \mathcal{F} \to i^* R j_* \mathcal{G}$ a morphism. Using this description, they construct a sheaf on $\bar{X}_{\text{syn}-\text{ét}}$ by defining a certain morphism (see [40, §5.1])

$$\alpha : S_n^r \to i^* R j_* \mu_{p^n}^{\otimes r}.$$

The resulting sheaf $\tilde{S}_n^r$ has

$$j^* \tilde{S}_n^r \cong \mu_{p^n}^{\otimes r}, \quad i^* \tilde{S}_n^r \cong S_n^r.$$

It follows from proper base-change (see the proof of [40, Proposition 6.2]) that the restriction map $H^*(\bar{X}_{\text{syn}-\text{ét}}, \tilde{S}_n^r) \to H^*(\mathcal{Z}_{\text{syn}-\text{ét}}, S_n^r)$ is an isomorphism, and we also have

$$\Big(\lim_{\substack{\leftarrow \\ n}} H^*(\mathcal{Z}_{\text{syn}-\text{ét}}, S_n^r)\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \big(\mathrm{Fil}^r\big(B_{\text{crys}}^+ \otimes_K H_{\text{dR}}^m(V/K)\big)\big)^{\phi=p^r}.$$

Moreover, the restriction map j^* gives

$$j^* : H^*(\bar{X}_{\text{syn}-\text{ét}}, \tilde{S}_n^r) \to H_{\text{ét}}^*(V_{\bar{K}}, \mu_{p^n}^{\otimes r}),$$

so passing to the limit, we have the map

$$\beta : \big(\mathrm{Fil}^r\big(B_{\text{crys}}^+ \otimes_K H_{\text{dR}}^m(V/K)\big)\big)^{\phi=p^r} \to H_{\text{ét}}^m\big(V_{\bar{K}}, \mathbb{Q}_p(r)\big),$$

which they show is an isomorphism.

This gives a twisted version of the result announced at the beginning of our discussion. To recover the untwisted version, they define a map

$$\mathbb{Q}_p(1)(\bar{K}) \to B_{\text{crys}}^+$$

by sending a p^n-root of unity ε in $\bar{K}$ to the logarithm of the Teichmüller lift of the mod p reduction of ε, and passing to the limit in n. Let $t \in \mathbb{Q}_p(1)(\bar{K})$ be a nonzero element and define $B_{\text{crys}} = B_{\text{crys}}^+[1/t]$, with induced filtration and Galois action. Twisting with respect to t translates the twisted version to the untwisted one.

The sheaf $\tilde{S}_n^r$ is only defined on $\bar{X}_{\text{syn}-\text{ét}}$ for X smooth over S and for $r < p$, and with base-ring Λ equal to $W(k)$, i.e., in the unramified case. Kato [79] studies the derived push-forward $\mathcal{S}_n(r)$ of the syntomic sheaf $\tilde{S}_n^r$ to $\mathrm{Sm}_{k,\text{ét}}$. Kurihara [82, §1, Theorem] considers the ramified case and also clarifies the relation of $\mathcal{S}_n(r)$ with the sheaf of log forms $W_n\Omega_{\log}^{r-1}$.

Theorem 6.5 (Kurihara). *Suppose that $[k : k^p] < \infty$. Let $X \to S$ be smooth and projective and suppose that $r < p-1$. Then there is a distinguished triangle in $D(Y_{ét})$,*

$$W_n\Omega^{r-1}_{\log}[-r-1] \to \mathcal{S}_n(r) \to i^*Rj_*\mu_{p^n}^{\otimes r} \to W_n\Omega^{r-1}_{\log}[-r].$$

Schneider [108] extends the construction of $\mathcal{S}_n(r)$ to all $r \geq 0$ by using the Bloch–Kato symbol map γ of (6.2) to give a map $s : \tau_{\leq r}Rj_*\mu_{p^n}^{\otimes r} \to i_*W_n\Omega^{r-1}_{\log}$ with i^*s the composition

$$i^*\tau_{\leq r}Rj_*\mu_{p^n}^{\otimes r} \to i^*R^r j_*\mu_{p^n}^{\otimes r}[-r] \xrightarrow{\gamma} W_n\Omega^{r-1}_{\log}[-r].$$

Schneider then defines the sheaf $\mathcal{S}_n(r)$ as $\mathrm{Cone}(s)[-1]$, giving the distinguished triangle

$$i_*W_n\Omega^{r-1}_{\log}[-r-1] \to \mathcal{S}_n(r) \to Rj_*\mu_{p^n}^{\otimes r} \xrightarrow{s} i_*W_n\Omega^{r-1}_{\log}[-r], \tag{6.3}$$

which recovers the one in Kurihara's theorem for $r < p-1$ by applying i^*.

Using a similar method, Schneider's construction was extended to the semi-stable case by Sato [106], who defines the object $\mathfrak{T}_n(r) \in D(X_{ét})$ with $\mathfrak{T}_n(r) \cong \mathcal{S}_n(r)$ in the smooth case.

6.2. Étale motivic cohomology

We return to algebraic cycles. As before, we consider a smooth separated finite type S-scheme $X \to S = \mathrm{Spec}\,\Lambda$ with generic fiber $j : V \to X$ and special fiber $i : Y \to X$, and with Λ a mixed characteristic $(0, p)$ dvr with perfect residue field.

Geisser [49] considers the motivic complex $\mathbb{Z}(r)_X$ on a smooth S-scheme $X \to S$ as a sheaf of complexes on X_{Nis}. Here we use the reindexed Bloch cycle complex to define $\mathbb{Z}(r)^*_X(U)$ as

$$\mathbb{Z}(r)^*_X(U) := z^r(U, 2r - *),$$

and define the motivic complexes $\mathbb{Z}(r)_V$ and $\mathbb{Z}(r)_Y$ on V and Y similarly.

Let $\alpha : (-)_{ét} \to (-)_{\mathrm{Nis}}$ be the change of topology map. Sheafifying for the étale topology gives complexes $\mathbb{Z}(r)_{ét,X}$, $\mathbb{Z}(r)_{ét,V}$, and $\mathbb{Z}(r)_{ét,Y}$. Geisser shows that various known properties of $\mathbb{Z}(r)_X$, $\mathbb{Z}(r)_V$, and $\mathbb{Z}(r)_Y$, such as the purity isomorphism [84, THEOREM 1.7]

$$i^!\mathbb{Z}(r)_X \cong \mathbb{Z}(r-1)_Y[-2],$$

the theorem of Geisser–Levine [52]

$$\mathbb{Z}/p^n(r)_Y \cong W_n\Omega^r_{\log,Y}[-r],$$

the Suslin–Voevodsky isomorphism in $D^b(V_{ét})$ (Beilinson's axiom (iv)(a))

$$j^*\mathbb{Z}/p^n(r)_{ét,X} \cong \mathbb{Z}/p^n(n)_{ét,V} \cong \mu_{p^n}^{\otimes r},$$

and the Beilinson–Lichtenbaum conjectures (now a theorem)

$$\mathbb{Z}/p^n(r)_V \cong \tau_{\leq r}R\alpha_*\mu_{p^n}^{\otimes r}, \quad R^{r+1}\alpha_*\mathbb{Z}(r)_{ét,V} = 0$$

have as consequence

Theorem 6.6 (Geisser [49, THEOREM 1.3]). *Let $X \to \operatorname{Spec}\Lambda$ be smooth and essentially of finite type, with Λ a complete discrete valuation ring of mixed characteristic $(0,p)$. Then there is a distinguished triangle in $D^b(X_{ét})$,*

$$i_* W_n \Omega^{r-1}_{\log}[-r-1] \to \mathbb{Z}/p^n(r)_{ét} \to \tau_{\le r} R j_* \mu_{p^n}^{\otimes r} \to i_* W_n \Omega^{r-1}_{\log}[-r],$$

and an isomorphism $\mathbb{Z}/p^n(r)_{ét} \cong \mathcal{S}_n(r)$ in $D^b(X_{ét})$ that transforms this triangle to Schneider's defining triangle (6.3).

Zhong has extended this to the semi-stable case, establishing an isomorphism with Sato's construction $\mathfrak{T}_n(r)$ after a truncation [128, PROPOSITION 4.5]:

$$\tau_{\le r}\mathbb{Z}/p^n(r)_{ét} \cong \mathfrak{T}_n(r).$$

Assuming a "weak Gersten conjecture" for $\mathbb{Z}/p^n(r)_{ét}$, the truncation is removed [128, THEOREM 4.8].

6.3. The theorems of Geisser–Hesselholt

The construction of a motivic tower for integral p-adic Hodge theory by Bhatt–Morrow–Scholze relies on properties of p-completed topological cyclic homology, including the results of Geisser–Hesselholt identifying this with the p-completed étale K-theory. We give a brief résumé of these constructions. Fix as before our mixed characteristic dvr Λ with perfect residue field k.

Topological cyclic homology for a fixed prime p is a spectrum refined version of Connes' cyclic homology and is defined for a scheme X with a topology $\tau \in \{ét, \mathrm{Nis}, \mathrm{Zar}\}$; we use the étale topology throughout. There is an inverse system of spectra $\{\mathrm{TC}^m(X,p)\}_{m\in\mathbb{N}}$ defining $\mathrm{TC}(X;p)$ as the homotopy inverse limit

$$\mathrm{TC}(X;p) := \operatorname*{holim}_m \mathrm{TC}^m(X,p).$$

Let $\mathcal{TC}^{\cdot}_i$ denote the étale sheaf associated to the presheaf of the ith pro-homotopy groups $U \mapsto \pi_i \mathrm{TC}^{\cdot}(U;p)$. There is a descent spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{cont}}(X, \mathcal{TC}^{\cdot}_{-t}) \Rightarrow \mathrm{TC}_{-s-t}(X;p)$$

and a cyclotomic trace map

$$\mathrm{trc} : K(X) \to \mathrm{TC}(X;p).$$

Let k be a perfect field of characteristic $p > 0$. It follows from a result of Hesselholt [60, THEOREM B] that there is a isomorphism of pro-sheaves on $\mathrm{Sm}_{két}$

$$\mathcal{TC}^{\cdot}_r \cong W_{\cdot}\Omega^r_{\log}. \tag{6.4}$$

The map trc induces the map of pro-sheaves on $\mathrm{Sm}_{két}$

$$\mathrm{trc} : \mathcal{K}_i(\mathbb{Z}/p^{\cdot}) \to \mathcal{TC}^{\cdot}_i;$$

where $\mathcal{K}_i(\mathbb{Z}/p^{\cdot})$ is the pro-étale sheaf associated to the system of presheaves $U \mapsto \{K_i(U, \mathbb{Z}/p^\nu)\}_\nu$. Relying on the main theorem of [52] and the isomorphism (6.4), Geisser and Hesselholt show

Theorem 6.7 (Geisser–Hesselholt [50, COROLLARY 4.2.5, THEOREM 4.2.6]).

1. *The trace map* $\mathrm{trc} : \mathcal{K}_i(\mathbb{Z}/p^\cdot) \to \mathcal{TC}_i^\cdot$ *is an isomorphism of pro-sheaves on* $\mathrm{Sm}_{k\acute{e}t}$.

2. *For* $Y \in \mathrm{Sm}_k$, $\mathrm{TC}(Y; p)$ *is weakly equivalent to the* p*-completed étale* K*-theory spectrum of* Y,

$$K^{\acute{e}t}(Y)^{\wedge p} := \operatorname*{holim}_m K^{\acute{e}t}(Y, \mathbb{Z}/p^n) \cong \mathrm{TC}(Y; p),$$

and this weak equivalence arises from the weak equivalences at the finite level

$$\mathrm{trc} : K^{\acute{e}t}(Y, \mathbb{Z}/p^\nu) \xrightarrow{\sim} \mathrm{TC}(Y; p, \mathbb{Z}/p^\nu).$$

Now consider a smooth finite type scheme $X \to \operatorname{Spec} \Lambda$ with special fiber $i : Y \to X$ and generic fiber $j : V \to X$, as before.

Theorem 6.8 (Geisser–Hesselholt [51, THEOREMS A AND B]). *Suppose* Λ *is henselian.*

A. *Suppose* $X \to \operatorname{Spec} \Lambda$ *is smooth and proper. Then*

$$\mathrm{trc} : K_q^{\acute{e}t}(X, \mathbb{Z}/p^\nu) \to \mathrm{TC}_q(X; p, \mathbb{Z}/p^\nu)$$

is an isomorphism for all $q \in \mathbb{Z}$ *and* $\nu \geq 1$.

B. *Suppose that* $X \to \operatorname{Spec} \Lambda$ *is smooth and finite type. Then the map of pro-sheaves on* $Y_{\acute{e}t}$,

$$i^*\mathcal{K}_q(\mathbb{Z}/p^\nu) \to \left\{ i^*\mathcal{TC}_q^m(p, \mathbb{Z}/p^\nu) \right\}_{m \in \mathbb{N}},$$

is an isomorphism for all $q \in \mathbb{Z}$ *and all* $\nu \geq 1$.

Remark 6.9. To pass from the isomorphism of Theorem 6.7 to that of Theorem 6.8(B), Geisser–Hesselholt rely on the theorem of McCarthy [88], stating that the cyclotomic trace map from relative K-theory to relative TC,

$$\mathrm{trc} : K_q(X/\pi^n, X/\pi^{n-r}, \mathbb{Z}/p^\nu) \to \mathrm{TC}_q(X/\pi^n, X/\pi^{n-r}, \mathbb{Z}/p^\nu),$$

is an isomorphism for affine X. Thus, the K-theory and topological cyclic homology of non-reduced schemes play a central role in the proof of Theorem 6.8.

6.4. Integral p-adic Hodge theory and the motivic filtration

Bhatt–Morrow–Scholze [17, 18] have constructed integral versions of p-adic Hodge theory. Here we discuss some aspects of the theory of [18] and its relation to p-adic étale motivic cohomology. This uses (p-completed) topological Hochschild homology $\mathrm{THH}(-, \mathbb{Z}_p)$, topological negative cyclic homology $\mathrm{TC}^-(-, \mathbb{Z}_p)$, and topological periodic cyclic homology $\mathrm{TP}(-, \mathbb{Z}_p)$. For a nice, quick overview of these theories, we refer the reader to [18, §1.2, §2.3], and to [18, THEOREM 1.12] for their relation to $\mathrm{TC}(-, \mathbb{Z}_p)$.

Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_{\mathbb{C}_p}$. As in our review of the work of Fontaine–Messing, we have the $\mathbb{F}_p$-algebra $\mathcal{O}_{\mathbb{C}_p}/p$, its perfection $\mathcal{O}_{\mathbb{C}_p}^\flat$ and the ring of Witt vectors $A_{\inf}(\mathcal{O}_{\mathbb{C}_p}) := W(\mathcal{O}_{\mathbb{C}_p}^\flat)$. Hesselholt has connected this with negative cyclic homology TC^-, constructing an isomorphism

$$\pi_0\mathrm{TC}^-(\mathcal{O}_{\mathbb{C}_p}, \mathbb{Z}_p) \cong A_{\inf}(\mathcal{O}_{\mathbb{C}_p}).$$

This has been generalized by Bhatt–Morrow–Scholze in the setting of *perfectoid* rings (see [17, DEFINITION 3.5]). For a perfectoid ring R, we have Scholze's ring $R^\flat$, defined as for $\mathcal{O}_{\mathbb{C}_p}^\flat$ by taking the perfection of R/p,

$$R^\flat := \lim_{\leftarrow \mathrm{Frob}} R/p.$$

This gives the ring of Witt vectors $A_{\inf}(R) := W(R^\flat)$ with Frobenius ϕ induced by the Frobenius on $R^\flat$.

Theorem 6.10 (Bhatt–Morrow–Scholze [18, THEOREM 1.6]). *Let R be a perfectoid ring. Then there is a canonical ϕ-equivariant isomorphism $\pi_0\mathrm{TC}^-(R, \mathbb{Z}_p) \cong A_{\inf}(R)$.*

Fix a discretely valued extension K of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ having perfect residue field k. Let C be the completed algebraic closure of K, with ring of integers $\mathcal{O}_C$. Let $A_{\inf} := A_{\inf}(\mathcal{O}_C)$.

Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_C$. In [17], Bhatt–Morrow–Scholze construct a presheaf of complexes of $A_{\inf}$-algebras on $\mathcal{X}_{\mathrm{Zar}}$, $A\Omega_{\mathcal{X}}$, whose Zariski hypercohomology specializes to crystalline cohomology, p-adic étale cohomology and de Rham cohomology via base-change with respect to suitable ring homomorphisms out of $A_{\inf}$, replacing the ring homomorphism $A_{\inf}(\mathcal{O}_{\mathbb{C}_p}) \to B_{\mathrm{crys}}$ used in the Fontaine–Messing theory. In [18], they refine and reinterpret this theory using TC^-. They define the notion of a quasi-syntomic ring and the associated quasi-syntomic site [18, DEFINITION 1.7]; this gives the presheaf $\pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)$ on the quasi-syntomic site $\mathrm{qsyn}_{\tilde{A}}$ over a quasi-syntomic ring $\tilde{A}$ and the associated derived global sections functor $R\Gamma_{\mathrm{syn}}(\tilde{A}, -)$.

Theorem 6.11 ([18, THEOREM 1.8]). *Let $\tilde{A}$ be an $\mathcal{O}_C$-algebra that can be written as the p-adic completion of a smooth $\mathcal{O}_C$-algebra. There is a functorial (in $\tilde{A}$) ϕ-equivariant isomorphism of E_∞-$A_{\inf}$-algebras*

$$A\Omega_{\tilde{A}} \cong R\Gamma_{\mathrm{syn}}\big(\tilde{A}, \pi_0\mathrm{TC}^-(-;\mathbb{Z}_p)\big).$$

The Postnikov tower $\tau_{\geq *}\mathrm{TC}(-;\mathbb{Z}_p)$ for the presheaf of spectra $\mathrm{TC}(-;\mathbb{Z}_p)$ on $\tilde{A}_{\mathrm{qsyn}}$ induces the tower over $\mathrm{TC}(\tilde{A};\mathbb{Z}_p$:

$$\cdots \to \mathrm{Fil}^{n+1}\mathrm{TC}(\tilde{A};\mathbb{Z}_p) \to \mathrm{Fil}^n\mathrm{TC}(\tilde{A};\mathbb{Z}_p) \to \cdots \to \mathrm{TC}(\tilde{A};\mathbb{Z}_p, \tag{6.5}$$

by setting

$$\mathrm{Fil}^n\mathrm{TC}(\tilde{A};\mathbb{Z}_p) := R\Gamma_{\mathrm{syn}}\big(\tilde{A}, \tau_{\geq 2n}\mathrm{TC}(-;\mathbb{Z}_p)\big)$$

(see [18, §1.4]). Define the sheaves $\mathbb{Z}_p^{\mathrm{BMS}}(n)$ by sheafifying the presheaf

$$\tilde{A} \mapsto \mathbb{Z}_p^{\mathrm{BMS}}(n)(\tilde{A}) := \mathrm{gr}^n_{\mathrm{Fil}}\mathrm{TC}(\tilde{A};\mathbb{Z}_p)[-2n],$$

where $\mathrm{gr}^n_{\mathrm{Fil}}\mathrm{TC}(\tilde{A};\mathbb{Z}_p)$ is the homotopy cofiber of $\mathrm{Fil}^{n+1}\mathrm{TC}(\tilde{A};\mathbb{Z}_p) \to \mathrm{Fil}^n\mathrm{TC}(\tilde{A};\mathbb{Z}_p)$.

Theorem 6.12 ([18, THEOREM 1.15]). (1) *Let k be a perfect field of characteristic $p > 0$, let A be a smooth k-algebra and let $X = \operatorname{Spec} A$. Then there is an isomorphism in the derived category of sheaves on the pro-étale site of X,*

$$\mathbb{Z}_p^{\mathrm{BMS}}(r)_X \cong W\Omega^r_{X,\log}[-r].$$

(2) *Let C be an algebraically closed complete extension of $\mathbb{Q}_p$, let A be the completion of a smooth $\mathcal{O}_C$-algebra, and let $\mathcal{X} = \operatorname{Spf} A$. Then there is an isomorphism in the derived category of sheaves on the pro-étale site of $\mathcal{X}$,*

$$\mathbb{Z}_p^{\mathrm{BMS}}(r)_{\mathcal{X}} \cong \tau_{\leq r} R\psi \mathbb{Z}_p(r)_{X,ét}.$$

Here $\mathbb{Z}_p(r)_{X,ét}$ denotes the pro-étale sheaf $\{\mu_{p^n}^{\otimes r}\}_n$ on the rigid analytic generic fiber X of $\mathcal{X}$ and $R\psi$ is the nearby cycles functor.

The isomorphism in (1) above, combined with the main result of [52], gives us the identification of pro-objects

$$\mathbb{Z}_p^{\mathrm{BMS}}(r)_X \cong \{\mathbb{Z}/p^n(r)_{X,\text{ét}}\}_n$$

in the setting of (1).

Consider the case of a smooth $\mathcal{O}_K$-scheme X as before. Bhatt–Morrow–Scholze suggest in [18, REMARK 1.16] that $\mathbb{Z}_p^{\mathrm{BMS}}/p^n(r)_X$ should be Schneider's sheaf $\mathcal{S}_n(r)$, and by passage to the limit, there should be a distinguished triangle

$$i_*W\Omega^{r-1}_{\log}[-r-1] \to \mathbb{Z}_p^{\mathrm{BMS}}(r) \to \tau_{\leq r} Rj_*\big(\mathbb{Z}_p(r)_{V,\text{ét}}\big) \to i_*W\Omega^{r-1}_{\log}[-r]. \tag{6.6}$$

For X a smooth $\mathcal{O}_K$-scheme with associated formal scheme $\mathcal{X}$ and special fiber $i : Y \to \mathcal{X}$, this would give an isomorphism of $i^*\mathbb{Z}_p^{\mathrm{BMS}}/p^n(r)_{\mathcal{X}}$ with the étale motivic complex $i^*\mathbb{Z}/p^n(r)_{\text{ét}}$ on $Y_{\text{ét}}$ considered by Geisser.

This has been proven in a work–in–progress by Bhargav Bhatt and Akhil Mathew [16]. They construct an isomorphism of a version of $\mathbb{Z}_p^{\mathrm{BMS}}/p^n(r)_X$ with Sato's sheaf $\mathfrak{T}_n(r)_X$ in the semi-stable case; using Zhong's extension of Geisser's results, this gives an isomorphism

$$i^*\mathbb{Z}_p^{\mathrm{BMS}}/p^n(r)_{\mathcal{X}} \cong i^*\tau_{\leq r}\mathbb{Z}/p^n(r)_{\text{ét}}$$

in the semi-stable case.

One has the Geisser–Hesselholt isomorphism (Theorem 6.8) of étale K-theory and topological cyclic homology given by the cyclotomic trace map. Perhaps one can compare the localization pro-distinguished triangle

$$K(Y;\mathbb{Z}_p) \to K(X;\mathbb{Z}_p) \to K(X \setminus Y;\mathbb{Z}_p)$$

with the distinguished triangle (6.6). Assuming one does have the pro-isomorphism $\mathbb{Z}_p(r)_{\text{ét}} \cong \mathbb{Z}_p^{\mathrm{BMS}}(r)$ as suggested above, it would be interesting to see if the identification of the sheaves $\mathcal{S}_n(r)$ with the étale motivic complexes $\mathbb{Z}/p^n(r)_{\text{ét}}$ and the Atiyah–Hirzebruch spectral sequence from motivic cohomology to K-theory could yield a comparison with the spectral sequence corresponding to the motivic tower $\mathrm{Fil}^*\mathrm{TC}(\mathcal{X};\mathbb{Z}_p)$ described above.

The sheaf $\mathbb{Z}_p^{\mathrm{BMS}}(r)$ is built from $\mathrm{TC}(-;\mathbb{Z}_p)$, which by the Geisser–Hesselholt theorem is p-completed étale K-theory. As we mentioned before, the Geisser–Hesselholt isomorphism arises at least in part from McCarthy's theorem identifying the relative K-theory and relative TC of the nilpotent thickenings $X/(\pi^n)$ of the special fiber Y. However, the motivic cohomology complexes do not detect the difference between $X/(\pi^n)$ and Y. Supposing again that one does have a pro-isomorphism $\mathbb{Z}_p(r)_{\text{ét}} \cong \mathbb{Z}_p^{\mathrm{BMS}}(r)$, this says that in mixed characteristic $(0, p)$, one can still see the K-theory of the thickened fibers $X/(\pi^n)$ reflected in the motivic complexes $\mathbb{Z}_p(r)_{\text{ét}}$.

I am not aware of a categorical framework for the tower $\mathrm{Fil}^n\mathrm{TC}(\tilde{A};\mathbb{Z}_p)$ and its layers $\mathbb{Z}_p^{\mathrm{BMS}}(r)$, analogous to the framework for Voevodsky's slice tower for K-theory given by $\mathrm{SH}(k)$. As $\mathbb{A}^1$-homotopy invariance fails for these theories, one would need a stable homotopy theory with a weaker invariance property, perhaps modeled on the one of the categories of motives with modulus discussed in the previous section, for these theories to find a home, in which the Bhatt–Morrow–Scholze tower (6.5) would be seen as a parallel to Voevodsky's slice tower.

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