K-THEORY OF LARGE CATEGORIES

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ABSTRACT

We give a short overview of a new notion of *continuous* K-theory, which is defined for a certain class of large (enhanced) triangulated categories. For compactly generated triangulated categories, this continuous K-theory gives the usual nonconnective K-theory of the category of compact objects. We formulate a general theorem about the computation of continuous K-theory for the category of sheaves (of modules) on a locally compact Hausdorff space. This result already gives a surprising answer for the category of sheaves on the real line.

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1. INTRODUCTION

This paper is a very short introduction to a recent new notion of K-theory for a certain class of "large" (enhanced) triangulated categories. We call it "continuous K-theory." We explain the general idea and formulate some results, including the computation of K-theory for categories of sheaves on locally compact (Hausdorff) spaces. The detailed study with complete proofs will appear in [2].

It is well known that the usual Grothendieck group $K_0(\mathcal{A})$ of an additive category $\mathcal{A}$ vanishes when $\mathcal{A}$ has countable direct sums (Eilenberg swindle). More generally, for any enhanced triangulated category $\mathcal{C}$, its (nonconnective) K-theory spectrum $\mathbb{K}(\mathcal{C})$ is contractible. In particular, when $\mathcal{C} = D(R)$ is the unbounded derived category of modules over a ring, we get $\mathbb{K}_n(D(R)) = 0$ for $n \in \mathbb{Z}$.

This very observation has been used by M. Schlichting to define the negative K-theory. Namely, a short exact sequence of enhanced triangulated categories

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

gives rise to the long exact sequence of K-groups. If $\mathcal{B}$ has countable direct sums, we have $\mathbb{K}_n(\mathcal{A}) \cong \mathbb{K}_{n+1}(\mathcal{C})$. In particular, ignoring the set-theoretic issues, we can take $\mathcal{B} = \mathrm{Ind}(\mathcal{A})$ and $\mathcal{C} = \mathrm{Calk}_{\mathcal{A}} := (\mathrm{Ind}(\mathcal{A})/\mathcal{A})^{\mathrm{Kar}}$, which is the Calkin category of $\mathcal{A}$ (an algebraic analogue of the usual Calkin algebra of a Hilbert space, namely bounded operators modulo compact operators).

One characterization of dualizable presentable categories is that they can be represented as a kernel of a localization, $\mathcal{C} = \ker(\mathrm{Ind}(\mathcal{A}) \xrightarrow{F} \mathrm{Ind}(\mathcal{B}))$, where the functor F commutes with direct sums and takes compact objects to compact objects. The idea is to define K-theory of such a category using the localization property for K-theory. Namely, $\mathbb{K}^{\mathrm{cont}}(\mathcal{C}) := \mathrm{Fiber}(\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{B}))$. However, such an approach would require checking the independence on the choice of the representation of $\mathcal{C}$ as such a kernel (and also the functoriality of $\mathbb{K}^{\mathrm{cont}}$ is not really immediate from such a definition).

An alternative characterization of dualizable categories is the following: these are presentable categories $\mathcal{C}$ such that the Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ has a twice left adjoint. Using it, one can define the continuous Calkin category $\mathrm{Calk}_{\mathcal{C}}^{\mathrm{cont}}$, namely the "virtual quotient by compact objects." Then we simply put

$$\mathbb{K}^{\mathrm{cont}}(\mathcal{C}) := \Omega\mathbb{K}\big(\mathrm{Calk}_{\mathcal{C}}^{\mathrm{cont}}\big).$$

Continuous K-theory is functorial with respect to *strongly continuous* functors, that is, the functors whose right adjoint commutes with infinite direct sums.

We have the following computation of the continuous K-theory for categories of sheaves on locally compact Hausdorff spaces.

Theorem 1.1. *Let X be a locally compact Hausdorff space, and let $\underline{R}$ be a presheaf of DG rings on X. Then we have a natural isomorphism*

$$\mathbb{K}^{\mathrm{cont}}\big(\mathrm{Shv}(X, \text{Mod-}\underline{R})\big) \cong \Gamma_c\big(X, \mathbb{K}(\underline{R})\big).$$

In particular, for any $n \in \mathbb{Z}_{\geq 0}$ and any DG ring A, we have

$$\mathbb{K}^{\mathrm{cont}}\big(\mathrm{Shv}(\mathbb{R}^n, \mathrm{Mod}\text{-}A)\big) \cong \Omega^n \mathbb{K}(A),$$

hence

$$\mathbb{K}_0^{\mathrm{cont}}\big(\mathrm{Shv}(\mathbb{R}^n, \mathrm{Mod}\text{-}A)\big) \cong \mathbb{K}_n(A).$$

Another interesting class of dualizable categories for which it would be very interesting to compute the continuous K-theory comes from the theory of condensed mathematics due to Clausen and Scholze [17,18]. These are the so-called categories of nuclear modules. We only briefly mention them in the case of affine formal schemes.

Although it would be natural to consider stable ∞-categories, in this paper we will restrict to DG categories to fix the ideas.

The paper is organized as follows. In Section 2 we recall the basic notions about DG categories. In Section 3 we recall presentable and dualizable DG categories, and give some examples.

Section 4 is devoted to the classical algebraic K-theory. In Section 5 we introduce the notion of continuous K-theory of dualizable categories.

Finally, in Section 6 we discuss in some detail Theorem 1.1, giving a sketch of its proof in the case of $X = \mathbb{R}$ and a constant presheaf of DG rings.

2. PRELIMINARIES ON DG CATEGORIES

We refer to [8,10] for a general introduction and overview of DG algebras, DG categories, and the derived categories of DG modules. We refer to [1] for the notion of an enhanced triangulated category. We refer to [20,21] for the model structures on the category of small DG categories.

For a small DG category $\mathcal{A}$ over a base ring k, we denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules. We will denote by $\mathrm{Mod}\text{-}\mathcal{A}$ the "correct" DG category of $\mathcal{A}$-modules, for example, the DG category of semi-free $\mathcal{A}$-modules. Moreover, for a usual associative ring R (considered as a DG algebra over $\mathbb{Z}$ concentrated in degree zero), we write $\mathrm{Mod}\text{-}R$ for the (correct) category of DG R-modules, i.e., complexes of usual R-modules

In some cases we will also write $\mathrm{Ind}(\mathcal{A})$ (ind-objects) instead of $\mathrm{Mod}\text{-}\mathcal{A}$, provided that $\mathcal{A}$ is pretriangulated. We will denote by $\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{tr}}$ the ∞-category of pretriangulated Karoubi complete small DG categories.

We denote by $\mathrm{Perf}(\mathcal{A}) \subset \mathrm{Mod}\text{-}\mathcal{A}$ the full DG subcategory of perfect A-modules. We denote by $\mathrm{PsPerf}(\mathcal{A}) \subset \mathrm{Mod}\text{-}\mathcal{A}$ the full DG subcategory of *pseudoperfect* modules, i.e., A-modules which are perfect over k.

The tensor product of DG categories will always be derived over k. Given small DG categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ the "correct" DG category of functors $\mathcal{A} \to \mathcal{B}$, i.e., the internal Hom in the symmetric monoidal homotopy category of DG categories (with inverted quasiequivalences).

Although in this paper we do not need the notions of smoothness and properness of DG algebras and DG categories, we recall them here for completeness.

Definition 2.1. A DG algebra A over k is called proper if A is perfect as a complex of k-modules.

Definition 2.2. A DG algebra A over k is called smooth if $A \in \mathrm{Perf}(A \otimes A^{op})$, i.e., the diagonal A-A-bimodule is perfect.

Note that for a proper (resp. smooth) DG algebra A, we have $\mathrm{Perf}(A) \subseteq \mathrm{PsPerf}(A)$ (resp. $\mathrm{PsPerf}(A) \subseteq \mathrm{Perf}(A)$). If X is a separated scheme of finite type over a field k, and $\mathrm{Perf}(X) \simeq \mathrm{Perf}(A)$, then X is smooth (resp. proper) if and only if such is A.

A morphism of DG algebras $f : A \to B$ is called a *homological epimorphism* if the map $B \overset{\mathbf{L}}{\otimes}_A B \to B$ is an isomorphism in $D(\mathrm{k})$. This is equivalent to f being a homotopy epimorphism, i.e., the map $B \sqcup_A^h B \to B$ is an isomorphism in $\mathrm{Ho}(\mathrm{dgalg}_{\mathrm{k}})$.

The notion of a homological epimorphism for a functor between small DG categories is defined similarly; it is also equivalent to the property of being a homotopy epimorphism (in the Morita model structure).

3. PRESENTABLE AND DUALIZABLE DG CATEGORIES

We fix some base commutative ring k. In what follows, we will mostly ignore set-theoretic issues. Recall that a DG category $\mathcal{C}$ is *presentable* if

1) $\mathcal{C}$ is pretriangulated;
2) the homotopy category $H^0(\mathcal{C})$ has small direct sums;
3) there exists a regular cardinal κ such that the triangulated category $H^0(\mathcal{C})$ is generated by κ-compact objects.

Recall that an object x of a triangulated category $\mathcal{T}$ is κ-compact if for any small family of objects $\{y_j\}_{j \in J}$, and for any morphism $x \xrightarrow{f} \bigoplus_{j \in J} y_j$, there exists a subset $I \subseteq J$ with $|I| < \kappa$ such that f factors through $\bigoplus_{i \in I} y_i$.

In other words, conditions 2) and 3) mean that the triangulated category $H^0(\mathcal{C})$ is well generated in the terminology of Neeman [14].

We will call a functor $F : \mathcal{C} \to \mathcal{D}$ between such DG categories *continuous* if it commutes with small direct sums. We denote by $\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{cont}}$ the ∞-category of presentable DG categories and continuous functors.

There is a natural symmetric monoidal structure on $\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{cont}}$—the Lurie tensor product. It is uniquely determined by the internal Hom, given by $\mathrm{Fun}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D})$—the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ formed by continuous functors. Denoting this tensor product by $-\widehat{\otimes}-$, we thus have for any $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D} \in \mathrm{dgcat}_{\mathrm{k}}^{\mathrm{cont}}$ a full embedding

$$\mathrm{Fun}^{\mathrm{cont}}(\mathcal{C}_1 \widehat{\otimes}_{\mathrm{k}} \mathcal{C}_2, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}_1 \otimes_{\mathrm{k}} \mathcal{C}_2, \mathcal{D}),$$

and the essential image consists of bicontinuous bifunctors.

Definition 3.1. A presentable DG category $\mathcal{C}$ is dualizable if it is a dualizable object in the symmetric monoidal category $(\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{cont}}, \widehat{\otimes}_{\mathrm{k}})$.

Remark 3.2. In fact, dualizability of a presentable DG category is independent of the base ring k.

The following is due to Lurie.

Theorem 3.3 ([12]). *Let $\mathcal{C}$ be a presentable DG category. The following are equivalent:*

(i) *$\mathcal{C}$ is dualizable.*

(ii) *There is a short exact sequence*

$$0 \to \mathcal{C} \to \mathrm{Mod}\text{-}A \xrightarrow{-\otimes_A B} \mathrm{Mod}\text{-}B \to 0,$$

where $A \to B$ is a homological epimorphism of DG algebras, i.e., $B \overset{\mathbf{L}}{\otimes}_A B \xrightarrow{\sim} B$.

(iii) *Same as* (ii) *with small DG categories instead of DG algebras.*

(iv) *$\mathcal{C}$ is a retract in* $\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{cont}}$ *of a compactly generated category.*

(v) *The Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ has a twice left adjoint.*

(vi) *Any continuous localization of presentable DG categories $\mathcal{D} \to \mathcal{C}$ has a continuous section (not necessarily fully faithful).*

We call a continuous functor $F : \mathcal{C} \to \mathcal{D}$ *strongly continuous* if its right adjoint is continuous. We denote by $\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{dual}} \subset \mathrm{dgcat}_{\mathrm{k}}^{\mathrm{cont}}$ the (nonfull) subcategory formed by dualizable categories and strongly continuous functors. We have a fully faithful embedding

$$\mathrm{dgcat}_{\mathrm{k}}^{\mathrm{tr}} \hookrightarrow \mathrm{dgcat}_{\mathrm{k}}^{\mathrm{dual}}, \quad \mathcal{A} \mapsto \mathrm{Mod}\text{-}\mathcal{A}.$$

Moreover, we have

$$(\mathrm{Mod}\text{-}\mathcal{A})\widehat{\otimes}_{\mathrm{k}}(\mathrm{Mod}\text{-}\mathcal{B}) \simeq \mathrm{Mod}\text{-}(\mathcal{A} \otimes_{\mathrm{k}} \mathcal{B}).$$

For a homological epimorphism $A \to B$, we have

$$\ker(\mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B)^{\vee} \simeq \ker(A\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}).$$

Example: derived categories of almost modules. The first class of examples is given by (a not necessarily commutative version of) the basic setup for almost mathematics [3, 4]. Let R be an associative ring, and $I \subset R$ an ideal such that $I^2 = I$ and I is flat as a left or right R-module. Then $I \overset{\mathbf{L}}{\otimes}_R I \cong I$, hence $R \to R/I$ is a homological epimorphism. We get a dualizable category

$$\mathcal{C}(R, I) := \ker(\mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R/I).$$

If $1 + I \subset R^{\times}$, then $\mathcal{C}(R, I)$ has no nonzero compact objects. This observation is originally due to Keller [9].

Example: sheaves on exponentiable spaces. Recall that a topological space X is exponentiable if the functor $X \times - : \mathrm{Top} \to \mathrm{Top}$ commutes with colimits. This is equivalent to X being *core-compact*: for any point $x \in X$ and any open neighborhood U of x, there exists an open neighborhood V of x such that $V \ll U$. Here $V \ll U$ means that V is relatively compact in U, i.e., any open cover of U admits a finite subcover of V. In particular, any locally compact Hausdorff space is exponentiable, and such is any spectral space.

Let R be a DG ring and denote by $\mathrm{Shv}(X, \mathrm{Mod}\text{-}R)$ the DG category of sheaves of R-modules. Then the category $\mathrm{Shv}(X, \mathrm{Mod}\text{-}R)$ is dualizable [12]. Similarly, one can replace a single DG ring R with a presheaf of DG rings $\underline{\mathcal{R}}$.

Remark 3.4. We consider sheaves of R-modules which are not necessarily hypercomplete. In particular, even if X is sober (that is, any irreducible closed subset has a unique generic point), there might be some sheaves $\mathcal{F}$ which have zero stalks at all points of X. For example, this happens when $R = \mathbb{Z}$, $X = [0,1]^{\mathbb{N}}$ is the Hilbert cube, and $\mathcal{F}$ is the sheaf such that $\mathcal{F}(U)$ is the (complex computing the) Borel–Moore homology $H_{\bullet}^{\mathrm{BM}}(U, \mathbb{Z})$. See [11] for details.

Below we will formulate a result on the computation of the continuous K-theory of such categories when X is locally compact Hausdorff.

Example: sheaves with condition on the singular support. Let X be a (paracompact Hausdorff) C^1-manifold, and R a DG ring. Recall that the singular support $SS(\mathcal{F}) \subset T^*X$ of a sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathrm{Mod}\text{-}R)$ is defined as follows (see [6] for details). For a point $(x_0, \xi_0) \in T^*X$, we have $(x_0, \xi_0) \notin SS(\mathcal{F})$ if and only if there exists an open neighborhood $(x_0, \xi_0) \in U \subset T^*X$ such that, for any point $(x_1, \xi_1) \in U$ and for any C^1-function $f : V \to \mathbb{R}$, $x_1 \in V \subset \pi(U)$ such that $df(x_1) = \xi_1$, we have $\Gamma_{\{f(x) \geq f(x_1)\}}(\mathcal{F})_{x_1} = 0$. Hence, $SS(\mathcal{F})$ is a conical closed subset.

Now, let $\Lambda \subset T^*X$ be any conical closed subset. Denote by $\mathrm{Shv}_{\Lambda}(X, \mathrm{Mod}\text{-}R) \subset \mathrm{Shv}(X, \mathrm{Mod}\text{-}R)$ the full subcategory of sheaves R such that $SS(\mathcal{F}) \subset \Lambda$. Then the inclusion functor $\mathrm{Shv}_{\Lambda}(X, \mathrm{Mod}\text{-}R) \hookrightarrow \mathrm{Shv}(X, \mathrm{Mod}\text{-}R)$ has a left adjoint. In particular, the category $\mathrm{Shv}_{\Lambda}(X, \mathrm{Mod}\text{-}R)$ is dualizable.

Example: nuclear modules. The following class of examples comes from the theory of condensed mathematics due to Clausen and Scholze [17,18]. These are the categories of nuclear modules on sufficiently nice analytic spaces.

We mention here the following characterization of the category of nuclear modules on an affine formal scheme (for simplicity assumed to be noetherian).

Theorem 3.5. *Let R be a commutative noetherian ring, and $I \subset R$ an ideal. Then the following hold:*

1) *The inverse limit* $\varprojlim_n (\mathrm{Mod}\text{-}R/I^n)$ *exists in* $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{dual}}$ *(and is totally different from the usual inverse limit).*

2) *The category of nuclear modules on* $\mathrm{Spf}(R_{\hat{I}})$ *defined in* [17] *embeds fully faithfully (and strongly continuously) into the above inverse limit.*

Nonexample. One of the characterizations of dualizable categories (condition (ii) of Theorem 3.3) is via taking kernels of extension of scalars for a homological epimorphism of DG algebras $A \to B$. However, for a general morphism $A \to B$ the kernel of the functor $- \otimes_A B$ is some "random" presentable DG category which does not have to be dualizable. For example, take $A = \mathrm{k}[x, y]$, $B = \mathrm{k}[t, t^{-1}]$ ($\deg(x) = \deg(y) = \deg(t) = 0$), and consider the morphism

$$f : A \to B, \quad f(x) = t,\ f(y) = 0.$$

Then one can show that the kernel $\ker(\text{Mod-}A \xrightarrow{-\otimes_A B} \text{Mod-}B)$ is not dualizable (a pleasant exercise).

4. K-THEORY OF RINGS, ABELIAN CATEGORIES, AND TRIANGULATED CATEGORIES

In this section we recall the notion of a Grothendieck group in various contexts, and very briefly recall the higher K-theory, as well as negative K-theory.

Recall that for an associative ring R, the Grothendieck group $K_0(R)$ is generated by the isomorphism classes $[P]$ of finitely generated (right) projective R-modules, subject to relation $[P \oplus Q] = [P] + [Q]$.

For an abelian category $\mathcal{A}$, the group $K_0(\mathcal{A})$ is defined similarly, but now we have a relation $[Y] = [X] \oplus [Z]$ for each short exact sequence

$$0 \to X \to Y \to Z \to 0.$$

For a triangulated category $\mathcal{T}$, we obtain the group $K_0(\mathcal{T})$ by replacing short exact sequences with exact triangles

$$0 \to X \to Y \to Z \to X[1].$$

For an abelian category $\mathcal{A}$, we have a natural isomorphism

$$K_0\big(D^b(\mathcal{A})\big) \xrightarrow{\sim} K_0(\mathcal{A}), \quad [X] \mapsto \sum_i (-1)^i \big[H^i(X)\big].$$

For a ring R, we have an isomorphism

$$K_0\big(\mathrm{Perf}(R)\big) \xrightarrow{\sim} K_0(R), \quad \big[P^\bullet\big] \mapsto \sum_i \big[P^i\big],$$

for a bounded complex $P^\bullet$ of finitely generated projective R-modules.

In particular, if R is right noetherian and has finite homological dimension, then

$$K_0(R) \cong K_0\big(\mathrm{Perf}(R)\big) \cong K_0(\mathrm{Mod}_{f.g.}\text{-}R).$$

For a scheme X (quasicompact, quasiseparated), one defines

$$K_0(X) := K_0(D_{\mathrm{perf}}(X)).$$

If X is noetherian, we have $G_0(X) = K_0(\mathrm{Coh}(X)) = K_0(D^b_{\mathrm{coh}}(X))$.

It is easy to check that for a noetherian scheme X and for a closed subset Z, we have an exact sequence

$$G_0(Z) \cong K_0\big(\mathrm{Coh}_Z(X)\big) \to G_0(X) \to G_0(X - Z) \to 0.$$

More generally, for an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B} \subset \mathcal{A}$ (that is, a full subcategory closed under subobjects, quotient objects, and extensions), we have an exact sequence

$$K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

Similarly, for a triangulated category $\mathcal{T}$ and a full idempotent complete triangulated subcategory $\mathcal{S}$, we have an exact sequence

$$K_0(\mathcal{S}) \to K_0(\mathcal{T}) \to K_0(\mathcal{T}/\mathcal{S}) \to 0.$$

It is well known that the above definition of K_0 can be extended to higher K-groups $K_n(-)$: for associative rings (via Quillen's plus construction [15]), for abelian categories (via Quillen's Q-construction [15]) and for *enhanced* triangulated categories (via Waldhausen's S-construction [23]). These are again compatible with each other: $K_n(R) \cong K_n(\mathrm{Perf}(R))$, $K_n(\mathcal{A}) \cong K_n(D^b(\mathcal{A}))$.

One of the most important properties of higher algebraic K-theory is the localization sequence. Namely, for an enhanced triangulated category $\mathcal{T}$ and a full idempotent complete triangulated subcategory $\mathcal{S} \subset \mathcal{T}$, we have a long exact sequence

$$\cdots \to K_1(\mathcal{S}) \to K_1(\mathcal{T}) \to K_1(\mathcal{T}/\mathcal{S}) \to \\ \to K_0(\mathcal{S}) \to K_0(\mathcal{T}) \to K_0(\mathcal{T}/\mathcal{S}) \to 0.$$

Now, an important observation is that if a triangulated category $\mathcal{T}$ has at least countable direct sums, then $K_0(\mathcal{T}) = 0$. Indeed, for any object $X \in \mathcal{T}$, we have

$$[X] + \big[X^{(\mathbb{N})}\big] = \big[X \oplus X^{(\mathbb{N})}\big] = \big[X^{(\mathbb{N})}\big], \quad \text{hence } [X] = 0$$

(Eilenberg swindle).

Essentially the same argument (applied to the identity functor) shows that, for an enhanced triangulated category $\mathcal{T}$ with countable direct sums, we have $K_n(\mathcal{T}) = 0$, $n \geq 0$. This actually allows defining negative K-theory, which was studied by Schlichting [16].

Namely, let $\mathcal{T} = \mathrm{Perf}(\mathcal{B})$ for a DG category $\mathcal{B}$. Let us define the Calkin category $\mathrm{Calk}_{\mathcal{B}}$ as the (Karoubi completion of) the quotient $(\text{Mod-}\mathcal{B}/\mathrm{Perf}(\mathcal{B}))^{\mathrm{Kar}}$. Then we have a short exact sequence

$$0 \to \mathrm{Perf}(\mathcal{B}) \to \text{Mod-}\mathcal{B} \to \mathrm{Calk}_{\mathcal{B}} \to 0,$$

which forces the definition $K_{-1}(\mathrm{Perf}(\mathcal{B})) := K_0(\mathrm{Calk}_{\mathcal{B}})$ (ignoring set theory for simplicity). Iterating, one gets the groups $K_n(\mathrm{Perf}(\mathcal{B}))$ for all negative integers n.

Let us remark that the terminology "Calkin category" comes from functional analysis. Namely, for a field k and a vector space V, we have $\mathrm{End}_{\mathrm{Calk}_{\mathrm{k}}}(V) = \mathrm{End}_{\mathrm{k}}(V)/V^* \otimes V$.

This is an algebraic version of the usual Calkin algebra of a Hilbert space $\mathcal{H}$, defined as $B(\mathcal{H})/C(\mathcal{H})$. Here $C(\mathcal{H})$ is the ideal of compact operators, and we have $C(\mathcal{H}) = \overline{\mathcal{H}^* \otimes \mathcal{H}}$.

Summarizing, for an enhanced triangulated category $\mathcal{T}$, one has K-groups $K_n(\mathcal{T})$, which are stable homotopy groups of the nonconnective K-theory spectrum $\mathbb{K}(\mathcal{T})$. For a short exact sequence of enhanced triangulated categories

$$0 \to \mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3 \to 0,$$

we have an exact triangle of spectra

$$\mathbb{K}(\mathcal{T}_1) \to \mathbb{K}(\mathcal{T}_2) \to \mathbb{K}(\mathcal{T}_3) \to \mathbb{K}(\mathcal{T}_1)[1].$$

5. CONTINUOUS K-THEORY

Condition (v) of Theorem 3.3 gives a canonical short exact sequence (strictly speaking, we have to choose a regular cardinal to get presentable compactly generated categories)

$$0 \to \mathcal{C} \xrightarrow{Y_{\mathcal{C}}^{LL}} \text{Mod-}\mathcal{C} \to \text{Mod-Calk}^{\text{cont}}(\mathcal{C}) \to 0.$$

Here $\text{Calk}^{\text{cont}}$ is the Karoubi closure of the image of $\text{Cone}(Y_{\mathcal{C}}^{LL} \to Y_{\mathcal{C}}) : \mathcal{C} \to \text{Mod-}\mathcal{C}$. This allows defining continuous K-theory for dualizable categories (and a continuous version of any localizing invariant):

$$\mathbb{K}^{\text{cont}} : \text{dgcat}_{\text{k}}^{\text{dual}} \to \text{Sp}, \quad \mathbb{K}^{\text{cont}}(\mathcal{C}) := \Omega\mathbb{K}\big(\text{Calk}^{\text{cont}}(\mathcal{C})\big).$$

Moreover, this is the only way to extend K-theory to dualizable categories such that

- For a small DG category $\mathcal{A}$, we have $\mathbb{K}(\text{Mod-}\mathcal{A}) \cong \mathbb{K}(\mathcal{A})$.
- $\mathbb{K}^{\text{cont}}$ is a localizing invariant.

In particular, if a dualizable category is represented as a kernel of extension of scalars for a homological epimorphism $A \to B$, then we have

$$\mathbb{K}^{\text{cont}}\big(\ker(\text{Mod-}A \to \text{Mod-}B)\big) \cong \text{Fiber}\big(\mathbb{K}(A) \to \mathbb{K}(B)\big).$$

The independence of this fiber of the choice of a homological epimorphism $A \to B$ is closely related with (and, in fact, reproves) the excision theorem of Tamme [22].

It is not hard to deduce the general properties of continuous K-theory from the corresponding properties of the usual nonconnective K-theory. In particular, continuous K-theory commutes with filtered colimits in $\text{dgcat}_{\mathbb{Z}}^{\text{dual}}$. Furthermore, one can deduce from the results of [7] that continuous K-theory commutes with the products in $\text{dgcat}_{\mathbb{Z}}^{\text{dual}}$ (which are quite different from the products in $\text{dgcat}_{\mathbb{Z}}^{\text{cont}}$).

6. CONTINUOUS K-THEORY OF CATEGORIES OF SHEAVES

This section is devoted to Theorem 1.1 from the introduction. We first recall its formulation. Let X be a locally compact Hausdorff space. Let $\underline{\mathcal{R}}$ be a presheaf of DG rings on X.

Theorem 6.1. *We have a natural isomorphism $\mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}(X, \text{Mod-}\underline{R})) \cong \Gamma_c(X, \mathbb{K}(\underline{R}))$. In particular, for any $n \in \mathbb{Z}_{\geq 0}$ and any DG ring A, we have*

$$\mathbb{K}^{\mathrm{cont}}\big(\mathrm{Shv}(\mathbb{R}^n, \text{Mod-}A)\big) \cong \Omega^n \mathbb{K}(A),$$

hence

$$\mathbb{K}_0^{\mathrm{cont}}\big(\mathrm{Shv}(\mathbb{R}^n, \text{Mod-}A)\big) \cong \mathbb{K}_n(A).$$

6.1. The case of the real line

Below we sketch the proof of Theorem 6.1 in the special case when $X = \mathbb{R}$ and $\underline{R}$ is the constant presheaf of DG rings with value A. We consider the subcategories $\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A)$, $\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\leq 0}}(\mathbb{R}, \text{Mod-}A)$, and $\mathrm{Shv}_{\mathbb{R}\times\{0\}}(\mathbb{R}, \text{Mod-}A) \simeq \text{Mod-}A$. Here, of course, we identify $T^*\mathbb{R}$ with $\mathbb{R} \times \mathbb{R}$. The following assertion is standard.

Proposition 6.2. *For a sheaf $\mathcal{F} \in \mathrm{Shv}(\mathbb{R}, \text{Mod-}A)$, the following are equivalent:*

(i) $SS(\mathcal{F}) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$*;*

(ii) *for any real numbers $a < b$, the natural map $\Gamma((-\infty, b), \mathcal{F}) \to \Gamma((a, b), \mathcal{F})$ is an isomorphism.*

It is not hard to check the following "gluing" statement.

Proposition 6.3. *We have a homotopy Cartesian square of DG categories*

$$\begin{array}{ccc} \mathrm{Shv}(\mathbb{R}, \text{Mod-}A) & \longrightarrow & \mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A) \\ \downarrow & & \downarrow \\ \mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\leq 0}}(\mathbb{R}, \text{Mod-}A) & \longrightarrow & \mathrm{Shv}_{\mathbb{R}\times\{0\}}(\mathbb{R}, \text{Mod-}A). \end{array} \tag{6.1}$$

Here each of the functors is left adjoint to the inclusion.

Since all the functors in the diagram (6.1) are localizations, we obtain the homotopy Cartesian square of (continuous) K-theory spectra

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}, \text{Mod-}A)) & \longrightarrow & \mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A)) \\ \downarrow & & \downarrow \\ \mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\leq 0}}(\mathbb{R}, \text{Mod-}A)) & \longrightarrow & \mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}_{\mathbb{R}\times\{0\}}(\mathbb{R}, \text{Mod-}A)) = \mathbb{K}(A). \end{array} \tag{6.2}$$

The special case of Theorem 6.1 states that $\mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}, \text{Mod-}A)) \cong \Omega\mathbb{K}(A)$. Using the commutative diagram (6.2), we observe that this assertion reduces to the vanishing of $\mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A))$ (note that, by symmetry, this implies the vanishing of $\mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\leq 0}}(\mathbb{R}, \text{Mod-}A))$).

Sketch of the proof of the vanishing of $\mathbb{K}^{\mathrm{cont}}(\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A))$. We define two auxiliary DG categories $\mathcal{B}$ and $\mathcal{C}$ as follows. We have $Ob(\mathcal{B}) = Ob(\mathcal{C}) = \mathbb{R}$, and the morphisms are given by

$$\mathcal{B}(a, b) = \begin{cases} A, & \text{for } a \leq b, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{C}(a, b) = \begin{cases} A, & \text{for } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

The compositions are induced by the multiplication in A. Denote by $F : \mathcal{B} \to \mathcal{C}$ the obvious functor given by $F(a) = a$. We claim that this functor is a homological epimorphism. This is straightforward to check. The only property of $\mathbb{R}$ we need here is that $\mathbb{R}$ is a dense linearly ordered set, i.e., for any $a < b$ there exists c such that $a < c < b$.

The above description of sheaves with singular support in $\mathbb{R} \times \mathbb{R}_{\geq 0}$ (Proposition 6.2) implies a short exact sequence in $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{dual}}$, namely

$$0 \to \mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A) \to \text{Mod-}\mathcal{B} \xrightarrow{F^*} \text{Mod-}\mathcal{C} \to 0.$$

Indeed, condition (ii) of Proposition 6.2 means that a sheaf $\mathcal{F} \in \mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A)$ is determined by its sections on the negative rays $(-\infty, a)$, together with compatible restriction maps $\Gamma((-\infty, b), \mathcal{F}) \to \Gamma((-\infty, a), \mathcal{F})$ subject to the following condition: for any $a \in \mathbb{R}$, the natural map

$$\Gamma\big((-\infty, a), \mathcal{F}\big) \to \varprojlim_{b<a} \Gamma\big((-\infty, b), \mathcal{F}\big)$$

is an isomorphism.

This exactly identifies $\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A)$ with the right orthogonal complement $F_*(\text{Mod-}\mathcal{C})^{\perp} \subset \text{Mod-}\mathcal{B}$. But the full subcategory $F_*(\text{Mod-}\mathcal{C}) \subset \text{Mod-}\mathcal{B}$ is both left and right admissible, hence its right orthogonal complement is equivalent to the left orthogonal complement, which exactly equals the kernel $\ker(\text{Mod-}\mathcal{B} \to \text{Mod-}\mathcal{C})$.

Finally, we obtain an exact triangle of spectra

$$\mathbb{K}^{\mathrm{cont}}\big(\mathrm{Shv}_{\mathbb{R}\times\mathbb{R}_{\geq 0}}(\mathbb{R}, \text{Mod-}A)\big) \to \mathbb{K}(\mathcal{B}) \to \mathbb{K}(\mathcal{C}),$$

so we are reduced to showing that the map $\mathbb{K}(\mathcal{B}) \to \mathbb{K}(\mathcal{C})$ is an isomorphism. This is immediate from the two standard properties of K-theory:

1) it is additive with respect to finite semiorthogonal decompositions;

2) it commutes with filtered colimits.

This finishes the (sketch of the) proof. ■

6.2. Reduction to the hypercomplete case

We recall the definition of a hypercomplete space in a slightly different form to avoid going into details.

Definition 6.4. A sober topological space X is hypercomplete if for any nonzero (homotopy) sheaf of spectra $\mathcal{F}$ on X there is a point $x \in X$ such that the stalk $\mathcal{F}_x$ is nonzero.

A paracompact space of finite covering dimension is hypercomplete. One can prove Theorem 6.1 in the hypercomplete case indirectly by "sheafifying" the assertion and reducing to stalks.

The general case can be reduced to the hypercomplete case using Urysohn's lemma. Namely, each compact Hausdorff space can be embedded into a product of an infinite number of copies of the closed unit interval $[0, 1]$ (for example, one can take the product over all continuous functions $X \to [0, 1]$). This allows representing X as a cofiltered limit of spaces X_i which are closed subsets of finite-dimensional cubes, hence are hypercomplete. This allows reducing Theorem 6.1 to the hypercomplete case.

6.3. Continuous presheaves and continuous partially ordered sets

We now sketch another more conceptual approach to computing the K-theory of the category of sheaves, and also recall the relevant notion of a continuous poset and a continuous category.

The notion of a continuous (abstract, discrete) category $\mathcal{C}$ is due to Johnstone and Joyal **[5]**. This means that the Yoneda embedding functor from $\mathcal{C}$ the the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects has a twice left adjoint (as in condition (v) of Theorem 3.3). Note that having a single left adjoint means exactly that the category $\mathcal{C}$ has small filtered colimits. It is proved in **[5]** that a category $\mathcal{C}$ with small filtered colimits is continuous if and only if there is some category $\mathcal{D}$ and a pair of functors $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{D})$, $G : \mathrm{Ind}(\mathcal{D}) \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$, and both F and G commute with filtered colimits (as in condition (iv) of Theorem 3.3).

If one considers a partially ordered set P as a category, then we recover the notion of a continuous poset **[13]**, which generalizes the notion of a continuous lattice due to Scott **[19]**. Namely, recall that in a partially ordered set an element x is *way below* y, written as $x \ll y$, if for any directed family of elements $\{z_i \in P\}_{i \in I}$ such that the supremum (join) $\sup\{z_i, i \in I\}$ exists and $\sup\{z_i, i \in I\} \geq y$, there exists some $i \in I$ such that $z_i \geq x$.

Definition 6.5. A partially ordered set P is continuous if the following conditions hold:

1) any directed subset of elements of P has a supremum;
2) for any element $x \in P$, the set $\{y \in P : y \ll x\}$ is directed and $\sup\{y \in P : y \ll x\} = x$.

It is proved in **[5]** that a poset P is continuous if and only if the associated category is continuous. The collection of open subsets of a locally compact Hausdorff space X is a continuous poset. Here we have $U \ll V$ if and only if $\bar{U} \subset V$ and $\bar{U}$ is compact. More generally, a topological space is exponentiable (= core-compact) if and only if the collection of its open subsets is a continuous poset.

Let now $\underline{\mathcal{R}}$ be a presheaf of DG rings on a continuous poset P. Let us call a presheaf $\mathcal{F}$ of $\underline{\mathcal{R}}$-modules a sheaf if for any $x \in P$ we have an isomorphism

$$\mathcal{F}(x) \xrightarrow{\sim} \varprojlim_{y \ll x} \mathcal{F}(y).$$

Denote by $\operatorname{Shv}(P, \text{Mod-}\underline{\mathcal{R}})$ the DG category of sheaves (they are actually the sheaves with respect to a suitable Grothendieck topology). We have the following result.

Proposition 6.6.

1) *The category* $\operatorname{Shv}(P, \text{Mod-}\underline{\mathcal{R}})$ *is dualizable.*

2) *We have a natural isomorphism*

$$\mathbb{K}^{\mathrm{cont}}\big(\operatorname{Shv}(P, \text{Mod-}\underline{\mathcal{R}})\big) \cong \bigoplus_{x \ll x} \mathbb{K}\big(\mathcal{R}(x)\big).$$

Recall that an element $x \in P$ such that $x \ll x$ is called *compact*, and this is the special case of the usual notion of a compact object of a category with small filtered colimits.

Now, let us return to a locally compact Hausdorff space X with a presheaf of DG rings $\underline{\mathcal{R}}$. For a presheaf $\mathcal{F}$ of $\underline{\mathcal{R}}$-modules, the sheafiness condition can be separated into the following two conditions:

(i) for any open $U \subset X$, we have an isomorphism

$$\mathcal{F}(U) \xrightarrow{\sim} \varprojlim_{V \ll U} \mathcal{F}(V),$$

(ii) for any pair of open subsets $U_1, U_2 \subset X$, we have a Cartesian square

$$\begin{array}{ccc} \mathcal{F}(U_1 \cup U_2) & \longrightarrow & \mathcal{F}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_2) & \longrightarrow & \mathcal{F}(U_1 \cap U_2). \end{array}$$

Let us call a presheaf $\mathcal{F}$ of $\underline{\mathcal{R}}$-modules *continuous* if it satisfies condition (i) (but not necessarily condition (ii)). We denote by $\operatorname{PSh}^{\mathrm{cont}}(X, \text{Mod-}\underline{\mathcal{R}})$ the DG category of continuous presheaves. We get the following corollary of Proposition 6.6.

Corollary 6.7.

1) *The category* $\operatorname{PSh}^{\mathrm{cont}}(X, \text{Mod-}\underline{\mathcal{R}})$ *is dualizable.*

2) *We have*

$$\mathbb{K}^{\mathrm{cont}}\big(\operatorname{PSh}^{\mathrm{cont}}(X, \text{Mod-}\underline{\mathcal{R}})\big) \cong \bigoplus_{U \ll U} \mathbb{K}\big(\mathcal{R}(U)\big).$$

Here the summation is taken over open-compact subsets $U \subset X$.

This computation leads to another way of computing the continuous K-theory of sheaves of $\underline{\mathcal{R}}$-modules. Namely, assuming X is compact for simplicity, one can "approximate" the category $\operatorname{Shv}(X, \text{Mod-}\underline{\mathcal{R}})$ by certain finite limits of categories of continuous presheaves on various closed subsets of X, and eventually reduce the statement of Theorem 6.1 to Corollary 6.7.

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