HODGE THEORY, BETWEEN ALGEBRAICITY AND TRANSCENDENCE

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ABSTRACT

The Hodge theory of complex algebraic varieties is at heart a transcendental comparison of two algebraic structures. We survey the recent advances bounding this transcendence, mainly due to the introduction of o-minimal geometry as a natural framework for Hodge theory.

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1. INTRODUCTION

Let X be a smooth connected projective variety over $\mathbb{C}$, and X^{an} its associated compact complex manifold. Classical Hodge theory [52] states that the Betti (i.e., singular) cohomology group $H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Z})$ is a *polarizable $\mathbb{Z}$-Hodge structure of weight k*: there exists a canonical decomposition (called the Hodge decomposition) of complex vector spaces

$$H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X^{\mathrm{an}}) \quad \text{satisfying} \quad \overline{H^{p,q}(X^{\mathrm{an}})} = H^{q,p}(X^{\mathrm{an}})$$

and a $(-1)^k$-symmetric bilinear pairing $q_k : H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Z}) \times H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Z}) \to \mathbb{Z}$ whose complexification makes the above decomposition orthogonal, and satisfies the positivity condition (the signs are complicated but are imposed to us by geometry)

$$\mathrm{i}^{p-q} q_{k,\mathbb{C}}(\alpha, \overline{\alpha}) > 0 \quad \text{for any nonzero } \alpha \in H^{p,q}(X^{\mathrm{an}}).$$

Deligne [29] vastly generalized Hodge's result, showing that the cohomology $H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Z})$ of *any* complex algebraic variety X is functorially endowed with a slightly more general *graded polarizable mixed $\mathbb{Z}$-Hodge structure*, that makes, after tensoring with $\mathbb{Q}$, $H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q})$ a successive extension of polarizable $\mathbb{Q}$-Hodge structures, with weights between 0 and $2k$. As mixed $\mathbb{Q}$-Hodge structures form a Tannakian category $\mathrm{MHS}_{\mathbb{Q}}$, one can conveniently (although rather abstractly) summarize the Hodge–Deligne theory as functorially assigning to any complex algebraic variety X a $\mathbb{Q}$-algebraic group: *the Mumford–Tate group* $\mathbf{MT}_X$ *of* X, defined as the Tannaka group of the Tannakian subcategory $\langle H^\bullet_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q})\rangle$ of $\mathrm{MHS}_{\mathbb{Q}}$ generated by $H^\bullet_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q})$. The knowledge of the group $\mathbf{MT}_X$ is equivalent to the knowledge of all *Hodge tensors* for the Hodge structure $H^\bullet_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q})$.

These apparently rather innocuous semilinear algebra statements are anything but trivial. They have become the main tool for analyzing the topology, geometry and arithmetic of complex algebraic varieties. Let us illustrate what we mean with regard to topology, which we will not go into later. The existence of the Hodge decomposition for smooth projective complex varieties, which holds more generally for compact Kähler manifolds, imposes many constraints on the cohomology of such spaces, the most obvious being that their odd Betti numbers have to be even. Such constraints are not satisfied even by compact complex manifolds as simple as the Hopf surfaces, quotients of $\mathbb{C}^2 \setminus \{0\}$ by the action of $\mathbb{Z}$ given by multiplication by $\lambda \neq 0$, $|\lambda| \neq 1$, whose first Betti number is one. Characterizing the homotopy types of compact Kähler manifolds is an essentially open question, which we will not discuss here.

The mystery of the Hodge–Deligne theory lies in the fact that it is at heart *not* an algebraic theory, but rather the transcendental comparison of two algebraic structures. For simplicity, let X be a smooth connected projective variety over $\mathbb{C}$. The Betti cohomology $H^\bullet_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q})$ defines a $\mathbb{Q}$-structure on the complex vector space of the algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X/\mathbb{C}) := H^\bullet(X, \Omega^\bullet_{X/\mathbb{C}})$ under the transcendental comparison isomorphism:

$$\varpi : H^\bullet_{\mathrm{dR}}(X/\mathbb{C}) \xrightarrow{\sim} H^\bullet\big(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}\big) =: H^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\sim} H^\bullet_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}, \qquad (1.1)$$

where the first canonical isomorphism is the comparison between algebraic and analytic de Rham cohomology provided by GAGA, and the second one is provided by integrating complex C^∞ differential forms over cycles (de Rham's theorem). The Hodge filtration F^p on $H^\bullet_B(X^{an}, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}$ is the image under ϖ of the algebraic filtration $F^p = \mathrm{Im}(H^\bullet(X, \Omega^{\bullet \geq p}_{X/\mathbb{C}}) \to H^\bullet_{dR}(X/\mathbb{C}))$ on the left-hand side.

The surprising power of the Hodge–Deligne theory lies in the fact that, although the comparison between the two algebraic structures is transcendental, this transcendence should be severely constrained, as predicted, for instance, by the Hodge conjecture and the Grothendieck period conjecture:

- For X smooth projective, it is well known that the cycle class $[Z]$ of any codimension k algebraic cycle on X with $\mathbb{Q}$ coefficients is a Hodge class in the Hodge structure $H^{2k}(X^{an}, \mathbb{Q})(k)$. Hodge [52] famously conjectured that the converse holds true: any Hodge class in $H^{2k}(X, \mathbb{Q})(k)$ should be such a cycle class.
- For X smooth and defined over a number field $K \subset \mathbb{C}$, its *periods* are the coefficients of the matrix of Grothendieck's isomorphism (generalizing (1.1))

$$\varpi : H^\bullet_{dR}(X/K) \otimes_K \mathbb{C} \xrightarrow{\sim} H^\bullet_B(X^{an}, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}$$

  with respect to bases of $H^\bullet_{dR}(X/K)$ and $H^\bullet_B(X^{an}, \mathbb{Q})$. The Grothendieck period conjecture (combined with the Hodge conjecture) predicts that the transcendence degree of the field $k_X \subset \mathbb{C}$ generated by the periods of X coincides with the dimension of $\mathbf{MT}_X$.

This tension between algebraicity and transcendence is perhaps best revealed when considering Hodge theory *in families*, as developed by Griffiths [43]. Let $f : X \to S$ be a smooth projective morphism of smooth connected quasiprojective varieties over $\mathbb{C}$. Its complex analytic fibers X_s^{an}, $s \in S^{an}$, are diffeomorphic, hence their cohomologies $\mathbb{V}_{\mathbb{Z},s} := H^\bullet_B(X_s^{an}, \mathbb{Z})$, $s \in S^{an}$ are all isomorphic to a fixed abelian group $V_\mathbb{Z}$ and glue together into a locally constant sheaf $\mathbb{V}_\mathbb{Z} := R^\bullet f^{an}{}_* \mathbb{Z}$ on S^{an}. However, the complex algebraic structure on X_s, hence also the Hodge structure on $\mathbb{V}_{\mathbb{Z},s}$, varies with s, making $R^\bullet f^{an}_* \mathbb{Z}$ a variation of $\mathbb{Z}$-Hodge structures ($\mathbb{Z}$VHS) $\mathbb{V}$ on S^{an}, which can be naturally polarized. One easily checks that the Mumford–Tate group $\mathbf{G}_s := \mathbf{MT}_{X_s}$, $s \in S^{an}$, is locally constant equal to the so-called *generic Mumford–Tate group* $\mathbf{G}$, outside of a meagre set $\mathrm{HL}(S, f) \subset S^{an}$, the *Hodge locus of the morphism* f, where it shrinks as exceptional Hodge tensors appear in $H^\bullet_B(X_s^{an}, \mathbb{Z})$. The variation $\mathbb{V}$ is completely described by its *period map*

$$\Phi : S^{an} \to \Gamma \backslash D.$$

Here the period domain D classifies all possible $\mathbb{Z}$-Hodge structure on the abelian group $V_\mathbb{Z}$, with a fixed polarization and Mumford–Tate group contained in $\mathbf{G}$; and Φ maps a point $s \in S^{an}$ to the point of D parameterizing the polarized $\mathbb{Z}$-Hodge structure on $V_\mathbb{Z}$ defined by $\mathbb{V}_{\mathbb{Z},s}$ (well defined up to the action of the arithmetic group $\Gamma := G \cap \mathbf{GL}(V_\mathbb{Z})$).

The transcendence of the comparison isomorphism (1.1) for each fiber X_s is embodied in the fact that the Hodge variety $\Gamma \backslash D$ is, in general, a mere complex analytic variety not admitting any algebraic structure; and that the period map Φ is a mere complex analytic map. On the other hand this transcendence is sufficiently constrained so that the following corollary of the Hodge conjecture [96] holds true, as proven by Cattani–Deligne–Kaplan [22]: the Hodge locus $\mathrm{HL}(S, f)$ is a countable union of *algebraic* subvarieties of S. Remarkably, their result is in fact valid for any polarized $\mathbb{Z}$VHS $\mathbb{V}$ on S^{an}, not necessarily coming from geometry: the Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is a countable union of algebraic subvarieties of S.

In this paper we report on recent advances in the understanding of this interplay between algebraicity and transcendence in Hodge theory, our main object of interest being period maps $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$. The paper is written for nonexperts: we present the mathematical objects involved, the questions, and the results but give only vague ideas of proofs, if any. It is organized as follows. After Section 2 presenting the objects of Hodge theory (which the advanced reader will skip to refer to on occasion), we present in Section 3 the main driving force behind the recent advances: although period maps are very rarely complex algebraic, their geometry is tame and does not suffer from any of the many possible pathologies of a general holomorphic map. In model-theoretic terms, period maps are definable in the *o-minimal structure* $\mathbb{R}_{\mathrm{an,exp}}$. In Section 4, we introduce the general format of *bialgebraic structures* for comparing the algebraic structure on S and that on (the compact dual $\check{D}$ of) the period domain D. The heuristic provided by this format, combined with o-minimal geometry, leads to a powerful functional transcendence result: the Ax–Schanuel theorem for polarized $\mathbb{Z}$VHS. It also suggests to interpret variational Hodge theory as a special case of an *atypical intersection* problem. In Section 5 we describe how this viewpoint leads to a stunning improvement of the result of Cattani, Deligne, and Kaplan: in most cases $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is not only a countable union of algebraic varieties, but is actually algebraic on the nose (at least if we restrict to its components of positive period dimension). Finally, in Section 6 we turn briefly to some arithmetic aspects of the theory.

For the sake of simplicity, we focus on the case of pure Hodge structures, only mentioning the references dealing with the mixed case.

2. VARIATIONS OF HODGE STRUCTURES AND PERIOD MAPS

2.1. Polarizable Hodge structures

Let $n \in \mathbb{Z}$. Let $R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$. An *R-Hodge structure V of weight n* is a finitely generated R-module V_R together with one of the following equivalent data: a bigrading $V_{\mathbb{C}} (:= V_R \otimes_R \mathbb{C}) = \bigoplus_{p+q=n} V^{p,q}$, called the Hodge decomposition, such that $\overline{V^{p,q}} = V^{q,p}$ (the numbers $(\dim V^{p,q})_{p+q=n}$ are called the Hodge numbers of V); or a decreasing filtration $F^\bullet$ of $V_{\mathbb{C}}$, called the Hodge filtration, satisfying $F^p \oplus \overline{F^{n+1-p}} = V_{\mathbb{C}}$. One goes from one to the other through $F^p = \bigoplus_{r \geq p} V^{r,n-r}$ and $V^{p,q} = F^p \cap \overline{F^q}$. The following group-theoretic description will be most useful to us: a Hodge structure is an R-module V_R and a real algebraic representation $\varphi : \mathbf{S} \to \mathbf{GL}(V_{\mathbb{R}})$ whose restriction to $\mathbf{G}_{m,\mathbb{R}}$ is defined over $\mathbb{Q}$. Here the Deligne torus $\mathbf{S}$ denotes the real algebraic group $\mathbb{C}^*$ of invertible matrices of the forms $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, which contains the diagonal subgroup $\mathbf{G}_{m,\mathbb{R}}$. Being of weight n is the requirement that $\varphi_{|\mathbf{G}_{m,\mathbb{R}}}$ acts via the character $z \mapsto z^{-n}$. The space $V^{p,q}$ is recovered as the eigenspace for the character $z \mapsto z^{-p}\bar{z}^{-q}$ of $\mathbf{S}(\mathbb{R}) \simeq \mathbb{C}^*$. A *morphism of Hodge structures* is a morphism of R-modules compatible with the bigrading (equivalently, with the Hodge filtration or the $\mathbf{S}$-action).

Example 2.1. We write $R(n)$ for the unique R-Hodge structure of weight $-2n$, called the Tate–Hodge structure of weight $-2n$, on the rank-one free R-module $(2\pi \mathrm{i})^n R \subset \mathbb{C}$.

One easily checks that the category of R-Hodge structures is an abelian category (where the kernels and cokernels coincide with the usual kernels and cokernels in the category of R-modules, with the induced Hodge filtrations on their complexifications), with natural tensor products $V \otimes W$ and internal homs $\hom(V, W)$ (in particular, duals $V^\vee := \hom(V, R(0))$). For $R = \mathbb{Q}$, or $\mathbb{R}$, we obtain a Tannakian category, with an obvious exact faithful R-linear tensor functor $\omega : (V_R, \varphi) \mapsto V_R$. In particular, $R(n) = R(1)^{\otimes n}$. If V is an R-Hodge structure, we write $V(n) := V \otimes R(n)$ its nth Tate twist.

If $V = (V_R, \varphi)$ is an R-Hodge structure of weight n, a *polarization for* V is a morphism of R-Hodge structures $q : V^{\otimes 2} \to R(-n)$ such that $(2\pi \mathrm{i})^n q(x, \varphi(\mathrm{i})y)$ is a positive-definite bilinear form on $V_\mathbb{R}$, called the *Hodge form* associated with the polarization. If there exists a polarization for V then V is said *polarizable*. One easily checks that the category of polarizable $\mathbb{Q}$-Hodge structures is semisimple.

Example 2.2. Let M be a compact complex manifold. If M admits a Kähler metric, the singular cohomology $H^n_{\mathrm{B}}(M, \mathbb{Z})$ is naturally a $\mathbb{Z}$-Hodge structure of weight n, see [52], [94, CHAP. 6]:

$$H^n_{\mathrm{B}}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^n_{dR}(M, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(M),$$

where $H^\bullet_{dR}(M, \mathbb{C})$ denotes the de Rham cohomology of the complex $(A^\bullet(M, \mathbb{C}), d)$ of C^∞ differential forms on M, the first equality is the canonical isomorphism obtained by integrating forms on cycles (de Rham theorem), and the complex vector subspace $H^{p,q}(M)$ of $H^n_{\mathrm{dR}}(M, \mathbb{C})$ is generated by the d-closed forms of type (p, q), and thus satisfies automatically $\overline{H^{p,q}(M)} = H^{q,p}(M)$. Although the second equality depends only on the complex structure on M, its proof relies on the choice of a Kähler form ω on M through the following sequence of isomorphisms:

$$H^n_{dR}(M, \mathbb{C}) \xrightarrow{\sim} \mathcal{H}^n_{\Delta_\omega}(M) = \bigoplus_{p+q=n} \mathcal{H}^{p,q}_{\Delta_\omega}(M) \xrightarrow{\sim} \bigoplus_{p+q=n} H^{p,q}(M),$$

where $\mathcal{H}^n_{\Delta_\omega}(M)$ denotes the vector space of Δ_ω-harmonic differential forms on M and $\mathcal{H}^{p,q}_{\Delta_\omega}(M)$ its subspace of Δ_ω-harmonic (p, q)-forms. The heart of Hodge theory is thus reduced to the statement that the Laplacian Δ_ω of a Kähler metric preserves the type of forms. The choice of a Kähler form ω on M also defines, through the hard Lefschetz theorem [94, THEOREM 6.25], a polarization *of the* $\mathbb{R}$*-Hodge structure* $H^n(M, \mathbb{R})$, see [94, THEOREM 6.32]. If $f : M \to N$ is any holomorphic map between compact complex manifolds admitting Kähler metrics then both $f^* : H^n_{\mathrm{B}}(N, \mathbb{Z}) \to H^n_{\mathrm{B}}(M, \mathbb{Z})$ and the Gysin morphism $f_* : H^n_{\mathrm{B}}(M, \mathbb{Z}) \to H^{n-2r}_{\mathrm{B}}(N, \mathbb{Z})(-r)$ are morphism of $\mathbb{Z}$-Hodge structures, where $r = \dim M - \dim N$.

Example 2.3. Suppose moreover that $M = X^{\mathrm{an}}$ is the compact complex manifold analytification of a smooth projective variety X over $\mathbb{C}$. In that case, $H^n_{\mathrm{B}}(X, \mathbb{Z})$ is a *polarizable* $\mathbb{Z}$*-Hodge structure*. Indeed, the Kähler class $[\omega]$ can be chosen as the first Chern class of an ample line bundle on X, giving rise to a rational Lefschetz decomposition and (after clearing denominators by multiplying by a sufficiently large integer) to an integral polarization. Moreover, the Hodge filtration $F^\bullet$ on $H^n_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C})$ can be defined algebraically: upon identifying $H^n_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C})$ with the algebraic de Rham cohomology $H^n_{\mathrm{dR}}(X/\mathbb{C}) := H^n(X, \Omega^\bullet_{X/\mathbb{C}})$, the Hodge filtration is given by $F^p = \mathrm{Im}(H^n(X, \Omega^{\bullet \geq p}_{X/\mathbb{C}}) \to H^n_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C}))$. It follows that if X is defined over a subfield K of $\mathbb{C}$, then the Hodge filtration $F^\bullet$ on $H^n_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C}) = H^n_{\mathrm{dR}}(X/K) \otimes_K \mathbb{C}$ is defined over K.

Example 2.4. The functor which assigns to a complex abelian variety A its $H^1_{\mathrm{B}}(A^{\mathrm{an}}, \mathbb{Z})$ defines an equivalence of categories between abelian varieties and polarizable $\mathbb{Z}$-Hodge structures of weight 1 and type $(1, 0)$ and $(0, 1)$.

2.2. Hodge classes and Mumford–Tate group

Let $R = \mathbb{Z}$ or $\mathbb{Q}$ and let V be an R-Hodge structure. A *Hodge class for* V is a vector in $V^{0,0} \cap V_{\mathbb{Q}} = F^0 V_{\mathbb{C}} \cap V_{\mathbb{Q}}$. For instance, any morphism of R-Hodge structures $f : V \to W$ defines a Hodge class in the internal $\mathrm{hom}(V, W)$. Let $T^{m,n} V_{\mathbb{Q}}$ denote the $\mathbb{Q}$-Hodge structure $V_{\mathbb{Q}}^{\otimes m} \otimes \mathrm{hom}(V, R(0))_{\mathbb{Q}}^{\otimes n}$. A *Hodge tensor for* V is a Hodge class in some $T^{m,n} V_{\mathbb{Q}}$.

The main invariant of an R-Hodge structure is its *Mumford–Tate group*. For any R-Hodge structure V we denote by $\langle V \rangle$ the Tannakian subcategory of the category of $\mathbb{Q}$-Hodge structures generated by $V_{\mathbb{Q}}$; in other words, $\langle V \rangle$ is the smallest full subcategory containing V, $\mathbb{Q}(0)$ and stable under $\oplus$, $\otimes$, and taking subquotients. If ω_V denotes the restriction of the tensor functor ω to $\langle V \rangle$, the functor $\mathrm{Aut}^{\otimes}(\omega_V)$ is representable by some closed $\mathbb{Q}$-algebraic subgroup $\mathbf{G}_V \subset \mathbf{GL}(V_{\mathbb{Q}})$, called the Mumford–Tate group of V, and ω_V defines an equivalence of categories $\langle V \rangle \simeq \mathrm{Rep}_{\mathbb{Q}} \mathbf{G}_V$. See [33, II, 2.11].

The Mumford–Tate group $\mathbf{G}_V$ can also be characterized as the fixator in $\mathbf{GL}(V_{\mathbb{Q}})$ of the Hodge tensors for V, or equivalently, writing $V = (V_R, \varphi)$, as the smallest $\mathbb{Q}$-algebraic subgroup of $\mathbf{GL}(V_{\mathbb{Q}})$ whose base change to $\mathbb{R}$ contains the image $\mathrm{Im}\, \varphi$. In particular φ factorizes as $\varphi : \mathbf{S} \to \mathbf{G}_{V,\mathbb{R}}$. The group $\mathbf{G}_V$ is thus connected, and reductive if V is polarizable. See [2, LEMMA 2].

Example 2.5. $\mathbf{G}_{\mathbb{Z}(n)} = \mathbf{G}_m$ if $n \neq 0$ and $\mathbf{G}_{\mathbb{Z}(0)} = \{1\}$.

Example 2.6. Let A be a complex abelian variety and let $V := H^1_{\mathrm{B}}(A^{\mathrm{an}}, \mathbb{Z})$ be the associated $\mathbb{Z}$-Hodge structure of weight 1. We write $\mathbf{G}_A := \mathbf{G}_V$. The choice of an ample line bundle on A defines a polarization q on V. On the one hand, the endomorphism algebra $D := \mathrm{End}^0(A) (:= \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q})$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra which, in view of Example 2.4, identifies with $\mathrm{End}(V_{\mathbb{Q}})^{\mathbf{G}_A}$. Thus $\mathbf{G}_A \subset \mathbf{GL}_D(V_{\mathbb{Q}})$. On the other hand, the polarization q defines a Hodge class in $\mathrm{hom}(V_{\mathbb{Q}}^{\otimes 2}, \mathbb{Q}(-1))$ thus $\mathbf{G}_A$ has to be contained in the group $\mathbf{GSp}(V_{\mathbb{Q}}, q)$ of symplectic similitudes of $V_{\mathbb{Q}}$ with respect to the symplectic form q. Finally, $\mathbf{G}_A \subset \mathbf{GL}_D(V_{\mathbb{Q}}) \cap \mathbf{GSp}(V_{\mathbb{Q}}, q)$.

If $A = E$ is an elliptic curve, it follows readily that either $D = \mathbb{Q}$ and $\mathbf{G}_E = \mathbf{GL}_2$, or D is an imaginary quadratic field (E has complex multiplication) and $\mathbf{G}_E = \mathbf{T}_D$, the $\mathbb{Q}$-torus defined by $\mathbf{T}_D(S) = (D \otimes_{\mathbb{Q}} S)^*$ for any $\mathbb{Q}$-algebra S.

2.3. Period domains and Hodge data

Let $V_{\mathbb{Z}}$ be a finitely generated abelian group $V_{\mathbb{Z}}$ of rank r. Fix a positive integer n, a $(-1)^n$-symmetric bilinear form $q_{\mathbb{Z}}$ on $V_{\mathbb{Z}}$ and a collection of nonnegative integers $(h^{p,q})$ $(p, q \geq 0,\ p + q = n)$ such that $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = r$. Associated with $(n, q_{\mathbb{Z}}, (h^{p,q}))$ we want to define a *period domain* D classifying $\mathbb{Z}$-Hodge structures of weight n on $V_{\mathbb{Z}}$, polarized by $q_{\mathbb{Z}}$, and with Hodge numbers $h^{p,q}$. Setting $f^p = \sum_{r \geq p} h^{r,n-r}$, we first define the *compact dual* $\check{D}$ parametrizing the finite decreasing filtrations $F^\bullet$ on $V_{\mathbb{C}}$ satisfying $(F^p)^{\perp_{q_{\mathbb{Z}}}} = F^{n+1-p}$ and $\dim F^p = f^p$. This is a closed algebraic subvariety of the product of Grassmannians $\prod_p \mathrm{Gr}(f^p, V_{\mathbb{C}})$. The period domain $D \subset \check{D}^{\mathrm{an}}$ is the open subset where the Hodge form is positive definite. If $\mathbf{G} := \mathbf{GAut}(V_{\mathbb{Q}}, q_{\mathbb{Q}})$ denotes the group of similitudes of $q_{\mathbb{Q}}$, one easily checks that $\mathbf{G}(\mathbb{C})$ acts transitively on $\check{D}^{\mathrm{an}}$, which is thus a flag variety for $\mathbf{G}_{\mathbb{C}}$; and that the connected component $G := \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ of the identity in the derived group $\mathbf{G}^{\mathrm{der}}(\mathbb{R})$ acts transitively on D, which identifies with an open G-orbit in $\check{D}$. If we fix a base point $o \in D$ and denote by P and M its stabilizer in $\mathbf{G}(\mathbb{C})$ and G, respectively, the period domain D is thus the homogeneous space

$$D = G/M \hookrightarrow \check{D}^{\mathrm{an}} = \mathbf{G}(\mathbb{C})/P.$$

The group P is a parabolic subgroup of $\mathbf{G}(\mathbb{C})$. Its subgroup $M = P \cap G$, consisting of real elements, not only fixes the filtration $F_o^\bullet$ but also the Hodge decomposition, hence the Hodge form, at o. It is thus a compact subgroup of G and D is an open elliptic orbit of G in $\check{D}$.

Example 2.7. Let $n = 1$, suppose that the only nonzero Hodge numbers are $h^{1,0} = h^{0,1} = g$, $q_{\mathbb{Z}}$ is a symplectic form and D is the subset of $\mathrm{Gr}(g, V_{\mathbb{C}})$ consisting of $q_{\mathbb{C}}$-Lagrangian subspaces F^1 on which $\mathrm{i}q_{\mathbb{C}}(u, \bar{u})$ is positive definite. In this case $\mathbf{G} = \mathbf{GSp}_{2g}$, $G = \mathbf{Sp}_{2g}(\mathbb{R})$, $M = \mathbf{SO}_{2g}(\mathbb{R})$ is a maximal compact subgroup of the connected Lie group G, and $D = G/M$ is a bounded symmetric domain naturally biholomorphic to Siegel's upper half-space $\mathfrak{H}_g$ of $g \times g$-complex symmetric matrices $Z = X + \mathrm{i}Y$ with Y positive definite. When $g = 1$, D is the Poincaré disk, biholomorphic to the Poincaré upper half-space $\mathfrak{H}$.

More generally, let $\mathbf{G}$ be a connected reductive $\mathbb{Q}$-algebraic group and let $\varphi : \mathbf{S} \to \mathbf{G}_{\mathbb{R}}$ be a real algebraic morphism such that $\varphi_{|\mathbf{G}_{m,\mathbb{R}}}$ is defined over $\mathbb{Q}$. We assume that $\mathbf{G}$ is the Mumford–Tate group of φ. The *period domain* (or Hodge domain) D associated with $\varphi : \mathbf{S} \to \mathbf{G}_{\mathbb{R}}$ is the connected component of the $\mathbf{G}(\mathbb{R})$-conjugacy class of $\varphi : \mathbf{S} \to \mathbf{G}_{\mathbb{R}}$ in $\mathrm{Hom}(\mathbf{S}, \mathbf{G}_{\mathbb{R}})$. Again, one easily checks that D is an open elliptic orbit of $G := \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ in the compact dual flag variety $\check{D}^{\mathrm{an}}$, the $\mathbf{G}(\mathbb{C})$-conjugacy class of $\varphi_{\mathbb{C}} \circ \mu : \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$, where $\mu : \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathbf{S}_{\mathbb{C}} = \mathbf{G}_{\mathrm{m},\mathbb{C}} \times \mathbf{G}_{\mathrm{m},\mathbb{C}}$ is the cocharacter $z \mapsto (z, 1)$. See [41] for details. The pair $(\mathbf{G}, D)$ is called a (connected) *Hodge datum*. A morphism of Hodge data $(\mathbf{G}, D) \to$

$(\mathbf{G}', D')$ is a morphism $\rho : \mathbf{G} \to \mathbf{G}'$ sending D to D'. Any linear representation $\lambda : \mathbf{G} \to \mathbf{GL}(V_{\mathbb{Q}})$ defines a $\mathbf{G}(\mathbb{Q})$-equivariant local system $\check{\mathbb{V}}_\lambda$ on $\check{D}^{\mathrm{an}}$. Moreover, each point $x \in D$, seen as a morphism $\varphi_x : \mathbf{S} \to \mathbf{G}_{\mathbb{R}}$, defines a $\mathbb{Q}$-Hodge structure $V_x := (V_{\mathbb{Q}}, \lambda \circ \varphi_x)$. The $\mathbf{G}(\mathbb{C})$-equivariant filtration $F^\bullet \check{\mathcal{V}}_\lambda := \mathbf{G}^{\mathrm{ad}}(\mathbb{C}) \times_{P,\lambda} F^\bullet V_{o,\mathbb{C}}$ of the holomorphic vector bundle $\check{\mathcal{V}}_\lambda := \mathbf{G}^{\mathrm{ad}}(\mathbb{C}) \times_{P,\lambda} V_{o,\mathbb{C}}$ on $\check{D}^{\mathrm{an}}$ induces the Hodge filtration on V_x for each $x \in D$. The Mumford–Tate group of V_x is $\mathbf{G}$ precisely when $x \in D \setminus \bigcup_\tau \tau(D')$, where τ ranges through the countable set of morphisms of Hodge data $\tau : (\mathbf{G}', D') \to (\mathbf{G}, D)$. The complex analytic subvarieties $\tau(D')$ of D are called the *special subvarieties* of D.

The following geometric feature of $\check{D}$ will be crucial for us. The algebraic tangent bundle $T\check{D}$ naturally identifies, as a $\mathbf{G}_{\mathbb{C}}$-equivariant bundle, with the quotient vector bundle $\check{\mathcal{V}}_{\mathrm{Ad}}/F^0\check{\mathcal{V}}_{\mathrm{Ad}}$, where $\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(\mathfrak{g})$ is the adjoint representation on the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. In particular, it is naturally filtered by the $F^iT\check{D} := F^i\check{\mathcal{V}}_{\mathrm{Ad}}/F^0\check{\mathcal{V}}_{\mathrm{Ad}}$, $i \leq -1$. The subbundle $F^{-1}T\check{D}$ is called the *horizontal tangent bundle* of $\check{D}$.

2.4. Hodge varieties

Let $(\mathbf{G}, D)$ be a Hodge datum as in Section 2.3. A *Hodge variety* is the quotient $\Gamma\backslash D$ of D by an arithmetic lattice Γ of $\mathbf{G}(\mathbb{Q})^+ := \mathbf{G}(\mathbb{Q}) \cap G$. It is thus naturally a complex analytic variety, which is smooth if Γ is torsion-free. The *special subvarieties* of $\Gamma\backslash D$ are the images of the special subvarieties of D under the projection $\pi : D \to \Gamma\backslash D$ (one easily checks these are closed complex analytic subvarieties of $\Gamma\backslash D$). For any algebraic representation $\lambda : \mathbf{G} \to \mathbf{GL}(V_{\mathbb{Q}})$, the $\mathbf{G}(\mathbb{Q})$-equivariant local system $\check{\mathbb{V}}_\lambda$ as well as the filtered holomorphic vector bundle $(\check{\mathcal{V}}_\lambda, F^\bullet)$ on $\check{D}$ are G-equivariant when restricted to D, hence descend to a triple $(\mathbb{V}_\lambda, (\mathcal{V}_\lambda, F^\bullet), \nabla)$ on $\Gamma\backslash D$. Similarly, the horizontal tangent bundle of $\check{D}$ defines the *horizontal tangent bundle* $T_h(\Gamma\backslash D) \subset T(\Gamma\backslash D)$ of the Hodge variety $\Gamma\backslash D$.

2.5. Polarized $\mathbb{Z}$-variations of Hodge structures

Hodge theory as recalled in Section 2.1 can be considered as the particular case over a point of Hodge theory over an arbitrary base. Again, the motivation comes from geometry. Let $f : Y \to B$ be a proper surjective complex analytic submersion from a connected Kähler manifold Y to a complex manifold B. It defines a locally constant sheaf $\mathbb{V}_{\mathbb{Z}} := R^\bullet f_*\mathbb{Z}$ of finitely generated abelian groups on B, gathering the cohomologies $H^\bullet_{\mathrm{B}}(Y_b, \mathbb{Z})$, $b \in B$. Upon choosing a base point $b_0 \in B$, the datum of $\mathbb{V}_{\mathbb{Z}}$ is equivalent to the datum of a *monodromy representation* $\rho : \pi_1(B, b_0) \to \mathbf{GL}(\mathbb{V}_{\mathbb{Z}, b_0})$. On the other hand, the de Rham incarnation of the cohomology of the fibers of f is the holomorphic flat vector bundle $(\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_B} \mathcal{O}_B \simeq R^\bullet f_*\Omega^\bullet_{Y/B}, \nabla)$, where $\mathcal{O}_B$ is the sheaf of holomorphic functions on B, $\Omega^\bullet_{Y/B}$ is the relative holomorphic de Rham complex and ∇ is the Gauss–Manin connection. The Hodge filtration on each $H^\bullet_{\mathrm{B}}(Y_b, \mathbb{C})$ is induced by the holomorphic subbundles $F^p := R^\bullet f_*\Omega^{\bullet \geq p}_{Y/B}$ of $\mathcal{V}$. The Hodge filtration is usually not preserved by the connection, but Griffiths [42] crucially observed that it satisfies the *transversality constraint* $\nabla F^p \subset \Omega^1_B \otimes_{\mathcal{O}_B} F^{p-1}$. More generally, *a variation of $\mathbb{Z}$-Hodge structures ($\mathbb{Z}$VHS)* on a connected complex manifold $(B, \mathcal{O}_B)$ is a pair $\mathbb{V} := (\mathbb{V}_{\mathbb{Z}}, F^\bullet)$, consisting of a locally constant sheaf of finitely generated abelian groups $\mathbb{V}_{\mathbb{Z}}$ on B and a (decreasing) filtration $F^\bullet$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_B} \mathcal{O}_B$ by holomorphic subbundles, called the Hodge filtration, satisfying the following conditions: for each $b \in B$, the pair $(\mathbb{V}_b, F_b^\bullet)$ is a $\mathbb{Z}$-Hodge structure; and the flat connection ∇ on $\mathcal{V}$ defined by $\mathbb{V}_{\mathbb{C}}$ satisfies Griffiths' transversality,

$$\nabla F^\bullet \subset \Omega_B^1 \otimes_{\mathcal{O}_B} F^{\bullet-1}. \tag{2.1}$$

A *morphism* $\mathbb{V} \to \mathbb{V}'$ *of* $\mathbb{Z}$*VHSs* on B is a morphism $f : \mathbb{V}_{\mathbb{Z}} \to \mathbb{V}'_{\mathbb{Z}}$ of local systems such that the associated morphism of vector bundles $f : \mathcal{V} \to \mathcal{V}'$ is compatible with the Hodge filtrations. If $\mathbb{V}$ has weight k, a *polarization of* $\mathbb{V}$ is a morphism $\mathrm{q} : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Z}_B(-k)$ inducing a polarization on each $\mathbb{Z}$-Hodge structure $\mathbb{V}_b$, $b \in B$. In the geometric situation, such a polarization exists if there exists an element $\eta \in H^2(Y, \mathbb{Z})$ whose restriction to each fiber Y_b defines a Kähler class, for instance if f is the analytification of a smooth projective morphism of smooth connected algebraic varieties over $\mathbb{C}$.

2.6. Generic Hodge datum and period map

Let S be a smooth connected quasiprojective variety over $\mathbb{C}$ and let $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS on S^{an}. Fix a base point $o \in S^{\mathrm{an}}$, let $p : \widetilde{S^{\mathrm{an}}} \to S^{\mathrm{an}}$ be the corresponding universal cover and write $V_{\mathbb{Z}} := \mathbb{V}_{\mathbb{Z},o}$, $q_{\mathbb{Z}} := \mathrm{q}_{\mathbb{Z},o}$. The pulled-back polarized $\mathbb{Z}$VHS $p^*\mathbb{V}$ is canonically trivialized as $(\widetilde{S^{\mathrm{an}}} \times V_{\mathbb{Z}}, (\widetilde{S^{\mathrm{an}}} \times V_{\mathbb{C}}, F^\bullet), \nabla = d, q_{\mathbb{Z}})$. In **[31, 7.5]**, Deligne proved that there exists a reductive $\mathbb{Q}$-algebraic subgroup $\iota : \mathbf{G} \hookrightarrow \mathbf{GL}(V_{\mathbb{Q}})$, called the *generic Mumford–Tate group* of $\mathbb{V}$, such that, for all points $\tilde{s} \in \widetilde{S^{\mathrm{an}}}$, the Mumford–Tate group $\mathbf{G}_{(V_{\mathbb{Z}}, F_{\tilde{s}}^\bullet)}$ is contained in $\mathbf{G}$, and is equal to $\mathbf{G}$ outside of a meagre set of $\widetilde{S^{\mathrm{an}}}$ (such points $\tilde{s}$ are said *Hodge generic* for $\mathbb{V}$). A closed irreducible subvariety $Y \subset S$ is said Hodge generic for $\mathbb{V}$ if it contains a Hodge generic point. The setup of Section 2.3 is thus in force. Without loss of generality, we can assume that the point $\tilde{o}$ is Hodge generic. Let $(\mathbf{G}, D)$ be the Hodge datum (called the *generic Hodge datum* of S^{an} for $\mathbb{V}$) associated with the polarized Hodge structure $(V_{\mathbb{Z}}, F_{\tilde{o}}^\bullet)$. The $\mathbb{Z}$VHS $p^*\mathbb{V}$ is completely described by a holomorphic map $\widetilde{\Phi} : \widetilde{S^{\mathrm{an}}} \to D$, which is naturally equivariant under the monodromy representation $\rho : \pi_1(S^{\mathrm{an}}, o) \to \Gamma := G \cap \mathbf{GL}(V_{\mathbb{Z}})$, hence descends to a holomorphic map $\Phi : S^{\mathrm{an}} \to \Gamma \backslash D$, called the *period map* of S for $\mathbb{V}$. We thus obtain the following commutative diagram in the category of complex analytic spaces:

$$\begin{array}{ccccc} \widetilde{S^{\mathrm{an}}} & \xrightarrow{\widetilde{\Phi}} & D & \hookrightarrow & \check{D}^{\mathrm{an}} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ S^{\mathrm{an}} & \xrightarrow{\Phi} & \Gamma\backslash D. & & \end{array} \tag{2.2}$$

Notice that the pair $(\mathbb{V}_{\mathbb{Q}}, (\mathcal{V}, F^\bullet))$ is the pullback under Φ of the pair $(\mathbb{V}_\iota, (\mathcal{V}_\iota, F^\bullet))$ on the Hodge variety $\Gamma \backslash D$ defined by the inclusion $\iota : \mathbf{G} \hookrightarrow \mathbf{GL}(V_{\mathbb{Q}})$. Griffiths' transversality condition is equivalent to the statement that Φ is *horizontal*, $d\Phi(TS^{\mathrm{an}}) \subset T_h(\Gamma \backslash D)$. By extension we call *period map* any holomorphic, horizontal, locally liftable map from S^{an} to a Hodge variety $\Gamma \backslash D$.

The *Hodge locus* $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ of S for $\mathbb{V}$ is the subset of points $s \in S^{\mathrm{an}}$ for which the Mumford–Tate group $\mathbf{G}_s$ is a strict subgroup of $\mathbf{G}$, or equivalently for which the Hodge structure $\mathbb{V}_s$ admits more Hodge tensors than the very general fiber $\mathbb{V}_{s'}$. Thus

$$\mathrm{HL}(S, \mathbb{V}^{\otimes}) = \bigcup_{(\mathbf{G}', D') \hookrightarrow (\mathbf{G}, D)} \Phi^{-1}(\Gamma' \backslash D'), \tag{2.3}$$

where the union is over all strict Hodge subdata and $\Gamma' \backslash D'$ is a slight abuse of notation for denoting the projection of $D' \subset D$ to $\Gamma \backslash D$.

Let $Y \subset S$ be a closed irreducible algebraic subvariety $i : Y \hookrightarrow S$. Let $(\mathbf{G}_Y, D_Y)$ be the generic Hodge datum of the $\mathbb{Z}$VHS $\mathbb{V}$ restricted to the smooth locus of Y. The algebraic monodromy group $\mathbf{H}_Y$ of Y for $\mathbb{V}$ is the identity component of the Zariski-closure in $\mathbf{GL}(V_{\mathbb{Q}})$ of the monodromy of the restriction to Y of the local system $\mathbb{V}_{\mathbb{Z}}$. It follows from Deligne's (in the geometric case) and Schmid's (in general) "Theorem of the fixed part" and "Semisimplicity Theorem" that $\mathbf{H}_Y$ is a normal subgroup of the derived group $\mathbf{G}_Y^{\mathrm{der}}$, see [2, THEOREM 1].

3. HODGE THEORY AND TAME GEOMETRY

3.1. Variational Hodge theory between algebraicity and transcendence

Let S be a smooth connected quasi-projective variety over $\mathbb{C}$ and let $\mathbb{V} = (\mathbb{V}_{\mathbb{Z}}, F^\bullet)$ be a polarized $\mathbb{Z}$VHS on S^{an}. Let $(\mathbf{G}, D)$ be the generic Hodge datum of S for $\mathbb{V}$ and let $\Phi : S^{\mathrm{an}} \to \Gamma \backslash D$ be the period map defined by $\mathbb{V}$.

The fact that Hodge theory is a transcendental theory is reflected in the following facts:

- First, the triplets $(\mathbb{V}_\lambda, (\mathcal{V}_\lambda, F^\bullet), \nabla)$ on $\Gamma \backslash D$ (for $\lambda : \mathbf{G} \to \mathbf{GL}(V_{\mathbb{Q}})$ an algebraic representation) do not in general satisfy Griffiths' transversality, hence do not define a $\mathbb{Z}$VHS on $\Gamma \backslash D$. They do if and only if $\mathbb{V}$ is *of Shimura type*, i.e., $(\mathbf{G}, D)$ is a (connected) *Shimura datum* (meaning that the weight zero Hodge structures on the fibers of $\mathbb{V}_{\mathrm{Ad}}$ are of type $\{(-1, 1), (0, 0), (1, -1)\}$); or equivalently, if the horizontal tangent bundle $T_h D$ coincides with TD. In other words, Hodge varieties are in general not classifying spaces for polarized $\mathbb{Z}$VHS.

- Second, and more importantly, the complex analytic Hodge variety $\Gamma \backslash D$ is in general not algebraizable (i.e., it is not the analytification of a complex quasiprojective variety). More precisely, let us write $D = G/M$ as in Section 2.3. A classical property of elliptic orbits like D is that there exists a unique maximal compact subgroup K of G containing M [46]. Supposing for simplicity that G is a real simple Lie group G, then $\Gamma \backslash D$ is algebraizable only if G/K is a hermitian symmetric domain and the projection $D \to G/K$ is holomorphic or antiholomorphic, see [45].

On the other hand, this transcendence is severely constrained, as shown by the following algebraicity results:

- If $(\mathbf{G}, D)$ is *of Shimura type*, then $\Gamma\backslash D = \mathrm{Sh}^{\mathrm{an}}$ is the analytification of an algebraic variety, called a Shimura variety Sh [8,30,32]. In that case Borel [17, THEOREM 3.10] proved that the complex analytic period map $\Phi : S^{\mathrm{an}} \to \mathrm{Sh}^{\mathrm{an}}$ is the analytification of an algebraic map.

- Let $S \subset \overline{S}$ be a log-smooth compactification of S by a simple normal crossing divisor Z. Following Deligne [28], the flat holomorphic connection ∇ on $\mathcal{V}$ defines a canonical extension $\overline{\mathcal{V}}$ of $\mathcal{V}$ to $\overline{S}$. Using GAGA for $\overline{S}$, this defines an algebraic structure on $(\mathcal{V}, \nabla)$, for which the connection ∇ is regular. Around any point of Z, the complex manifold S^{an} is locally isomorphic to a product $(\Delta^*)^k \times \Delta^l$ of punctured polydisks. Borel showed that the monodromy representation $\rho : \pi_1(S^{\mathrm{an}}, s_o) \to \Gamma \subset \mathbf{G}(\mathbb{Q})$ of $\mathbb{V}$ is "tame at infinity," that is, its restriction to $\mathbb{Z}^k = \pi_1((\Delta^*)^k \times \Delta^l)$ is quasiunipotent, see [82, (4.5)]. Using this result, Schmid showed that the Hodge filtration $F^\bullet$ extends holomorphically to the Deligne extension $\overline{\mathcal{V}}$. This is the celebrated Nilpotent Orbit theorem [82, (4.12)]. It follows, as noticed by Griffiths [82, (4.13)], that the Hodge filtration on $\mathcal{V}$ comes from an algebraic filtration on the underlying algebraic bundle, whether $\mathbb{V}$ is of geometric origin or not.

- More recently, an even stronger evidence came from the study of Hodge loci. Cattani, Deligne, and Kaplan proved the following celebrated result (generalized to the mixed case in [18–21]):

Theorem 3.1 ([22]). *Let S be a smooth connected quasiprojective variety over $\mathbb{C}$ and $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS over S. Then $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is a countable union of closed irreducible algebraic subvarieties of S.*

In view of this tension between algebraicity and transcendence, it is natural to ask if there is a framework, less strict than complex algebraic geometry but more constraining than complex analytic geometry, where to analyze period maps and explain its remarkable properties.

3.2. O-minimal geometry

Such a framework was in fact envisioned by Grothendieck in [47, §5] under the name "tame topology," as a way out of the pathologies of general topological spaces. Examples of pathologies are Cantor sets, space-filling curves but also much simpler objects like the graph $\Gamma := \{(x, \sin \frac{1}{x}),\ 0 < x \leq 1\} \subset \mathbb{R}^2$: its closure $\overline{\Gamma} := \Gamma \amalg \mathrm{I}$, where $\mathrm{I} := \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected but not arc-connected; $\dim(\overline{\Gamma} \setminus \Gamma) = \dim \Gamma$, which prevents any reasonable stratification theory; and $\Gamma \cap \mathbb{R}$ is not "of finite type." Tame geometry has been developed by model theorists as o-minimal geometry, which studies structures where every definable set has a finite geometric complexity. Its prototype is real semialgebraic geometry, but it is much richer. We refer to [34] for a nice survey.

Definition 3.2. A structure $\mathcal{S}$ expanding the real field is a collection $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$, where S_n is a set of subsets of $\mathbb{R}^n$ such that for every $n \in \mathbb{N}$:

(1) all algebraic subsets of $\mathbb{R}^n$ are in S_n.

(2) S_n is a boolean subalgebra of the power set of $\mathbb{R}^n$ (i.e., S_n is stable by finite union, intersection, and complement).

(3) If $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$.

(4) Let $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a linear projection. If $A \in S_{n+1}$ then $p(A) \in S_n$.

The elements of S_n are called the $\mathcal{S}$-definable sets of $\mathbb{R}^n$. A map $f : A \to B$ between $\mathcal{S}$-definable sets is said to be $\mathcal{S}$-definable if its graph is $\mathcal{S}$-definable.

A dual point of view starts from the functions, namely considers sets definable in a first-order structure $\mathcal{S} = \langle \mathbb{R}, +, \times, <, (f_i)_{i\in I}\rangle$ where I is a set and the $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$, $i \in I$, are functions. A subset $Z \subset \mathbb{R}^n$ is $\mathcal{S}$-definable if it can be defined by a formula

$$Z := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \phi(x_1, \dots, x_n) \text{ is true}\},$$

where ϕ is a first-order formula that can be written using only the quantifiers $\forall$ and $\exists$ applied to real variables; logical connectors; algebraic expressions written with the f_i; the order symbol $<$; and fixed parameters $\lambda_i \in \mathbb{R}$. When the set I is empty the $\mathcal{S}$-definable subsets are the semialgebraic sets. Semialgebraic subsets are thus always $\mathcal{S}$-definable.

One easily checks that the composite of $\mathcal{S}$-definable functions is $\mathcal{S}$-definable, as are the images and the preimages of $\mathcal{S}$-definable sets under $\mathcal{S}$-definable maps. Using that the euclidean distance is a real-algebraic function, one shows easily that the closure and interior of an $\mathcal{S}$-definable set are again $\mathcal{S}$-definable.

The following o-minimal axiom for a structure $\mathcal{S}$ guarantees the possibility of doing geometry using $\mathcal{S}$-definable sets as basic blocks.

Definition 3.3. A structure $\mathcal{S}$ is said to be o-minimal if S_1 consists precisely of the finite unions of points and intervals (i.e., the semialgebraic subsets of $\mathbb{R}$).

Example 3.4. The structure $\mathbb{R}_{\sin} := \langle \mathbb{R}, +, \times, <, \sin\rangle$ is not o-minimal. Indeed, the infinite union of points $\pi\mathbb{Z} = \{x \in \mathbb{R} \mid \sin x = 0\}$ is a definable subset of $\mathbb{R}$ in this structure.

Any o-minimal structure $\mathcal{S}$ has the following main tameness property: given finitely many $\mathcal{S}$-definable sets $U_1, \dots, U_k \subset \mathbb{R}^n$, there exists a definable cylindrical cellular decomposition of $\mathbb{R}^n$ such that each U_i is a finite union of cells. Such a decomposition is defined inductively on n. For $n = 1$, this is a finite partition of $\mathbb{R}$ into cells which are points or open intervals. For $n > 1$, it is obtained from a definable cylindrical cellular decomposition of $\mathbb{R}^{n-1}$ by fixing, for any cell $C \subset \mathbb{R}^{n-1}$, finitely many definable functions $f_{C,i} : C \to \mathbb{R}$, $1 \leq i \leq k_C$, with $f_{C,0} := -\infty < f_{C,1} < \cdots < f_{C,k_C} < f_{C,k_C+1} := +\infty$, and defining the cells of $\mathbb{R}^n$ as the graphs $\{(x, f_{C,i}(x)), x \in C\}$, $1 \leq i \leq k_C$, and the bands $\{(x, f_{C,i}(x) < y < f_{C,i+1}(x)), x \in C, y \in \mathbb{R}\}$, $0 \leq i \leq k_C$, for all cells C of $\mathbb{R}^{n-1}$.

The simplest o-minimal structure is the structure $\mathbb{R}_{\text{alg}}$ consisting of semialgebraic sets. It is too close to algebraic geometry to be used for studying transcendence phenomena. Luckily much richer o-minimal geometries do exist. A fundamental result of Wilkie, building on the result of Khovanskii [54] that any exponential set $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid P(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n)) = 0\}$ (where $P \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_n]$) has finitely many connected components, states:

Theorem 3.5 ([97]). *The structure* $\mathbb{R}_{\exp} := \langle \mathbb{R}, +, \times, <, \exp : \mathbb{R} \to \mathbb{R} \rangle$ *is o-minimal.*

In another direction, let us define

$$\mathbb{R}_{\text{an}} := \big\langle \mathbb{R}, +, \times, <, \{f\} \text{ for } f \text{ restricted real analytic function} \big\rangle,$$

where a function $f : \mathbb{R}^n \to \mathbb{R}$ is a restricted real analytic function if it is zero outside $[0, 1]^n$ and if there exists a real analytic function g on a neighborhood of $[0, 1]^n$ such that f and g are equal on $[0, 1]^n$. Gabrielov's result [37] that the difference of two subanalytic sets is subanalytic implies rather easily that the structure $\mathbb{R}_{\text{an}}$ is o-minimal. The structure generated by two o-minimal structures is not o-minimal in general, but Van den Dries and Miller [35] proved that the structure $\mathbb{R}_{\text{an,exp}}$ generated by $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\exp}$ is o-minimal. This is the o-minimal structure which will be mainly used in the rest of this text.

Let us now globalize the notion of definable set using charts:

Definition 3.6. A definable topological space $\mathcal{X}$ is the data of a Hausdorff topological space $\mathcal{X}$, a *finite* open covering $(U_i)_{1 \leq i \leq k}$ of $\mathcal{X}$, and homeomorphisms $\psi_i : U_i \to V_i \subset \mathbb{R}^n$ such that all V_i, $V_{ij} := \psi_i(U_i \cap U_j)$ and $\psi_i \circ \psi_j^{-1} : V_{ij} \to V_{ji}$ are definable. As usual the pairs (U_i, ψ_i) are called charts. A morphism of definable topological spaces is a continuous map which is definable when read in the charts. The definable site $\underline{\mathcal{X}}$ of a definable topological space $\mathcal{X}$ has for objects definable open subsets $U \subset X$ and admissible coverings are the finite ones.

Example 3.7. Let X be an algebraic variety over $\mathbb{R}$. Then $X(\mathbb{R})$ equipped with the euclidean topology carries a natural $\mathbb{R}_{\text{alg}}$-definable structure (up to isomorphism): one covers X by finitely many (Zariski) open affine subvarieties X_i and take $U_i := X_i(\mathbb{R})$ which is naturally a semialgebraic set. One easily check that any two finite open affine covers define isomorphic $\mathbb{R}_{\text{alg}}$-structures on $X(\mathbb{R})$. If X is an algebraic variety over $\mathbb{C}$ then $X(\mathbb{C}) = (\text{Res}_{\mathbb{C}/\mathbb{R}} X)(\mathbb{R})$ carries thus a natural $\mathbb{R}_{\text{alg}}$-structure. We call this the $\mathbb{R}_{\text{alg}}$-*definabilization* of X and denote it by $X^{\mathbb{R}_{\text{alg}}}$.

In the rest of this section, we fix an o-minimal structure $\mathcal{S}$ and write "definable" for $\mathcal{S}$-definable. Given a complex algebraic variety X we write X^{def} for the $\mathcal{S}$-definabilization $X^{\mathcal{S}}$.

3.3. O-minimal geometry and algebraization

Why should an algebraic geometer care about o-minimal geometry? Because o-minimal geometry provides strong algebraization results.

3.3.1. Diophantine criterion

The first algebraization result is the celebrated Pila–Wilkie theorem:

Theorem 3.8 ([77]). *Let $Z \subset \mathbb{R}^n$ be a definable set. We define Z^{alg} as the union of all connected* positive-dimensional *semialgebraic subsets of Z. Then, denoting by $H : \mathbb{Q}^n \to \mathbb{R}$ the standard height function:*

$$\forall\, \varepsilon > 0, \quad \exists\, C_\varepsilon > 0, \quad \forall\, T > 0, \quad \left|\left\{x \in (Z \setminus Z^{\mathrm{alg}}) \cap \mathbb{Q}^n,\ H(x) \leq T\right\}\right| < C_\varepsilon T^\varepsilon.$$

In words, if a definable set contains at least polynomially many rational points (with respect to their height), then it contains a positive dimensional semialgebraic set! For instance, if $f : \mathbb{R} \to \mathbb{R}$ is a real analytic function such that its graph $\Gamma_f \cap [0,1] \times [0,1]$ contains at least polynomially many rational points (with respect to their height), then the function f is real algebraic [15]. This algebraization result is a crucial ingredient in the proof of functional transcendence results for period maps, see Section 4.

3.3.2. Definable Chow and definable GAGA

In another direction, algebraicity follows from the meeting of o-minimal geometry with complex geometry. The motto is that o-minimal geometry is incompatible with the many pathologies of complex analysis. As a simple illustration, let $f : \Delta^* \to \mathbb{C}$ be a holomorphic function, and assume that f is definable (where we identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\Delta^* \subset \mathbb{R}^2$ is semialgebraic). Then f does not have any essential singularity at 0 (i.e., f is meromorphic). Otherwise, by the Big Picard theorem, the boundary $\overline{\Gamma_f} \setminus \Gamma_f$ of its graph would contain $\{0\} \times \mathbb{C}$, hence would have the same real dimension (two) as Γ_f, contradicting the fact that Γ_f is definable.

Let us first define a good notion of a definable topological space "endowed with a complex analytic structure." We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by taking real and imaginary parts. Given $U \subset \mathbb{C}^n$ a definable open subset, let $\mathcal{O}_{\mathbb{C}^n}(U)$ denote the $\mathbb{C}$-algebra of holomorphic definable functions $U \to \mathbb{C}$. The assignment $U \rightsquigarrow \mathcal{O}_{\mathbb{C}^n}(U)$ defines a sheaf $\mathcal{O}_{\mathbb{C}^n}$ on $\underline{\mathbb{C}^n}$ whose stalks are local rings. Given a finitely generated ideal $I \subset \mathcal{O}_{\mathbb{C}^n}(U)$, its zero locus $V(I) \subset U$ is definable and the restriction $\mathcal{O}_{V(I)} := (\mathcal{O}_U / I\mathcal{O}_U)_{|\underline{V(I)}}$ define a sheaf of local rings on $\underline{V(I)}$.

Definition 3.9. A definable complex analytic space is a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ consisting of a definable topological space $\mathcal{X}$ and a sheaf $\mathcal{O}_\mathcal{X}$ on $\underline{\mathcal{X}}$ such that there exists a finite covering of $\mathcal{X}$ by definable open subsets $\mathcal{X}_i$ on which $(\mathcal{X}, \mathcal{O}_\mathcal{X})_{|\mathcal{X}_i}$ is isomorphic to some $(V(I), \mathcal{O}_{V(I)})$.

Bakker et al. [10, THEOREM 2.16] show that this is a reasonable definition: the sheaf $\mathcal{O}_\mathcal{X}$, in analogy with the classical Oka's theorem, is a coherent sheaf of rings. Moreover, one has a natural definabilization functor $(X, \mathcal{O}_X) \rightsquigarrow (X^{\mathrm{def}}, \mathcal{O}_{X^{\mathrm{def}}})$ from the category of separated schemes (or algebraic spaces) of finite type over $\mathbb{C}$ to the category of definable complex analytic spaces, which induces a morphism $g : (\underline{X^{\mathrm{def}}}, \mathcal{O}_{X^{\mathrm{def}}}) \to (\underline{X}, \mathcal{O}_X)$ of locally ringed sites.

Let us now describe the promised algebraization results. The classical Chow's theorem states that a closed complex analytic subset Z of X^{an} for X smooth projective over $\mathbb{C}$ is in fact algebraic. This fails dramatically if X is only quasiprojective, as shown by the graph of the complex exponential in $(\mathbb{A}^2)^{\mathrm{an}}$. However, Peterzil and Starchenko, generalizing [36] in the $\mathbb{R}_{\mathrm{alg}}$ case, have shown the following:

Theorem 3.10 ([69,70]). *Let X be a complex quasiprojective variety and let $Z \subset X^{\mathrm{an}}$ be a closed analytic subvariety. If Z is definable in X^{def} then Z is complex algebraic in X.*

Chow's theorem, which deals only with spaces, was extended to sheaves by Serre [83]: when X is proper, the analytification functor $(\cdot)^{\mathrm{an}} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ defines an equivalence of categories between the categories of coherent sheaves $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X^{\mathrm{an}})$. In the definable world, let X be a separated scheme (or algebraic space) of finite type over $\mathbb{C}$. Associating with a coherent sheaf F on X the coherent sheaf $F^{\mathrm{def}} := F \otimes_{g^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{def}}}$ on the $\mathcal{S}$-definabilization X^{def} of X, one obtains a definabilization functor $(\cdot)^{\mathrm{def}} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{def}})$. Similarly there is an analytification functor $\mathcal{X} \rightsquigarrow \mathcal{X}^{\mathrm{an}}$ from complex definable analytic spaces to complex analytic spaces, that induces a functor $(\cdot)^{\mathrm{an}} : \mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{X}^{\mathrm{an}})$.

Theorem 3.11 ([10]). *For every separated algebraic space of finite type X, the definabilization functor $(\cdot)^{\mathrm{def}} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{def}})$ is exact and fully faithful (but it is not necessarily essentially surjective). Its essential image is stable under subobjects and subquotients.*

Using Theorem 3.11 and Artin's algebraization theorem for formal modification [4], one obtains the following useful algebraization result for definable images of algebraic spaces, which will be used in Section 3.6.2:

Theorem 3.12 ([10]). *Let X be a separated algebraic space of finite type and let $\mathcal{E}$ be a definable analytic space. Any proper definable analytic map $\Phi : X^{\mathrm{def}} \to \mathcal{E}$ factors uniquely as $\iota \circ f^{\mathrm{def}}$, where $f : X \to Y$ is a proper morphism of separated algebraic spaces (of finite type) such that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective, and $\iota : Y^{\mathrm{def}} \hookrightarrow \mathcal{E}$ is a closed immersion of definable analytic spaces.*

3.4. Definability of Hodge varieties

Let us now describe the first result establishing that o-minimal geometry is potentially interesting for Hodge theory.

Theorem 3.13 ([11]). *Any Hodge variety $\Gamma\backslash D$ can be naturally endowed with a functorial structure $(\Gamma\backslash D)^{\mathbb{R}_{\mathrm{alg}}}$ of $\mathbb{R}_{\mathrm{alg}}$-definable complex analytic space.*

Here "functorial" means that that any morphism $(\mathbf{G}', D') \to (\mathbf{G}, D)$ of Hodge data induces a definable map $(\Gamma'\backslash D')^{\mathbb{R}_{\mathrm{alg}}} \to (\Gamma\backslash D)^{\mathbb{R}_{\mathrm{alg}}}$ of Hodge varieties. Let us sketch the construction of $(\Gamma\backslash D)^{\mathbb{R}_{\mathrm{alg}}}$. Without loss of generality (replacing $\mathbf{G}$ by its adjoint group if necessary), we can assume that $\mathbf{G}$ is semisimple, $G = \mathbf{G}(\mathbb{R})^+$. For simplicity, let us assume that the arithmetic lattice Γ is torsion free. We choose a base point in $D = G/M$. Notice that

G and $G/M \subset \check{D}^{\mathbb{R}_{\mathrm{alg}}}$ are naturally endowed with a G-equivariant semialgebraic structure, making the projection $G \to G/M$ semialgebraic. To define an $\mathbb{R}_{\mathrm{alg}}$-structure on $\Gamma\backslash(G/M)$, it is thus enough to find a semialgebraic open fundamental set $F \subset G/M$ for the action of Γ and to write $\Gamma\backslash G/M = \Gamma\backslash F$, where the right-hand side is the quotient of F by the closed étale semialgebraic equivalence relation induced by the action of Γ on D. Here by fundamental set we mean that the set of $\gamma \in \Gamma$ such that $\gamma F \cap F \neq \emptyset$ is finite. We construct the fundamental set F using the reduction theory of arithmetic groups, namely the theory of Siegel sets. Let K be the unique maximal compact subgroup of G containing M. For any $\mathbb{Q}$-parabolic $\mathbf{P}$ of $\mathbf{G}$ with unipotent radical $\mathbf{N}$, the maximal compact subgroup K of G determines a real Levi $L \subset G$ which decomposes as $L = AQ$ where A is the center and Q is semisimple. A semialgebraic Siegel set of G associated to $\mathbf{P}$ and K is then a set of the form $\mathfrak{S} = U(aA_{>0})W$ where $U \subset \mathbf{N}(\mathbb{R})$, $W \subset QK$ are bounded semialgebraic subsets, $a \in A$, and $A_{>0}$ is the cone corresponding to the positive root chamber. By a Siegel set of G associated to K we mean a semialgebraic Siegel set associated to $\mathbf{P}$ and K for some $\mathbb{Q}$-parabolic $\mathbf{P}$ of $\mathbf{G}$. Suppose now that $\Gamma \subset G$ is an arithmetic group. A fundamental result of Borel [16] states that there exists finitely many Siegel sets $\mathfrak{S}_i \subset G$, $1 \leq i \leq s$, associated with K, whose images in $\Gamma\backslash G/K$ cover the whole space; and such that for any $1 \leq i \neq j \leq s$, the set of $\gamma \in \Gamma$ such that $\gamma\mathfrak{S}_i \cap \mathfrak{S}_j \neq \emptyset$ is finite. We call the images $\mathfrak{S}_{i,D} := \mathfrak{S}_i/M$ *Siegel sets for* D. Noticing that these Siegel sets for D are semialgebraic in D, we can take $F = \coprod_{i=1}^{s} \mathfrak{S}_{i,D}$. It is not difficult to show that the $\mathbb{R}_{\mathrm{alg}}$-structure thus constructed is independent of the choice of the base point $eM \in G/M$. The functoriality follows from a nontrivial property of Siegel sets with respect to morphisms of algebraic groups, due to Orr [68].

3.5. Definability of period maps

Once Theorem 3.13 is in place, the following result shows that o-minimal geometry is a natural framework for Hodge theory:

Theorem 3.14 ([11]). *Let S be a smooth connected complex quasiprojective variety. Any period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ is the analytification of a morphism $\Phi : S^{\mathbb{R}_{\mathrm{an,exp}}} \to (\Gamma\backslash D)^{\mathbb{R}_{\mathrm{an,exp}}}$ of $\mathbb{R}_{\mathrm{an,exp}}$-definable complex analytic spaces, where the $\mathbb{R}_{\mathrm{an,exp}}$-structures on $S(\mathbb{C})$ and $\Gamma\backslash D$ extend their natural $\mathbb{R}_{\mathrm{alg}}$-structures defined in Example 3.7 and Theorem 3.13, respectively.*

In down-to-earth terms, this means that we can cover S by finitely many open affine charts S_i such that Φ restricted to $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}} S_i)(\mathbb{R}) = S_i(\mathbb{C})$ and read in a chart of $\Gamma\backslash D$ defined by a Siegel set of D, can be written using only real polynomials, the real exponential function, and restricted real analytic functions! This statement is already nontrivial when $S = \mathrm{Sh}$ is a Shimura variety and $\Phi^{\mathrm{an}} : S^{\mathrm{an}} \to \Gamma\backslash D$ is the identity map coming from the uniformization $\pi : D \to S^{\mathrm{an}}$ of S^{an} by the hermitian symmetric domain $D = G/K$. In that case the $\mathbb{R}_{\mathrm{alg}}$-definable varieties $\mathrm{Sh}^{\mathbb{R}_{\mathrm{alg}}}$ and $(\Gamma\backslash D)^{\mathbb{R}_{\mathrm{alg}}}$ are not isomorphic, but Theorem 3.14 claims that their $\mathbb{R}_{\mathrm{an,exp}}$-extensions $\mathrm{Sh}^{\mathbb{R}_{\mathrm{an,exp}}}$ and $(\Gamma\backslash D)^{\mathbb{R}_{\mathrm{an,exp}}}$ are. This is equivalent to showing that the restriction $\pi_{|\mathfrak{S}_D} : \mathfrak{S}_D \to S^{\mathbb{R}_{\mathrm{an,exp}}}$ to a Siegel set for D can be written using only real polynomials, the real exponential function, and restricted real analytic functions. This is a nice exercise on the j-function when Sh is a modular curve, was done in [71] and [76] for $\mathrm{Sh} = \mathcal{A}_g$, and [58] in general.

Let us sketch the proof of Theorem 3.14. We choose a log-smooth compactification of S, hence providing us with a definable cover of $S^{\mathbb{R}_{\mathrm{an}}}$ by punctured polydisks $(\Delta^*)^k \times \Delta^l$. We are reduced to showing that the restriction of Φ to such a punctured polydisk is $\mathbb{R}_{\mathrm{an,exp}}$-definable. This is clear if $k = 0$, as in this case $\varphi : \Delta^{k+l} \to \Gamma\backslash D$ is even $\mathbb{R}_{\mathrm{an}}$-definable. For $k > 0$, let $\mathrm{e} : \exp(2\pi \mathrm{i} \cdot) : \mathfrak{H} \to \Delta^*$ be the universal covering map. Its restriction to a sufficiently large bounded vertical strip $V := [a, b] \times]0, +\infty[\subset \mathfrak{H} = \{x + \mathrm{i}y,\ y > 0\}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. Considering the following commutative diagram:

$$\begin{array}{ccccc} V^k \times \Delta^l & \xrightarrow{\quad\widetilde{\Phi}\quad} & D & \hookleftarrow & F \\ \downarrow{\scriptstyle \mathrm{e}} & & \downarrow{\scriptstyle \pi} & & \\ (\Delta^*)^k \times \Delta^l \hookrightarrow S^{\mathrm{an}} & \xrightarrow[\Phi]{} & \Gamma\backslash D, & & \end{array}$$

it is thus enough to show that $\pi \circ \widetilde{\Phi} : V^k \times \Delta^l \to \Gamma\backslash D$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Let the coordinates of $(\Delta^*)^k \times \Delta^l$ be t_i, $1 \leq i \leq k + l$, those of $\mathfrak{H}^k$ be z_i, $1 \leq i \leq k$, so that $\mathrm{e}(z_i) = t_i$. Let T_i be the monodromy at infinity of Φ around the hyperplane $(z_i = 0)$, boundary component of $\overline{S} \setminus S$. By Borel's theorem T_i is quasiunipotent. Replacing S by a finite étale cover, we can without loss of generality assume that each $T_i = \exp(N_i)$, with $N_i \in \mathfrak{g}$ nilpotent. The Nilpotent Orbit Theorem of Schmid is equivalent to saying that $\widetilde{\Phi} : V^k \times \Delta^l \to D$ can be written as $\widetilde{\Phi}(z_1, \ldots, z_k, t_{k+1}, \ldots, t_{k+l}) = \exp(\sum_{i=1}^k z_i N_i) \cdot \Psi(t_1, \ldots, t_{k+l})$ for $\Psi : \Delta^k \times \Delta^l \to \check{D}^{\mathrm{an}}$ a holomorphic map. On the one hand, Ψ is $\mathbb{R}_{\mathrm{an}}$-definable as a function of the variables t_i, hence $\mathbb{R}_{\mathrm{an,exp}}$-definable as a function of the variables z_i, $1 \leq i \leq k$, and the variables t_j, $k + 1 \leq j \leq k + l$. On the other hand, $\exp(\sum_{i=1}^k z_i N_i) \in \mathbf{G}(\mathbb{C})$ is polynomial in the variables z_i, as the monodromies N_i are nilpotent and commute pairwise. As the action of $\mathbf{G}(\mathbb{C})$ on $\check{D}$ is algebraic, it follows that $\widetilde{\Phi} : V^k \times \Delta^l \to D$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. The proof of Theorem 3.14 is thus reduced to the following, proven by Schmid when $k = 1$, $l = 0$ [82, 5.29]:

Theorem 3.15 ([11]). *The image $\widetilde{\Phi}(V^k \times \Delta^l)$ lies in a finite union of Siegel sets of D.*

This can be interpreted as showing that, possibly after passing to a definable cover of $V^k \times \Delta^l$, the Hodge form of $\widetilde{\Phi}$ is Minkowski reduced with respect to a flat frame. This is done using the hard analytic theory of Hodge forms estimates for degenerations of variations of Hodge structure, as in [53, THEOREMS 3.4.1 AND 3.4.2] and [23, THEOREM 5.21].

Remark 3.16. Theorems 3.13 and 3.14 have been extended to the mixed case in [9].

3.6. Applications

3.6.1. About the Cattani–Deligne–Kaplan theorem

As a corollary of Theorems 3.14 and 3.10 one obtains the following, which, in view of (2.3), implies immediately Theorem 3.1:

Theorem 3.17 ([11]). *Let S be a smooth quasiprojective complex variety. Let $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS on S^{an} with period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$. For any special subvariety $\Gamma'\backslash D' \subset \Gamma\backslash D$, its preimage $\Phi^{-1}(\Gamma'\backslash D')$ is a finite union of irreducible algebraic subvarieties of S.*

Indeed, it follows from Theorem 3.13 that $\Gamma'\backslash D'$ is definable in $(\Gamma\backslash D)^{\mathbb{R}_{\mathrm{alg}}}$. By Theorem 3.14, its preimage $\Phi^{-1}(\Gamma'\backslash D')$ is definable in $S^{\mathbb{R}_{\mathrm{an,exp}}}$. As Φ is holomorphic and $\Gamma'\backslash D' \subset \Gamma\backslash D$ is a closed complex analytic subvariety, $\Phi^{-1}(\Gamma'\backslash D')$ is also a closed complex analytic subvariety of S^{an}. By Theorem 3.10, it is thus algebraic in S.

Remark 3.18. Theorem 3.17 has been extended to the mixed case in [9], thus recovering [18–21].

Let $Y \subset S$ be a closed irreducible algebraic subvariety. Let $(\mathbf{G}_Y, D_Y) \subset (\mathbf{G}, D)$ be the generic Hodge datum of $\mathbb{V}$ restricted to the smooth locus of of Y. There exist a smallest Hodge subvariety $\Gamma_Y\backslash D_Y$ of $\Gamma\backslash D$ containing $\Phi(Y^{\mathrm{an}})$. The following terminology will be convenient:

Definition 3.19. Let S be a smooth quasiprojective complex variety. Let $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS on S^{an} with period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$. A closed irreducible subvariety $Y \subset S$ is called a *special subvariety* of S for $\mathbb{V}$ if it coincides with an irreducible component of the preimage $\Phi^{-1}(\Gamma_Y\backslash D_Y)$.

Equivalently, a special subvariety of S for $\mathbb{V}$ is a closed irreducible algebraic subvariety $Y \subset S$ maximal among the closed irreducible algebraic subvarieties Z of S such that the generic Mumford–Tate group $\mathbf{G}_Z$ of $\mathbb{V}_{|Z}$ equals $\mathbf{G}_Y$.

3.6.2. A conjecture of Griffiths

Combining Theorem 3.14 this time with Theorem 3.12 leads to a proof of an old conjecture of Griffiths [44], claiming that the image of any period map has a natural structure of quasiprojective variety (Griffiths proved it when the target Hodge variety is compact):

Theorem 3.20 ([10]). *Let S be a smooth connected quasiprojective complex variety and let $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ be a period map. There exists a unique dominant morphism of complex algebraic varieties $f : S \to T$, with T quasiprojective, and a closed complex analytic immersion $\iota : T^{\mathrm{an}} \hookrightarrow \Gamma\backslash D$ such that $\Phi = \iota \circ f^{\mathrm{an}}$.*

Let us sketch the proof. As before, let $S \subset \overline{S}$ be a log-smooth compactification by a simple normal crossing divisor Z. It follows from a result of Griffiths [43, PROP. 9.11I)] that Φ extends to a *proper* period map over the components of Z around which the monodromy is finite. Hence, without loss of generality, we can assume that Φ is proper. The existence of f in the category of algebraic spaces then follows immediately from Theorems 3.14 and 3.12 (for $\mathcal{S} = \mathbb{R}_{\mathrm{an,exp}}$). The proof that T is in fact quasiprojective exploits a crucial observation of Griffiths that $\Gamma\backslash D$ carries a positively curved $\mathbb{Q}$-line bundle $\mathcal{L} := \bigotimes_p \det(F^p)$. This line bundle is naturally definable on $(\Gamma\backslash D)^{\mathrm{def}}$. Using the definable GAGA Theorem 3.11, one shows that its restriction to T^{def} comes from an algebraic $\mathbb{Q}$-line bundle L_T on T, which one manages to show to be ample.

4. FUNCTIONAL TRANSCENDENCE

4.1. Bialgebraic geometry

As we saw, Hodge theory, which compares the Hodge filtration on $H^{\bullet}_{\mathrm{dR}}(X/\mathbb{C})$ with the rational structure on $H^{\bullet}_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C})$, gives rise to variational Hodge theory, whose fundamental diagram (2.2) compares the algebraic structure of S with the algebraic structure on the dual period domain $\check{D}$. As such, it is a partial answer to one of the most classical problem of complex algebraic geometry: the transcendental nature of the topological universal cover of complex algebraic varieties. If S is a connected complex algebraic variety, the universal cover $\widetilde{S^{\mathrm{an}}}$ has usually no algebraic structure as soon as the topological fundamental group $\pi_1(S^{\mathrm{an}})$ is infinite. As an aside, let us mention an interesting conjecture of Kóllar and Pardon **[60]**, predicting that if X is a normal projective irreducible complex variety whose universal cover $\widetilde{X^{\mathrm{an}}}$ is biholomorphic to a semialgebraic open subset of an algebraic variety then $\widetilde{X^{\mathrm{an}}}$ is biholomorphic to $\mathbb{C}^n \times D \times F^{\mathrm{an}}$, where D is a bounded symmetric domain and F is a normal, projective, irreducible, topologically simply connected, complex algebraic variety. We want to think of variational Hodge theory as an attempt to provide a partial *algebraic uniformization*: the period map emulates an algebraic structure on $\widetilde{S^{\mathrm{an}}}$, modeled on the flag variety $\check{D}$. The remaining task is then to describe the transcendence properties of the complex analytic uniformization map $p : \widetilde{S^{\mathrm{an}}} \to S^{\mathrm{an}}$ with respects to the emulated algebraic structure on $\widetilde{S^{\mathrm{an}}}$ and the algebraic structure S on S^{an}. A few years ago, the author **[55]**, together with Ullmo and Yafaev **[59]**, introduced a convenient format for studying such questions, which encompasses many classical transcendence problems and provides a powerful heuristic.

Definition 4.1. A bialgebraic structure on a connected quasiprojective variety S over $\mathbb{C}$ is a pair

$$\big(f : \widetilde{S^{\mathrm{an}}} \to Z^{\mathrm{an}},\ \rho : \pi_1(S^{\mathrm{an}}) \to \mathrm{Aut}(Z)\big)$$

where Z denotes an algebraic variety (called the *algebraic model* of $\widetilde{S^{\mathrm{an}}}$), $\mathrm{Aut}(Z)$ is its group of algebraic automorphisms, ρ is a group morphism (called the monodromy representation) and f is a ρ-equivariant holomorphic map (called the developing map).

An irreducible analytic subvariety $Y \subset \widetilde{S^{\mathrm{an}}}$ is said to be an *algebraic subvariety of* $\widetilde{S^{\mathrm{an}}}$ for the bialgebraic structure (f, ρ) if Y is an analytic irreducible component of $f^{-1}(\overline{f(Y)}^{\mathrm{Zar}})$ (where $\overline{f(Y)}^{\mathrm{Zar}}$ denotes the Zariski-closure of $f(Y)$ in Z). An irreducible algebraic subvariety $Y \subset \widetilde{S^{\mathrm{an}}}$, resp. $W \subset S$, is said to be *bialgebraic* if $p(Y)$ is an algebraic subvariety of S, resp. any (equivalently one) analytic irreducible component of $p^{-1}(W)$ is an irreducible algebraic subvariety of $\widetilde{S^{\mathrm{an}}}$. The bialgebraic subvarieties of S are precisely the ones where the emulated algebraic structure on $\widetilde{S^{\mathrm{an}}}$ and the one on S interact nontrivially.

Example 4.2. (a) *tori*, $S = (\mathbb{C}^*)^n$. The uniformization map is the multiexponential

$$p := \big(\exp(2\pi i \cdot), \dots, \exp(2\pi i \cdot)\big) : \mathbb{C}^n \to (\mathbb{C}^*)^n,$$

and f is the identity morphism of $\mathbb{C}^n$. An irreducible algebraic subvariety $Y \subset \mathbb{C}^n$ (resp. $W \subset (\mathbb{C}^*)^n$) is bialgebraic if and only if Y is a translate of a rational linear subspace of $\mathbb{C}^n = \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{C}$ (resp. W is a translate of a subtorus of $(\mathbb{C}^*)^n$).

(b) *abelian varieties*, $S = A$ is a complex abelian variety of dimension n. Let $p : \mathrm{Lie}\, A \simeq \mathbb{C}^n \to A$ be the uniformizing map of a complex abelian variety A of dimension n. Once more $\widetilde{S^{\mathrm{an}}} = \mathbb{C}^n$ and f is the identity morphism. One checks easily that an irreducible algebraic subvariety $W \subset A$ is bialgebraic if and only if W is the translate of an abelian subvariety of A.

(c) *Shimura varieties*, $(\mathbf{G}, D)$ is a Shimura datum. The quotient $S^{\mathrm{an}} = \Gamma \backslash D$ (for $\Gamma \subset G := \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ a congruence torsion-free lattice) is the complex analytification of a (connected) Shimura variety Sh, defined over a number field (a finite extension of the reflex field of $(\mathbf{G}, D)$). And f is the open embedding $D \hookrightarrow \check{D}^{\mathrm{an}}$.

Let us come back to the case of the bialgebraic structure on S

$$\big(\widetilde{\Phi} : \widetilde{S^{\mathrm{an}}} \to \check{D}^{\mathrm{an}},\ \rho : \pi_1(S^{\mathrm{an}}) \to \Gamma \subset \mathbf{G}(\mathbb{Q})\big)$$

defined by a polarized $\mathbb{Z}$VHS $\mathbb{V}$ and its period map $\Phi : S^{\mathrm{an}} \to \Gamma \backslash D$ with monodromy $\rho : \pi_1(S^{\mathrm{an}}) \to \Gamma \subset \mathbf{G}(\mathbb{Q})$ (in fact, all the examples above are of this form if we consider more generally graded-polarized variations of mixed $\mathbb{Z}$-Hodge structures). What are its bialgebraic subvarieties? To answer this question, we need to define the *weakly special* subvarieties of $\Gamma \backslash D$, as either a special subvariety or a subvariety of the form

$$\Gamma_{\mathbf{H}} \backslash D_H \times \{t\} \subset \Gamma_{\mathbf{H}} \backslash D_H \times \Gamma_{\mathbf{L}} \backslash D_L \subset \Gamma \backslash D,$$

where $(\mathbf{H} \times \mathbf{L}, D_H \times D_L)$ is a Hodge subdatum of $(\mathbf{G}^{\mathrm{ad}}, D)$ and $\{t\}$ is a Hodge generic point in $\Gamma_{\mathbf{L}} \backslash D_L$. Generalizing Theorem 3.17, the preimage under Φ of any weakly special subvariety of $\Gamma \backslash D$ is an algebraic subvariety of S **[56]**. An irreducible component of such a preimage is called a *weakly special* subvariety of S for $\mathbb{V}$ (or Φ).

Theorem 4.3 (**[56]**). *Let $\Phi : S^{\mathrm{an}} \to \Gamma \backslash D$ be a period map. The bialgebraic subvarieties of S for the bialgebraic structure defined by Φ are precisely the weakly special subvarieties of S for Φ. In analogy with Definition 3.19, they are also the closed irreducible algebraic subvarieties $Y \subset S$ maximal among the closed irreducible algebraic subvarieties Z of S whose algebraic monodromy group $\mathbf{H}_Z$ equals $\mathbf{H}_Y$.*

When $S = \mathrm{Sh}$ is a Shimura variety, these results are due to Moonen **[65]** and **[91]**. In that case the weakly special subvarieties are also the irreducible algebraic subvarieties of Sh whose smooth locus is totally geodesic in $\mathrm{Sh}^{\mathrm{an}}$ for the canonical Kähler–Einstein metric on $\mathrm{Sh}^{\mathrm{an}} = \Gamma \backslash D$ coming from the Bergman metric on D, see **[65]**.

To study not only functional transcendence but also arithmetic transcendence, we enrich bialgebraic structures over $\overline{\mathbb{Q}}$. A $\overline{\mathbb{Q}}$-bialgebraic structure on a quasi-projective variety S defined over $\overline{\mathbb{Q}}$ is a bialgebraic structure $(f : \widetilde{S^{\mathrm{an}}} \to Z^{\mathrm{an}}, h : \pi_1(S^{\mathrm{an}}) \to \mathrm{Aut}(Z))$ such that Z is defined over $\overline{\mathbb{Q}}$ and the homomorphism h takes values in $\mathrm{Aut}_{\overline{\mathbb{Q}}} Z$. An algebraic subvariety $Y \subset \widetilde{S^{\mathrm{an}}}$ is said to be defined over $\overline{\mathbb{Q}}$ if its model $\overline{f(Y)}^{\mathrm{Zar}} \subset Z$ is. A $\overline{\mathbb{Q}}$-bialgebraic subvariety $W \subset S$ is an algebraic subvariety of S defined over $\overline{\mathbb{Q}}$ and such that any (equivalently one) of the analytic irreducible components of $p^{-1}(W)$ is an algebraic subvariety of

$\widetilde{S^{\mathrm{an}}}$ defined over $\overline{\mathbb{Q}}$. A $\overline{\mathbb{Q}}$-bialgebraic point $s \in S(\overline{\mathbb{Q}})$ is also called an *arithmetic point*. Example 4.2a) is naturally defined over $\overline{\mathbb{Q}}$, with arithmetic points the torsion points of $(\mathbb{C}^*)^n$. In Example 4.2b) the bialgebraic structure can be defined over $\overline{\mathbb{Q}}$ if the abelian variety A has CM, and its arithmetic points are its torsion points, see **[90]**. Example 4.2c) is naturally a $\overline{\mathbb{Q}}$-bialgebraic structure, with arithmetic points the *special points* of the Shimura variety (namely the special subvarieties of dimension zero), at least when the pure part of the Shimura variety is of Abelian type, see **[84]**. In all these cases it is interesting to notice that the $\overline{\mathbb{Q}}$-bialgebraic subvarieties are the bialgebraic subvarieties containing one arithmetic point (in Example 4.2c) these are the special subvarieties of the Shimura variety).

The bi-algebraic structure associated with a period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ is defined over $\overline{\mathbb{Q}}$ as soon as S is. In this case, we expect the $\overline{\mathbb{Q}}$-bi-algebraic subvarieties to be precisely the special subvarieties, see **[55, 2.6 AND 3.4]**.

4.2. The Ax–Schanuel theorem for period maps

The geometry of bialgebraic structures is controlled by the following functional transcendence heuristic, whose idea was introduced by Pila in the case of Shimura varieties, see **[73,74]**:

Ax–Schanuel principle. Let S be an irreducible algebraic variety endowed with a non-trivial bialgebraic structure. Let $U \subset \widetilde{S^{\mathrm{an}}} \times S^{\mathrm{an}}$ be an algebraic subvariety (for the product bialgebraic structure) and let W be an analytic irreducible component of $U \cap \Delta$, where Δ denotes the graph of $p : \widetilde{S^{\mathrm{an}}} \to S^{\mathrm{an}}$. Then $\operatorname{codim}_U W \geq \dim \overline{W}^{\mathrm{bi}}$, where $\overline{W}^{\mathrm{bi}}$ denotes the smallest bialgebraic subvariety of S containing $p(W)$.

When applied to a subvariety $U \subset \widetilde{S^{\mathrm{an}}} \times S^{\mathrm{an}}$ of the form $Y \times \overline{p(Y)}^{\mathrm{Zar}}$ for $Y \subset \widetilde{S^{\mathrm{an}}}$ algebraic, the Ax–Schanuel principle specializes to the following:

Ax–Lindemann principle. Let S be an irreducible algebraic variety endowed with a nontrivial bialgebraic structure. Let $Y \subset \widetilde{S^{\mathrm{an}}}$ be an algebraic subvariety. Then $\overline{p(Y)}^{\mathrm{Zar}}$ is a bialgebraic subvariety of S.

Ax **[5, 6]** showed that the abstract Ax–Schanuel principle holds true for Example 4.2a) and Example 4.2b) above, using differential algebra. Notice that the Ax–Lindemann principle in Example 4.2a) is the functional analog of the classical Lindemann theorem stating that if $\alpha_1, \dots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$. This explains the terminology. The Ax–Lindemann principle in Example 4.2c) was proven by Pila **[72]** when S is a product $Y(1)^n \times (\mathbb{C}^*)^k$, by Ullmo–Yafaev **[92]** for projective Shimura varieties, by Pila–Tsimerman **[76]** for $\mathcal{A}_g$, and by Klingler–Ullmo–Yafaev **[58]** for any pure Shimura variety. The full Ax–Schanuel principle was proven by Mok–Pila–Tsimerman for pure Shimura varieties **[64]**.

We conjectured in **[55, CONJ. 7.5]** that the Ax-Schanuel principle holds true for the bi-algebraic structure associated to a (graded-)polarized variation of (mixed) $\mathbb{Z}$HS on an arbitrary quasiprojective variety S. Bakker and Tsimerman proved this conjecture in the pure case:

Theorem 4.4 (Ax–Schanuel for $\mathbb{Z}$VHS, [12]). *Let $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ be a period map. Let $V \subset S \times \check{D}$ be an algebraic subvariety. Let U be an irreducible complex analytic component of $W \cap (S \times_{\Gamma\backslash D} D)$ such that*

$$\operatorname{codim}_{S\times D} U < \operatorname{codim}_{S\times \check{D}} W + \operatorname{codim}_{S\times D}(S \times_{\Gamma\backslash D} D). \tag{4.1}$$

Then the projection of U to S is contained in a strict weakly special subvariety of S for Φ.

Remark 4.5. The results of [64] were extended by Gao [39] to mixed Shimura varieties of Kuga type. Recently the full Ax–Schanuel [55, CONJ. 7.5] for variations of mixed Hodge structures has been fully proven independently in [40] and [26].

The proof of Theorem 4.4 follows a strategy started in [58] and fully developed in [64] in the Shimura case, see [88] for an introduction. It does not use Theorem 3.14, but only a weak version equivalent to the Nilpotent Orbit Theorem, and relies crucially on the definable Chow Theorem 3.10, the Pila–Wilkie Theorem 3.8, and the proof that the volume (for the natural metric on $\Gamma\backslash D$) of the intersection of a ball of radius R in $\Gamma\backslash D$ with the horizontal complex analytic subvariety $\Phi(S^{\mathrm{an}})$ grows exponentially with R (a negative curvature property of the horizontal tangent bundle).

4.3. On the distribution of the Hodge locus

Theorem 4.4 is most useful, even in its simplest version of the Ax–Lindemann theorem. After Theorem 3.1 one would like to understand the distribution in S of the special subvarieties for $\mathbb{V}$. For instance, are there any geometric constraints on the Zariski closure of $\mathrm{HL}(S, \mathbb{V}^{\otimes})$? To approach this question, let us decompose the adjoint group $\mathbf{G}^{\mathrm{ad}}$ into a product $\mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ of its simple factors. It gives rise (after passing to a finite étale covering if necessary) to a decomposition of the Hodge variety $\Gamma\backslash D$ into a product of Hodge varieties $\Gamma_1\backslash D_1 \times \cdots \times \Gamma_r\backslash D_r$. A special subvariety Z of S for $\mathbb{V}$ is said of *positive period dimension* if $\dim_{\mathbb{C}} \Phi(Z^{\mathrm{an}}) > 0$; and of *factorwise positive period dimension* if, moreover, the projection of $\Phi(Z^{\mathrm{an}})$ on each factor $\Gamma_i\backslash D_i$ has positive dimension. The *Hodge locus of factorwise positive period dimension* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{fpos}}$ is the union of the strict special subvarieties of positive period dimension, it is contained in the *Hodge locus of positive period dimension* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ union of the strict special subvarieties of positive period dimension, and the two coincide if $\mathbf{G}^{\mathrm{ad}}$ is simple.

Using the Ax–Lindemann theorem special case of Theorem 4.4 and a global algebraicity result in the total bundle of $\mathcal{V}$, Otwinowska and the author proved the following:

Theorem 4.6 ([56]). *Let $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS on a smooth connected complex quasi-projective variety S. Then either $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{fpos}}$ is Zariski-dense in S; or it is an algebraic subvariety of S (i.e., the set of strict special subvarieties of S for $\mathbb{V}$ of factorwise positive period dimension has only finitely many maximal elements for the inclusion).*

Example 4.7. The simplest example of Theorem 4.6 is the following. Let $S \subset \mathcal{A}_g$ be a Hodge-generic closed irreducible subvariety. Either the set of positive-dimensional closed irreducible subvarieties of S which are not Hodge generic has finitely many maximal elements (for the inclusion), or their union is Zariski-dense in S.

Example 4.8. Let $B \subset \mathbb{P}H^0(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}(d))$ be the open subvariety parametrizing the smooth surfaces of degree d in $\mathbb{P}^3_{\mathbb{C}}$. Suppose $d > 3$. The classical Noether theorem states that any surface $Y \subset \mathbb{P}^3_{\mathbb{C}}$ corresponding to a very general point $[Y] \in B$ has Picard group $\mathbb{Z}$: every curve on Y is a complete intersection of Y with another surface in $\mathbb{P}^3_{\mathbb{C}}$. The countable union $\mathrm{NL}(B)$ of closed algebraic subvarieties of B corresponding to surfaces with bigger Picard group is called the Noether–Lefschetz locus of B. Let $\mathbb{V} \to B$ be the $\mathbb{Z}$VHS $R^2 f_* \mathbb{Z}_{\mathrm{prim}}$, where $f : \mathcal{Y} \to B$ denotes the universal family of surfaces of degree d. Clearly $\mathrm{NL}(B) \subset \mathrm{HL}(B, \mathbb{V}^{\otimes})$. Green (see [94, PROP. 5.20]) proved that $\mathrm{NL}(B)$, hence also $\mathrm{HL}(B, \mathbb{V}^{\otimes})$, is analytically dense in B. Now Theorem 4.6 implies the following: Let $S \subset B$ be a Hodge-generic closed irreducible subvariety. Either $S \cap \mathrm{HL}(B, \mathbb{V}^{\otimes})_{\mathrm{fpos}}$ contains only finitely many maximal positive-dimensional closed irreducible subvarieties of S, or the union of such subvarieties is Zariski-dense in S.

5. TYPICAL AND ATYPICAL INTERSECTIONS: THE ZILBER–PINK CONJECTURE FOR PERIOD MAPS

5.1. The Zilber–Pink conjecture for $\mathbb{Z}$VHS: Conjectures

In the same way that the Ax–Schanuel principle controls the geometry of bialgebraic structures, the diophantine geometry of $\overline{\mathbb{Q}}$-bialgebraic structures is controlled by the following heuristic:

Atypical intersection principle. Let S be an irreducible algebraic $\overline{\mathbb{Q}}$-variety endowed with a $\overline{\mathbb{Q}}$-bialgebraic structure. Then the union S_{atyp} of atypical $\overline{\mathbb{Q}}$-bialgebraic subvarieties of S is an algebraic subvariety of S (i.e., it contains only finitely many atypical $\overline{\mathbb{Q}}$-bialgebraic subvarieties maximal for the inclusion).

Here a $\overline{\mathbb{Q}}$-bialgebraic subvariety $Y \subset S$ is said to be *atypical* for the given bialgebraic structure on S if it is obtained as an excess intersection of $f(\widetilde{S^{\mathrm{an}}})$ with its model $\overline{f(\tilde{Y})}^{\mathrm{Zar}} \subset Z$; and S_{atyp} denotes the union of all atypical subvarieties of S. As a particular case of the atypical intersection principle:

Sparsity of arithmetic points principle. Let S be an irreducible algebraic $\overline{\mathbb{Q}}$-variety endowed with a $\overline{\mathbb{Q}}$-bialgebraic structure. Then any irreducible algebraic subvariety of S containing a Zariski-dense set of atypical arithmetic points is a $\overline{\mathbb{Q}}$-bialgebraic subvariety.

This principle that arithmetic points are sparse is a theorem of Mann [63] in Example 4.2a). For abelian varieties over $\overline{\mathbb{Q}}$ (Example 4.2b)), this is the Manin–Mumford conjecture proven first by Raynaud [80], saying that an irreducible subvariety of an abelian variety over $\overline{\mathbb{Q}}$ containing a Zariski-dense set of torsion point is the translate of an abelian subvariety by a torsion point. For Shimura varieties of abelian type (Example 4.2c)), this is the classical André–Oort conjecture [1, 67] stating that an irreducible subvariety of a Shimura variety containing a Zariski-dense set of special points is special. It has been proven in this case using tame geometry and following the strategy proposed by Pila–Zannier [78] (let us mention [3,58,72,76,87,89,98]; and [38] in the mixed case; see [59] for a survey). Recently the André–Oort conjecture in full generality has been obtained in [75], reducing to the case of abelian type using ingredients from p-adic Hodge theory. We refer to [99] for many examples of atypical intersection problems.

In the case of Shimura varieties (Example 4.2c)) the general atypical intersection principle is the Zilber–Pink conjecture [51,79,100]. Only very few instances of the Zilber–Pink conjecture are known outside of the André–Oort conjecture, see [27,49,50], for example.

For a general polarized $\mathbb{Z}$VHS $\mathbb{V}$ with period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$, which we can assume to be proper without loss of generality, we already mentioned that even the geometric characterization of the $\overline{\mathbb{Q}}$-bialgebraic subvarieties as the special subvarieties is unknown. Replacing the $\overline{\mathbb{Q}}$-bialgebraic subvarieties of S by the special ones, we define:

Definition 5.1. A special subvariety $Z = \Phi^{-1}(\Gamma_Z\backslash D_Z)^0 \subset S$ is said *atypical* if either Z is *singular for* $\mathbb{V}$ (meaning that $\Phi(Z^{\mathrm{an}})$ is contained in the singular locus of the complex analytic variety $\Phi(S^{\mathrm{an}})$), or if $\Phi(S^{\mathrm{an}})$ and $\Gamma_Z\backslash D_Z$ do not intersect generically along $\Phi(Z)$:

$$\operatorname{codim}_{\Gamma\backslash D} \Phi(Z^{\mathrm{an}}) < \operatorname{codim}_{\Gamma\backslash D} \Phi(S^{\mathrm{an}}) + \operatorname{codim}_{\Gamma\backslash D} \Gamma_Z\backslash D_Z.$$

Otherwise, it is said to be *typical*.

Defining the *atypical Hodge locus* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{atyp}} \subset \mathrm{HL}(S, \mathbb{V}^{\otimes})$ as the union of the atypical special subvarieties of S for $\mathbb{V}$, we obtain the following precise atypical intersection principle for $\mathbb{Z}$VHS, first proposed in [55] in a more restrictive form:

Conjecture 5.2 (Zilber–Pink conjecture for $\mathbb{Z}$VHS, [13,55]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on an irreducible smooth quasiprojective variety S. The atypical Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{atyp}}$ is a finite union of atypical special subvarieties of S for $\mathbb{V}$. Equivalently, the set of atypical special subvarieties of S for $\mathbb{V}$ has finitely many maximal elements for the inclusion.*

Notice that this conjecture is in some sense more general than the above atypical intersection principle, as we do not assume that S is defined over $\overline{\mathbb{Q}}$; this has to be compared to the fact that the Manin–Mumford conjecture holds true for every complex abelian variety, not necessarily defined over $\overline{\mathbb{Q}}$.

Example 5.3. Recently Baldi and Ullmo [14] proved a special case of Conjecture 5.2 of much interest. Margulis' arithmeticity theorem states that any lattice in a simple real Lie group G of real rank at least 2 is arithmetic: it is commensurable with a group $\mathbf{G}(\mathbb{Z})$, for $\mathbf{G}$ a $\mathbb{Q}$-algebraic group such that $\mathbf{G}(\mathbb{R}) = G$ up to a compact factor. On the other hand, the structure of lattices in a simple real Lie group of rank 1, like the group $\mathrm{PU}(n, 1)$ of holomorphic isometries of the complex unit ball $\mathbf{B}^n_{\mathbb{C}}$ endowed with its Bergman metric, is an essentially open question. In particular, there exist nonarithmetic lattices in $\mathrm{PU}(n, 1)$, $n = 2, 3$. Let $\iota : \Lambda \hookrightarrow \mathrm{PU}(n, 1)$ be a lattice. The ball quotient $S^{\mathrm{an}} := \Lambda\backslash\mathbf{B}^n_{\mathbb{C}}$ is the analytification of a complex algebraic variety S. By results of Simpson and Esnault-Groechenig, there exists a $\mathbb{Z}$VHS $\Phi : S^{\mathrm{an}} \to \Gamma\backslash(\mathbf{B}^n_{\mathbb{C}} \times D')$ with monodromy representation $\rho : \Lambda \to \mathrm{PU}(n, 1) \times G'$

whose first factor $\Lambda \to \mathrm{PU}(n,1)$ is the rigid representation ι. The special subvarieties of S for $\mathbb{V}$ are the totally geodesic complex subvarieties of S^{an}. When Λ is nonarithmetic, they are automatically atypical. In accordance with Conjecture 5.2 in this case, Baldi and Ullmo prove that if Λ is nonarithmetic, then S^{an} contains only finitely many maximal totally geodesic subvarieties. This result has been proved independently by Bader, Fisher, Miller, and Stover [7], using completely different methods from homogeneous dynamics.

Among the special points for a $\mathbb{Z}$VHS $\mathbb{V}$, the CM-points (i.e., those for which the Mumford–Tate group is a torus) are always atypical except if the generic Hodge datum $(\mathbf{G}, D)$ is of Shimura type and the period map Φ is dominant. Hence, as explained in [55, SECTION 5.2], Conjecture 5.2 implies the following:

Conjecture 5.4 (André–Oort conjecture for $\mathbb{Z}$VHS, [55]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on an irreducible smooth quasiprojective variety S. If S contains a Zariski-dense set of CM-points then the generic Hodge datum $(\mathbf{G}, D)$ of $\mathbb{V}$ is a Shimura datum, and the period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ is an algebraic map, dominant on the Shimura variety $\Gamma\backslash D$.*

Example 5.5. Consider the Calabi–Yau Hodge structure V of weight 3 with Hodge numbers $h^{3,0} = h^{2,1} = 1$ given by the mirror dual quintic. Its universal deformation space S is the projective line minus 3 points, which carries a $\mathbb{Z}$VHS $\mathbb{V}$ of the same type. This gives a nontrivial period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$, where $D = \mathbf{Sp}(4,\mathbb{R})/U(1) \times U(1)$ is a 4-dimensional period domain. This period map is known not to factorize through a Shimura subvariety (its algebraic monodromy group is $\mathbf{Sp}_4$). Conjecture 5.4 in that case predicts that S contains only finitely many points CM-points s. A version of this prediction already appears in [48]. The more general Conjecture 5.2 also predicts that S contains only finitely many points s where $\mathbb{V}_s$ splits as a direct sum of two (Tate twisted) weight one Hodge structures $(\mathbb{V}_s^{2,1} \oplus \mathbb{V}_s^{1,2})$ and its orthogonal for the Hodge metric $(\mathbb{V}_s^{3,0} \oplus \mathbb{V}_s^{0,3})$ (the so-called "rank two attractors" points, see [66]).

Conjecture 5.2 about the atypical Hodge locus takes all its meaning if we compare it to the expected behavior of its complement, the *typical Hodge locus* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}} := \mathrm{HL}(S, \mathbb{V}^{\otimes}) \setminus \mathrm{HL}(S, \mathbb{V}^{\otimes}_{\mathrm{atyp}})$:

Conjecture 5.6 (Density of the typical Hodge locus, [13]). *If $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}}$ is not empty then it is dense (for the analytic topology) in S^{an}.*

Conjectures 5.2 and 5.6 imply immediately the following, which clarifies the possible alternatives in Theorem 4.6:

Conjecture 5.7 ([13]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on an irreducible smooth quasiprojective variety S. If $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}}$ is empty then $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is algebraic; otherwise, $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is analytically dense in S^{an}.*

5.2. The Zilber–Pink conjecture for $\mathbb{Z}$VHS: Results

In [13] Baldi, Ullmo, and I establish the *geometric part* of Conjecture 5.2: the maximal atypical special subvarieties *of positive period dimension* arise in a finite number of families whose geometry is well understood. We cannot say anything on the atypical locus of zero period dimension (for which different ideas are certainly needed):

Theorem 5.8 (Geometric Zilber–Pink, [13]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth connected complex quasiprojective variety S. Let Z be an irreducible component of the Zariski closure of* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{pos,atyp}} := \mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{pos}} \cap \mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{atyp}}$ *in S. Then:*

(a) *Either Z is a maximal atypical special subvariety;*

(b) *Or the generic adjoint Hodge datum $(\mathbf{G}_Z^{\mathrm{ad}}, D_{G_Z})$ decomposes as a nontrivial product $(\mathbf{G}', D') \times (\mathbf{G}'', D'')$, inducing (after replacing S by a finite étale cover if necessary)*

$$\Phi_{|Z^{\mathrm{an}}} = (\Phi', \Phi'') : Z^{\mathrm{an}} \to \Gamma_{\mathbf{G}_Z} \backslash D_{G_Z} = \Gamma' \backslash D' \times \Gamma'' \backslash D'' \subset \Gamma \backslash D,$$

such that Z contains a Zariski-dense set of atypical special subvarieties for Φ'' of zero period dimension. Moreover, Z is Hodge generic in the special subvariety $\Phi^{-1}(\Gamma_{\mathbf{G}_Z} \backslash D_{G_Z})^0$ of S for Φ, which is typical.

Conjecture 5.2, which also takes into account the atypical special subvarieties of zero period dimension, predicts that the branch (b) of the alternative in the conclusion of Theorem 5.8 never occurs. Theorem 5.8 is proven using properties of definable sets and the Ax–Schanuel Theorem 4.4, following an idea originating in [89].

As an application of Theorem 5.8, let us consider the *Shimura locus* of S for $\mathbb{V}$, namely the union of the special subvarieties of S for $\mathbb{V}$ which are of Shimura type (but not necessarily with dominant period maps). In [55], I asked (generalizing the André–Oort conjecture for $\mathbb{Z}$VHS) whether a polarizable $\mathbb{Z}$VHS $\mathbb{V}$ on S whose Shimura locus in Zariski-dense in S is necessarily of Shimura type. As a corollary of Theorem 5.8 we obtain:

Theorem 5.9 ([13]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth irreducible complex quasi-projective variety S, with generic Hodge datum $(\mathbf{G}, D)$. Suppose that the Shimura locus of S for $\mathbb{V}$* of positive period dimension *is Zariski-dense in S. If $\mathbf{G}^{\mathrm{ad}}$ is simple then $\mathbb{V}$ is of Shimura type.*

5.3. On the algebraicity of the Hodge locus

In view of Conjecture 5.7, it is natural to ask if there a simple combinatorial criterion on $(\mathbf{G}, D)$ for deciding whether $\mathrm{HL}(S, \mathbb{V})_{\mathrm{typ}}$ is empty. Intuitively, one expects that the more “complicated” the Hodge structure is, the smaller the typical Hodge locus should be, due to the constraint imposed by Griffiths’ transversality. Let us measure the complexity of $\mathbb{V}$ by its *level*: when $\mathbf{G}^{\mathrm{ad}}$ is simple, it is the greatest integer k such that $\mathfrak{g}^{k,-k} \neq 0$ in the Hodge decomposition of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$; in general one takes the minimum of these integers obtained for each simple $\mathbb{Q}$-factor of $\mathbf{G}^{\mathrm{ad}}$. While strict typical special subvarieties usually abound for $\mathbb{Z}$VHSs of level one (e.g., families of abelian varieties, see Example 4.7; or families of K3 surfaces) and can occur in level two (see Example 4.8), they do not exist in level at least three!

Theorem 5.10 ([13]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth connected complex quasi-projective variety S. If $\mathbb{V}$ is of level at least 3 then $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}} = \emptyset$ (and thus $\mathrm{HL}(S, \mathbb{V}^{\otimes}) = \mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{atyp}}$).*

The proof of Theorem 5.10 is purely Lie-theoretic. Let $(\mathbf{G}, D)$ be the generic Hodge datum of $\mathbb{V}$ and $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ its period map. Suppose that $Y \subset S$ is a typical special subvariety, with generic Hodge datum $(\mathbf{G}_Y, D_Y)$. The typicality condition and the horizontality of the period map Φ imply that $\mathfrak{g}_Y^{-i,i} = \mathfrak{g}^{-i,i}$ for all $i \geq 2$ (for the Hodge structures on the Lie algebras $\mathfrak{g}_Y$ and $\mathfrak{g}$ defined by some point of D_Y). Under the assumption that $\mathbb{V}$ has level at least 3, we show that this is enough to ensure that $\mathfrak{g}_Y = \mathfrak{g}$, hence $Y = S$. Hence there are no strict typical special subvariety.

Notice that Conjecture 5.2 and Theorem 5.10 imply:

Conjecture 5.11 (Algebraicity of the Hodge locus in level at least 3, [13]). *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth connected complex quasiprojective variety S. If $\mathbb{V}$ is of level at least 3 then $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is algebraic.*

The main result of [13], which follows immediately from Theorems 5.8 and 5.10, is the following stunning geometric reinforcement of Theorems 3.1 and 4.6:

Theorem 5.12 ([13]). *If $\mathbb{V}$ is of level at least 3 then $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{fpos}}$ is algebraic.*

As a simple geometric illustration of Theorem 5.12, we prove the following, to be contrasted with the $n = 2$ case (see Example 4.8):

Corollary 5.13. *Let $\mathbf{P}_{\mathbb{C}}^{N(n,d)}$ be the projective space parametrizing the hypersurfaces X of $\mathbf{P}_{\mathbb{C}}^{n+1}$ of degree d (where $N(n,d) = \binom{n+d+1}{d} - 1$). Let $U_{n,d} \subset \mathbf{P}_{\mathbb{C}}^{N(n,d)}$ be the Zariski-open subset parametrizing the smooth hypersurfaces X and let $\mathbb{V} \to U_{n,d}$ be the $\mathbb{Z}$VHS corresponding to the primitive cohomology $H^n(X, \mathbb{Z})_{\mathrm{prim}}$. If $n \geq 3$ and $d > 5$, then $\mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}} \subset U_{n,d}$ is algebraic.*

5.4. On the typical Hodge locus in level one and two

In the direction of Conjecture 5.6, we obtain:

Theorem 5.14 (Density of the typical locus, [13]). *Let $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS on a smooth connected complex quasiprojective variety S. If the typical Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}}$ is nonempty (hence the level of $\mathbb{V}$ is one or two by Theorem 5.10) then $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is analytically (hence Zariski) dense in S.*

Notice that, in Theorem 5.14, we also treat the typical Hodge locus of zero period dimension. Theorem 5.14 is new even for S a subvariety of a Shimura variety. Its proof is inspired by the arguments of Chai [24] in that case.

It remains to find a criterion for deciding whether, in level one or two, the typical Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}}$ is empty or not. We refer to [57, THEOREM 2.15] and [85, 86] for results in this direction.

6. ARITHMETIC ASPECTS

We turn briefly to some arithmetic aspects of period maps.

6.1. Field of definition of special subvarieties

Once more the geometric case provides us with a motivation and a heuristic. Let $f : X \to S$ be a smooth projective morphism of connected algebraic varieties defined over a number field $L \subset \mathbb{C}$ and let $\mathbb{V}$ be the natural polarizable $\mathbb{Z}$VHS on S^{an} with underlying local system $R^{\bullet} f_*^{\mathrm{an}} \mathbb{Z}$. In that case, the Hodge conjecture implies that each special subvariety Y of S for $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$ and that each of the $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-conjugates of Y is again a special subvariety of S for $\mathbb{V}$. More generally, let us say that a polarized $\mathbb{Z}$VHS $\mathbb{V} = (\mathbb{V}_{\mathbb{Z}}, (\mathcal{V}, F^{\bullet}, \nabla), \mathrm{q})$ on S^{an} is *defined over a number field* $L \subset \mathbb{C}$ if S, $\mathcal{V}$, $F^{\bullet}$ and ∇ are defined over L (with the obvious compatibilities).

Conjecture 6.1. *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS defined over a number field $L \subset \mathbb{C}$. Then any special subvariety of S for $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$, and any of its finitely many* $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$*-conjugates is a special subvariety of S for $\mathbb{V}$.*

There are only few results in that direction: see [95, THEOREM 0.6] for a proof under a strong geometric assumption; and [81], where it is shown that when S (not necessarily $\mathbb{V}$) is defined over $\overline{\mathbb{Q}}$, then a special subvariety of S for $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$ if and only if it contains a $\overline{\mathbb{Q}}$-point of S. In [57] Otwinowska, Urbanik, and I provide a simple geometric criterion for a special subvariety of S for $\mathbb{V}$ to satisfy Conjecture 6.1. In particular we obtain:

Theorem 6.2 ([57]). *Let $\mathbb{V}$ be a polarized $\mathbb{Z}$VHS on a smooth connected complex quasi-projective variety S. Suppose that the adjoint generic Mumford–Tate group $\mathbf{G}^{\mathrm{ad}}$ of $\mathbb{V}$ is simple. If S is defined over a number field L, then any maximal (strict) special subvariety $Y \subset S$ of positive period dimension is defined over $\overline{\mathbb{Q}}$. If, moreover, $\mathbb{V}$ is defined over L then the finitely many* $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$*-translates of Y are special subvarieties of S for $\mathbb{V}$.*

As a corollary of Theorems 5.12 and 6.2, one obtains the following, which applies for instance in the situation of Corollary 5.13.

Corollary 6.3. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth connected quasiprojective variety S. Suppose that $\mathbb{V}$ is of level at least* 3*, and that it is defined over $\overline{\mathbb{Q}}$. Then* $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{fpos}}$ *is an algebraic subvariety of S, defined over $\overline{\mathbb{Q}}$.*

It is interesting to notice that Conjecture 5.11, which is stronger than Theorem 5.12, predicts the existence of a Hodge generic $\overline{\mathbb{Q}}$-point in S for $\mathbb{V}$ in the situation of Corollary 6.3.

As the criterion given in [57] is purely geometric, it says nothing about fields of definitions of special points. It is, however, strong enough to reduce the first part of Conjecture 6.1 to this particular case:

Theorem 6.4. *Special subvarieties for $\mathbb{Z}$VHSs defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$ if and only if it holds true for special points.*

6.2. Absolute Hodge locus

Interestingly, Conjecture 6.1 in the geometric case follows from an *a priori* much weaker conjecture than the Hodge conjecture. Let $f : X \to S$ be a smooth projective morphism of smooth connected complex quasiprojective varieties. For any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we can consider the algebraic family $f^\sigma : X^\sigma \to S^\sigma$, where $\sigma^{-1} : S^\sigma = S \times_{\mathbb{C},\sigma} \mathbb{C} \xrightarrow{\sim} S$ is the natural isomorphism of abstract schemes; and the attached polarizable $\mathbb{Z}$VSH $\mathbb{V}^\sigma = (\mathbb{V}^\sigma_{\mathbb{Z}}, \mathcal{V}^\sigma, F^{\bullet\sigma}, \nabla^\sigma)$ with underlying local system $\mathbb{V}^\sigma_{\mathbb{Z}} = R{f^{\sigma}}^{\mathrm{an}}_{*}\mathbb{Z}$ on $(S^\sigma)^{\mathrm{an}}$. The algebraic construction of the algebraic de Rham cohomology provides compatible canonical comparison isomorphisms $\iota^\sigma : (\mathcal{V}^\sigma, F^{\bullet\sigma}, \nabla^\sigma) \xrightarrow{\sim} {\sigma^{-1}}^*(\mathcal{V}, F^\bullet, \nabla)$ of the associated algebraic filtered vector bundles with connection. More generally, a collection of $\mathbb{Z}$VHS $(\mathbb{V}^\sigma)_\sigma$ with such compatible comparison isomorphisms is called a *(de Rham) motivic variation of Hodge structures* on S, in which case we write $\mathbb{V} := \mathbb{V}^{\mathrm{Id}}$. Following Deligne (see [25] for a nice exposition), an *absolute Hodge tensor* for such a collection is a Hodge tensor α for $\mathbb{V}_s$ such that the conjugates ${\sigma^{-1}}^*\alpha_{\mathrm{dR}}$ of the de Rham component of α defines a Hodge tensor in $\mathbb{V}^\sigma_{\sigma(s)}$ for all σ. The *generic absolute Mumford–Tate group* for $(\mathbb{V}^\sigma)_\sigma$ is defined in terms of the absolute Hodge tensors as the generic Mumford–Tate group is defined in terms of the Hodge tensors. Thus $\mathbf{G} \subset \mathbf{G}^{\mathrm{AH}}$. In view of Definition 3.19 the following is natural:

Definition 6.5. Let $(\mathbb{V}^\sigma)_\sigma$ be a (de Rham) motivic variation of Hodge structure on a smooth connected complex quasiprojective variety S. A closed irreducible algebraic subvariety Y of S is called *absolutely special* if it is maximal among the closed irreducible algebraic subvarieties Z of S satisfying $\mathbf{G}^{\mathrm{AH}}_Z = \mathbf{G}^{\mathrm{AH}}_Y$.

In the geometric case, the Hodge conjecture implies, since any automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ maps algebraic cycles in X to algebraic cycles on X^σ, the following conjecture of Deligne:

Conjecture 6.6 ([33]). *Let $(\mathbb{V}^\sigma)_\sigma$ be a (de Rham) motivic variation of Hodge structure on S. Then all Hodge tensors are absolute Hodge tensors, i.e., $\mathbf{G} = \mathbf{G}^{\mathrm{AH}}$.*

This conjecture immediately implies:

Conjecture 6.7. *Let $(\mathbb{V}^\sigma)_\sigma$ be a (de Rham) motivic variation of Hodge structure on S. Then any special subvariety of S for $\mathbb{V}$ is absolutely special for $(\mathbb{V}^\sigma)_\sigma$.*

Let us say that a (de Rham) motivic variation $(\mathbb{V}^\sigma)_\sigma$ is defined over $\overline{\mathbb{Q}}$ if $\mathbb{V}^\sigma = \mathbb{V}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$. In the geometric case, any morphism $f : X \to S$ defined over $\overline{\mathbb{Q}}$ defines such a (de Rham) motivic variation $(\mathbb{V}^\sigma)_\sigma$ over $\overline{\mathbb{Q}}$. Notice that the absolutely special subvarieties of S for $(\mathbb{V}^\sigma)_\sigma$ are then by their very definition defined over $\overline{\mathbb{Q}}$, and their Galois conjugates are also special. In particular, Conjecture 6.7 implies Conjecture 6.1 in the geometric case. As proven in [95], Deligne's conjecture is actually equivalent to a much stronger version of Conjecture 6.1, where one replaces the special subvarieties of S (components of the Hodge locus) with the special subvarieties in the total bundle of $\mathcal{V}^\otimes$ (components of the locus of Hodge tensors).

Recently T. Kreutz, using the same geometric argument as in [57], justified Theorem 6.2 by proving:

Theorem 6.8 ([62]). *Let $(\mathbb{V}^\sigma)_\sigma$ be a (de Rham) motivic variation of Hodge structure on S. Suppose that the adjoint generic Mumford–Tate group $\mathbf{G}^{\mathrm{ad}}$ is simple. Then any strict maximal special subvariety $Y \subset S$ of positive period dimension for $\mathbb{V}$ is absolutely special.*

We refer the reader to [61], as well as [93], for other arithmetic aspects of Hodge loci taking into account not only the de Rham incarnation of absolute Hodge classes but also their ℓ-adic components.

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