DEVELOPMENTS IN 3D RICCI FLOW SINCE PERELMAN

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ABSTRACT

This is a report on progress in 3D Ricci flow since Perelman's work 20 years ago.

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1. INTRODUCTION

A smooth family of Riemannian metrics $(g(t))_{t\in[0,T)}$ is a *Ricci flow* if it satisfies the equation

$$\partial_t g(t) = -2\,\mathrm{Ric}\big(g(t)\big)$$

for every $t \in [0,T)$, where $\mathrm{Ric}(g(t))$ denotes the Ricci tensor of $g(t)$ [34]. The Ricci flow equation is a fundamental partial differential equation in mathematics—it is the natural analog of the heat equation for Riemannian metrics, just as mean curvature flow and harmonic map heat flow are the heat equation analogs for submanifolds and mappings, and the (elliptic) Einstein equation, minimal surface equation, and harmonic map equation are the respective analogs of Laplace's equation. Ricci flow may be used to canonically smooth a metric, and, in favorable situations, deform it into an optimal shape. For this reason, it has had a profound impact on geometry and topology as a powerful tool for solving many problems, including many longstanding conjectures which had resisted all other techniques. Singularity formation has been a great challenge central to the topic, one which has required a wide range of ingredients from PDE, differential geometry, metric geometry, and topology. While Ricci flow is fascinating from many points of view, it is especially interesting from the PDE viewpoint because it has some features in common with other geometric evolution equations (e.g., mean curvature flow and harmonic map heat flow); in particular, for the last 40 years the treatment of singularities has been a common theme, and has led to important cross-fertilization.

In his preprints from 2002–2003, Perelman made a series of landmark contributions to Ricci flow, some specific to flow on 3-manifolds, and some applicable in any dimension. He introduced a number of new ingredients which opened the way to subsequent progress in many directions, including (in particular) flow on 3-manifolds, Kähler–Ricci flow, and Ricci flow under certain curvature assumptions. The aim of this article is to present the advances in 3D Ricci flow from a bird's-eye view, for a general mathematical audience. The technical nature of the subject forces some compromises in the exposition, both in the precision of statements and in the coverage of accompanying history and conceptual background. Also, in writing for a broad audience it was unavoidable to make some choices of material and emphasis which may be unsatisfactory to the experts; I hope that any such readers will be understanding.

By convention, all 3-manifolds will be orientable.

2. PERELMAN'S WORK ON 3D RICCI FLOW

In this section, we briefly review what was known about 3D Ricci flow up through 2003, when Perelman posted his preprints. We refer the interested reader to the introductions in [21,28,39,50,51] for more detailed overviews.

Hamilton showed that if h is a smooth Riemannian metric on a compact n-manifold M, then there exists a unique solution $(g(t))_{t\in[0,T)}$ to the Ricci flow equation

$$\partial_t g = -2\,\mathrm{Ric}\big(g(t)\big)$$

with initial condition $g(0) = h$, and which is defined on a maximal time interval $[0, T)$; moreover, if the time T is finite, then the norm of the curvature tensor Rm becomes unbounded as t approaches T [34]. When M is a 2-sphere, the behavior of the Ricci flow $(g(t))_{[0,T)}$ is very simple: it blows up in finite time, and as $t \to T$ the volume-renormalized metric $\hat{g}(t) := (\mathrm{vol}(g(t)))^{-\frac{2}{n}} g(t)$ converges smoothly as $t \to T$ to a metric of constant curvature 4π [29, 35]. Ricci flow in 3D is more complicated. Consider, for instance, the case when $(M, g(0))$ is obtained from two compact Riemannian 3-manifolds (M, h_M), (N, h_N) by performing a geometric connected sum, i.e., by choosing $r > 0$ small, removing r-balls, and attaching a round cylindrical "neck" (interpolating appropriately near the gluing locus). Heuristically, one expects that when r is small, the Ricci flow $(g(t))_{t\in[0,T)}$ will blow up after a short time due to the large positive Ricci curvature in the neck region, and that away from the neck region $g(t)$ will have a smooth limit as $t \to T$. The occurrence of such localized "neck pinch" singularities leads to the idea of prolonging the evolution by performing "surgery", i.e., by removing an open set U diffeomorphic to $(0, 1) \times S^2$ which contains the set where the metric goes singular as $t \to T$, gluing approximate hemispherical caps onto the two 2-sphere boundary components, and restarting the Ricci flow from the resulting compact smooth Riemannian manifold. By iterating this procedure, one might hope to obtain a Ricci flow with surgery defined for all time, allowing for the possibility that the manifold may be empty from some time onward. Noting that the surgery has the potential to simplify topology by undoing connected sums, around the time of Hamilton's original paper Yau suggested that Ricci flow with surgery might be used to address fundamental questions in three-dimensional topology—the Poincaré Conjecture and, more generally, Thurston's Geometrization Conjecture. Pursuing this idea, Hamilton developed many tools for analyzing singularities, and implemented a version of Ricci flow with surgery in an analogous 4-dimensional setting.

In 2003, Perelman completed the program in a breakthrough result:

Theorem 2.1 (Informal statement). *For any compact smooth Riemannian 3-manifold (M, h), there exists a Ricci flow with surgery with initial condition (M, h) which is defined for all time.*

In addition to many fresh insights, the proof involved numerous ingredients, including most of Hamilton's prior results on Ricci flow, as well as a variety of other tools from geometric analysis. We will only touch on a few key points here, treating some aspects differently from Perelman.

Perelman's Ricci flow with surgery consists of a sequence of Ricci flows

$$\big(M_1, \big(g_1(t)\big)_{t\in[T_0,T_1)}\big),\ \big(M_2, \big(g_2(t)\big)\big)_{t\in[T_1,T_2)}\big),\ \big(M_3, \big(g_3(t)\big)_{t\in[T_2,T_3)}\big),\ \ldots$$

where the time intervals $[T_{i-1}, T_i)$ are consecutive and $\bigcup_i [T_{i-1}, T_i) = [0, \infty)$. For every $0 < T_i < \infty$, the Ricci flow $(g_i(t))_{t\in[T_{i-1},T_i)}$ goes singular as $t \to T_i$ and has a smooth limit $\bar{g}_i$ on an open (possibly empty) subset $\Omega_i \subset M_i$. The initial condition $(M_{i+1}, g_{i+1}(T_i))$ for the next flow is obtained from $(\Omega_i, \bar{g}_i)$ by a geometric surgery procedure—cutting along 2-spheres, capping off boundary components, and throwing away some connected components—which generalizes the simple neck removal described above. The cumulative effect of the surgery process on the topology is easy to describe: for every $i > 1$, the original manifold M_1 is diffeomorphic to a connected sum

$$M \overset{\text{diff}}{\simeq} M_i \#(\#_j N_j) \tag{2.2}$$

where for every j the summand N_j is either a copy of $S^2 \times S^1$ or a spherical space form; recall that a spherical space form is a manifold of the form S^3/Γ where $\Gamma \subset O(4)$ is a finite subgroup acting freely on S^3.

A central issue in Perelman's argument is controlling the structure of the flow near singularities. This control is implemented as a set of conditions collectively referred to as the *Canonical Neighborhood Assumption*. Informally speaking, the Canonical Neighborhood Assumption asserts that near points with large curvature the flow has a restricted form, i.e., it is well approximated by a flow belonging to a family of model Ricci flows Perelman called *κ-solutions*; examples include:

(A) A shrinking round metric on S^3 or a spherical space form;

(B) A shrinking round cylindrical metric on $S^2 \times \mathbb{R}$ or the quotient $(S^2 \times \mathbb{R})/\mathbb{Z}_2$;

(C) A special Ricci flow solution $(g_{\mathrm{Bry}}(t))_{t\in\mathbb{R}}$ on $\mathbb{R}^3$ called the *Bryant soliton*.

In the schematic diagram shown in Figure 1, these provide models near the points A, B, and C, respectively.

After proving the existence of a Ricci flow with surgery, Perelman analyzed the behavior as $t \to \infty$, and used the geometry of the flow to deduce a topological conclusion:

Theorem 2.3 ([54]). *For every t sufficiently large, if $t \in [T_{i-1}, T_i)$, there is a finite disjoint collection $\{N_j\}$ of embedded incompressible tori in M_i such that each connected component of $M_i \setminus \bigcup_j N_j$ is diffeomorphic to either a complete, finite-volume hyperbolic manifold or a graph manifold.*

A *hyperbolic manifold* is a Riemannian manifold with universal cover isometric to hyperbolic 3-space $\mathbb{H}^3$. A connected embedded surface N in a 3-manifold X is *incompressible* if the inclusion map $N \to X$ induces an injective homomorphism of fundamental groups. A 3-manifold X is a *graph manifold* if there is a finite disjoint collection $\{N_k\}$ of embedded tori such that every connected component of $X \setminus \bigcup_k N_k$ is diffeomorphic to (the total space of) a circle bundle over a surface.

A few years prior to the appearance of Perelman's preprints, Hamilton proved an assertion roughly similar to Theorem 2.3, assuming an additional bound on the curvature tensor [36]. The proof of Theorem 2.3 uses several key contributions from [36] in identifying the hyperbolic piece, as well as several fundamental new ideas.

The results on Ricci flow with surgery have many applications to problems in geometry and topology. Combining Theorem 2.3 with well-known results from 3-manifold topology, Perelman proved Thurston's Geometrization Conjecture:

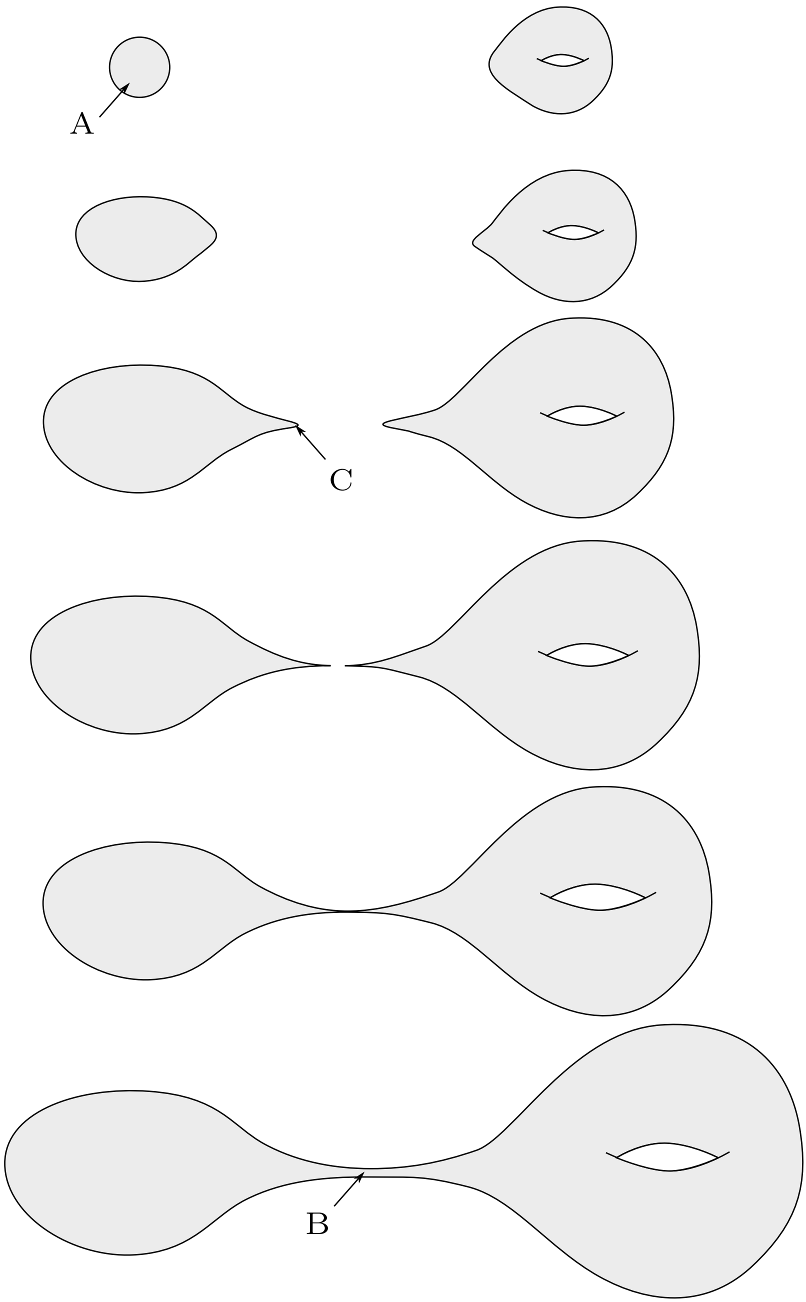

FIGURE 1
A Ricci flow with surgery with a neckpinch.

Theorem 2.4. *Every closed 3-manifold is a connected sum of manifolds that can be cut along embedded, incompressible copies of T^2 into geometrizable pieces.*

A connected 3-manifold is *geometrizable* if it admits a finite-volume Riemannian metric with universal cover isometric to one of the eight Thurston geometries S^3, H^3, $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, Nil, Solv, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ [61].

The Poincaré Conjecture is an immediate corollary:

Theorem 2.5. *Any closed, simply connected* 3*-manifold is diffeomorphic to* S^3.

Perelman and Colding–Minicozzi showed that if the initial manifold M_1 of a Ricci flow with surgery has no aspherical summands in its prime decomposition, then the flow eventually becomes extinct, i.e., for some i, we have $[T_{i-1}, T_i) = [T_{i-1}, \infty)$ and $M_i = \emptyset$ [30,53]; it then follows from (2.2) that M_1 is a connected sum of spherical space forms and copies of $S^2 \times S^1$. This gives an alternative approach to the Poincaré Conjecture avoiding Theorem 2.3.

In addition to settling central conjectures in topology, Perelman solved longstanding problems in geometry:

Theorem 2.6. *Let M be a closed* 3*-manifold, and $\sigma(M)$ denote the Yamabe invariant of M* [43,55].

- *Manifold M admits a Riemannian metric with positive scalar curvature if and only if it is a connected sum of spherical space forms and copies of $S^2 \times S^1$* [33,56].
- *If M is irreducible and $\sigma(M) \le 0$, then $(-\frac{1}{6}\sigma(M))^{\frac{3}{2}}$ is total volume of the hyperbolic pieces appearing in the geometric decomposition of M, as in Theorem* 2.4 [2,39].

3. DEVELOPMENTS BASED ON QUESTIONS RAISED BY PERELMAN'S WORK

In this section we review the progress on some fundamental questions arising in Perelman's papers [52,54].

3.1. Large-time behavior

Let $\{(M_i, ((g_i(t))_{t\in[T_{i-1},T_i)}\}$ be a Ricci flow with surgery as constructed by Perelman.

One basic question concerns the set of surgery times $\{T_i \mid 0 < T_i < \infty\}$; in the statement of Theorem 2.1, this set could potentially be infinite. Although Perelman discussed finiteness of surgeries in his preprints, he did not settle the issue or give any indication how it might be addressed, because he was able to find an approach to Theorem 2.3 (and the Geometrization Conjecture) which circumvented the matter altogether. The problem of finiteness of surgeries was also noted earlier: Hamilton had expressed the hope that it would be possible to define a Ricci flow with surgery such that only finitely many surgeries would be necessary and that the curvature would then remain bounded for large t, after appropriate normalization [36]. In a tour de force, Bamler was able to confirm this:

Theorem 3.1 ([8–12]). *In any Ricci flow with surgery as constructed by Perelman, there are only finitely many surgery times, and there exist $C, T \in (0, \infty)$ such that the curvature tensor satisfies the bound $|\mathrm{Rm}_{g(t)}| < Ct^{-1}$ for all $t > T$.*

In the theorem and in what follows, we let $g(t) := g_i(t)$ for $t \in [T_{i-1}, T_i)$.

It is natural to ask: beyond the assertions in Theorem 3.1, how much more can be said about the asymptotic behavior of the Ricci flow at $t \to \infty$? First, the results of [30,53] imply that each connected component of M_i is prime and aspherical when $T_i > T$. On general principles, one might expect that Ricci flow improves the geometry, and therefore as $t \to \infty$ the asymptotic behavior should be very simple. For instance, when M is geometrizable, then one might expect that as $t \to \infty$ the Ricci flow would converge (in some appropriate sense) to the geometric structure, and when M is not geometrizable, then the Ricci flow would construct the JSJ decomposition—a system of embedded incompressible tori which are canonical up to isotopy—as well the geometric structure on the pieces. This speculation has been confirmed only when M admits a hyperbolic metric, in which case Perelman's proof of Theorem 2.3 implies that $\frac{1}{4}t^{-1}g(t)$ converges to a hyperbolic metric as $t \to \infty$. In other cases there has been progress in this direction. For instance, Lott has shown:

Theorem 3.2 ([48]). *Let N be a connected component of M_i, where $[T_{i-1}, T_i) = [T_{i-1}, \infty)$. If the quantity $t^{-\frac{1}{2}} \operatorname{diam}(N, g(t))$ remains bounded as $t \to \infty$, then the pullback of the rescaled metric $t^{-1}g(t)$ to the universal cover $\tilde{N}$ converges to a homogeneous expanding soliton.*

Bamler has a number of results covering both geometrizable and nongeometrizable cases. The simplest case is the torus:

Theorem 3.3 ([12]). *If M is diffeomorphic to T^3, then either $g(t)$ converges to a flat metric as $t \to \infty$, or the quantity $t^{-\frac{1}{2}} \operatorname{diam}(g(t))$ is unbounded and for large t the metric $g(t)$ is well approximated by another metric $g'(t)$ with T^2-symmetry and T^2-orbits of diameter $\ll t^{\frac{1}{2}}$.*

A similar alternative holds for 3-manifolds modeled on Thurston's Nil or Solv geometries. We refer the reader to [12] for these and other results, as well as a discussion of open questions.

3.2. Classification of singularity models

As described in Section 2, Perelman's treatment of Ricci flow with surgery involved a family of Ricci flows called κ-solutions, which model the formation of singularities. Building on Hamilton's work on singularity formation, in his first preprint Perelman established many properties of κ-solutions, including:

(a) (Topological classification) Every κ-solution is diffeomorphic to a spherical space form S^3/Γ, the cylinder $S^2 \times \mathbb{R}$, the $\mathbb{Z}_2$-quotient $(S^2 \times \mathbb{R})/\mathbb{Z}_2$, or $\mathbb{R}^3$.

(b) Any κ-solution not diffeomorphic to $\mathbb{R}^3$, S^3, or RP^3 is isometrically covered by a shrinking round sphere or shrinking round cylinder.

(c) In a quantitative sense, κ-solutions are "mostly necklike." For instance, any κ-solution diffeomorphic to $\mathbb{R}^3$ is asymptotically cylindrical near infinity.

In the exceptional cases in (b), Perelman's work provided both qualitative and quantitative information, but not a complete classification. In the $\mathbb{R}^3$ case, he made the following conjecture:

Conjecture 3.4 ([52]). *Any κ-solution diffeomorphic to $\mathbb{R}^3$ is isometric to a Bryant soliton, up to rescaling.*

He also constructed a κ-solution on S^3 which is $O(3)$-symmetric, and becomes more and more elongated as $t \to -\infty$; this descends to a κ-solution on RP^3.

Recently, in the culmination of a long development in the theory of ancient solutions, Brendle, Daskalopoulos, and Sesum have completed the classification of κ-solutions:

Theorem 3.5 ([25]). *Conjecture 3.4 holds.*

Theorem 3.6 ([26]). *Any compact κ-solution is isometrically covered by a shrinking round metric or a rescaling of the nonround κ-solution constructed by Perelman.*

3.3. Ricci flow through singularities

Although the construction of Ricci flow with surgery had a spectacular impact on mathematics, in both of his preprints Perelman indicated that he had a further objective in mind:

> *"It is likely that by passing to the limit in this construction [of Ricci flow with surgery] one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that."* [52, P. 37]

> *"Our approach …is aimed at eventually constructing a canonical Ricci flow, defined on a largest possible subset of space-time,—a goal, that has not been achieved yet in the present work."* [54, P. 1]

From the PDE perspective, one may interpret Perelman's notion of a "Ricci flow through singularities" as a kind of generalized solution to the Ricci flow equation; his stated goal then fits into a long-established theme in PDE—the existence and uniqueness of weak or generalized solutions. A further motivation for pursuing such a program comes from applications in geometry and topology involving families of Ricci flows depending continuously on a parameter, which necessitate well-behaved unique solutions.

In recent years Perelman's goal was attained in the papers [16,41]. The first step was a definition Ricci flow through singularities, which was given in [41]. This is uses the following spacetime version of Ricci flow:

Definition 3.7 ([41]). A *Ricci flow spacetime* is a tuple $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$, where:

- $\mathcal{M}$ is a smooth 4-manifold with boundary.
- $\mathfrak{t} : \mathcal{M} \to [0, \infty)$ is a smooth function called the *time function*; its level sets $\mathcal{M}_t := \mathfrak{t}^{-1}(t)$ are called *time-slices*.
- $\partial_\mathfrak{t}$ is a smooth vector field satisfying $\partial_t \mathfrak{t} \equiv 1$; it is called the *time vector field*, and its trajectories are called *worldlines*.
- g is a Riemannian metric on the subbundle of the tangent bundle $T\mathcal{M}$ defined by $\ker d\mathfrak{t}$, and hence induces a Riemannian metric g_t on the time slice $\mathcal{M}_t$.
- g satisfies the Ricci flow equation
$$\mathcal{L}_{\partial_t} g = -2\,\mathrm{Ric}(g).$$
- The time slice $\mathcal{M}_0$ is the boundary of $\mathcal{M}$.

For brevity, we typically denote the entire spacetime by $\mathcal{M}$.

An ordinary Ricci flow $(g(t))_{t\in[0,T)}$ on a manifold M gives rise to a Ricci flow spacetime $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ where $\mathcal{M} = M \times [0, T)$, the time function $\mathfrak{t}$ is projection onto the second factor, the time vector field $\partial_\mathfrak{t}$ projects to the unit vector field ∂_x on the second factor, and g induces the metric $g(t)$ on the time slice $\mathcal{M}_t = M \times \{t\}$ corresponding to $g(t)$. Up to diffeomorphism, a general Ricci flow spacetime looks locally like such a product Ricci flow spacetime.

A Ricci flow spacetime by itself is too general to be useful; one obtains a good notion of Ricci flow through singularities by imposing some extra conditions on a Ricci flow spacetime:

Definition 3.8 ([41]). A *singular Ricci flow* is a Ricci flow spacetime $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ where:

(1) The initial time slice $\mathcal{M}_0$ is compact.

(2) $\mathcal{M}$ satisfies the Canonical Neighborhood Assumption.

(3) $\mathcal{M}$ is 0-complete.

Here condition (2) is similar to the Canonical Neighborhood Assumption in Perelman's Ricci flow with surgery, and asserts that around a point $x \in \mathcal{M}_t$ with large curvature, the time slice $\mathcal{M}_t$ is well approximated by a κ-solution. The 0-completeness requirement in condition (3) is a replacement for the conventional notion of completeness. A generic neck pinch gives rise to a Ricci flow spacetime exhibiting both spatial and temporal incompleteness: if T is the time at which the pinch occurs, then the time slice $\mathcal{M}_T$ will be an incomplete Riemannian manifold, and the trajectories of the time vector field ∂_t which go into the singularity are incomplete.

Definition 3.9. A Ricci flow spacetime $\mathcal{M}$ is 0-*complete* if the following holds. Suppose $\gamma : [0, s_0) \to \mathcal{M}$ is either an integral curve of $\pm\partial_t$, or a unit speed curve in some time slice of $\mathcal{M}$. If $\sup |\operatorname{Rm}|(\gamma(s)) < \infty$, then $\lim_{s\to s_0} \gamma(s)$ exists.

Ricci flow in dimension 3 is globally well posed in the setting of singular Ricci flows [16,41]:

Theorem 3.10. (1) *If (N, h) is a compact Riemannian 3-manifold, then there exists a singular Ricci flow $\mathcal{M}$ with initial time slice $\mathcal{M}_0$ isometric to (N, h).*

(2) *A singular Ricci flow is determined uniquely by its initial condition: if $\mathcal{M}$, $\mathcal{M}'$ are singular Ricci flows then any isometry $\mathcal{M}_0 \to \mathcal{M}'_0$ extends to an isometry of $\mathcal{M} \to \mathcal{M}'$ of Ricci flow spacetimes (i.e., a diffeomorphism respecting the tuples).*

The methods of [16] also imply that singular Ricci flows depend continuously on their initial condition. Perelman's assertion about convergence of Ricci flow with surgery also holds:

Theorem 3.11 ([16]). *Let (N, h) be a compact Riemannian 3-manifold, which by Theorem 3.10(1) we may identify with the time 0 slice $\mathcal{M}_0$ of some singular Ricci flow $\mathcal{M}$. Then the family of Ricci flows with surgery with initial condition $\mathcal{M}_0$ converges to $\mathcal{M}$ as the surgery parameter δ tends to zero.*

Here δ is a parameter appearing in Perelman's construction of Ricci flow with surgery; when δ is small then in particular the surgery process involves cutting along necks with small cross-section.

The results above show that there is a well-behaved notion of Ricci flow through singularities in dimension three, for arbitrary smooth initial conditions. It is natural to ask:

Question 3.12. Is there a good notion of Ricci flow through singularities in higher dimensions, for arbitrary initial conditions?

This currently seems to be a significant challenge already in dimension 4; see, however, [4–6] and the references therein for recent progress in this direction. Note that the answer to Question 3.12 is "yes" if one imposes restrictions the initial condition (see, for instance, [37,60]); also, starting in dimension 5 there are examples of Angenent–Knopf showing that one should not expect uniqueness [3]. We remark that the problem of constructing a well-behaved generalized solutions to a closely related PDE—the mean curvature flow equation—has been a major topic of research in geometric analysis for more than 40 years [24].

We now state a few results concerning the structure of singular Ricci flows.

Theorem 3.13 ([15,41,42]). *Let $\mathcal{M}$ be a singular Ricci flow.*

- *If C is a connected component of some time slice, then C is diffeomorphic to a compact manifold punctured at finitely many points.*
- *Let $\hat{M}_t$ be the manifold obtained from some time slice $\mathcal{M}_t$ by filling in punctures and throwing away components diffeomorphic to S^3. Then $\hat{M}_t$ is compact and its prime decomposition is part of the prime decomposition of $\mathcal{M}_0$.*
- *The set of times $t \in [0, \infty)$ such that the time slice $\mathcal{M}_t$ is noncompact has Minkowski dimension $\leq \frac{1}{2}$.*

At present it is unknown if time slices could have infinitely many connected components, or if there could be uncountably many noncompact time slices. In this direction we have the following conjecture:

Conjecture 3.14. *If $\mathcal{M}$ is a singular Ricci flow, then the set of times t for which $\mathcal{M}_t$ is noncompact is finite. Moreover, if $\mathcal{M}_t$ is noncompact, then as $\bar{t} \nearrow t$ each connected component of $\mathcal{M}_{\bar{t}}$ either goes extinct, or experiences finitely many (possibly degenerate) neckpinch singularities.*

4. FURTHER RESULTS

We conclude by listing a number of other directions which have seen progress involving 3D Ricci flow.

- Ricci flow with surgery and/or singular Ricci flow can be extended, or partially extended, to noncompact manifolds [19, 20, 22, 47].
- There is a large literature on various types of special Ricci flow solutions, including shrinking, expanding, and steady solitons, ancient solutions, and eternal solutions. Many of these solutions arise as potential singularity models for finite time singularities or as blow-up limits of type I, II, or III [36]. There does not seem to be a good single source covering these developments, so we recommend searching the internet for "Ricci soliton."
- Perelman's results on Ricci flow with surgery (Theorems 2.1 and 2.3) extend to orbifolds, giving a Ricci flow proof of the Orbifold Theorem, see [23, 40].
- Singular Ricci flow may be used to understand the topology of the space $\mathrm{Diff}(M)$ of diffeomorphisms $M \to M$ with the smooth topology, in particular, settling the Generalized Smale Conjecture, and completing the determination of the topology of $\mathrm{Diff}(M)$ when M is a prime 3-manifold. See [13, 15, 17, 18], and also [38] for history and background.
- Ricci flow with surgery and singular Ricci flow may be used to study the topology of the space $\mathrm{Met}_{\mathrm{PSC}}(M)$ of Riemannian metrics of positive scalar curvature on a 3-manifold M, and the moduli space $\mathrm{Met}_{\mathrm{PSC}}(M)/\,\mathrm{Diff}(M)$. It was shown in [49]

that the moduli space $\mathrm{Met}_{\mathrm{PSC}}(M)/\operatorname{Diff}(M)$ is empty or path-connected, and [17] extended this, proving that $\mathrm{Met}_{\mathrm{PSC}}(M)$ is empty or contractible. See [17, 49] for more background and references.

- In [1] Ricci flow with surgery was used to give sharp volume estimates for hyperbolic 3-manifolds with minimal surface boundary.
- Although the topics are not specific to dimension 3, we mention that there are a number of papers studying Ricci flow starting from rough initial conditions [14, 31, 44–46, 57–59], and papers using Ricci flow to study scalar curvature lower bounds in a C^0-setting [7, 27, 32].

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REFERENCES

[1] I. Agol, P. A. Storm, and W. P. Thurston, Lower bounds on volumes of hyperbolic Haken 3-manifolds. With an appendix by Nathan Dunfield. *J. Amer. Math. Soc.* **20** (2007), no. 4, 1053–1077.

[2] M. T. Anderson, Scalar curvature and geometrization conjectures for 3-manifolds. In *Comparison geometry (Berkeley, CA, 1993–94)*, pp. 49–82, Math. Sci. Res. Inst. Publ. 30, Cambridge Univ. Press, Cambridge, 1997.

[3] S. Angenent and D. Knopf, Ricci solitons, conical singularities, and nonuniqueness. 2021, arXiv:1909.08087.

[4] R. Bamler, Compactness theory of the space of super Ricci flows. 2020, arXiv:2008.09298.

[5] R. Bamler, Entropy and heat kernel bounds on a Ricci flow background. 2020, arXiv:2008.07093.

[6] R. Bamler, Structure theory of non-collapsed limits of Ricci flows. 2020, arXiv:2009.03243.

[7] R. H. Bamler, A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature. *Math. Res. Lett.* **23** (2016), no. 2, 325–337.

[8] R. H. Bamler, Long-time behavior of 3-dimensional Ricci flow: A: generalizations of Perelman's long-time estimates. *Geom. Topol.* **22** (2018), no. 2, 775–844.

[9] R. H. Bamler, Long-time behavior of 3-dimensional Ricci flow: B: evolution of the minimal area of simplicial complexes under Ricci flow. *Geom. Topol.* **22** (2018), no. 2, 845–892.

[10] R. H. Bamler, Long-time behavior of 3-dimensional Ricci flow: C: 3-manifold topology and combinatorics of simplicial complexes in 3-manifolds. *Geom. Topol.* **22** (2018), no. 2, 893–948.

[11] R. H. Bamler, Long-time behavior of 3-dimensional Ricci flow: D: proof of the main results. *Geom. Topol.* **22** (2018), no. 2, 949–1068.

[12] R. H. Bamler, Long-time behavior of 3-dimensional Ricci flow: introduction. *Geom. Topol.* **22** (2018), no. 2, 757–774.

[13] R. H. Bamler, Some recent developments in Ricci flow. In *ICM 2022 Proceedings, Vol. 4*, pp. 2432–2455, EMS Press, 2022.

[14] R. H. Bamler, E. Cabezas-Rivas, and B. Wilking, The Ricci flow under almost non-negative curvature conditions. *Invent. Math.* **217** (2019), no. 1, 95–126.

[15] R. Bamler and B. Kleiner, Ricci flow and diffeomorphism groups of 3-manifolds. 2017, arXiv:1712.06197.

[16] R. Bamler and B. Kleiner, Uniqueness and stability of Ricci flow through singularities. 2018, arXiv:1709.04122.

[17] R. Bamler and B. Kleiner, Ricci flow and contractibility of spaces of metrics. 2019, arXiv:1909.08710.

[18] R. Bamler and B. Kleiner, Diffeomorphism groups of prime 3-manifolds. 2021, arXiv:2108.03302.

[19] L. Bessières, G. Besson, and S. Maillot, Ricci flow on open 3-manifolds and positive scalar curvature. *Geom. Topol.* **15** (2011), no. 2, 927–975.

[20] L. Bessières, G. Besson, and S. Maillot, Long time behaviour of Ricci flow on open 3-manifolds. *Comment. Math. Helv.* **90** (2015), no. 2, 377–405.

[21] L. Bessières, G. Besson, S. Maillot, M. Boileau, and J. Porti, *Geometrisation of 3-manifolds*. EMS Tracts Math. 13, European Mathematical Society (EMS), Zürich, 2010.

[22] L. Bessières, G. Besson, S. Maillot, and F. C. Marques, Deforming 3-manifolds of bounded geometry and uniformly positive scalar curvature. *J. Eur. Math. Soc. (JEMS)* **23** (2021), no. 1, 153–184.

[23] M. Boileau, B. Leeb, and J. Porti, Geometrization of 3-dimensional orbifolds. *Ann. of Math. (2)* **162** (2005), no. 1, 195–290.

[24] K. A. Brakke, *The motion of a surface by its mean curvature*. Math. Notes 20, Princeton University Press, Princeton, NJ, 1978.

[25] S. Brendle, Ancient solutions to the Ricci flow in dimension 3. *Acta Math.* **225** (2020), no. 1, 1–102.

[26] S. Brendle, P. Daskalopoulos, and N. Sesum, Uniqueness of compact ancient solutions to three-dimensional Ricci flow. 2020, arXiv:2002.12240.

[27] P. Burkhardt-Guim, Pointwise lower scalar curvature bounds for C^0 metrics via regularizing Ricci flow. *Geom. Funct. Anal.* **29** (2019), no. 6, 1703–1772.

[28] H.-D. Cao and X.-P. Zhu, A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton–Perelman theory of the Ricci flow. *Asian J. Math.* **10** (2006), no. 2, 165–492.

[29] B. Chow, The Ricci flow on the 2-sphere. *J. Differential Geom.* **33** (1991), no. 2, 325–334.

[30] T. H. Colding and W. P. Minicozzi II, Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman. *J. Amer. Math. Soc.* **18** (2005), no. 3, 561–569.

[31] P. Gianniotis and F. Schulze, Ricci flow from spaces with isolated conical singularities. *Geom. Topol.* **22** (2018), no. 7, 3925–3977.

[32] M. Gromov, Dirac and Plateau billiards in domains with corners. *Cent. Eur. J. Math.* **12** (2014), no. 8, 1109–1156.

[33] M. Gromov and H. Blaine Lawson Jr., Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)* **111** (1980), no. 2, 209–230.

[34] R. S. Hamilton, Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), no. 2, 255–306.

[35] R. S. Hamilton, The Ricci flow on surfaces. In *Mathematics and general relativity (Santa Cruz, CA, 1986)*, pp. 237–262, Contemp. Math. 71, Amer. Math. Soc., Providence, RI, 1988.

[36] R. S. Hamilton, Non-singular solutions of the Ricci flow on three-manifolds. *Comm. Anal. Geom.* **7** (1999), no. 4, 695–729.

[37] R. Haslhofer, Uniqueness and stability of singular Ricci flows in higher dimensions. 2021, arXiv:2110.03412.

[38] S. Hong, J. Kalliongis, D. McCullough, and J. H. Rubinstein, *Diffeomorphisms of elliptic 3-manifolds*. Lecture Notes in Math. 2055, Springer, Heidelberg, 2012.

[39] B. Kleiner and J. Lott, Notes on Perelman's papers. *Geom. Topol.* **12** (2008), no. 5, 2587–2855.

[40] B. Kleiner and J. Lott, Geometrization of three-dimensional orbifolds via Ricci flow. *Astérisque* **365** (2014), 101–177.

[41] B. Kleiner and J. Lott, Singular Ricci flows I. *Acta Math.* **219** (2017), no. 1, 65–134.

[42] B. Kleiner and J. Lott, Singular Ricci flows II. 2018, arXiv:1804.03265.

[43] O. Kobayashi, Scalar curvature of a metric with unit volume. *Math. Ann.* **279** (1987), no. 2, 253–265.

[44] H. Koch and T. Lamm, Geometric flows with rough initial data. *Asian J. Math.* **16** (2012), no. 2, 209–235.

[45] H. Koch and T. Lamm, Parabolic equations with rough data. *Math. Bohem.* **140** (2015), no. 4, 457–477.

[46] Y. Lai, Ricci flow under local almost non-negative curvature conditions. *Adv. Math.* **343** (2019), 353–392.

[47] Y. Lai, Producing 3d Ricci flows with non-negative Ricci curvature via singular Ricci flows. 2020, arXiv:2004.05291.

[48] J. Lott, Dimensional reduction and the long-time behavior of Ricci flow. *Comment. Math. Helv.* **85** (2010), no. 3, 485–534.

[49] F. Marques, Deforming three-manifolds with positive scalar curvature. *Ann. of Math. (2)* **176** (2012), no. 2, 815–863.

[50] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*. Clay Math. Monogr. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[51] J. Morgan and G. Tian, *The geometrization conjecture*. Clay Math. Monogr. 5, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014.
[52] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. 2002, arXiv:math/0211159.
[53] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. 2003, arXiv:math/0307245.
[54] G. Perelman, Ricci flow with surgery on three-manifolds. 2003, arXiv:math/0303109v1.
[55] R. M. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, pp. 120–154, Lecture Notes in Math. 1365, Springer, Berlin, 1989.
[56] R. Schoen and S. T. Yau, On the structure of manifolds with positive scalar curvature. *Manuscripta Math.* **28** (1979), no. 1–3, 159–183.
[57] M. Simon, Ricci flow of almost non-negatively curved three manifolds. *J. Reine Angew. Math.* **630** (2009), 177–217.
[58] M. Simon, Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below. *J. Reine Angew. Math.* **662** (2012), 59–94.
[59] M. Simon and P. M. Topping, Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces. *Geom. Topol.* **25** (2021), no. 2, 913–948.
[60] J. Song and G. Tian, The Kähler–Ricci flow through singularities. *Invent. Math.* **207** (2017), no. 2, 519–595.
[61] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*. Princeton Math. Ser. 35, Princeton University Press, Princeton, NJ, 1997.

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