SOME RECENT DEVELOPMENTS IN RICCI FLOW

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ABSTRACT

In this article we survey some of the recent developments in Ricci flow. We present a new theory of weak, 3-dimensional Ricci flows "through singularities," which can be viewed as an improvement of Perelman's Ricci flow with surgery. We point out two topological applications: the resolution of the Generalized Smale Conjecture regarding the diffeomorphism groups of 3-manifolds and the resolution of a conjecture regarding the space of positive scalar curvature metrics on 3-manifolds. We also describe ongoing research on the formation of singularities in higher dimensions, which may yield further interesting applications in the future.

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1. INTRODUCTION

The Ricci flow has proven to be a powerful tool, as it was used by Perelman in the early 2000s to resolve two of the most important conjectures in 3-manifold topology: the Poincaré Conjecture and the Geometrization Conjecture [46–48]. These applications were far from coincidental, as they provide a new perspective on 3-manifold topology using the geometric-analytic language of Ricci flow. Since then there have been further advances in the study of Ricci flow, which have led to new topological applications in dimension 3. In addition, more applications in higher dimensions may be forthcoming. The goal of this article is to survey some of these developments, particularly those relating to questions in (geometric) topology.

This article[1] is structured as follows. We will first provide a brief introduction to Ricci flow, review some of the earlier results in dimension 2 and Perelman's work in dimension 3. Next, we will discuss more recent results in dimension 3 regarding singular "Ricci flows through singularities," their uniqueness and continuous dependence on the initial data and describe their topological applications. Lastly, we present a new approach towards the study of Ricci flows in higher dimensions and point out potential future directions and applications.

2. RICCI FLOW

A Ricci flow (introduced by Hamilton [31]) on a manifold M is given by a smooth family $g(t)$, $t \in [0, T)$, of Riemannian metrics satisfying the evolution equation

$$\partial_t g(t) = -2 \operatorname{Ric}_{g(t)}, \tag{2.1}$$

where $\operatorname{Ric}_{g(t)}$ denotes the Ricci curvature of the metric $g(t)$, i.e., the trace of its Riemann curvature tensor $\operatorname{Rm}_{g(t)}$. Equation (2.1) is weakly parabolic and it implies an evolution equation for the curvature tensor $\operatorname{Rm}_{g(t)}$ of the form

$$\partial_t \operatorname{Rm}_{g(t)} = \Delta \operatorname{Rm}_{g(t)} + Q(\operatorname{Rm}_{g(t)}), \tag{2.2}$$

where the last term denotes a quadratic term; its exact form will not be important for this survey. Equation (2.2) suggests that the metric $g(t)$ becomes "smoother" or "more homogeneous" as time moves on, similar to solutions of heat equations. On the other hand, the last term in (2.2) seems to indicate that – possibly at larger scales or in regions of large curvature – this diffusion property may be outweighed by some other nonlinear effects, which could lead to singularities.

If M is compact, then for any initial metric g_0 the Ricci flow equation (2.1) has a unique solution $g(t)$, $t \in [0, T)$, with initial condition $g(t) = g_0$ and for some maximal $T \in (0, \infty]$ [31]. If $T < \infty$, then the flow $g(t)$ must develop a singularity at time T and the curvature must blow up: $\max_M |\operatorname{Rm}_{g(t)}| \xrightarrow[t \nearrow T]{} \infty$.

1 This article has appeared in a modified form in the Notices of the AMS [8].

The most basic examples of Ricci flows are those in which g_0 is Einstein, i.e., $\mathrm{Ric}_{g_0} = \lambda g_0$. In this case the flow evolves by rescaling,

$$g(t) = (1 - 2\lambda t) g_0. \tag{2.3}$$

So, for example, a round sphere ($\lambda > 0$) shrinks under the flow, develops a singularity in finite time, and its diameter goes to 0. On the other hand, if we start with a hyperbolic metric ($\lambda < 0$), then the flow is immortal (i.e., $T = \infty$) and the metric expands linearly. In the following we will consider more general initial metrics g_0 and hope that – at least in some cases – the flow is asymptotic to a solution of the form (2.3). This will then allow us to understand the topology of the underlying manifold in terms of the limiting geometry.

3. DIMENSION 2

In dimension 2, Ricci flows are very well understood [24, 26, 32]:

Theorem 3.1. *Any Ricci flow on a compact 2-dimensional manifold converges, modulo rescaling, to a metric of constant curvature.*

In addition, one can show that the flow in dimension 2 preserves the conformal class, i.e., for all times t we have $g(t) = f(t) g_0$ for some smooth positive function $f(t)$ on M. This observation, combined with Theorem 3.1, can in fact be used to reprove the Uniformization Theorem:[2]

Theorem 3.2. *Each compact surface M admits a metric of constant curvature in each conformal class.*

In order to obtain *new* applications, however, we will need to study the flow in higher dimensions.

4. DIMENSION 3

In dimension 3, the behavior of the flow – and its singularity formation – becomes far more complicated. In the following, we will first review prior work on Ricci flow in dimension 3, which is mostly due to Hamilton and Perelman and which led to the resolution of the Poincaré and Geometrization Conjectures. We will keep this part short and only focus on aspects that will become important later; for a more in-depth discussion see, for example, [1]. Next, we will focus on more recent work by Kleiner, Lott, and the author on singular Ricci flows and their uniqueness and continuous dependence, which led to the resolution of several longstanding topological conjectures.

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2 The original proof of Theorem 3.1 relied on the Uniformization Theorem. This dependence was later removed by Chen, Lu, and Tian.

4.1. Singularity formation – an example

To get an idea of the possible singularity formation of 3-dimensional Ricci flows, it is useful to consider the famous dumbbell example [2, 3] (see Figure 1). In this example, the initial manifold (M, g_0) is the result of connecting two round spheres of radii r_1, r_3 by a certain type of rotationally symmetric neck of radius r_2 (see Figure 1). So $M \approx S^3$ and $g_0 = f^2(s)g_{S^2} + ds^2$ is a warped product away from the two endpoints.

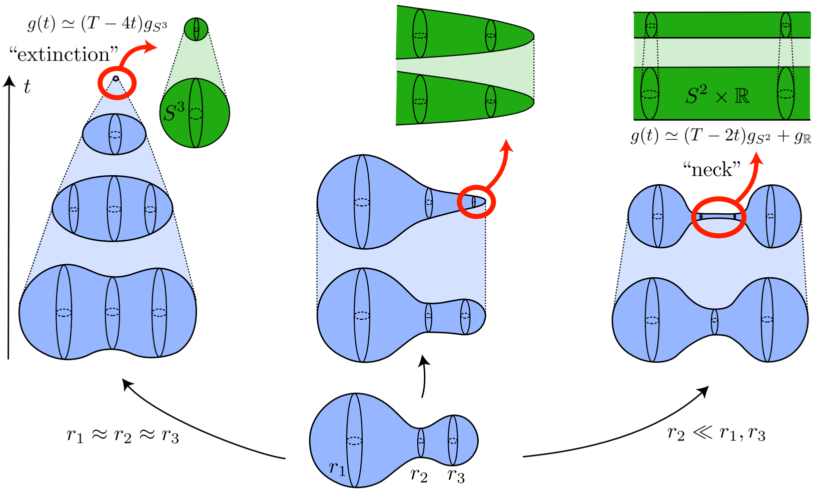

FIGURE 1
Different singularity formations in the rotationally symmetric case, depending on the choice of the radii r_1, r_2, r_3. The flows depicted on the top are the corresponding singularity models. These turn out to be the only singularity models, even in the nonrotationally symmetric case (see Section 4.3).

It can be shown that any flow starting from a metric of this form must develop a singularity in finite time. The singularity *type,* however, depends on the choice of the radii r_1, r_2, r_3. More specifically, if the radii r_1, r_2, r_3 are comparable (Figure 1, left), then the diameter of the manifold converges to zero and, after rescaling, the flow becomes asymptotically round – just as in Theorem 3.1. This case is called *extinction*. On the other hand, if $r_2 \ll r_1, r_3$ (Figure 1, right), then the flow develops a *neck singularity*, which looks like a round cylinder ($S^2 \times \mathbb{R}$) at small scales. Note that in this case the singularity only occurs in a certain region of the manifold, while the metric converges to a smooth limit everywhere else. Lastly, there is also an intermediate case (Figure 1, center), in which the flow develops a singularity that is modeled on the Bryant soliton – a one-ended paraboloid-like model [16].[3]

3 We have omitted a less important nongeneric case, called the *peanut solution*. In this case the diameter converges to zero in finite time. However, after rescaling, the metric looks like a long cylinder with a slight indentation that is capped off on each side by regions whose geometry is close to Bryant soliton. See [2] for more details.

4.2. Blow-up analysis

Perelman's work implied that the previous example is in fact prototypical for the singularity formation of *general* (not necessarily rotationally symmetric) 3-dimensional Ricci flow. In order to make this statement more precise, let us first recall a method called *blow-up analysis,* which is used frequently to study singularities in geometric analysis.

Suppose that $(M, (g(t))_{t\in[0,T)})$ is a Ricci flow that develops a singularity at time $T < \infty$ (see Figure 2). Then we can find a sequence of spacetime points $(x_i, t_i) \in M \times [0, T)$ such that $\lambda_i := |\mathrm{Rm}|(x_i, t_i) \to \infty$ and $t_i \nearrow T$. Our goal will be to understand the local geometry at small scales near (x_i, t_i), for large i. For this purpose, we consider the sequence of pointed, parabolically rescaled flows

$$\big(M, \big(g'_i(t) := \lambda_i g\big(\lambda_i^{-1} t + t_i\big)\big)_{t\in[-\lambda_i t_i, 0]}, (x_i, 0)\big).$$

Geometrically, the flows $(g'_i(t))$ are the result of rescaling distances by $\lambda_i^{1/2}$, times by λ_i and an application of a time-shift such that the points $(x_i, 0)$ in the new flows corresponds to the points (x_i, t_i) in the old flows. The new flows $(g'_i(t))$ still satisfy the Ricci flow equation and are defined on larger and larger backwards time-intervals of size $\lambda_i t_i \to \infty$. Moreover, we have $|\mathrm{Rm}|(x_i, 0) = 1$ on these new flows. Observe also that the geometry of the original flows near (x_i, t_i) at scale $\lambda_i^{-1/2} \ll 1$ is a rescaling of the geometry of $(g'_i(t))$ near $(x_i, 0)$ at scale 1.

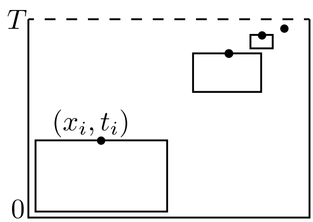

FIGURE 2

A Ricci flow $M \times [0, T)$ that develops a singularity at time T and a sequence of points (x_i, t_i) that "run into a singularity." The geometry in the parabolic neighborhoods around (x_i, t_i) (rectangles) is close to the singularity model modulo rescaling if $i \gg 1$.

Under certain additional assumptions, we may now apply a compactness theorem (à la Arzela–Ascoli) such that, after passing to a subsequence, we have convergence

$$\big(M, \big(g'_i(t)\big)_{t\in[-\lambda_i^2 t_i, 0]}, (x_i, 0)\big) \xrightarrow[i\to\infty]{} \big(M_\infty, \big(g_\infty(t)\big)_{t\leq 0}, (x_\infty, 0)\big). \tag{4.1}$$

The limit is called a *blow-up* or *singularity model*, as it gives valuable information on the singularity formation near the points (x_i, t_i). This model is a Ricci flow that is defined for all times $t \leq 0$; it is therefore called *ancient.* So in summary, a blow-up analysis reduces the study of singularity *formation* to the classification of ancient singularity *models.*

The notion of the convergence in (4.1) is a generalization of Cheeger–Gromov convergence to Ricci flows; it is due to Hamilton [33]. Instead of demanding global convergence of the metric tensors, as in Theorem 3.1, we only require convergence up to diffeomorphisms here. More specifically, we roughly require that we have convergence

$$\phi_i^* g_i'(t) \xrightarrow[i\to\infty]{C^\infty_{\mathrm{loc}}} g_\infty \tag{4.2}$$

on $M_\infty \times (-\infty, 0]$ of the pullbacks of $g_i'(t)$ via (time-independent) diffeomorphisms $\phi_i : U_i \to V_i \subset M$ that are defined over larger and larger subsets $U_i \subset M_\infty$ and satisfy $\phi_i(x_\infty) = x_i$. We will see later (in Section 5) that this notion of convergence is too strong to capture the more subtle singularity formation of higher dimensional Ricci flows and we will discuss necessary refinements. Luckily, in dimension 3 the current notion is still sufficient for our purposes, though.

4.3. Singularity models and canonical neighborhoods

One of key discoveries of Perelman's work was the classification of singularity models of 3-dimensional Ricci flows and the resulting structural description of the flow near a singularity. The following theorem[4] summarizes this classification.

Theorem 4.1. *Any singularity model $(M_\infty, (g_\infty(t))_{t\le 0})$ obtained as in (4.1) is isometric, modulo rescaling, to one the following:*

(1) *a quotient of the round shrinking sphere $(S^3, (1-4t)g_{S^3})$,*

(2) *the Bryant soliton $(M_{\mathrm{Bry}}, (g_{\mathrm{Bry}}(t)))$,*

(3) *the round shrinking cylinder $(S^2 \times \mathbb{R}, (1-2t)g_{S^2} + g_{\mathbb{R}})$ or its quotient $(S^2 \times \mathbb{R})/\mathbb{Z}_2$.*

Note that these three models correspond to the three cases in the rotationally symmetric dumbbell example from Section 4.1 (see Figure 1). The Bryant soliton in (2) is a rotationally symmetric solution to the Ricci flow on $\mathbb{R}^3$ with the property that all its time-slices are isometric to a metric of the form

$$g_{\mathrm{Bry}} = f^2(r) g_{S^2} + dr^2, \quad f(r) \sim \sqrt{r}.$$

The name *soliton* refers to the fact that all time-slices of the flow are isometric, so the flow merely evolves by pullbacks of a family of diffeomorphisms.

The next theorem describes the structure of the flow near *any* point of the flow that is close to a singularity – not just along a single blow-up sequence. In order to state this result efficiently, we will need to consider the class of *κ-solutions.* This class consists of all solutions listed in Theorem 4.1, plus an additional compact, ellipsoidal solution **[15]** (the details of this solution won't be important here[5]). Then we have:

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4 Perelman proved a version of Theorem 4.1 that contained a more qualitative characterization in Case (2), which was sufficient for most applications. Later, Brendle **[14]** showed that the only possibility in Case (2) is the Bryant soliton.

5 This solution does not occur as a singularity of a single flow, but can be observed as a transitional model in families of flows that interpolate between two different singularity models.

Theorem 4.2 (Canonical neighborhood theorem). *If* $(M, (g(t))_{t\in[0,T)})$, $T < \infty$, *is a 3-dimensional Ricci flow and* $\varepsilon > 0$, *then there is a constant* $r_{\mathrm{can}}(g(0), T, \varepsilon) > 0$ *such that for any* $(x, t) \in M \times [0, T)$ *with the property that*

$$r := |\mathrm{Rm}|^{-1/2}(x,t) \leq r_{\mathrm{can}}$$

the geometry of the metric $g(t)$ *restricted to the ball* $B_{g(t)}(x, \varepsilon^{-1} r)$ *is* ε*-close*[6] *to a time-slice of a* κ*-solution.*

4.4. Ricci flow with surgery

Our understanding of the structure of the flow near a singularity allows us to carry out a so-called *surgery construction.* Under this construction, (almost) singularities of the flow are removed, resulting in a "less singular" geometry, from which the flow can be restarted. This leads to a new type of flow that is defined beyond its singularities and which will provide important information on the underlying manifold.

Let us be more precise. A *(3-dimensional) Ricci flow with surgery* (see Figure 3) consists of a sequence of Ricci flows

$$\big(M_1, (g_1(t))_{t\in[0,T_1]}\big), \quad \big(M_2, (g_2(t))_{t\in[T_1,T_2]}\big), \quad \big(M_3, (g_3(t))_{t\in[T_2,T_3]}\big), \quad \dots,$$

which live on manifolds $M_1, M_2, \dots$ of possibly different topology and are parameterized by consecutive time-intervals of the form $[0, T_1], [T_1, T_2], \dots$ whose union equals $[0, \infty)$.

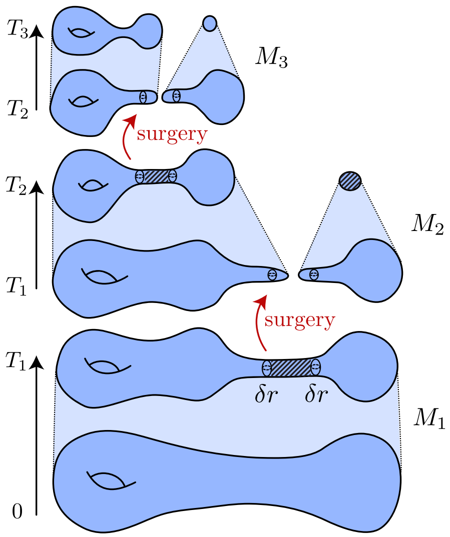

FIGURE 3
A schematic depiction of a Ricci flow with surgery. The almost-singular parts $M_{\text{almost-sing}}$, i.e., the parts that are discarded under each surgery construction, are hatched.

6 Similar to the definition of (4.2), this roughly means that there is a diffeomorphism between an ε^{-1}-ball in a κ-solution and this ball such that the pullback of $r^{-2} g(t)$ is ε-close in the $C^{[\varepsilon^{-1}]}$-sense to the metric on the κ-solution.

The time-slices $(M_i, g_i(T_i))$ and $(M_{i+1}, g_{i+1}(T_i))$ are related by a *surgery process,* which can be roughly summarized as follows. Consider the set $M_{\text{almost-sing}} \subset M_i$ of all points of high enough curvature, such that they have a canonical neighborhood as in Theorem 4.2. Cut M_i open along approximate cross-sectional 2-spheres of diameter $r_{\text{surg}}(T_i) \ll 1$ near the cylindrical ends of $M_{\text{almost-sing}}$, discard most of the high-curvature components (including the closed, spherical components of $M_{\text{almost-sing}}$), and glue in cap-shaped 3-disks to the cutting surfaces. In doing so we have constructed a new, "less singular," Riemannian manifold $(M_{i+1}, g_{i+1}(T_i))$, from which we can restart the flow. Stop at some time $T_{i+1} > T_i$, shortly before another singularity occurs and repeat the process.

The precise surgery construction is quite technical and more delicate than presented here. The main difficulty in this construction is to ensure that the surgery times T_i do not accumulate, i.e., that the flow can be extended for all times. It was shown by Perelman that this and other difficulties can, indeed, be overcome:

Theorem 4.3. *Let (M, g) be a closed, 3-dimensional Riemannian manifold. If the surgery scales $r_{\text{surg}}(T_i) > 0$ are chosen sufficiently small (depending on (M, g) and T_i), then a Ricci flow with surgery with initial condition $(M_1, g_1(0)) = (M, g)$ can be constructed.*

Note that the topology of the underlying manifold M_i may change in the course of a surgery, but only in a controlled way. In particular, it is possible to show that for any i the initial manifold M_1 is diffeomorphic to a connected sum of components of M_i and copies of spherical space forms S^3/Γ and $S^2 \times S^1$. So if the flow goes *extinct* in finite time, meaning that $M_i = \emptyset$ for some large i, then

$$M_1 \approx \#_{j=1}^{k} (S^3/\Gamma_j) \# m(S^2 \times S^1). \tag{4.3}$$

Perelman, moreover, showed that if M_1 is simply connected, then the flow *must* go extinct and therefore M_1 must be of the form (4.3). This implies the Poincaré Conjecture:

Theorem 4.4 (Poincaré Conjecture). *Any simply connected, closed 3-manifold is diffeomorphic to S^3.*

On the other hand, Perelman showed that if the Ricci flow with surgery does not go extinct, meaning if it exists for all times, then for large times $t \gg 1$ the flow decomposes the manifold (at time t) into a thick and a thin part:

$$M_{\text{thick}}(t) \mathbin{\dot\cup} M_{\text{thin}}(t), \tag{4.4}$$

such that the metric on $M_{\text{thick}}(t)$ is asymptotic to a hyperbolic metric, the metric on $M_{\text{thin}}(t)$ is locally collapsed and the boundary of $M_{\text{thick}}(t)$ consists of incompressible 2-tori. A further topological analysis of this collapse implied the Geometrization Conjecture:

Theorem 4.5 (Geometrization Conjecture). *Every closed 3-manifold is a connected sum of manifolds that can be cut along embedded, incompressible copies of T^2 into pieces which each admit a locally homogeneous geometry.*

4.5. Ricci flows through singularities

Despite their spectacular applications, Ricci flows with surgery have one major drawback: their construction is not canonical. In other words, each surgery step depends on a number of auxiliary parameters, for which there does not seem to be a canonical choice, such as:

- The surgery scales $r_{\text{surg}}(T_i)$, i.e., the diameters of the cross-sectional spheres along which the manifold is cut open. These scales need to be positive and small.
- The precise locations of these surgery spheres.

Different choices of these parameters may influence the future development of the flow significantly (as well as the space of future surgery parameters). Hence a Ricci flow with surgery is not *uniquely* determined by its initial metric.

This disadvantage was already recognized in Perelman's work, where he conjectured that there should be another flow, in which surgeries are effectively carried out automatically at an infinitesimal scale (think "$r_{\text{surg}} = 0$"), or which, in other words, "flows through singularities."

Perelman's conjecture was recently resolved by Kleiner, Lott, and the author (see [39] for the "Existence" and [10] for the "Uniqueness" part; part (2) of Theorem 4.6 follows from a combination of both papers):

Theorem 4.6. *There is a notion of singular Ricci flow (through singularities) such that:*

(1) *For any compact, 3-dimensional Riemannian manifold* (M, g)*, there is a unique singular Ricci flow* $\mathcal{M}$ *whose initial time-slice* $(\mathcal{M}_0, g_0)$ *is* (M, g).

(2) *Any Ricci flow with surgery starting from* (M, g) *can be viewed as an approximation of* $\mathcal{M}$*. More specifically, if we consider a sequence of Ricci flows with surgery starting from* (M, g) *with surgery scales* $\max_t r_{\text{surg}}(t) \to 0$*, then these flows converge to* $\mathcal{M}$ *in the local* C^∞*-sense. (More details on this convergence will be given in the end of this subsection.)*

Before continuing, let us compare this result with past work on the mean curvature flow – a close cousin of the Ricci flow. There are two important weak formulations of the mean curvature flow, namely Brakke and level set flows. Existence theories [13,25,28,38] rely heavily on the fact that a mean curvature flow concerns *embedded* submanifold in an ambient space. Uniqueness of these flows, on the other hand, is false in general [50], but true in the mean convex case [51]; so the analogous statement to Part (1) holds in this case. Moreover, under the more restrictive condition of 2-convexity, which guarantees the existence of a surgery procedure, an equivalent of Part (2) holds as well [35,41].

The concept of a singular Ricci flow is less technical than that of a Ricci flow with surgery—in fact, we will be able to state its full definition here. To do this, we will first define the concept of a Ricci flow spacetime. In short, this is a smooth 4-manifold that locally looks like a Ricci flow, but which may have non-trivial *global* topology (see Figure 4).

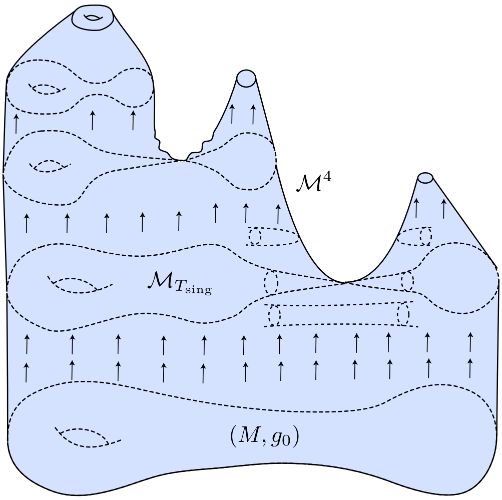

FIGURE 4
Illustration of a singular Ricci flow given by a Ricci flow spacetime. The arrows indicate the time-vector field $\partial_{\mathfrak{t}}$.

Definition 4.7. A *Ricci flow spacetime* consists of:

(1) a smooth 4-dimensional manifold $\mathcal{M}$ with boundary, called *spacetime;*

(2) a *time-function* $\mathfrak{t} : \mathcal{M} \to [0, \infty)$; its level sets $\mathcal{M}_t := \mathfrak{t}^{-1}(t)$ are called *time-slices* and we require that $\mathcal{M}_0 = \partial \mathcal{M}$;

(3) a *time-vector field* $\partial_{\mathfrak{t}}$ on $\mathcal{M}$ with $\partial_{\mathfrak{t}} \cdot \mathfrak{t} \equiv 1$; trajectories of $\partial_{\mathfrak{t}}$ are called *world-lines;*

(4) a family g of inner products on $\ker d\mathfrak{t} \subset T\mathcal{M}$, which induce a Riemannian metric g_t on each time-slice $\mathcal{M}_t$; we require that the Ricci flow equation holds:

$$\mathcal{L}_{\partial_{\mathfrak{t}}} g_t = -2 \operatorname{Ric}_{g_t} .$$

By abuse of notation, we will often write $\mathcal{M}$ instead of $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$.

A classical, 3-dimensional Ricci flow $(M, (g(t))_{t \in [0,T)})$ can be converted into a Ricci flow spacetime by setting $\mathcal{M} := M \times [0, T)$, letting $\mathfrak{t}, \partial_{\mathfrak{t}}$ be the projection onto the second factor and the pullback of the unit vector field on the second factor, respectively, and letting g_t be the metric corresponding to $g(t)$ on $M \times \{t\} \approx M$. Hence worldlines correspond to curves of the form $t \mapsto (x, t)$.

Likewise, a Ricci flow with surgery, given by flows

$$\big(M_1, \big(g_1(t)\big)_{t \in [0, T_1]}\big), \quad \big(M_2, \big(g_2(t)\big)_{t \in [T_1, T_2]}\big), \quad \ldots,$$

can be converted into a Ricci flow spacetime as follows. Consider first the Ricci flow spacetimes $M_1 \times [0, T_1], M_2 \times [T_1, T_2], \ldots$ arising from each single flow. We can now glue these flows together by identifying the set of points $U_i^- \subset M_i \times \{T_i\}$ and $U_i^+ \subset M_{i+1} \times \{T_i\}$ that survive each surgery step via maps $\phi_i : U_i^- \to U_i^+$. The resulting space has a boundary that consists of the time-0-slice $M_1 \times \{0\}$ and the points

$$\mathcal{S}_i = \big(M_i \times \{T_i\} \setminus U_i^-\big) \cup \big(M_{i+1} \times \{T_i\} \setminus U_i^+\big),$$

which were removed and added during each surgery step. After removing these points, we obtain a Ricci flow spacetime of the form:

$$\mathcal{M} = \big(M_1 \times [0, T_1] \cup_{\phi_1} M_2 \times [T_1, T_2] \cup_{\phi_2} \cdots\big) \setminus (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots). \tag{4.5}$$

Note that, for any regular time $t \in (T_{i-1}, T_i)$, the time-slice $\mathcal{M}_t$ is isometric to $(M_i, g_i(t))$. On the other hand, the time-slices $\mathcal{M}_{T_i}$ corresponding to surgery times are incomplete; they have cylindrical open ends of scale $\approx r_{\text{surg}}(T_i)$.

The following definition captures this incompleteness:

Definition 4.8. A Ricci flow spacetime is r*-complete,* for some $r \geq 0$, if the following holds. Consider a smooth path $\gamma : [0, s_0) \to \mathcal{M}$ with the property that

$$\inf_{s \in [0, s_0)} |\mathrm{Rm}|^{-1/2}\big(\gamma(s)\big) > r$$

and:

(1) $\gamma([0, l)) \subset \mathcal{M}_t$ is contained in a single time-slice and its length measured with respect to the metric g_t is finite, or

(2) γ is a worldline, i.e., a trajectory of $\pm\partial_t$.

Then the limit $\lim_{s \nearrow s_0} \gamma(s)$ exists.

So $\mathcal{M}$ being r-complete roughly means that it has only "holes" of scale $\lesssim r$. For example, the flow from (4.5) is $C \max_t r_{\text{surg}}(t)$-complete for some universal $C < \infty$.

In addition, Theorem 4.2 motivates the following definition:

Definition 4.9. A Ricci flow spacetime is said to satisfy the *ε-canonical neighborhood assumption at scales* (r_1, r_2) if for any point $x \in \mathcal{M}_t$ with $r := |\mathrm{Rm}|^{-1/2}(x) \in (r_1, r_2)$ the metric g_t restricted to the ball $B_{g_t}(x, \varepsilon^{-1} r)$ is ε-close, after rescaling by r^{-2}, to a time-slice of a κ-solution.

We can finally define singular Ricci flows (through singularities), as used in Theorem 4.6:

Definition 4.10. A *singular Ricci flow* is a Ricci flow spacetime $\mathcal{M}$ with the following two properties:

(1) It is 0-complete.

(2) For any $\varepsilon > 0$ and $T < \infty$, there is an $r(\varepsilon, T) > 0$ such that the flow $\mathcal{M}$ restricted to $[0, T)$ satisfies the ε-canonical neighborhood assumption at scales $(0, r)$.

See again Figure 4 for a depiction of a singular Ricci flow. The time-slices $\mathcal{M}_t$ for $t < T_{\text{sing}}$ develop a cylindrical region, which collapses to some sort of topological double cone singularity in $\mathcal{M}_{T_{\text{sing}}}$ at time T_{sing}. This singularity is immediately resolved and the flow is smooth for some $t > T_{\text{sing}}$.

Let us digest the definition of a singular Ricci flow a bit more. It is tempting to think of the time function $\mathfrak{t}$ as a Morse function and compare critical points with infinitesimal surgeries. However, this comparison is flawed: First, by definition $\mathfrak{t}$ cannot have critical points since $\partial_{\mathfrak{t}}\, \mathfrak{t} = 1$. In fact, a singular Ricci flow is a completely smooth object. The "singular points" of the flow are not part of $\mathcal{M}$, but can be obtained after metrically completing each time-slice by adding a discrete set of points. Second, it is currently unknown whether the set of singular times, i.e., the set of times whose time-slices are incomplete, is discrete. In addition, the approach taken in the definition of singular Ricci flows is different from that of weak solutions to the mean curvature flow. While the Brakke and level set flows characterize the flow equation at singular points via integral or barrier conditions, a singular Ricci flow only characterizes the flow on its regular part. In lieu of a weak formulation of the Ricci flow equation on the singular set, we have to impose the canonical neighborhood assumption, which serves as an asymptotic characterization near the incomplete ends.

Finally, let us briefly explain how singular Ricci flows are constructed and convey the meaning of Part (2) of Theorem 4.6. Fix an initial time-slice (M, g) and consider a sequence of Ricci flow spacetimes $\mathcal{M}^j$ that correspond to Ricci flows with surgery starting from (M, g), with surgery scale $\max_t r_{\text{surg},j}(t) \to 0$. It can be shown that these flows are $C \max_t r_{\text{surg},j}(t)$-complete and satisfy the ε-canonical neighborhood assumption at scales $(C_\varepsilon \max_t r_{\text{surg},j}(t), r_\varepsilon)$, where $C, C_\varepsilon, r_\varepsilon$ do not depend on j. A compactness theorem implies that a subsequence of the spacetimes $\mathcal{M}^j$ converges to a spacetime $\mathcal{M}$, which is a singular Ricci flow. This implies the existence of $\mathcal{M}$; the proof of uniqueness uses other techniques, which are outside the scope of this article.

4.6. Continuous dependence

The proof of the uniqueness property in Theorem 4.6, due to Kleiner and the author, implies an important continuity property, which leads to further topological applications. To state this property, let M be a compact 3-manifold and for every Riemannian metric g on M let $\mathcal{M}^g$ be the singular Ricci flow with initial condition $(\mathcal{M}^g_0, g) = (M, g)$.

Theorem 4.11 ([11]). *The flow $\mathcal{M}^g$ depends continuously on g.*

Together with Theorem 4.6, this implies that the Ricci flow equation in dimension 3 is well-posed within the class of singular Ricci flows.

Note that the topology of the flow $\mathcal{M}^g$ may change as we vary g and the continuity holds for the entire flows – past potential singularities. We therefore have to choose an appropriate sense of continuity in Theorem 4.11 that allows such a topological change. This is roughly done via a topology and lamination structure on the disjoint union $\bigsqcup_g \mathcal{M}^g$, transverse to which the variation of the flow can be studied locally.

Instead of elaborating on these technicalities, let us discuss the example illustrated in Figure 5. In this example $(g_s)_{s \in [0,1]}$ denotes a continuous family of metrics on S^3 such that the corresponding flows $\mathcal{M}^s := \mathcal{M}^{g_s}$ interpolate between a round and a cylindrical sin-

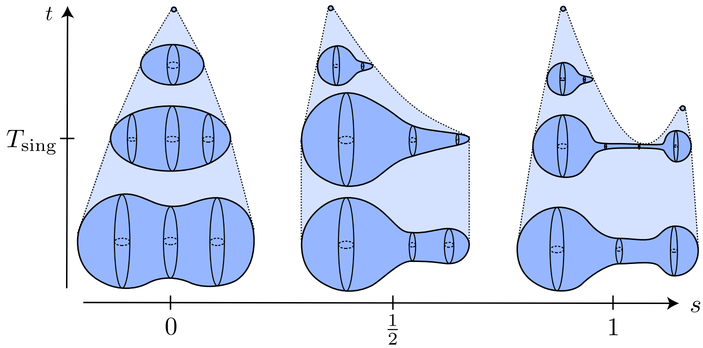

FIGURE 5
A family of singular Ricci flows starting from a continuous family of initial conditions.

gularity. For $s \in [0, \frac{1}{2})$, the flow $\mathcal{M}^s$ can be described in terms of a conventional, nonsingular Ricci flow (g_t^s) on M and the continuity statement in Theorem 4.11 is equivalent to continuous dependence of this flow on s. Likewise, the flows $\mathcal{M}^s$ restricted to $[0, T_{\text{sing}})$ can again be described by a continuous family of conventional Ricci flows. The question is now what happens at the critical parameter $s = \frac{1}{2}$, where the type of singularity changes. The uniqueness property guarantees that the flows $\mathcal{M}^s$ for $s \nearrow \frac{1}{2}$ and $s \searrow \frac{1}{2}$ must limit to the same flow $\mathcal{M}^{\frac{1}{2}}$. The convergence is locally smooth, but the topology of the spacetime manifold $\mathcal{M}^s$ may still change.

4.7. Topological applications

Theorem 4.11 provides us a tool to deduce the first topological applications of Ricci flow since Perelman's work. Before stating these, let us make the following definitions. We denote by $\text{Met}(M)$ the space of all Riemannian metrics on a manifold M, equipped with the C^∞-topology. Let $\text{Met}_{K\equiv k}(M), \text{Met}_{\text{PSC}}(M) \subset \text{Met}(M)$ be the subsets of metrics of constant sectional curvature k and of positive scalar curvature, respectively. Furthermore, we denote by $\text{Diff}(M)$ the group of diffeomorphisms $\phi : M \to M$, again equipped with the C^∞-topology, and for a Riemannian metric $g \in \text{Met}(M)$ we denote by $\text{Isom}(M, g) \subset \text{Diff}(M)$ the isometry group of (M, g).

Theorem 4.12 ([11]). *For any closed 3-manifold M, the space* $\text{Met}_{\text{PSC}}(M)$ *is either contractible or empty.*

Theorem 4.13 (Generalized Smale Conjecture, [9, 11]). *Suppose that (M^3, g) is a closed manifold of constant curvature $K \equiv \pm 1$. Then the inclusion map* $\text{Isom}(M, g) \hookrightarrow \text{Diff}(M)$ *is a homotopy equivalence.*

The study of the spaces $\text{Met}_{\text{PSC}}(M)$ was initiated by Hitchin in the 1970s and has led to many interesting results – based on index theory – which show that these spaces have nontrivial topology when M is high dimensional. Theorem 4.12 provides the first examples of manifolds of dimension ≥ 3 for which the homotopy type of $\mathrm{Met}_{\mathrm{PSC}}(M)$ is completely understood; see also prior work by Marques [42].

The Generalized Smale Conjecture has had a long history and many interesting special cases have been established using topological methods, including the case $M = S^3$ by Hatcher [34] and the hyperbolic case by Gabai [30]. However, the full conjecture remained open until recently. For more background, see the first chapter of [37]. The proof of Theorem 4.13 is independent of Hatcher's and Gabai's proof, so it provides an alternative approach to the S^3 and hyperbolic case. In addition, it provides a uniform treatment of all topological cases and the same method can also be used characterize the homotopy type of other prime 3-manifolds (see, for example, [11] for the case $S^2 \times S^1$). For many of these manifolds, this was already accomplished using topological methods; however, the following result is new:

Theorem 4.14 ([12]). *Let g be a compact, orientable, non-Haken 3-manifold modeled on the Thurston geometry* Nil *and let g be a* Nil*-metric on M. Then the inclusion* $\mathrm{Isom}(M, g) \hookrightarrow \mathrm{Diff}(M)$ *is a homotopy equivalence.*

Combining Theorems 4.13, 4.14 with the previously known characterization of $\mathrm{Diff}(M)$ in all other cases, this completes the project of understanding the topology of $\mathrm{Diff}(M)$ when M is a closed 3-manifold.

There are two proofs for Theorem 4.13: a short proof and a long proof. The short proof [9] requires the additional assumption that $M \not\approx \mathbb{R}P^3$ and relies on Hatcher's resolution of the Smale Conjecture. The long proof [11] establishes both Theorems 4.12, 4.13 in their full form.

Both proofs rely on two basic observations:

- The positive scalar curvature condition is preserved by the flow.
- Theorem 4.13 is equivalent to the contractibility of the space $\mathrm{Met}_{K\equiv\pm 1}(M)$ of constant curvature metrics. This can be seen via a standard argument involving the long-exact homotopy sequence for the fiber bundle $\mathrm{Isom}(M) \to \mathrm{Diff}(M) \to \mathrm{Met}_{K\equiv\pm 1}(M)$.

Let us simplify our setting for a moment and suppose that M was the 2-dimensional sphere. Then by Theorem 3.1, Ricci flow can be seen as a deformation retraction of $\mathrm{Met}(M)$ or $\mathrm{Met}_{\mathrm{PSC}}(M)$ to $\mathrm{Met}_{K\equiv\pm 1}(M)$ – modulo rescaling and reparameterization. This shows that the spaces $\mathrm{Met}(M)$, $\mathrm{Met}_{\mathrm{PSC}}(M)$ and $\mathrm{Met}_{K\equiv\pm 1}(M)$ are homotopy equivalent, and since the first space is contractible (it is a convex subset of a vector space), we obtain that all spaces are contractible.

Unfortunately, the strategy in the 2-dimensional case does not readily generalize to dimension 3, because singular flows cannot be viewed as trajectories in $\mathrm{Met}(M)$ as they are defined by metrics on different time-slices – possibly of different topology. Therefore, the proofs of Theorems 4.12, 4.13 have to follow a different strategy, which we will outline now. To this end we first observe that, since $\mathrm{Met}(M)$ is contractible, it is enough to show that

$$\pi_k(\mathrm{Met}(M), \mathrm{Met}_X(M)) \text{ is trivial, where } X \text{ may stand for “PSC” or “} K \equiv \pm 1 \text{.”}$$

Let us now fix a representative $g : D^{k+1} \to \mathrm{Met}(M)$ of this relative homotopy group, i.e., $g(s) \in \mathrm{Met}_X(M)$ for all $s \in \partial D^{k+1}$. Our goal will be to construct a null-homotopy $\hat{g} : D^{k+1} \times [0,1] \to \mathrm{Met}(M)$, where $\hat{g}(\cdot, 0) = g$ and $\hat{g}(s,t) \in \mathrm{Met}_X(M)$ if $s \in \partial D^{k+1}$ or $t = 1$.

If all the Ricci flows starting from each metric $g(s)$ were to converge to a round metric (modulo rescaling), then $\hat{g}(s, \cdot)$ could simply be constructed using these flows (as we did in dimension 2). In general, however, the family g only induces a continuous family of singular Ricci flows $(\mathcal{M}^s := \mathcal{M}^{g(s)})_{s \in D^{k+1}}$. In a second step, this family of flows has to be converted to the desired null-homotopy $\hat{g}$ within $\mathrm{Met}(M)$. In the long proof, this is achieved via a new notion called *partial homotopy*. This notion is a hybrid between a null-homotopy in $\mathrm{Met}(M)$ and a continuous family of Ricci flows, which permits variation of underlying topology. A partial homotopy allows the construction of a null-homotopy via backwards induction in time via certain modification moves that roughly correspond to the singularities of the flows $\mathcal{M}^s$. The short proof, on the other hand, uses the flow of the time-vector field $\partial_{\mathfrak{t}}$ on each $\mathcal{M}^s$ to push forward the metrics g^s_t to its initial time-slice $\mathcal{M}^s_0 = M$. This flow is not defined everywhere and thus such a construction only offers a continuous family of metrics $\tilde{g}^s$ defined on open subsets $U^s \subset M$, where $M \setminus U^s$ can be covered by pairwise disjoint 3-disks. These metrics then have to be extended onto all of M via an obstruction theoretic argument, which relies on Hatcher's resultion of the S^3-case.

5. DIMENSIONS $n \geq 4$

For a long time, most of the known results of Ricci flows in higher dimensions concerned special cases, such as Kähler–Ricci flows or flows that satisfy certain preserved curvature conditions. *General* flows, on the other hand, were relatively poorly understood. Recently, however, there has been some movement on this topic – in part, thanks to a different geometric perspective on Ricci flows [5–7]. The goal of this section is to convey some of these new ideas and to provide an outlook on possible geometric and topological applications.

5.1. Gradient shrinking solitons

Gradient shrinking solitons (GSSs) comprise an important class of singularity models in Ricci flow, especially in higher dimensions. The GSS equation concerns Riemannian manifolds (M, g) equipped with a potential function $f \in C^\infty(M)$ and reads

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{2} g = 0.$$

This generalization of the Einstein equation gives rise to an associated self-similar Ricci flow

$$g(t) := |t| \phi_t^* g, \quad t < 0,$$

where $(\phi_t : M \to M)_{t<0}$ is the flow of the vector field $|t| \nabla f$.

A basic class of examples for GSSs are the round cylinders $S^{k \geq 2} \times \mathbb{R}^{n-k}$, where

$$g = 2(k-1) g_{S^k} + g_{\mathbb{R}^{n-k}}, \quad f = \frac{1}{4} \sum_{i=k+1}^{n} x_i^2.$$

In this case, $|t|\nabla f$ generates a family of dilations on the $\mathbb{R}^{n-k}$ factor and

$$g(t) = 2(k-1)|t| g_{S^k} + g_{\mathbb{R}^{n-k}},$$

which is isometric to $|t|g$. A special case of this is the round shrinking sphere ($k = n$). In dimensions $n \leq 3$, all nontrivial[7] GSSs are quotients of round spheres or cylinders. However, more complicated GSSs exist in dimensions $n \geq 4$ (see, for example, [29]).

By construction, GSSs (or their associated flows, to be precise) are invariant under parabolic rescaling. So the blow-up singularity model of the singularity at time 0 (taken along an appropriately chosen sequence of basepoints) is equal to the flow itself. Therefore every GSS does indeed occur as a singularity model, at least of its own flow.

Vice versa, the following conjecture, which will be kept vague for now, predicts that the converse should also be true in a certain sense.

Conjecture 5.1. *For any Ricci flow, "most" singularity models are gradient shrinking solitons.*

This conjecture has been implicit in Hamilton's work from the 1990s, and a similar result is known to be true for mean curvature flow. In the remainder of this section, we will present a resolution of a version of this conjecture.

5.2. Examples of singularity formation

Let us first discuss an example in order to adjust our expectations in regards to Conjecture 5.1. In [4], Appleton constructs a class of 4-dimensional Ricci flows[8] that develop a singularity in finite time, which can be studied via the blow-up technique from Section 4.2 – this time we even allow the rescaling factors to be *any* sequence of numbers $\lambda_i \to \infty$, not just $\lambda_i = |\mathrm{Rm}|^{1/2}(x_i, t_i)$. Appleton obtains the following classification of all nontrivial blow-up singularity models:

(1) the Eguchi–Hanson metric, which is Ricci flat and asymptotic to the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$,

(2) the flat orbifold $\mathbb{R}^4/\mathbb{Z}_2$,

(3) the quotient $M_{\mathrm{Bry}}/\mathbb{Z}_2$ of the Bryant soliton, which has an isolated orbifold singularity at its tip,

(4) the cylinder $\mathbb{R}P^3 \times \mathbb{R}$.

Here the models (1) and (2) *have* to occur as singularity models, and it is unknown whether the models (3) and (4) actually do show up. The only GSSs in this list are (2) and (4). Note that the flow on $\mathbb{R}^4/\mathbb{Z}_2$ is constant, but each time-slice is a metric cone, and therefore invariant

7 Euclidean space $\mathbb{R}^n$ equipped with $f = \frac{1}{4}r^2$ is called a trivial GSS.

8 The flows are defined on noncompact manifolds, but the geometry at infinity is well controlled.

under rescaling. So we may also view this model as a (degenerate) gradient shrinking soliton (in this case $f = \frac{1}{4}r^2$).

It is conceivable that there are Ricci flow singularities whose only blow-up models are of type (1) or (2). In addition, there are further examples in higher dimensions [49] whose only blow-up models that are GSSs must be singular and possibly degenerate. This motivates the following revision of Conjecture 5.1.

Conjecture 5.2. *For any Ricci flow, "most" singularity models are gradient shrinking solitons. These may be degenerate and may have a singular set of codimension* ≥ 4.

5.3. A compactness and partial regularity theory for Ricci flows

The previous example suggests that in higher dimensions we may need to consider *nonsmooth* blow-up limits. The usual convergence and compactness theory of Ricci flows due to Hamilton (see Section 4.2) is too restrictive for such purposes, as it relies on curvature bounds and only produces smooth limits. Instead, we need a fundamentally new *compactness and partial regularity theory* for Ricci flows, which will enable us to take limits of arbitrary Ricci flows and study their structural properties. This theory was recently found by the author [5–7] and will lie at the heart of a resolution of Conjecture 5.2.

An important related compactness and partial regularity theory is that for Einstein metrics due to Cheeger, Colding, Gromov, Naber, and Tian [17–23, 27]. This theory roughly states that any noncollapsed sequence of pointed Einstein metrics subsequentially converges in the pointed Gromov–Hausdorff sense to a metric space whose singular set has Minkowski dimension $\leq n-4$. Similar theories also exist for other geometric equations (e.g., minimal surfaces, harmonic maps, mean curvature flow). What these theories have in common is that their proofs all rely on only a few basic ingredients (e.g., a monotonicity formula, an almost cone rigidity theorem, and an ε-regularity theorem), which can be verified in each setting. A similar theory for Ricci flows, however, is more complicated, mainly due to two reasons:

- The basic ingredients mentioned above are – at least *a priori* – not available for Ricci flows. This necessitates a different approach for proving partial regularity.
- Parabolic versions of notions like "metric space", "Gromov–Hausdorff convergence", etc., did not exist until recently. So these – and a theory surrounding them – first had to be developed.

Let us now state the main compactness and partial regularity results for Ricci flows. We will remain somewhat vague on the new terminologies for now and defer a more detailed discussion to Section 5.5. Consider a sequence of pointed, n-dimensional Ricci flows

$$\big(M_i, (g_i(t))_{t\in(-T_i,0]}, (x_i,0)\big),$$

where we imagine the basepoints $(x_i, 0)$ to live in the final time-slices, and suppose that $T_\infty := \lim_{i\to\infty} T_i > 0$. Then we have:

Theorem 5.3. *After passing to a subsequence, these flows $\mathbb{F}$-converge to a pointed metric flow*

$$\big(M_i, (g_i(t)), (x_i, 0)\big) \xrightarrow[i\to\infty]{\mathbb{F}} \big(\mathcal{X}, (\nu_{x_\infty;t})\big).$$

Here the terms "metric flow" and "$\mathbb{F}$-convergence" can be thought as a parabolic versions of "metric space" and "Gromov–Hausdorff convergence," respectively.

Next, we impose the following noncollapsing condition:

$$\mathcal{N}_{x_i,0}\big(r_*^2\big) \geq -Y_*. \tag{5.1}$$

Here $\mathcal{N}_{x,t}(r^2)$ is the pointed Nash-entropy, which is a natural quantity in Ricci flow and related to Perelman's $\mathcal{W}$-functional and rediscovered by work of Hein and Naber [36]. It and can be thought of as the parabolic analogue of the normalized volume of a ball.

Theorem 5.4. *Assuming* (5.1)*, we have a regular–singular decomposition*

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S}$$

such that:

(1) *The flow on $\mathcal{R}$ can be described by a smooth Ricci flow spacetime structure (see Definition* 4.7*). The entire flow $\mathcal{X}$ is uniquely determined by this structure.*

(2) *We have the following dimensional estimate on the singular set:*

$$\dim_{\mathcal{M}^*} \mathcal{S} \leq (n+2) - 4.$$

(3) *Tangent flows (i.e., blow-ups based at a fixed point of $\mathcal{X}$) are (possibly singular) gradient shrinking solitons.*

(4) *There is a filtration $\mathcal{S}^0 \subset \cdots \subset \mathcal{S}^{n-2} = \mathcal{S}$ such that $\dim_{\mathcal{H}^*} \mathcal{S}^k \leq k$ and such that every $x \in \mathcal{S}^k \setminus \mathcal{S}^{k-1}$ has a tangent cone that either splits off an $\mathbb{R}^k$-factor or it splits off an $\mathbb{R}^{k-2}$-factor and is static.*

Let us make a few remarks. First, note that the fact that $\mathcal{X}$ is uniquely determined by the smooth Ricci flow spacetime structure on $\mathcal{R}$ is comparable to what we have observed in dimension 3 (see Section 4.5), where we did not even *consider* the entire flow $\mathcal{X}$.

Second, property (2) involves a parabolic version of the Minkowski dimension that is natural for Ricci flows; a precise definition would be beyond the scope of this article. Note that the time direction accounts for 2 dimensions, which is natural. In dimension 3, this implies that the set of singular times has dimension $\leq \frac{1}{2}$; this what was previously known in this dimension [40]. In Appleton's 4-dimensional example, the singular set $\mathcal{S}$ may consist of an isolated orbifold point in every time-slice; so its parabolic dimension is $2 = (4+2) - 4$. On the other hand, a flow on $S^2 \times T^2$ develops a singularity at a single time and collapses to the 2-torus T^2, which again has parabolic dimension 2. This shows that the dimensional bounds in Theorem 5.4 are optimal.

Lastly, the "tangent flows" in property (3) can be viewed as parabolic versions of "tangent cones," as both are invariant under rescaling.

5.4. Applications

Theorems 5.3 and 5.4 enable us to study the finite-time singularity formation and long-time behavior of Ricci flows in higher dimensions.

Regarding Conjecture 5.2, we roughly obtain:

Theorem 5.5. *Suppose that $(M, (g(t))_{t\in[0,T)})$ develops a singularity at time $T < \infty$. Then we can extend this flow by a "singular time-T-slice" (M_T, d_T) such that the tangent flows at any $(x, T) \in M_T$ are (possibly singular) gradient shrinking solitons.*

Regarding the long-time asymptotics, we obtain the following picture, which closely resembles that in dimension 3; compare with (4.4) in Section 4.4:

Theorem 5.6. *Suppose that $(M, (g(t))_{t\geq 0})$ is immortal. Then for $t \gg 1$ we have a thick–thin decomposition*

$$M = M_{\text{thick}}(t) \,\dot\cup\, M_{\text{thin}}(t)$$

such that the flow on $M_{\text{thick}}(t)$ converges, after rescaling, to a singular Einstein metric ($\mathrm{Ric}_{g_\infty} = -g_\infty$) and the flow on $M_{\text{thin}}(t)$ is collapsed in the opposite sense of (5.1).

In dimension 3, these theorems essentially recover Perelman's results; so they can be seen as generalizations to higher dimensions.

5.5. Metric flows

The definition of a metric flow and associated concepts require a new perspective on the geometry of Ricci flows. In the following we will briefly convey some of the rough ideas behind this perspective.

Let us first imitate the process of passing from a (smooth) Riemannian manifold (M, g) to its metric length space (M, d_g). So our goal will be to turn a Ricci flow $(M, (g(t))_{t\in I})$ into a synthetic object, which we call "metric flow." To do this, we consider the spacetime $\mathcal{X} := X \times I$ and the time-slices $\mathcal{X}_t := X \times \{t\}$ equipped with the length metrics $d_t := d_{g(t)}$. It may be tempting to retain the product structure $X \times I$ on $\mathcal{X}$, i.e., to record the set of worldlines $t \mapsto (x, t)$. However, this turns out to be unnatural. Instead, we will view the time-slices $(\mathcal{X}_t, d_t)$ as separate metric spaces whose points may not even be in 1–1 correspondence to some given space X.

It remains to record some relation between these metric spaces $(\mathcal{X}_t, d_t)$. This will be done via the conjugate heat kernel $K(x, t; y, s)$ – an important object in the study of Ricci flows. For fixed $(x, t) \in M \times I$ and $s < t$, this kernel satisfies the backwards conjugate[9] heat equation on a Ricci flow background,

$$(-\partial_s - \triangle_{g(s)} + R_{g(s)})K(x, t; \cdot, s) = 0, \tag{5.2}$$

9 Equation (5.2) is the L^2-conjugate of the standard (forward) heat equation and $K(\cdot, \cdot; y, t)$ is a heat kernel centered at (y, t).

centered at (x,t). This kernel has the property that, for any (x,t) and $s<t$,

$$\int_M K(x,t;\cdot,s)dg(s)=1,$$

which motivates the definition of the following probability measures:

$$d\nu_{(x,t);s}:=K(x,t;\cdot,s)dg(s),\quad \nu_{(x,t);t}=\delta_x.$$

This is the additional information that we will record. So we define:

Definition 5.7. A *metric flow* is (essentially[10]) given by a pair

$$\big((\mathcal{X}_t,d_t)_{t\in I},(\nu_{x;s})_{x\in\mathcal{X}_t,s<t,s\in I}\big)$$

consisting of a family of metric spaces $(\mathcal{X}_t,d_t)$ and probability measures $\nu_{x;s}$ on $\mathcal{X}_s$, which satisfy certain (basic) compatibility relations.

So given points $x\in\mathcal{X}_t$, $y\in\mathcal{X}_s$ at two times $s<t$, it is not possible to say whether "y corresponds to x." Instead, we only know that "y belongs to the past of x with a probability density of $d\nu_{x;s}(y)$." This definition is surprisingly fruitful. For example, it is possible to use the measures $\nu_{x;s}$ to define a natural topology on $\mathcal{X}$ and to understand when and in what sense the geometry of time-slices $\mathcal{X}_t$ depends continuously on t.

The concept of metric flows also allows the definition of a natural notion of geometric convergence – $\mathbb{F}$-convergence – which is similar to Gromov–Hausdorff convergence. Even better, this notion can be phrased on terms of a certain $d_{\mathbb{F}}$-distance, which is similar to the Gromov–Hausdorff distance, and the Compactness Theorem 5.3 can be expressed as a statement on the compactness of a certain subset of metric flow (pairs),[11] similar to the definition of Gromov–Hausdorff compactness.

Lastly, we sketch an example that illustrates why it was so important that we have divorced ourselves from the concept of worldlines. Consider the Bryant soliton $(M_{\mathrm{Bry}}, (g_{\mathrm{Bry}}(t))_{t\le 0})$ (see Figure 6). Recall that every time-slice $(M_{\mathrm{Bry}}, g_{\mathrm{Bry}}(t))$ is isometric to the same rotationally symmetric model with center x_{Bry}. By Theorems 5.3 and 5.4, any pointed sequence of blow-*downs* $(\lambda_i\to 0)$,

$$\big(M_{\mathrm{Bry}},\big(\lambda_i^2 g_{\mathrm{Bry}}\big(\lambda_i^{-2}t\big)\big)_{t\le 0},(x_{\mathrm{Bry}},0)\big),$$

$\mathbb{F}$-converges to a pointed metric flow $\mathcal{X}$ that is regular on a large set. What is this $\mathbb{F}$-limit $\mathcal{X}$? For any fixed time $t<0$, the sequence of pointed Riemannian manifolds $(M_{\mathrm{Bry}}, \lambda_i^2 g_{\mathrm{Bry}}(\lambda_i^{-2}t), x_{\mathrm{Bry}})$ converges to a pointed ray of the form $([0,\infty),0)$. This seems to contradict Theorem 5.4. However, here we have implicitly used the concept of worldlines, because we have used the point (x_{Bry},t) corresponding to the "official" basepoint $(x_{\mathrm{Bry}},0)$ at time t. Instead, we have to focus on the "past" of $(x_{\mathrm{Bry}},0)$, i.e., the region of $(M_{\mathrm{Bry}}, \lambda_i^2 g_{\mathrm{Bry}}(\lambda_i^{-2}t))$ where the conjugate heat kernel $\nu_{(x_{\mathrm{Bry}},0);\lambda_i^{-2}t}$ is concentrated. This region is cylindrical of

10 This is a simplified definition.

11 Strictly speaking, $\mathbb{F}$-convergence and $d_{\mathbb{F}}$-distance concern metric flow *pairs*, $(\mathcal{X},(\nu_{x;t}))$, where the second entry serves as some kind of substitute of a basedpoint.

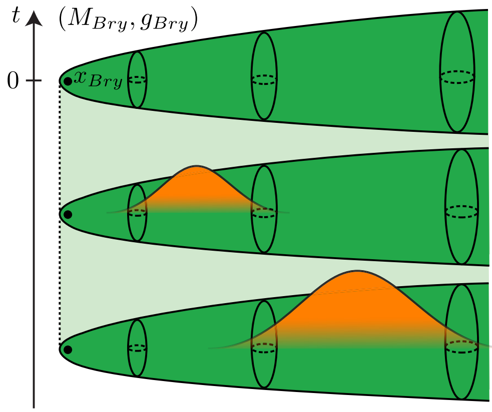

FIGURE 6
The Bryant soliton (green) and a conjugate (backward) heat kernel (orange) starting at the point x_{Bry} at time 0.

scale $\sim \sqrt{|t|}$, because the conjugate heat kernel "drifts away from the tip" at an approximate linear rate. In fact, one can show that the blow-down limit $\mathcal{X}$ is isometric to a round shrinking cylinder that develops a singularity at time 0. While this may seem slightly less intuitive at first, it turns out to be a much more natural way of looking at it.

5.6. Outlook

Our new theory of higher-dimensional Ricci flows demonstrates that, at least on an analytical level, Ricci flows behave similarly in higher dimension as they do in dimension 3. However, while there are only a handful of possible singularity models in dimension 3, gaining a full understanding of all such models in higher dimensions (e.g., classifying gradient shrinking solitons) may be impossible. Some past work in dimension 4 (e.g., by Munteanu and Wang [43–45]) has demonstrated that most *noncompact* gradient shrinking solitons have ends that are either cylindrical or conical. This motivates the following conjecture:

Conjecture 5.8. *Given a closed Riemannian 4-manifold (M, g), there is a "Ricci flow through singularities" in which topological change occurs along cylinders or cones and in which time-slices are allowed to have isolated orbifold singularities.*

The term "Ricci flow through singularities" is still left somewhat vague. Most likely, it should denote an object that is similar to a metric flow and that has the same partial regularity properties as described in Theorem 5.4, but with the exception that time-slices may consist of several components (i.e., we allow distances to be infinite). It may also be useful to require some sort of topological monotonicity property, meaning that the topology becomes "simpler" after the resolution of a singularity.

The existence of such a flow may have interesting consequences. For example, it may be used to decompose 4-manifolds with positive scalar curvature into certain building blocks. It may also offer an approach to proving the $\frac{11}{8}$-Conjecture. Note here that this conjecture holds for both important asymptotic models – gradient shrinking solitons and Einstein metrics – due to Lichnerowicz' Theorem and the Hitchin–Thorpe inequality. Lastly, there also seems to be potential applications in Kähler geometry, for example towards the Minimal Model Program and the Abundance Conjecture, assuming a similar flow could be constructed in higher dimensions.

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